Nonlinear Pricing with Arbitrage: On the Role of Correlation

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Abstract

In nonlinear pricing environment with correlated types, we characterize optimal selling mechanisms when buyers could form a coalition to coordinate their reports and to arbitrage on the goods. We find that when the types of agents are weakly positively correlated, the optimal weakly collusion-proof mechanism calls for distortions away from efficiency obtained without arbitrage; when the types are weakly negatively correlated, the monopolist can achieve the same profit regardless of whether or not buyers can arbitrage on their goods. Allowing arbitrage within coalitions result in right discontinuity between the correlated and uncorrelated environment, but the left continuity is still available.

Keywords: Nonlinear pricing, weakly collusion-proof, arbitrage, correlated types

Journal of Economic Literature Classification Number: D42, D62, D82

1 Introduction

The theory of nonlinear pricing studies a monopolist's optimal pricing scheme when she has incomplete information about buyers' individual preferences. It explains how a monopolist seller who offers a pricing contract to buyers should structure his contract to overcome, at least partially, the asymmetries of information he faces. The Revelation Principle tells us that, in the absence of restrictions on contracts, but with decentralized information, any organization

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is equivalent to a centralized organization in which information must be communicated in an incentive-compatible way to a center that transmits back to the agents instructions about the actions to be implemented. In accordance with this principle, the principal can maximize her profit by a incentive compatible and self-selected revelation mechanism. But the Revelation Principle relies for its validity on the absence of collusion among agents, that is, the agents behave in a non-cooperative way. [1] It is a rather unrealistic assumption except that the principal has a complete control over communication between the agents or if the agents' communication costs are formidable large. Agents often engage in communication at low cost and this might open room for collusive behavior to promote their joint interests at loss of that of the principal. The possibility that some agents may collude is an important concern in mechanism design theory. Typically, collusion imposes severe limits on what can be achieved. The possibility of collusion between agents is generally regarded as a factor that reduces the principal's payoff in addition to the mere asymmetric information. A notable example is that of auctions, for which the occurrence of cooperative behavior between bidders is a general and well-documented phenomenon: see Porter and Zona (1993,1999), Baldwin et al. (1997), Pesendorfer (2000), etc.

When the agents' types are correlated and there is no collusion between agents, Crémer and McLean [hereafter CM] (1985, 1988) show that the principal can obtain the complete-information optimum by fully extracting the information rents of agents. This Full Surplus Extraction (FSE) result hold for any degree of correlation, even if it is very close to zero. However, when it is zero, rents are given up by the principal, only the second-best allocation is achievable. Therefore, a notable discontinuity occurs at the point with zero correlation.

CM's FSE mechanism is vulnerable to collusion. If the agents may coordinate their reports, it is impossible for the principal to elicit truthtelling at no cost. Laffont and Mortimort [hereafter LM] (2000) restor the right continuity of mechanism at zero point by allowing collusion between agents. They offered a tractable modeling framework for analyzing the role of colluders' information asymmetry in collusion-proof mechanism design. An uninformed third party is assumed to plays the role of proposer and organizer of collusion between agents. He maximizes the sum of agents' utilities under the agents' incentive and participation constraints. The agents are offered a side contract [2] which entails manipulation of reports in the mechanism offered by the principal and balanced side payments between the agents. This side contract has to be such that

[1] Other assumptions underlying the Revelation Principle include: there is no communication cost, information processing costs and contract complexity cost; the principal has ability to commit upfront to a mechanism and will not renegotiate it later, etc.

[2] In this paper we use "contract" and "mechanism" interchangeably

the combination of the principal's mechanism and the third party's side payment is incentive compatible for all agents and offers them interim expected utilities larger than what they would get from playing the principal's mechanism non-cooperatively. The asymmetric information between agents imposes transaction costs on their ability to carry out collusive arrangements. In procurement and public good provision settings, LM (1997, 2000) obtain the following result: with independent types, collusion imposes no additional distortion on the second best allocation attained under asymmetric information. That is, optimal outcome can be made collusion-proof at no cost to the principal if the agent's types are uncorrelated. Moreover, under asymmetric information with positive correlated types, preventing collusion imposes strict cost to the mechanism designer. The greater the correlation of types, the greater the required distortion of the allocation. When the positive correlation goes to zero, the allocative efficiency implemented by a collusion-proof mechanism approaches the second-best one. Therefore, the collusion-proof mechanism is right continuous at zero point.

Yeon-Koo Che and Jinwoo Kim (2007) unifies several scattered observations made in the literature and provides a general insight into how the transaction cost associated with agents' private information can be exploited to overcome collusion. They show that agents' collusion can be prevented at no cost in a broad class of circumstances—virtually all circumstances with uncorrelated agents and a broad set of circumstances with correlated types which satisfies certain rank conditions. Their result is an extension of CM's FSE theorem and the weakly collusion-proof mechanism of LM (1997,2000). Notably, their result is robust to the formation of side mechanism and can even be extended to a setting in which only a subset of agents may collude.

In monopolist pricing environment, the buyers could form coalitions to conduct arbitrage, that is, to reallocate the goods they bought among themselves at the loss of the seller's profit. Thus, the mechanism designer should make optimal contractual response preventing the agents from conducting reallocation as well as reports manipulation and side transfers. LM's result are partially reproduced by Jeon and Menicucci [hereafter JM](2005) in a nonlinear pricing model. They modifies the side contract of LM (1997, 2000) by allowing collusive buyers to arbitrage on their purchase, that is the goods is allowed to be reallocated within the coalition. They show that with independently distributed types, the monopolist can achieve the same profit regardless of whether or not buyers can form a coalition. That is, with uncorrelated types, collusion with arbitrage is preventable at no cost, the buyers cannot gain additional profit from it.

Considering that JM's result crucially relies on the agents' types being uncorrelated, we extend the nonlinear pricing model to the case with correlated types in this paper. Its contributions are twofold. First, applying the methodology developed by LM (1997,2000), in a nonlinear pricing setting, we extend JM's result by showing that when the types of consumers are weakly correlated, the optimal weakly collusion-proof mechanism calls for distortions away from allocation efficiency obtained without collusion. Second, we modify the result of LM (2000) by showing that the reallocation-preventing mechanism is not right but left continuous at zero point. When the correlation goes to zero from right, the mechanism does not approach that of JM (2005); but when it goes to zero from left, JM (2005)'s result is approached. We show that it is the non-arbitrage-constraint that result in this discontinuity.

The rest of this paper is organized as follows. Section 2 introduces the model. Section3 reviews as a benchmark the optimal pricing mechanism without collusion. Section 4 characterize the coalition incentive compatible constraints which must be satisfied by a weakly collusion-proof mechanism. Section 5 describes the optimal weakly collusion-proof mechanism. Section 6 gives conclusions.

2 The model

2.1 Preferences, information, and mechanisms

A monopolist seller can produce any amount of goods at constant marginal cost c and sells the goods to two buyers whose consumptions are $q_i, i \in \{1,2\}$. Buyer i obtains utility $\theta_i V(q_i) - t_i$ from consuming q_i units of goods and paying t_i units of money to the seller. $V(\cdot)$ is a increasing concave function with $V'(\cdot) > 0, V''(\cdot) < 0$. In order to obtain the analytical solution, we assume in this paper that consumers have CRRA (constant-relative-risk-aversion) utility function: $V(q) = \frac{q^{1-\gamma}}{1-\gamma}$, with $\gamma \in [0,1)$.

The consumer privately observes his own type $\theta_i \in \Theta = \{\underline{\theta}, \overline{\theta}\}$, where $\Delta\theta \equiv \overline{\theta} - \underline{\theta}$. The probabilities $p(\theta_1,\theta_2)$ of each state (θ_1,θ_2) for $(\theta_1,\theta_2) \in \Theta^2$, are common knowledge prior beliefs. For simplicity, we write

$$p_{11} = p(\underline{\theta},\underline{\theta}), p_{12} = p(\underline{\theta},\overline{\theta}) = p(\overline{\theta},\underline{\theta}), p_{22} = p(\overline{\theta},\overline{\theta}).$$

We also denote by

$$\rho \equiv \det \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} = p_{11}p_{22} - p_{12}^2$$

the degree of correlation between the agents' types.

The monopolist seller designs a grand sale mechanism $\mathcal{M}$ to maximize her expected profit. Considering the Revelation Principle, we can restrict our attention to direct revelation mechanism which maps any pair of reported types $(\widehat{\theta}_1, \widehat{\theta}_2)$ into a combination of consumptions and payments:

$$\mathcal{M} = \left\{ q_1(\widehat{\theta}_1, \widehat{\theta}_2), q_2(\widehat{\theta}_1, \widehat{\theta}_2), t_1(\widehat{\theta}_1, \widehat{\theta}_2), t_2(\widehat{\theta}_1, \widehat{\theta}_2) \right\}, (\widehat{\theta}_1, \widehat{\theta}_2) \in \Theta^2$$

Since buyers are ex ante identical, without loss of generality, we focus on anonymous mechanism in which the consumption and payment of a buyer depend only on the reports and not on his identity. Then we can introduce the following simplified notations:

$$q_{11} = q_1(\underline{\theta}, \underline{\theta}) = q_2(\underline{\theta}, \underline{\theta}); t_{11} = t_1(\underline{\theta}, \underline{\theta}) = t_2(\underline{\theta}, \underline{\theta})$$

$$q_{12} = q_1(\underline{\theta}, \overline{\theta}) = q_2(\overline{\theta}, \underline{\theta}); t_{12} = t_1(\underline{\theta}, \overline{\theta}) = t_2(\overline{\theta}, \underline{\theta})$$

$$q_{21} = q_1(\overline{\theta}, \underline{\theta}) = q_2(\underline{\theta}, \overline{\theta}); t_{21} = t_1(\overline{\theta}, \underline{\theta}) = t_2(\underline{\theta}, \overline{\theta})$$

$$q_{22} = q_1(\overline{\theta}, \overline{\theta}) = q_2(\overline{\theta}, \overline{\theta}); t_{22} = t_1(\overline{\theta}, \overline{\theta}) = t_2(\overline{\theta}, \overline{\theta})$$

Let $\mathbb{q} = (q_{11}, q_{12}, q_{21}, q_{22}) \in \mathbb{R}^4_{++}$ and $\mathbb{t} = (t_{11}, t_{12}, t_{21}, t_{22}) \in \mathbb{R}^4$ denote the vector of quantities and transfers respectively.

2.2 Coalition formation

Applying the methodology of LM (1997, 2000), we model the buyers' coalition formation by a side contract, denoted by $\mathcal{S}$, offered by a benevolent uninformed third party. The third party organize the buyers into collusion in order to maximize the sum of their payoffs subject to incentive compatibility and participation constraints written with respect to the utility a buyer obtains when the grand mechanism $\mathcal{M}$ is played non-cooperatively. The timing of the overall game of contract offer cum coalition formation is as follows:

- Stage 1: Buyers learn their respective "types".
- Stage 2: The seller proposes a grand sale mechanism $\mathcal{M}$. If an buy vetoes the grand mechanism, all buyers get their reservation utility normalized exogenously at zero and the following stages do not occur.
- Stage 3: The third party proposes a side mechanism $\mathcal{S}$ to the buyers. If anyone refuses this side mechanism, $\mathcal{M}$ is played non-cooperatively. If both buyers accept $\mathcal{S}$, they report their types to the third party who enforce manipulation of report into $\mathcal{M}$ and commits to enforce the corresponding side transfers and reallocation within coalition.
- Stage 4: Reports are sent into the grand mechanism. Quantities and payments specified in $\mathcal{M}$ are enforced. Quantities reallocation and side transfers specified in $\mathcal{S}$, if any, are implemented.

Formally, a side mechanism $\mathcal{S}$ takes the following form:

$$\mathcal{S} = \left\{\phi(\widetilde{\theta_1}, \widetilde{\theta_2}), x_1(\widetilde{\theta_1}, \widetilde{\theta_2}, \phi), x_2(\widetilde{\theta_1}, \widetilde{\theta_2}, \phi), y_1(\widetilde{\theta_1}, \widetilde{\theta_2}), y_2(\widetilde{\theta_1}, \widetilde{\theta_2})\right\}, (\widetilde{\theta_1}, \widetilde{\theta_2}) \in \Theta^2.$$

$\widetilde{\theta_i}$ is buyer $i's$ report to the third party. $\phi(\cdot)$ is the manipulated report to the grand mechanism. $y_i(\widetilde{\theta_1}, \widetilde{\theta_2})$ denotes the monetary transfer from buyer i to the third party. $x_i(\widetilde{\theta_1}, \widetilde{\theta_2}, \phi)$ represents the quantity of goods buyer i received from the third party when ϕ is reported to the seller. Such a reallocation rule maximize the joint surplus of the buyers subject to the total amount of the goods being allocated to all consumers. Since the third party is not a source of goods or money, we assume that a side mechanism should satisfy the ex post budget-balance constraints for the reallocation of goods and for the side transfers, respectively $\sum_{i=1}^{2} y_i(\widetilde{\theta_1}, \widetilde{\theta_2}) = 0$, and $\sum_{i=1}^{2} x_i(\widetilde{\theta_1}, \widetilde{\theta_2}, \phi) = 0, \forall(\widetilde{\theta_1}, \widetilde{\theta_1}) \in \Theta^2, \forall \phi \in \Theta^2$. Let $U^{\mathcal{M}}(\theta_i)$ denote the expected payoff of a θ_i type in truthful equilibrium of $\mathcal{M}$. The side mechanism must guarantee to an agent an utility level greater than what he expects from playing non-cooperatively the grand mechanism and then getting a utility $U^{\mathcal{M}}(\theta_i)$.

3 The optimal grand-mechanism without buyer coalition

3.1 The case with correlated types

In this subsection, we study, as a benchmark, the optimal grand-mechanism without side-contracting with correlated types. The seller's expected profit is

$$\Pi(\mathbb{t}, \mathbb{q}) \equiv 2p_{11}\left(t_{11} - cq_{11}\right) + 2p_{12}\left(t_{12} + t_{21} - cq_{12} - cq_{21}\right) + 2p_{22}\left(t_{22} - cq_{22}\right).$$

The following Bayesian incentive-compatibility constraints should be satisfied. For a $\underline{\theta}$ type buyer

$$\begin{aligned} BIC(\underline{\theta}) :& p_{11}\left[\underline{\theta}V(q_{11}) - t_{11}\right] + p_{12}[\underline{\theta}V(q_{12}) - t_{12}] \\ \geq & p_{11}[\underline{\theta}V(q_{21}) - t_{21}] + p_{12}[\underline{\theta}V(q_{22}) - t_{22}]; \end{aligned} \tag{1}$$

for a $\overline{\theta}$ type buyer

$$\begin{aligned} BIC(\overline{\theta}) :& p_{12}[\overline{\theta}V(q_{21}) - t_{21}] + p_{22}[\overline{\theta}V(q_{22}) - t_{22}] \\ \geq & p_{12}[\overline{\theta}V(q_{11}) - t_{11}] + p_{22}[\overline{\theta}V(q_{12}) - t_{12}]. \end{aligned} \tag{2}$$

The mechanism should also satisfy the following individual-rationality constraints. For a $\underline{\theta}$ and $\overline{\theta}$ type,respectively,

$$BIR(\underline{\theta}) : p_{11}[\underline{\theta}V(q_{11}) - t_{11}] + p_{12}[\underline{\theta}V(q_{12}) - t_{12}] \geq 0 \tag{3}$$

$$BIR(\overline{\theta}) : p_{12}[\overline{\theta}V(q_{21}) - t_{21}] + p_{22}[\overline{\theta}V(q_{22}) - t_{22}] \geq 0 \tag{4}$$

The seller maximizes his expected profit $\Pi(\mathbb{t}, \mathbb{q})$ subject to constraints (1) to (4).

We look for the transfers such that the four constraints are all binding, i.e., which satisfy incentive compatibility without leaving any expected rent at the interim stage to any buyer.[3] Indeed, for $\rho \neq 0$, the equation system of (1) to (4) are invertible since the determinant is ρ^2. Thus, the transfers are determined uniquely.

$$t_{11} = \frac{\left(p_{11}p_{22}\underline{\theta} - p_{12}^2\overline{\theta}\right) V(q_{11}) - p_{12}p_{22}\Delta\theta V(q_{12})}{\rho} \tag{5}$$

$$t_{12} = \frac{\left(p_{11}p_{22}\overline{\theta} - p_{12}^2\underline{\theta}\right) V(q_{12}) + p_{11}p_{12}\Delta\theta V(q_{11})}{\rho} \tag{6}$$

$$t_{21} = \frac{\left(p_{11}p_{22}\underline{\theta} - p_{12}^2\overline{\theta}\right) V(q_{21}) - p_{12}p_{22}\Delta\theta V(q_{22})}{\rho} \tag{7}$$

$$t_{22} = \frac{\left(p_{11}p_{22}\overline{\theta} - p_{12}^2\underline{\theta}\right) V(q_{22}) + p_{11}p_{12}\Delta\theta V(q_{21})}{\rho} \tag{8}$$

Substituting these transfers into the the seller's expected welfare $\Pi(\mathbb{t}, \mathbb{q})$ then optimizing this expression yields the first-best consumptions: $\mathbb{q}^N(\rho) = \mathbb{q}^{FB} = (q_{11}^{FB}, q_{12}^{FB}, q_{21}^{FB}, q_{22}^{FB})$, where

$$q_{11}^{FB} = q_{12}^{FB} = \left(\frac{\underline{\theta}}{c}\right)^{\frac{1}{\gamma}} ; q_{21}^{FB} = q_{22}^{FB} = \left(\frac{\overline{\theta}}{c}\right)^{\frac{1}{\gamma}} \tag{9}$$

From the expressions of transfers (5) to (8), we observe that, if ρ is positive and goes to zero, then t_{11}, t_{21} go to $-\infty$ and t_{12}, t_{22} go to $+\infty$. The consumers' quasilinear utility function suggests that they are risk neutral on transfers. The selling mechanism exploits the risk neutrality of the agents by specifying extreme rewards and penalties. A $\overline{\theta}-$ agent faces, when he tell the truth, an extreme reward if the other agent is a $\underline{\theta}$ type and extreme penalties if the other agent is a $\overline{\theta}$ type. Similarly for $\underline{\theta}-$ agent. If the correlation ρ is negative, the sign of all the transfers will be altered, which implies the opposite directions of awards and penalties. Given such a mechanism, the buyers will always tell the truth. The weaker is the correlation, the larger penalties or awards are needed to elicit revelation. It may not be surprising that when buyers are very similar, it is relatively simple to extract their rents by "cross-checking" . The more interesting point is that the above first-best result hold for any degree of correlation of types, even if it is infinitesimal.

The above analysis is in line with the well known Full Surplus Extraction theorem that optimal mechanism achieves the first-best outcome in the correlated environment when the implementation concept is Bayesian-Nash equilibrium, even with interim individual rationality constraints (CM (1985, 1988)).

[3] CM (1988) show that incentive constraints can be slack.

3.2 The case with independent types

For the independent case, we define $v = \Pr(\theta_i = \underline{\theta}), i = 1, 2$. Then

$$p_{11} = v^2, p_{12} = p_{21} = v(1-v), p_{22} = (1-v)^2.$$

The principal's objective function is:

$$\Pi(\mathbb{t}, \mathbb{q}) = 2v^2 (t_{11} - cq_{11}) + 2v(1-v)(t_{12} + t_{21} - cq_{12} - cq_{21}) + 2(1-v)^2 (t_{22} - cq_{22})$$

The constraints (1) to (4) can be rewritten as follows:

$$\begin{aligned} BIC(\underline{\theta}) :& v[\underline{\theta}V(q_{11}) - t_{11}] + (1-v)[\underline{\theta}V(q_{12}) - t_{12}] \\ \geq & v[\underline{\theta}V(q_{21}) - t_{21}] + (1-v)[\underline{\theta}V(q_{22}) - t_{22}] && (10) \\ BIC(\overline{\theta}) :& v[\overline{\theta}V(q_{21}) - t_{21}] + (1-v)[\overline{\theta}V(q_{22}) - t_{22}] \\ \geq & v[\overline{\theta}V(q_{11}) - t_{11}] + (1-v)[\overline{\theta}V(q_{12}) - t_{12}] && (11) \\ BIR(\underline{\theta}) :& v[\underline{\theta}V(q_{11}) - t_{11}] + (1-v)[\underline{\theta}V(q_{12}) - t_{12}] \geq 0 && (12) \\ BIR(\overline{\theta}) :& v[\overline{\theta}V(q_{21}) - t_{21}] + (1-v)[\overline{\theta}V(q_{22}) - t_{22}] \geq 0 && (13) \end{aligned}$$

With $\rho = 0$, the system of binding constraints can no more be inverted. The standard method for solving single-agent adverse selection model shows that $BIC(\overline{\theta})$ and $BIR(\underline{\theta})$ bind in the optimum. When the transfers in $\Pi(\mathbb{t}, \mathbb{q})$ are replaced with those obtained from $BIC(\overline{\theta})$ and $BIR(\underline{\theta})$ written with equality, the solution of the principal's program are characterized as $\mathbb{q}^N(0) = \mathbb{q}^{SB} = (q_{11}^{SB}, q_{12}^{SB}, q_{21}^{SB}, q_{22}^{SB})$, where

$$q_{22}^{SB} = q_{21}^{SB} = \left(\frac{\overline{\theta}}{c}\right)^{\frac{1}{\gamma}}; q_{11}^{SB} = q_{12}^{SB} = \left(\frac{\underline{\theta} - \frac{1-v}{v}\Delta\theta}{c}\right)^{\frac{1}{\gamma}}. \qquad (14)$$

From the above two results we can see that a striking discontinuity occurs at $\rho = 0$, since $\lim_{\rho \to 0} \mathcal{M}^N(\rho) \neq \mathcal{M}^N(0)$[4]. Indeed, for correlated types the seller can exploit yardstick competition between buyers to induce their revelation at no cost, while for uncorrelated types, he cannot do that since the report of one consumer is uninformative signal for the other consumer's type. The complete information optimum is thus not achievable when $\rho = 0$. Seller should give up information rents to the $\overline{\theta}$ buyer and, to decrease those rents, distort the quantities of the $\underline{\theta}$ buyer downward.

[4] $\mathcal{M}^N(\rho)$ and $\mathcal{M}^N(0)$ denote the non-collusion mechanism with correlated and independent types respectively.

4 Coalition formation under asymmetric information

The above analysis shows that the agents gets zero rent from playing non-cooperatively the grand mechanism proposed by the buyer with correlated types. So the optimal grand mechanism with a noncooperative behavior creates endogenously the stakes for collusive behavior. In this section, we study formally the third party's optimizing problem and derive the coalition incentive constraints which must be satisfied in the optimal collusion-proof grand mechanism under asymmetric information.

Definition 1 *a A side mechanism*

$$\mathcal{S} = \left\{ \phi(\widetilde{\theta}_1, \widetilde{\theta}_2), x_1(\widetilde{\theta}_1, \widetilde{\theta}_2, \phi), x_2(\widetilde{\theta}_1, \widetilde{\theta}_2, \phi), y_1(\widetilde{\theta}_1, \widetilde{\theta}_2), y_2(\widetilde{\theta}_1, \widetilde{\theta}_2) \right\} \forall (\widetilde{\theta}_1, \widetilde{\theta}_2) \in \Theta^2$$

is coalition-interim-efficient with respect to an incentive-compatible grand mechanism $\mathcal{M}$ providing the reservation utilities $\left\{U^{\mathcal{M}}(\underline{\theta}), U^{\mathcal{M}}(\overline{\theta})\right\}$ if and only if it solves the following program:

$$[PT] : \underset{\phi(\cdot), x_i(\cdot), y_i(\cdot)}{Max} \sum_{(\theta_1, \theta_2) \in \Theta^2} p(\theta_1, \theta_2) \left[U^1(\theta_1) + U^2(\theta_2) \right]$$

subject to :

$$U^i(\theta_i) = \sum_{\theta_j \in \Theta} p(\theta_j|\theta_i) \left[\theta_i V \left(x_i\big(\theta_i, \theta_j, \phi(\theta_i, \theta_j)\big) + q_i\big(\phi(\theta_i, \theta_j)\big) \right) + y_i(\theta_i, \theta_j) - t_i\big(\phi(\theta_i, \theta_j)\big) \right]$$

for any $\theta_i \in \Theta$ and $i, j = 1, 2$ with $i \neq j$;

$$\left(BIC_i^{\mathcal{S}}\right) : U^i(\theta_i) \geq U^i(\widetilde{\theta}_i \mid \theta_i)$$

where

$$U^i(\widetilde{\theta}_i \mid \theta_i) = \sum_{\theta_j \in \Theta} p(\theta_j|\theta_i) \left[\theta_i V \left(x_i\big(\widetilde{\theta}_i, \theta_j, \phi(\widetilde{\theta}_i, \theta_j)\big) + q_i\big(\phi(\widetilde{\theta}_i, \theta_j)\big) \right) + y_i(\widetilde{\theta}_i, \theta_j) - t_i\big(\phi(\widetilde{\theta}_i, \theta_j)\big) \right]$$

for any $\left(\theta_i, \widetilde{\theta}_i\right) \in \Theta^2$ and $i, j = 1, 2$ with $i \neq j$;

$$\left(BIR_i^{\mathcal{S}}\right) : U^i(\theta_i) \geq U^{\mathcal{M}}(\theta_i)$$

for any $\theta_i \in \Theta$ and $i = 1, 2$;

$$(BB : y) : \sum_{i=1}^{2} y_i(\theta_1, \theta_2) = 0$$

$$(BB : x) : \sum_{i=1}^{2} x_i(\theta_1, \theta_2, \widetilde{\phi}) = 0$$

for any $(\theta_1, \theta_2) \in \Theta^2$ and $\widetilde{\phi} \in \Theta^2$.

Let

$$\mathcal{S}^0 \equiv \left\{ \phi(\cdot) = Id(\cdot), x_1(\cdot) = x_2(\cdot) = 0, y_1(\cdot) = y_2(\cdot) = 0 \right\}$$

denote the null contract that implements no manipulation of reports, no reallocation of quantities, and no side transfers. Hence, $\mathcal{M}$ is not affected by buyer coalition if the third-party proposes $\mathcal{S}^0$.

Definition 2 *An incentive-compatible grand mechanism $\mathcal{M}$ is weakly collusion-proof if and only it is a truthtelling direct mechanism and the null side mechanism $\mathcal{S}^0$ is coalition-interim-efficient with respect to $\mathcal{M}$.*

In other words, a truthtelling direct mechanism $\mathcal{M}$ is weakly collusion-proof if and only if the third party's best response to it is to do nothing but executing it truthfully in a noncooperative way.

Proposition 1 *(Weakly Collusion-Proofness Principle, WCPC). There is no loss of generality in restricting the seller to offer weakly collusion-proof mechanisms in order to characterize the outcome of any perfect Bayesian equilibrium of the game of sellers mechanism offer cum coalition formation such that a collusive equilibrium occurs on the equilibrium path.*

Proof. The proof is omitted since it is straightforward adaptation of proposition 3 of LM (2000). ■

The intuition behind WCPC is similar to that underlying the standard Revelation Principle: since the uninformed third party has no informational or instrumental advantage over the seller, any equilibrium of the overall game of grand mechanism offer cum side contracting gives an allocation that can be replicated with a direct grand mechanism offered by the seller himself.

The next proposition characterize the coalition incentive constraints which must be satisfied by the weakly collusion-proof grand mechanism.

Proposition 2 *A symmetric Bayesian incentive compatible grand mechanism $\mathcal{M}$ such that the $\underline{\theta}-$ type's incentive constraints is not binding is weakly collusion-proof if and only if there exist $\epsilon \in [0, 1)$ such that:*

- *The following coalition incentive constraints are satisfied:*

for a $(\underline{\theta},\underline{\theta})$ *coalition,*

$$CIC(\underline{\theta},\underline{\theta};\underline{\theta},\overline{\theta}): \quad 2\left(\underline{\theta}-\frac{p_{12}^2\epsilon\Delta\theta}{p_{11}p_{12}+\rho\epsilon}\right)V(q_{11})-2t_{11} \geq 2\left(\underline{\theta}-\frac{p_{12}^2\epsilon\Delta\theta}{p_{11}p_{12}+\rho\epsilon}\right)V\Big(\frac{q_{12}+q_{21}}{2}\Big)-t_{12}-t_{21} \tag{15}$$

$$CIC(\underline{\theta},\underline{\theta};\overline{\theta},\overline{\theta}): \quad 2\left(\underline{\theta}-\frac{p_{12}^2\epsilon\Delta\theta}{p_{11}p_{12}+\rho\epsilon}\right)V(q_{11})-2t_{11} \geq 2\left(\underline{\theta}-\frac{p_{12}^2\epsilon\Delta\theta}{p_{11}p_{12}+\rho\epsilon}\right)V(q_{22})-2t_{22} \tag{16}$$

for a $(\underline{\theta},\overline{\theta})$ *coalition,*

$$CIC(\underline{\theta},\overline{\theta};\underline{\theta},\underline{\theta}): \left(\underline{\theta}-\frac{p_{22}\epsilon\Delta\theta}{p_{12}}\right)V\big(\lambda^\epsilon(q_{12}+q_{21})\big)+\overline{\theta}V\big((1-\lambda^\epsilon)(q_{12}+q_{21})\big)-t_{12}-t_{21} \geq \left(\underline{\theta}-\frac{p_{22}\epsilon\Delta\theta}{p_{12}}\right)V\big(\lambda^\epsilon(2q_{11})\big)+\overline{\theta}V\big((1-\lambda^\epsilon)(2q_{11})\big)-2t_{11} \tag{17}$$

$$CIC(\underline{\theta},\overline{\theta};\overline{\theta},\overline{\theta}): \left(\underline{\theta}-\frac{p_{22}\epsilon\Delta\theta}{p_{12}}\right)V\big(\lambda^\epsilon(q_{12}+q_{21})\big)+\overline{\theta}V\big((1-\lambda^\epsilon)(q_{12}+q_{21})\big)-t_{12}-t_{21} \geq \left(\underline{\theta}-\frac{p_{22}\epsilon\Delta\theta}{p_{12}}\right)V\big(\lambda^\epsilon(2q_{22})\big)+\overline{\theta}V\big((1-\lambda^\epsilon)(2q_{22})\big)-2t_{22} \tag{18}$$

for an $(\overline{\theta},\overline{\theta})$ *coalition,*

$$CIC(\overline{\theta},\overline{\theta};\underline{\theta},\underline{\theta}): 2\overline{\theta}V(q_{22})-2t_{22} \geq 2\overline{\theta}V(q_{11})-2t_{11} \tag{19}$$

$$CIC(\overline{\theta},\overline{\theta};\underline{\theta},\overline{\theta}): 2\overline{\theta}V(q_{22})-2t_{22} \geq 2\overline{\theta}V\Big(\frac{q_{12}+q_{21}}{2}\Big)-t_{12}-t_{21} \tag{20}$$

Where $(\lambda^\epsilon, 1-\lambda^\epsilon) = \left(\frac{\left(\underline{\theta}-\frac{p_{22}\epsilon\Delta\theta}{p_{12}}\right)^{\frac{1}{\gamma}}}{\left(\underline{\theta}-\frac{p_{22}\epsilon\Delta\theta}{p_{12}}\right)^{\frac{1}{\gamma}}+\overline{\theta}^{\frac{1}{\gamma}}}, \frac{\overline{\theta}^{\frac{1}{\gamma}}}{\left(\underline{\theta}-\frac{p_{22}\epsilon\Delta\theta}{p_{12}}\right)^{\frac{1}{\gamma}}+\overline{\theta}^{\frac{1}{\gamma}}}\right)$ *is the optimal splitting rule within a heterogenous coalition.*

- *The following no-arbitrage-constraint is satisfied:*

$$NAC: \frac{q_{12}}{q_{21}} = \frac{\left(\underline{\theta}-\frac{p_{22}\epsilon\Delta\theta}{p_{12}}\right)^{\frac{1}{\gamma}}}{\overline{\theta}^{\frac{1}{\gamma}}}. \tag{21}$$

- *if* $\epsilon > 0$, *the* $\overline{\theta}-$*type's incentive compatibility constraint is binding in the side-contract. If it is slack,* $\epsilon = 0$.

Proof. See appendix A. ■

Remark 4.1 *If all the coalition incentive constraints hold, the third-party has no incentive to manipulate the agent's report. For instance, if* $CIC(\underline{\theta},\underline{\theta};\underline{\theta},\overline{\theta})$ *is satisfied, an* $(\underline{\theta},\underline{\theta})$ *coalition* *prefers truthtelling to reporting* $(\underline{\theta}, \overline{\theta})$. *If* NAC *is satisfied, the third party has no incentive to reallocate the goods in a heterogenous coalition; the symmetric assumption* $q_1(\underline{\theta}, \underline{\theta}) = q_2(\underline{\theta}, \underline{\theta})$ *and* $q_1(\overline{\theta}, \overline{\theta}) = q_2(\overline{\theta}, \overline{\theta})$ *guarantee that there is generically no arbitrage in homogenous coalitions* $(\underline{\theta}, \underline{\theta})$ *or* $(\overline{\theta}, \overline{\theta})$. *Therefore, the symmetric assumption and conditions (15) to (21) characterize the collusion-proof constraints.*

Remark 4.2 *The coalition incentive constraints under asymmetric information are written by replacing the real valuations with virtual valuations. A* $\overline{\theta}-$ *type's virtual valuation is always equal to the real one:* $\overline{\theta}^v = \overline{\theta}$. *A* $\underline{\theta}-$ *type's virtual valuation is lower than the real one:* $\underline{\theta}^v \leq \underline{\theta}$. *In a* $(\underline{\theta}, \underline{\theta})$ *coalition,* $\underline{\theta}^v = \underline{\theta} - \frac{p_{12}^2 \epsilon \Delta\theta}{p_{11}p_{12}+\rho\epsilon}$*; in a* $(\underline{\theta}, \overline{\theta})$ *coalition,* $\underline{\theta}^v = \underline{\theta} - \frac{p_{22}\epsilon\Delta\theta}{p_{12}}$. *If* $\rho = 0$, *the virtual valuations in different coalitions are identical.*

Remark 4.3 *Parameter* ϵ *is positive if the* $\overline{\theta}-$ *type's incentive compatibility constraint is binding in the third party's program, it is zero if the constraint is slack.* ϵ *represents the friction in coalition formation, the colluding partners often cannot reliably share their private information with each other. This will create a possible transaction cost in the formation of the coalition as the individual incentive constraints may become binding. In the absence of any asymmetric information at coalition formation stage, collusion would be efficient. Letting* $\epsilon = 0$ *in (15) to (21), we get the coalition incentive constraints and non-arbitrage constraint under symmetric information.*

In the sequel, we consider the case with weak correlation, only the downward local coalition incentive constraints (17), (20) or the upward local coalition incentive constraints (15), (18) are of particular interest. The logic of CM's Full Surplus Extraction Mechanism described in subsection (3.1) is to offer large penalties or rewards depending on the states of nature to induce revelation. For instance, the expressions for transfers (5) to (8) show that when correlation ρ is positive and goes to zero, $t_{22} \to +\infty$, $t_{11} \to -\infty$, the transfer when both agents announce $\overline{\theta}$ (resp. $\underline{\theta}$) may become extremely large and positive (resp. negative). This suggests that the coalition incentive constraint (20) (resp. (17)) is likely to be binding in the principal's problem. In contrast, if the correlation ρ is negative and goes to zero, $t_{22} \to -\infty$, $t_{11} \to +\infty$, the upward local coalition incentive constraints (15) and (18) is likely to be binding. These idea will be confirmed in proposition 4 and 5.

5 The optimal weakly collusion-proof mechanism

In order to make comparison, we first restate as a benchmark the collusion proof mechanism with uncorrelated types put forth by JM(2005).

Proposition 3 *There exists a transfer scheme t^* such that $\mathcal{M}^N(0) = (t^*, q^N(0))$ is an optimal mechanism in the absence of buyer coalition and is also weakly collusion-proof.*

Proof. The proof is omitted since it is the straightforward adaptation of Proposition 4 of JM (2005). ■

The above proposition state that collusion with manipulation and arbitrage is preventable at no cost with uncorrelated types. What will happen in the case with correlated types? We will answer this question in the following analysis.

The optimal weakly collusion-proof mechanism maximizes the seller's expected profit subject to Bayesian individual incentive, interim participation, coalition incentive and no- arbitrage constraints. The difficulty, as usual, is to determine which are binding constraints. To simplify the constraints system, it is useful to derive the implementability conditions.

Lemma 1 *Under coalition incentive constraints (15) to (20), for a weak correlation, that is, ρ is close enough to zero, the schedule of implementability consumptions satisfy the following inequality:*

$$q_{11} \leq \frac{q_{12} + q_{21}}{2} \leq q_{22} \tag{22}$$

for all $\epsilon \in [0, 1)$.

Proof. See appendix B. ■

If the weak correlation condition ensuring the implementability condition holds, we will only consider the two upward local coalition incentive constraints or the two downward local coalition incentive constraints with the implementability condition which implies the global constraints. The neglected coalition incentive constraints will be automatically satisfied.

This simplification in the set of incentive constraints being made, all relevant constraints for the seller reduce to the $\overline{\theta}-$ type's individual incentive constraint (2); $\underline{\theta}-$ type's individual rationality constraint (3); local coalition incentive constraints (15), (18) or (17), (20); non-arbitrage constraint (21) and the implementability condition (22). Then the optimization program of the principal can be represented as

$$(\text{P3}) \begin{cases} \underset{\{\mathbf{t},\mathbf{q},\epsilon\}}{Max} \Pi(\mathbf{t}, \mathbf{q}, \epsilon) \\ s.t. \ (2), (3), (17), (20), (21), (22), \epsilon \in [0, 1) \end{cases}.$$

or

$$
(\mathrm{P4}) \begin{cases} \underset{\{\mathbf{t},\mathbf{q},\epsilon\}}{Max} \Pi(\mathbf{t},\mathbf{q},\epsilon) \\ s.t.\ (2),(3),(15),(18),(21),(22), \epsilon \in [0,1) \end{cases}
$$

The following two propositions state that the solutions to (P3) and (P4) give weakly collusion-proof selling mechanism with weakly positive and negative correlation respectively.

Proposition 4 *If the correlation is weakly positive , that is, ρ is bigger than and is close enough to zero, constraints (2),(3),(17),(18), (20),(21), and the second inequality of (22) are binding, $\epsilon = 0$ is the principal's best choice to weaken constraint, the weakly collusion-proof mechanism $\mathcal{M}^C(\rho)$* [5] *entails a partial bunching consumptions $q_{11} < \frac{q_{12}+q_{21}}{2} = q_{22}$ where*

$$
q_{11}^C = \left(\frac{\underline{\theta}\left(\frac{p_{12}+p_{22}}{\rho+p_{12}}\right) - \overline{\theta}\left(\frac{p_{12}^2}{p_{11}(\rho+p_{12})}\right) - \frac{\rho(1-p_{11})}{p_{11}(\rho+p_{12})}\left(\frac{\underline{\theta}^{\frac{1}{\gamma}}+\overline{\theta}^{\frac{1}{\gamma}}}{2}\right)^{\gamma}}{c} \right)^{\frac{1}{\gamma}} \tag{23}
$$

$$
q_{12}^C = \left(\frac{\frac{\underline{\theta}p_{12}-p_{22}\Delta\theta}{\rho+p_{12}} + \overline{\theta}\frac{p_{12}\left(\frac{\overline{\theta}}{\underline{\theta}}\right)^{\frac{1-\gamma}{\gamma}} + p_{22}\left(\frac{\overline{\theta}^{\frac{1}{\gamma}}+\underline{\theta}^{\frac{1}{\gamma}}}{2\underline{\theta}^{\frac{1}{\gamma}}}\right)^{1-\gamma}}{\rho+p_{12}} + \frac{\rho\left(\frac{\underline{\theta}^{\frac{1}{\gamma}}+\overline{\theta}^{\frac{1}{\gamma}}}{\underline{\theta}^{\frac{1-\gamma}{\gamma}}}\right)}{2(\rho+p_{12})p_{12}}}{c} \right)^{\frac{1}{\gamma}} \tag{24}
$$

$$
q_{21}^C = \left(\frac{\overline{\theta}}{\underline{\theta}}\right)^{\frac{1}{\gamma}} \left(\frac{\frac{\underline{\theta}p_{12}-p_{22}\Delta\theta}{\rho+p_{12}} + \overline{\theta}\frac{p_{12}\left(\frac{\overline{\theta}}{\underline{\theta}}\right)^{\frac{1-\gamma}{\gamma}} + p_{22}\left(\frac{\overline{\theta}^{\frac{1}{\gamma}}+\underline{\theta}^{\frac{1}{\gamma}}}{2\underline{\theta}^{\frac{1}{\gamma}}}\right)^{1-\gamma}}{\rho+p_{12}} + \frac{\rho\left(\frac{\underline{\theta}^{\frac{1}{\gamma}}+\overline{\theta}^{\frac{1}{\gamma}}}{\underline{\theta}^{\frac{1-\gamma}{\gamma}}}\right)}{2(\rho+p_{12})p_{12}}}{c} \right)^{\frac{1}{\gamma}} \tag{25}
$$

$$
q_{22}^C = \left(\frac{\overline{\theta}^{\frac{1}{\gamma}}+\underline{\theta}^{\frac{1}{\gamma}}}{2\underline{\theta}^{\frac{1}{\gamma}}}\right) \left(\frac{\frac{\underline{\theta}p_{12}-p_{22}\Delta\theta}{\rho+p_{12}} + \overline{\theta}\frac{p_{12}\left(\frac{\overline{\theta}}{\underline{\theta}}\right)^{\frac{1-\gamma}{\gamma}} + p_{22}\left(\frac{\overline{\theta}^{\frac{1}{\gamma}}+\underline{\theta}^{\frac{1}{\gamma}}}{2\underline{\theta}^{\frac{1}{\gamma}}}\right)^{1-\gamma}}{\rho+p_{12}} + \frac{\rho\left(\frac{\underline{\theta}^{\frac{1}{\gamma}}+\overline{\theta}^{\frac{1}{\gamma}}}{\underline{\theta}^{\frac{1-\gamma}{\gamma}}}\right)}{2(\rho+p_{12})p_{12}}}{c} \right)^{\frac{1}{\gamma}} \tag{26}
$$

Proof. See appendix C. ■

Remark 5.1 *The fact that both coalitions $(\overline{\theta},\overline{\theta})$ and $(\overline{\theta},\underline{\theta})$ are prevented from misreporting limits the feasible awards or penalties that could be used by the principal to elicit the agents' revelation. t_{22} cannot be made arbitrary largely positive as it is in the collusion-free mechanism without*

[5] The superscript C denote "collusion".

violating the coalition incentives (20). So (20) must be binding for the weakly collusion-proof mechanism. Similarly, a $(\overline{\theta}, \underline{\theta})$ *coalition could like to mimic a* $(\underline{\theta}, \underline{\theta})$ *one to get large reward requested in the collusion-free mechanism since* t_{11} *is large and negative. Hence, (17) must also be binding.*

Remark 5.2 *If the third party has no incentive to reallocate goods within the coalition, constraint (21) is redundant, the collusion-proof mechanism is as same as the mechanism put forth by LM (2000). The conflict between (21) and the remained constraints call for a further distortion away from the efficiency obtained by LM's mechanism. We get a partial pooling mechanism in which the constraints (20) , (18) and the second inequality of (22) are both binding. The fact that the third party could conduct arbitrage within coalition exert an additional constraint on the principal's program, which makes the optimal mechanism discontinuous from right side.*

Corollary 1 *The weakly collusion-proof mechanism* $\mathcal{M}^C(\rho)$ *is right discontinuous at* $\rho = 0$.

Proof. From Proposition 3 we get: $q_{ij}^C(0) = q_{ij}^N(0), \forall i, j = 1, 2$. (23) implies

$$\lim_{\rho \to 0^+} q_{11}^C(\rho) = \left(\frac{\underline{\theta} - \frac{1-v}{v} \Delta\theta}{c} \right)^{\frac{1}{\gamma}} = q_{11}^N(0) = q_{11}^C(0).$$

Hence, $q_{11}^C(\rho)$ is right continuous at $\rho = 0$. But

$$\lim_{\rho \to 0^+} q_{12}^C(\rho) = \left(\frac{\frac{\underline{\theta} p_{12} - p_{22}\Delta\theta}{p_{12}} + \overline{\theta} \frac{p_{12}\left(\frac{\overline{\theta}}{\underline{\theta}}\right)^{\frac{1-\gamma}{\gamma}} + p_{22}\left(\frac{\overline{\theta}^{\frac{1}{\gamma}} + \underline{\theta}^{\frac{1}{\gamma}}}{2\underline{\theta}^{\frac{1}{\gamma}}}\right)^{1-\gamma}}{p_{12}}}{c} \right)^{\frac{1}{\gamma}} \neq q_{12}^C(0) = q_{12}^N(0),$$

$\lim_{\rho \to 0^+} q_{21}^C(\rho) \neq q_{21}^C(0)$, $\lim_{\rho \to 0^+} q_{22}^C(\rho) \neq q_{22}^C(0)$. Therefore, $\mathcal{M}^C(\rho)$ is right discontinuous at $\rho = 0$.
■

Proposition 5 *If the correlation is weakly negative , that is,* ρ *is smaller than and is close enough to zero, constraints (2),(3),(15),(18),(21), (22) are binding,* $\epsilon = 1$ *is the best choice to weaken constraint, the weakly collusion-proof mechanism* $\mathcal{M}^C(\rho)$ *entails a strictly decreasing of consumptions* $q_{11} < \frac{q_{12}+q_{21}}{2} < q_{22}$ *with*

$$\left\{ \left(\frac{\overline{\theta} p_{12}}{\rho + p_{12}} \right) + \left(\frac{\left(\underline{\theta} - \frac{p_{22}\Delta\theta}{p_{12}}\right)^{\frac{1}{\gamma}} + \overline{\theta}^{\frac{1}{\gamma}}}{2} \right)^{\gamma} \frac{\rho}{\rho + p_{12}} \right\} q_{22}^{-\gamma} = c \tag{27}$$

$$\left(\frac{\overline{\theta}p_{12}}{\rho+p_{12}}\right)q_{21}^{-\gamma}+\left\{\frac{1-p_{11}}{2}\left(\underline{\theta}-\frac{p_{12}^2\Delta\theta}{p_{11}p_{12}+\rho}\right)-\frac{p_{22}}{2}\left(\frac{\left(\underline{\theta}-\frac{p_{22}\Delta\theta}{p_{12}}\right)^{\frac{1}{\gamma}}+\overline{\theta}^{\frac{1}{\gamma}}}{2}\right)^{\gamma}\right\} \times\frac{\rho}{(\rho+p_{12})p_{12}}\left(\frac{q_{12}+q_{21}}{2}\right)^{-\gamma}=c \tag{28}$$

$$\left(\frac{\underline{\theta}p_{12}-p_{22}\Delta\theta}{\rho+p_{12}}\right)q_{12}^{-\gamma}+\left\{\frac{1-p_{11}}{2}\left(\underline{\theta}-\frac{p_{12}^2\Delta\theta}{p_{11}p_{12}+\rho}\right)-\frac{p_{22}}{2}\left(\frac{\left(\underline{\theta}-\frac{p_{22}\Delta\theta}{p_{12}}\right)^{\frac{1}{\gamma}}+\overline{\theta}^{\frac{1}{\gamma}}}{2}\right)^{\gamma}\right\} \times\frac{\rho}{(\rho+p_{12})p_{12}}\left(\frac{q_{12}+q_{21}}{2}\right)^{-\gamma}=c \tag{29}$$

$$\left\{\underline{\theta}\left(\frac{p_{12}+p_{22}}{\rho+p_{12}}\right)-\overline{\theta}\left(\frac{p_{12}^2}{p_{11}\left(\rho+p_{12}\right)}\right)-\frac{\rho(1-p_{11})}{p_{11}(\rho+p_{12})}\left(\underline{\theta}-\frac{p_{12}^2\Delta\theta}{p_{11}p_{12}+\rho}\right)\right\}q_{11}^{-\gamma}=c \tag{30}$$

Proof. See appendix D. ■

The binding and slack coalition incentive constraints stated in proposition 4 and 5 can be depicted in figures of Appendix E.

Corollary 2 *The weakly collusion-proof mechanism $\mathcal{M}^C(\rho)$ is left continuous at $\rho=0$.*

Proof. From $\lim_{\rho\to 0^-} q_{ij}^C(\rho)=q_{ij}^N(0)=q_{ij}^C(0)$ for all $i=1,2$ and $j=1,2$, the left continuity is obtained. ■

6 Conclusion

Using CM's FSE mechanism , in nonlinear pricing setting, seller may exploit the correlation between buyers to elicit their private information at no cost. In order to protect their rents, buyers have incentive to collude at the sellers loss by coordinating their reports and arbitrage on their purchases. So the seller faces two tasks when designing the sale mechanism: to prevent both manipulation and arbitrage. This makes the optimal weakly collusion-proof mechanism distorted away from LM's mechanism which focus only on the first task. We find that in the presence of arbitrage the weakly collusion-proof sale mechanism depends crucially on the degree of correlation. In uncorrelated settings buyer coalitions do not hurt the seller, collusion and arbitrage are preventable at no cost. When the consumers' types are weakly correlated, the allocations will be distorted away from the first-best efficiency in collusion-free case and distortion patterns with positive and negative correlations are quite different. If correlation is weakly positive, the collusion-proof sale mechanism obtained using LM's methodology is not arbitrage-preventable. The possibility of arbitrage makes the weakly collusion-proof mechanism partial bunching and right discontinuous at zero point. If correlation is weakly negative, LM's mechanism is arbitrage preventable. Hence, the left continuity between the correlated and the uncorrelated environment is still available.

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Appendix A: Proof of Proposition 2

Proof. Let us denote

$$\phi_{11} = \phi(\underline{\theta}, \underline{\theta}), \phi_{12} = \phi(\underline{\theta}, \overline{\theta}), \phi_{21} = \phi(\overline{\theta}, \underline{\theta}), \phi_{22} = \phi(\overline{\theta}, \overline{\theta}).$$

The third-party's problem can be written as:

$$\underset{\phi(\cdot),x_i(\cdot),y_i(\cdot)}{Max} \sum_{(\theta_1,\theta_2)\in\Theta^2} p(\theta_1,\theta_2) \left\{ \sum_{i=1,2} \Big[\theta_i V\big(x_i(\theta_1,\theta_2,\phi(\theta_1,\theta_2)) + q_i(\phi(\theta_1,\theta_2))\big) - t_i\big(\phi(\theta_1,\theta_2)\big) \Big] \right\}$$

subject to the following constraints.

- Budget balance:

$$(BB:y) \sum_{k=1,2} y_k(\theta_1,\theta_2) = 0, \forall(\theta_1,\theta_2) \in \Theta^2 \tag{31}$$

$$(BB:x) \sum_{k=1,2} x_k(\theta_1,\theta_2,\phi) = 0, \forall(\theta_1,\theta_2) \in \Theta^2, \forall\phi \in \Theta^2. \tag{32}$$

- Incentive constraints for respectively the $\overline{\theta}$ 1 and 2:

$$\begin{aligned} BIC_1^{\mathcal{S}}(\overline{\theta}) : p_{12}&\Big[\overline{\theta}V\big(x_1(\overline{\theta},\underline{\theta},\phi_{21}) + q_1(\phi_{21})\big) + y_1(\overline{\theta},\underline{\theta}) - t_1(\phi_{21})\Big] \\ +p_{22}&\Big[\overline{\theta}V\big(x_1(\overline{\theta},\overline{\theta},\phi_{22}) + q_1(\phi_{22})\big) + y_1(\overline{\theta},\overline{\theta}) - t_1(\phi_{22})\Big] \\ \geq p_{12}&\Big[\overline{\theta}V\big(x_1(\underline{\theta},\underline{\theta},\phi_{11}) + q_1(\phi_{11})\big) + y_1(\underline{\theta},\underline{\theta}) - t_1(\phi_{11})\Big] \\ +p_{22}&\Big[\overline{\theta}V\big(x_1(\underline{\theta},\overline{\theta},\phi_{12}) + q_1(\phi_{12})\big) + y_1(\underline{\theta},\overline{\theta}) - t_1(\phi_{12})\Big] \end{aligned} \tag{33}$$

$$\begin{aligned} BIC_2^{\mathcal{S}}(\overline{\theta}) : p_{12}&\Big[\overline{\theta}V\big(x_2(\underline{\theta},\overline{\theta},\phi_{12}) + q_2(\phi_{12})\big) + y_2(\underline{\theta},\overline{\theta}) - t_2(\phi_{12})\Big] \\ +p_{22}&\Big[\overline{\theta}V\big(x_2(\overline{\theta},\overline{\theta},\phi_{22}) + q_2(\phi_{22})\big) + y_2(\overline{\theta},\overline{\theta}) - t_2(\phi_{22})\Big] \\ \geq p_{12}&\Big[\overline{\theta}V\big(x_2(\underline{\theta},\underline{\theta},\phi_{11}) + q_2(\phi_{11})\big) + y_2(\underline{\theta},\underline{\theta}) - t_2(\phi_{11})\Big] \\ +p_{22}&\Big[\overline{\theta}V\big(x_2(\overline{\theta},\underline{\theta},\phi_{21}) + q_2(\phi_{21})\big) + y_2(\overline{\theta},\underline{\theta}) - t_2(\phi_{21})\Big]. \end{aligned} \tag{34}$$

- Participation constraints for respectively the $\overline{\theta}$ 1 and 2:

$$\begin{aligned} BIR_1^{\mathcal{S}}(\overline{\theta}) : &p_{12}\Big[\overline{\theta}V\big(x_1(\overline{\theta},\underline{\theta},\phi_{21}) + q_1(\phi_{21})\big) + y_1(\overline{\theta},\underline{\theta}) - t_1(\phi_{21})\Big] \\ &+p_{22}\Big[\overline{\theta}V\big(x_1(\overline{\theta},\overline{\theta},\phi_{22}) + q_1(\phi_{22})\big) + y_1(\overline{\theta},\overline{\theta}) - t_1(\phi_{22})\Big] \\ &\geq (p_{21}+p_{22})U_1^{\mathcal{M}}(\overline{\theta}) \end{aligned} \tag{35}$$

$$
\begin{aligned}
BIR_2^S(\overline{\theta}) :& p_{12}\Big[\overline{\theta}V\big(x_2(\underline{\theta},\overline{\theta},\phi_{12})+q_2(\phi_{12})\big)+y_2(\underline{\theta},\overline{\theta})-t_2(\phi_{12})\Big] \\
&+p_{22}\Big[\overline{\theta}V\big(x_2(\overline{\theta},\overline{\theta},\phi_{22})+q_2(\phi_{22})\big)+y_2(\overline{\theta},\overline{\theta})-t_2(\phi_{22})\Big] \\
&\geq (p_{21}+p_{22})U_2^{\mathcal{M}}(\overline{\theta})
\end{aligned} \tag{36}
$$

- Participation constraints for respectively the $\underline{\theta}$ agents 1 and 2:

$$
\begin{aligned}
BIR_1^S(\underline{\theta}) :& p_{11}\Big[\underline{\theta}V\big(x_1(\underline{\theta},\underline{\theta},\phi_{11})+q_1(\phi_{11})\big)+y_1(\underline{\theta},\underline{\theta})-t_1(\phi_{11})\Big] \\
&+p_{12}\Big[\underline{\theta}V\big(x_1(\underline{\theta},\overline{\theta},\phi_{12})+q_1(\phi_{12})\big)+y_1(\underline{\theta},\overline{\theta})-t_1(\phi_{12})\Big] \\
&\geq (p_{11}+p_{12})U_1^{\mathcal{M}}(\underline{\theta})
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
BIR_2^S(\underline{\theta}) :& p_{11}\Big[\underline{\theta}V\big(x_2(\underline{\theta},\underline{\theta},\phi_{11})+q_2(\phi_{11})\big)+y_2(\underline{\theta},\underline{\theta})-t_2(\phi_{11})\Big] \\
&+p_{12}\Big[\underline{\theta}V\big(x_2(\overline{\theta},\underline{\theta},\phi_{21})+q_2(\phi_{21})\big)+y_2(\overline{\theta},\underline{\theta})-t_2(\phi_{21})\Big] \\
&\geq (p_{11}+p_{12})U_2^{\mathcal{M}}(\underline{\theta}).
\end{aligned} \tag{38}
$$

Let us introduce the following multipliers $\rho(\theta_1,\theta_2),\tau(\theta_1,\theta_2),\delta_1,\delta_2,\underline{\nu}_1,\underline{\nu}_2,\overline{\nu}_1,\overline{\nu}_2$, associate with constraints (31) to (38) respectively. We write the Lagrangian function of the above maximization problem as:

$$
\begin{aligned}
L =& E(U_1+U_2)+\sum_{i=1,2}\delta_i BIC_i^S(\overline{\theta})+\sum_{i=1,2}\overline{\nu}_i BIR_i^S(\overline{\theta})+\sum_{i=1,2}\underline{\nu}_i BIR_i^S(\underline{\theta}) \\
&+\sum_{(\theta_1,\theta_2)\in\Theta^2}\rho(\theta_1,\theta_2)(BB:y)(\theta_1,\theta_2)+\sum_{(\theta_1,\theta_2)\in\Theta^2}\tau(\theta_1,\theta_2)(BB:x)(\theta_1,\theta_2).
\end{aligned}
$$

- Maximizing with respect to $y_1(\cdot,\cdot),y_2(\cdot,\cdot)$ yields

$$
\begin{aligned}
y_1(\underline{\theta},\underline{\theta}):&\quad \rho(\underline{\theta},\underline{\theta})-p_{12}\delta_1+p_{11}\underline{\nu}_1 &= 0 \qquad (39)\\
y_2(\underline{\theta},\underline{\theta}):&\quad \rho(\underline{\theta},\underline{\theta})-p_{12}\delta_2+p_{11}\underline{\nu}_2 &= 0 \qquad (40)\\
y_1(\underline{\theta},\overline{\theta}):&\quad \rho(\underline{\theta},\overline{\theta})-p_{22}\delta_1+p_{12}\underline{\nu}_1 &= 0 \qquad (41)\\
y_2(\underline{\theta},\overline{\theta}):&\quad \rho(\underline{\theta},\overline{\theta})+p_{12}\left(\delta_2+\overline{\nu}_2\right) &= 0 \qquad (42)\\
y_1(\overline{\theta},\underline{\theta}):&\quad \rho(\overline{\theta},\underline{\theta})+p_{12}\left(\delta_1+\overline{\nu}_1\right) &= 0 \qquad (43)\\
y_2(\overline{\theta},\underline{\theta}):&\quad \rho(\overline{\theta},\underline{\theta})+p_{12}\underline{\nu}_2-p_{22}\delta_2 &= 0 \qquad (44)\\
y_1(\overline{\theta},\overline{\theta}):&\quad \rho(\overline{\theta},\overline{\theta})+p_{22}\left(\delta_1+\overline{\nu}_1\right) &= 0 \qquad (45)\\
y_2(\overline{\theta},\overline{\theta}):&\quad \rho(\overline{\theta},\overline{\theta})+p_{22}\left(\delta_2+\overline{\nu}_2\right) &= 0. \qquad (46)
\end{aligned}
$$

Expressions (39) and (40) imply

$$
-p_{12}\delta_1+p_{11}\underline{\nu}_1=-p_{12}\delta_2+p_{11}\underline{\nu}_2. \tag{47}
$$

(41) and (42) imply

$$\delta_2 + \overline{\nu}_2 = \underline{\nu}_1 - \frac{p_{22}}{p_{12}}\delta_1. \tag{48}$$

(43) and (44) imply

$$\delta_1 + \overline{\nu}_1 = \underline{\nu}_2 - \frac{p_{22}}{p_{12}}\delta_2. \tag{49}$$

(45) and (46) imply

$$\delta_1 + \overline{\nu}_1 = \delta_2 + \overline{\nu}_2. \tag{50}$$

In what follows, without loss of generality, we consider the symmetric multipliers

$$\delta_1 = \delta_2 \equiv \delta, \underline{\nu}_1 = \underline{\nu}_2 \equiv \underline{\nu}, \overline{\nu}_1 = \overline{\nu}_2 \equiv \overline{\nu}.$$

- Maximizing with respect to $x_1(\cdot), x_2(\cdot)$ yields

$$\begin{aligned}
x_1(\underline{\theta},\underline{\theta},\phi_{11}):&\quad \tau(\underline{\theta},\underline{\theta}) + (p_{11}\underline{\theta} - p_{12}\delta_1\overline{\theta} + p_{11}\underline{\nu}_1\underline{\theta})V'\big(x_1(\underline{\theta},\underline{\theta},\phi_{11}) + q_1(\phi_{11})\big) &= 0 \quad (51)\\
x_2(\underline{\theta},\underline{\theta},\phi_{11}):&\quad \tau(\underline{\theta},\underline{\theta}) + (p_{11}\underline{\theta} - p_{12}\delta_2\overline{\theta} + p_{11}\underline{\nu}_2\underline{\theta})V'\big(x_2(\underline{\theta},\underline{\theta},\phi_{11}) + q_2(\phi_{11})\big) &= 0 \quad (52)\\
x_1(\underline{\theta},\overline{\theta},\phi_{12}):&\quad \tau(\underline{\theta},\overline{\theta}) + (p_{12}\underline{\theta} - p_{22}\overline{\theta}\delta_1 + p_{12}\underline{\nu}_1\underline{\theta})V'\big(x_1(\underline{\theta},\overline{\theta},\phi_{12}) + q_1(\phi_{12})\big) &= 0 \quad (53)\\
x_2(\underline{\theta},\overline{\theta},\phi_{12}):&\quad \tau(\underline{\theta},\overline{\theta}) + (p_{12}\overline{\theta} + p_{12}\overline{\theta}\delta_2 + p_{12}\overline{\nu}_2\overline{\theta})V'\big(x_2(\underline{\theta},\overline{\theta},\phi_{12}) + q_2(\phi_{12})\big) &= 0 \quad (54)\\
x_1(\overline{\theta},\underline{\theta},\phi_{21}):&\quad \tau(\overline{\theta},\underline{\theta}) + (p_{12}\overline{\theta} + p_{12}\overline{\theta}\delta_1 + p_{12}\overline{\nu}_1\overline{\theta})V'\big(x_1(\overline{\theta},\underline{\theta},\phi_{21}) + q_1(\phi_{21})\big) &= 0 \quad (55)\\
x_2(\overline{\theta},\underline{\theta},\phi_{21}):&\quad \tau(\overline{\theta},\underline{\theta}) + (p_{12}\underline{\theta} - p_{22}\overline{\theta}\delta_2 + p_{12}\underline{\nu}_2\underline{\theta})V'\big(x_2(\overline{\theta},\underline{\theta},\phi_{21}) + q_2(\phi_{21})\big) &= 0 \quad (56)\\
x_1(\overline{\theta},\overline{\theta},\phi_{22}):&\quad \tau(\overline{\theta},\overline{\theta}) + (p_{22}\overline{\theta} + p_{22}\overline{\theta}\delta_1 + p_{22}\overline{\theta}\overline{\nu}_1)V'\big(x_1(\overline{\theta},\overline{\theta},\phi_{22}) + q_1(\phi_{22})\big) &= 0 \quad (57)\\
x_2(\overline{\theta},\overline{\theta},\phi_{22}):&\quad \tau(\overline{\theta},\overline{\theta}) + (p_{22}\overline{\theta} + p_{22}\overline{\theta}\delta_2 + p_{22}\overline{\theta}\overline{\nu}_2)V'\big(x_2(\overline{\theta},\overline{\theta},\phi_{22}) + q_2(\phi_{22})\big) &= 0 \quad (58)
\end{aligned}$$

(51) and (52) imply

$$V'\big(x_1(\underline{\theta},\underline{\theta},\phi_{11}) + q_1(\phi_{11})\big) = V'\big(x_2(\underline{\theta},\underline{\theta},\phi_{11}) + q_2(\phi_{11})\big), \forall \phi_{11} \in \Theta^2. \tag{59}$$

(57) and (58) imply

$$V'\big(x_1(\overline{\theta},\overline{\theta},\phi_{22}) + q_1(\phi_{22}\big) = V'\big(x_2(\overline{\theta},\overline{\theta},\phi_{22}) + q_2(\phi_{22})\big), \forall \phi_{22} \in \Theta^2 \tag{60}$$

Since $x_1(\underline{\theta},\underline{\theta},\phi_{11}) + x_2(\underline{\theta},\underline{\theta},\phi_{11}) = 0, x_1(\overline{\theta},\overline{\theta},\phi_{22}) + x_2(\overline{\theta},\overline{\theta},\phi_{22}) = 0$ from a budget-balance constraint, we have

$$x_1(\underline{\theta},\underline{\theta},\phi_{11}) + q_1(\phi_{11}) = x_2(\underline{\theta},\underline{\theta},\phi_{11}) + q_1(\phi_{11}) = \frac{q_1(\phi_{11}) + q_2(\phi_{11})}{2}, \forall \phi_{11} \tag{61}$$

$$x_1(\overline{\theta},\overline{\theta},\phi_{22}) + q_1(\phi_{22}) = x_2(\overline{\theta},\overline{\theta},\phi_{22}) + q_1(\phi_{22}) = \frac{q_1(\phi_{22}) + q_2(\phi_{22})}{2}, \forall \phi_{22} \tag{62}$$

(53) and (54) imply

$$\left(\underline{\theta} - \frac{p_{22}}{p_{12}}\overline{\theta}\delta_1 + \underline{\nu}_1\underline{\theta}\right) V^{'}\big(x_1(\underline{\theta},\overline{\theta},\phi_{12}) + q_1(\phi_{12})\big) = (1+\delta_2+\overline{\nu}_2)\,\overline{\theta}V^{'}\big(x_2(\underline{\theta},\overline{\theta},\phi_{12}) + q_2(\phi_{12})\big) \tag{63}$$

Using (48), we obtain

$$\left(\underline{\theta} - \frac{p_{22}\epsilon}{p_{12}}\Delta\theta\right) V^{'}\big(x_1(\underline{\theta},\overline{\theta},\phi_{12}) + q_1(\phi_{12})\big) = \overline{\theta}V^{'}\big(x_2(\underline{\theta},\overline{\theta},\phi_{12}) + q_2(\phi_{12})\big), \forall\phi_{12} \tag{64}$$

where

$$\epsilon = \frac{\delta}{1+\delta+\overline{\nu}}.$$

Similarly, expressions (55) ,(56) and (49) imply

$$\overline{\theta}V^{'}\big(x_1(\overline{\theta},\underline{\theta},\phi_{21}) + q_1(\phi_{21})\big) = \left(\underline{\theta} - \frac{p_{22}\epsilon}{p_{12}}\Delta\theta\right) V^{'}\big(x_2(\overline{\theta},\underline{\theta},\phi_{21}) + q_2(\phi_{21})\big), \forall\phi_{21}. \tag{65}$$

With budget-balance constraints $x_1(\underline{\theta},\overline{\theta},\phi_{12}) + x_2(\underline{\theta},\overline{\theta},\phi_{12}) = 0, x_1(\overline{\theta},\underline{\theta},\phi_{21}) + x_2(\overline{\theta},\underline{\theta},\phi_{21}) = 0$ and CRRA utility function $V(q) = \frac{q^{1-\gamma}}{1-\gamma}$, the total quantity available to a heterogeneous coalition is split according to the following rule:

$$\begin{aligned}
x_1(\underline{\theta},\overline{\theta},\phi_{12}) + q_1(\phi_{12}) &= \lambda^{\epsilon}\big(q_1(\phi_{12}) + q_2(\phi_{12})\big), \forall\phi_{12} && (66)\\
x_2(\underline{\theta},\overline{\theta},\phi_{12}) + q_2(\phi_{12}) &= (1-\lambda^{\epsilon})\big(q_1(\phi_{12}) + q_2(\phi_{12})\big), \forall\phi_{12} && (67)\\
x_1(\overline{\theta},\underline{\theta},\phi_{21}) + q_1(\phi_{21}) &= (1-\lambda^{\epsilon})\big(q_1(\phi_{21}) + q_2(\phi_{21})\big), \forall\phi_{12} && (68)\\
x_2(\overline{\theta},\underline{\theta},\phi_{21}) + q_2(\phi_{21}) &= \lambda^{\epsilon}\big(q_1(\phi_{21}) + q_2(\phi_{21})\big), \forall\phi_{12} && (69)
\end{aligned}$$

where

$$\lambda^{\epsilon} = \frac{\left(\underline{\theta} - \frac{p_{22}\epsilon\Delta\theta}{p_{12}}\right)^{\frac{1}{\gamma}}}{\left(\underline{\theta} - \frac{p_{22}\epsilon\Delta\theta}{p_{12}}\right)^{\frac{1}{\gamma}} + \overline{\theta}^{\frac{1}{\gamma}}} < \frac{1}{2}.$$

For weakly collusion proof grand mechanism, $\phi(\theta_1,\theta_2) = (\theta_1,\theta_2), x_i\big(\theta_1,\theta_2,\phi(\theta_1,\theta_2)\big) = 0, i = 1,2$, so (61) and (62) imply $q_{11} = \frac{q_{11}+q_{11}}{2}$ and $q_{22} = \frac{q_{22}+q_{22}}{2}$, which is trivially satisfied. Any one of the four expressions (66) to (69) implies

$$\frac{q_{12}}{q_{21}} = \frac{\left(\underline{\theta} - \frac{p_{22}\epsilon\Delta\theta}{p_{12}}\right)^{\frac{1}{\gamma}}}{\overline{\theta}^{\frac{1}{\gamma}}}. \tag{70}$$

This condition ensures that the third party has no incentive to reallocate the quantities of agents, so we name it "no-arbitrage constraint (NAC)" .

- The optimal manipulation of report.

1. Optimizing with respect to ϕ_{11} yields

$$\begin{aligned}\phi_{11}^* \in \underset{\phi_{11}}{argmax}\Big\{ & p_{11}\Big[\underline{\theta}V\big(x_1(\underline{\theta},\underline{\theta},\phi_{11})+q_1(\phi_{11})\big)+\underline{\theta}V\big(x_2(\underline{\theta},\underline{\theta},\phi_{11})+q_2(\phi_{11})\big)\\ & -t_1(\phi_{11})-t_2(\phi_{11})\Big]\\ & +p_{11}\underline{\nu}_1\big[\underline{\theta}V\big(x_1(\underline{\theta},\underline{\theta},\phi_{11})+q_1(\phi_{11})\big)-t_1(\phi_{11})\big]\\ & +p_{11}\underline{\nu}_2\big[\underline{\theta}V\big(x_2(\underline{\theta},\underline{\theta},\phi_{11})+q_2(\phi_{11})\big)-t_2(\phi_{11})\big]\\ & -p_{12}\delta_1\big[\overline{\theta}V\big(x_2(\underline{\theta},\underline{\theta},\phi_{11})+q_2(\phi_{11})\big)-t_1(\phi_{11})\big]\\ & -p_{12}\delta_2\big[\overline{\theta}V\big(x_1(\underline{\theta},\underline{\theta},\phi_{11})+q_2(\phi_{11})\big)-t_1(\phi_{11})\big]\Big\}\end{aligned} \tag{71}$$

Note that at symmetric equilibrium $\delta_1=\delta_2=\delta;\underline{\nu}_1=\underline{\nu}_2=\underline{\nu};\overline{\nu}_1=\overline{\nu}_2=\overline{\nu}$, then from constraints (47) to (49) and (61),(62) we can write the objective function as

$$\begin{aligned}&(p_{11}+p_{11}\underline{\nu}_1-p_{12}\delta_1)\big[\underline{\theta}V\big(x_1(\underline{\theta},\underline{\theta},\phi_{11})+q_1(\phi_{11})\big)-t_1(\phi_{11})\big]\\ &+(p_{11}+p_{11}\underline{\nu}_2-p_{12}\delta_2)\big[\underline{\theta}V\big(x_2(\underline{\theta},\underline{\theta},\phi_{11})+q_2(\phi_{11})\big)-t_2(\phi_{11})\big]\\ &-p_{12}\Delta\theta\delta_1V\big(x_1(\underline{\theta},\underline{\theta},\phi_{11})+q_1(\phi_{11})\big)-p_{12}\Delta\theta\delta_2V\big(x_2(\underline{\theta},\underline{\theta},\phi_{11})+q_2(\phi_{11})\big)\\ &=(p_{11}+p_{11}\underline{\nu}-p_{12}\delta)\\ &\times\Bigg\{\left(\underline{\theta}-\frac{p_{12}^2\epsilon\Delta\theta}{p_{11}p_{12}+\rho\epsilon}\right)\Big[V\big(x_1(\underline{\theta},\underline{\theta},\phi_{11})+q_1(\phi_{11})\big)+V\big(x_2(\underline{\theta},\underline{\theta},\phi_{11})+q_2(\phi_{11})\big)\Big]\\ &-t_1(\phi_{11})-t_2(\phi_{11})\Bigg\}\\ &=(p_{11}+p_{11}\underline{\nu}-p_{12}\delta)\left\{2\left(\underline{\theta}-\frac{p_{12}^2\epsilon\Delta\theta}{p_{11}p_{12}+\rho\epsilon}\right)V\left(\frac{q_1(\phi_{11})+q_2(\phi_{11})}{2}\right)-t_1(\phi_{11})-t_2(\phi_{11})\right\}.\end{aligned} \tag{72}$$

So we have

$$\phi_{11}^* \in \underset{\phi_{11}}{argmax}\left\{2\left(\underline{\theta}-\frac{p_{12}^2\epsilon\Delta\theta}{p_{11}p_{12}+\rho\epsilon}\right)V\left(\frac{q_1(\phi_{11})+q_2(\phi_{11})}{2}\right)-t_1(\phi_{11})-t_2(\phi_{11})\right\}. \tag{73}$$

2. Similarly, optimizing with respect to ϕ_{12} and ϕ_{21} yield respectively

$$\begin{aligned}\phi_{12}^* \in \underset{\phi_{12}}{argmax}\Big\{ & \Big(\underline{\theta}-\frac{p_{22}\epsilon\Delta\theta}{p_{12}}\Big)V\big(x_1(\underline{\theta},\overline{\theta},\phi_{12})+q_1(\phi_{12})\big)\\ & +\overline{\theta}V\big(x_2(\underline{\theta},\overline{\theta},\phi_{12})+q_2(\phi_{12})\big)-t_1(\phi_{12})-t_2(\phi_{12})\Big\}\\ =\underset{\phi_{12}}{argmax}\Big\{ & \Big(\underline{\theta}-\frac{p_{22}\epsilon\Delta\theta}{p_{12}}\Big)V\Big(\lambda^\epsilon\big(q_1(\phi_{12})+q_1(\phi_{12})\big)\Big)\\ & +\overline{\theta}V\Big(\big(1-\lambda^\epsilon\big)(q_1(\phi_{12})+q_2(\phi_{12}))\Big)-t_1(\phi_{12})-t_2(\phi_{12})\Big\}\end{aligned} \tag{74}$$

and

$$\phi_{21}^{*} \in \underset{\phi_{21}}{argmax}\Big\{ \Big(\underline{\theta} - \frac{p_{22}\epsilon\Delta\theta}{p_{12}}\Big) V\big(x_2(\overline{\theta},\underline{\theta},\phi_{21}) + q_2(\phi_{21})\big) + \overline{\theta}V\big(x_1(\overline{\theta},\underline{\theta},\phi_{21}) + q_1(\phi_{21})\big) - t_1(\phi_{21}) - t_2(\phi_{21})\Big\}$$
$$= \underset{\phi_{21}}{argmax}\Big\{ \big(\underline{\theta} - \frac{p_{22}\epsilon\Delta\theta}{p_{21}}\big) V\Big(\lambda^{\epsilon}\big(q_1(\phi_{21}) + q_1(\phi_{21})\big)\Big) + \overline{\theta}V\Big(\big(1-\lambda^{\epsilon}\big)(q_1(\phi_{21}) + q_2(\phi_{21}))\Big) - t_1(\phi_{21}) - t_2(\phi_{21})\Big\}. \tag{75}$$

3. Optimizing with respect to ϕ_{22} yields

$$\phi_{22}^{*} \in \underset{\phi_{22}}{argmax}\Big\{\overline{\theta}V\big(x_1(\overline{\theta},\overline{\theta},\phi_{22}) + q_1(\phi_{22})\big) + \overline{\theta}V\big(x_2(\overline{\theta},\overline{\theta},\phi_{22}) + q_2(\phi_{22})\big) - t_1(\phi_{22}) - t_2(\phi_{22})\Big\}$$
$$= \underset{\phi_{22}}{argmax}\Big\{2\overline{\theta}V\left(\frac{q_1(\phi_{22}) + q_2(\phi_{22})}{2}\right) - t_1(\phi_{22}) - t_2(\phi_{22})\Big\} \tag{76}$$

- In a weakly collusion-proof mechanism $\phi(\theta_1, \theta_2) = (\theta_1, \theta_2)$, inserting into (73),(74),(75) and (76) yields constraints (15) to (20) in main text.
- Note that $\epsilon = \frac{\delta}{1+\delta+\overline{\nu}} \in [0, 1)$. Moreover, $\delta > 0$ when the Bayesian incentive constraints (33) and (34) are binding in the third party's optimizing problem.
- Note that participation constraints (35) to (38) are binding for a weakly collusion-proof mechanism. Hence the slackness condition obtained from the Lagrangean optimization do not give any information on ϵ. Therefore, ϵ is a free variable in the principal's programme.

■

Appendix B: Proof of Lemma 1.

Proof. Summing constraints (18) and (20) yields

$$\left\{\left[\frac{\left(\underline{\theta} - \frac{p_{22}\epsilon\Delta\theta}{p_{12}}\right)^{\frac{1}{\gamma}} + \overline{\theta}^{\frac{1}{\gamma}}}{2}\right]^{\gamma} - \overline{\theta}\right\}\left[\left(\frac{q_{12}+q_{21}}{2}\right)^{1-\gamma} - q_{22}^{1-\gamma}\right] \geq 0, \tag{77}$$

which is satisfied for $\frac{q_{12}+q_{21}}{2} \leq q_{22}$ since $\left[\left(\underline{\theta} - \frac{p_{22}\epsilon\Delta\theta}{p_{12}}\right)^{\frac{1}{\gamma}} + \overline{\theta}^{\frac{1}{\gamma}}\right]^{\gamma} - 2^{\gamma}\overline{\theta} < 0$.

Summing constraints (15) and (17) yields another revealed preference argument,

$$\left\{\left(\underline{\theta} - \frac{p_{12}^2\epsilon\Delta\theta}{p_{11}p_{12} + \rho\epsilon}\right) - \left[\frac{\left(\underline{\theta} - \frac{p_{22}\epsilon\Delta\theta}{p_{12}}\right)^{\frac{1}{\gamma}} + \overline{\theta}^{\frac{1}{\gamma}}}{2}\right]^{\gamma}\right\}\left[q_{11}^{1-\gamma} - \left(\frac{q_{12}+q_{21}}{2}\right)^{1-\gamma}\right] \geq 0 \tag{78}$$

Denote the first term on the left-hand-side of this inequality by

$$\pi(\epsilon)=\left(\underline{\theta}-\frac{p_{12}^2\epsilon\Delta\theta}{p_{11}p_{12}+\rho\epsilon}\right)-\left[\frac{\left(\underline{\theta}-\frac{p_{22}\epsilon\Delta\theta}{p_{12}}\right)^{\frac{1}{\gamma}}+\overline{\theta}^{\frac{1}{\gamma}}}{2}\right]^{\gamma}. \tag{79}$$

It follows from Jensen's inequality that:

$$\begin{aligned}\pi(\epsilon)&\leq\left(\underline{\theta}-\frac{p_{12}^2\epsilon\Delta\theta}{p_{11}p_{12}+\rho\epsilon}\right)-\left(\frac{\underline{\theta}-\frac{p_{22}\epsilon\Delta\theta}{p_{12}}+\overline{\theta}}{2}\right)\\&=-\left[1+\epsilon\frac{p_{12}^3-\rho(p_{12}+p_{22}\epsilon)}{p_{12}(p_{11}p_{12}+\rho\epsilon)}\right]\frac{\Delta\theta}{2}\end{aligned} \tag{80}$$

When ρ is close enough to zero, the second line of the above expression is negative, hence, $\pi(\epsilon)<0$ for weak correlation. So we find $q_{11}\leq\frac{q_{12}+q_{21}}{2}$ ■

Appendix C: Proof of Proposition 4

Proof. If $\rho>0$, we write the downward individual incentive compatible constraints (2), the $\underline{\theta}$ type's participation constraints (3), the downward local coalition constraints (17) and (20) as binding constraints by introducing nonnegative parameters $\varepsilon_i, i=2,3,17,20$ respectively. Then the constraint equations system can be written as:

$$\begin{pmatrix}p_{11}&p_{12}&0&0\\p_{12}&p_{22}&-p_{12}&-p_{22}\\2&-1&-1&0\\0&1&1&-2\end{pmatrix}\times\begin{pmatrix}t_{11}\\t_{12}\\t_{21}\\t_{22}\end{pmatrix}=\begin{pmatrix}\beta_3-\varepsilon_3\\\beta_2+\varepsilon_2\\\beta_{11}+\varepsilon_{17}\\\beta_{14}+\varepsilon_{20}\end{pmatrix} \tag{81}$$

with

$$\begin{pmatrix}\beta_3\\\beta_2\\\beta_{17}\\\beta_{20}\end{pmatrix}\equiv\begin{pmatrix}\frac{\underline{\theta}p_{11}q_{11}^{1-\gamma}+\underline{\theta}p_{12}q_{12}^{1-\gamma}}{1-\gamma}\\\frac{\overline{\theta}p_{12}\left(q_{11}^{1-\gamma}-q_{21}^{1-\gamma}\right)+\overline{\theta}p_{22}\left(q_{12}^{1-\gamma}-q_{22}^{1-\gamma}\right)}{1-\gamma}\\\left[\left(\underline{\theta}-\frac{p_{22}\epsilon\Delta\theta}{p_{12}}\right)^{\frac{1}{\gamma}}+\overline{\theta}^{\frac{1}{\gamma}}\right]^{\gamma}\frac{(2q_{11})^{1-\gamma}-(q_{12}+q_{21})^{1-\gamma}}{1-\gamma}\\2^{\gamma}\overline{\theta}\left[\frac{(q_{12}+q_{21})^{1-\gamma}-(2q_{22})^{1-\gamma}}{1-\gamma}\right]\end{pmatrix}$$

The parameters $\epsilon,\varepsilon_2,\varepsilon_3,\varepsilon_{17},\varepsilon_{20}$ are free variables to the principal, $0\leq\epsilon<1,\varepsilon_i\geq0$. Constraint i is binding if and only if the associated parameter $\varepsilon_i=0$. Solving the above equations system, we get the expected transfers:

$$\begin{aligned}\sum_{i=1,2}\sum_{j=1,2}p_{ij}t_{ij}&=\frac{p_{12}+p_{22}}{\rho+p_{12}}(\beta_3-\varepsilon_3)-\frac{p_{12}}{\rho+p_{12}}(\beta_2+\varepsilon_2)\\&-\frac{\rho(1-p_{11})}{2(\rho+p_{12})}(\beta_{17}+\varepsilon_{17})-\frac{\rho p_{22}}{2(\rho+p_{12})}(\beta_{20}+\varepsilon_{20})\end{aligned} \tag{82}$$

Since $q_{11} \leq \frac{q_{12}+q_{21}}{2}$ and $\rho > 0$, $\epsilon = \varepsilon_2 = \varepsilon_3 = \varepsilon_{17} = \varepsilon_{20} = 0$ are the best choices to maximize the expected transfer $\sum_{i,j} p_{ij} t_{ij}$. Therefore, it can be verified that constraints (2),(3),(17) and (20) are binding with weakly positive correlation. Substituting expression (82) into the objective function of (P3), then maximizing with respect to $q_{11}, q_{12}, q_{21}, q_{22}$ we obtain:

$$\left\{ \underline{\theta} \left(\frac{p_{12}+p_{22}}{\rho+p_{12}} \right) - \overline{\theta} \left[\frac{p_{12}^2}{p_{11}\left(\rho+p_{12}\right)} \right] - \frac{\rho(1-p_{11})}{p_{11}(\rho+p_{12})} \left(\frac{\underline{\theta}^{\frac{1}{\gamma}} + \overline{\theta}^{\frac{1}{\gamma}}}{2} \right)^{\gamma} \right\} q_{11}^{-\gamma} = c \tag{83}$$

$$\left(\frac{\underline{\theta} p_{12} - p_{22}\Delta\theta}{\rho+p_{12}} \right) q_{12}^{-\gamma} + \frac{\rho}{2(\rho+p_{12})p_{12}} \left[(1-p_{11}) \left(\frac{\underline{\theta}^{\frac{1}{\gamma}} + \overline{\theta}^{\frac{1}{\gamma}}}{2} \right)^{\gamma} - p_{22}\overline{\theta} \right] \left(\frac{q_{12}+q_{21}}{2} \right)^{-\gamma} = c \tag{84}$$

$$\left(\frac{\overline{\theta} p_{12}}{\rho+p_{12}} \right) q_{21}^{-\gamma} + \frac{\rho}{2(\rho+p_{12})p_{12}} \left[(1-p_{11}) \left(\frac{\underline{\theta}^{\frac{1}{\gamma}} + \overline{\theta}^{\frac{1}{\gamma}}}{2} \right)^{\gamma} - p_{22}\overline{\theta} \right] \left(\frac{q_{12}+q_{21}}{2} \right)^{-\gamma} = c \tag{85}$$

$$q_{22} = q_{22}^{FB} = \left(\frac{\overline{\theta}}{c} \right)^{\frac{1}{\gamma}}. \tag{86}$$

It is obvious that when ρ is close enough to zero, the monotonic implementability condition $q_{11} \leq \frac{q_{12}+q_{21}}{2} \leq q_{22}$ holds. (84) and (85) imply that $\frac{q_{12}}{q_{21}} = \frac{\left(\underline{\theta} - \frac{p_{22}\Delta\theta}{p_{12}}\right)^{\frac{1}{\gamma}}}{\overline{\theta}^{\frac{1}{\gamma}}}$, it fails to satisfy the no-arbitrage constraint $\frac{q_{12}}{q_{21}} = \frac{\underline{\theta}^{\frac{1}{\gamma}}}{\overline{\theta}^{\frac{1}{\gamma}}}$ since $\epsilon = 0$.

Substituting (82) and $q_{12} = \left(\frac{\underline{\theta}^{\frac{1}{\gamma}}}{\overline{\theta}^{\frac{1}{\gamma}}} \right) q_{21}$ into the objective function of (P3) then maximizing with respect to q_{ij}, we obtain:

$$\left\{ \underline{\theta} \left(\frac{p_{12}+p_{22}}{\rho+p_{12}} \right) - \overline{\theta} \left[\frac{p_{12}^2}{p_{11}\left(\rho+p_{12}\right)} \right] - \frac{\rho(1-p_{11})}{p_{11}(\rho+p_{12})} \left(\frac{\underline{\theta}^{\frac{1}{\gamma}} + \overline{\theta}^{\frac{1}{\gamma}}}{2} \right)^{\gamma} \right\} q_{11}^{-\gamma} = c \tag{87}$$

$$\begin{aligned} &\left[\frac{\underline{\theta} p_{12} - p_{22}\Delta\theta}{\rho+p_{12}} + \frac{p_{12}\overline{\theta}\left(\overline{\theta}/\underline{\theta}\right)^{\frac{1-\gamma}{\gamma}}}{\rho+p_{12}} \right] q_{12}^{-\gamma} + \frac{\rho}{2(\rho+p_{12})p_{12}} \left(\frac{\underline{\theta}^{\frac{1}{\gamma}} + \overline{\theta}^{\frac{1}{\gamma}}}{\underline{\theta}^{\frac{1}{\gamma}}} \right) \\ &\times \left\{ (1-p_{11}) \left(\frac{\underline{\theta}^{\frac{1}{\gamma}} + \overline{\theta}^{\frac{1}{\gamma}}}{2} \right)^{\gamma} - p_{22}\overline{\theta} \right\} \left(\frac{q_{12}+q_{21}}{2} \right)^{-\gamma} = c \end{aligned} \tag{88}$$

$$q_{21} = \left(\frac{\overline{\theta}}{\underline{\theta}} \right)^{\frac{1}{\gamma}} q_{12} \tag{89}$$

$$q_{22} = \left(\frac{\overline{\theta}}{c} \right)^{\frac{1}{\gamma}} \tag{90}$$

It can be verified that when ρ is close enough to zero, $q_{21} > q_{12} > q_{22}$, hence the monotonic implementability condition $\frac{q_{12}+q_{21}}{2} \leq q_{22}$ is violated. Therefore, the monotonic condition $\frac{q_{12}+q_{21}}{2} \leq q_{22}$ must be binding. Maximizing with respect to $q_{ij}, i = 1, 2$ subject to $\frac{q_{21}}{q_{12}} = \left(\frac{\overline{\theta}}{\underline{\theta}}\right)^{\frac{1}{\gamma}}$ and $q_{22} = \frac{q_{12}+q_{21}}{2} = \frac{\overline{\theta}^{\frac{1}{\gamma}}+\underline{\theta}^{\frac{1}{\gamma}}}{2\underline{\theta}^{\frac{1}{\gamma}}} q_{12}$, we obtain the partial bunching weakly collusion-proof mechanism with $q_{11} < \frac{q_{12}+q_{21}}{2} = q_{22}$:

$$\left\{\underline{\theta}\left(\frac{p_{12}+p_{22}}{\rho+p_{12}}\right) - \frac{\overline{\theta}p_{12}^2}{p_{11}\left(\rho+p_{12}\right)} - \frac{\rho(1-p_{11})}{p_{11}(\rho+p_{12})}\left(\frac{\underline{\theta}^{\frac{1}{\gamma}}+\overline{\theta}^{\frac{1}{\gamma}}}{2}\right)^{\gamma}\right\}(q_{11}^C)^{-\gamma} = c$$
$$\Rightarrow q_{11}^C = \left(\frac{\underline{\theta}\left(\frac{p_{12}+p_{22}}{\rho+p_{12}}\right) - \overline{\theta}\left(\frac{p_{12}^2}{p_{11}(\rho+p_{12})}\right) - \frac{\rho(1-p_{11})}{p_{11}(\rho+p_{12})}\left(\frac{\underline{\theta}^{\frac{1}{\gamma}}+\overline{\theta}^{\frac{1}{\gamma}}}{2}\right)^{\gamma}}{c}\right)^{\frac{1}{\gamma}} \tag{91}$$

$$\left\{\frac{\underline{\theta}p_{12}-p_{22}\Delta\theta}{\rho+p_{12}} + \overline{\theta}\frac{p_{12}\left(\frac{\underline{\theta}}{\overline{\theta}}\right)^{\frac{1-\gamma}{\gamma}} + p_{22}\left(\frac{\overline{\theta}^{\frac{1}{\gamma}}+\underline{\theta}^{\frac{1}{\gamma}}}{2\underline{\theta}^{\frac{1}{\gamma}}}\right)^{1-\gamma}}{\rho+p_{12}} + \frac{\rho}{2(\rho+p_{12})p_{12}}\left(\frac{\underline{\theta}^{\frac{1}{\gamma}}+\overline{\theta}^{\frac{1}{\gamma}}}{\underline{\theta}^{\frac{1-\gamma}{\gamma}}}\right)\right\}\left(q_{12}^C\right)^{-\gamma} = c$$
$$\Rightarrow q_{12}^C = \left(\frac{\frac{\underline{\theta}p_{12}-p_{22}\Delta\theta}{\rho+p_{12}} + \overline{\theta}\frac{p_{12}\left(\frac{\underline{\theta}}{\overline{\theta}}\right)^{\frac{1-\gamma}{\gamma}}+p_{22}\left(\frac{\overline{\theta}^{\frac{1}{\gamma}}+\underline{\theta}^{\frac{1}{\gamma}}}{2\underline{\theta}^{\frac{1}{\gamma}}}\right)^{1-\gamma}}{\rho+p_{12}} + \frac{\rho\left(\frac{\underline{\theta}^{\frac{1}{\gamma}}+\overline{\theta}^{\frac{1}{\gamma}}}{\underline{\theta}^{\frac{1-\gamma}{\gamma}}}\right)}{2(\rho+p_{12})p_{12}}}{c}\right)^{\frac{1}{\gamma}} \tag{92}$$

$$q_{21}^C = \left(\frac{\overline{\theta}}{\underline{\theta}}\right)^{\frac{1}{\gamma}} \times q_{12}^C = \left(\frac{\overline{\theta}}{\underline{\theta}}\right)^{\frac{1}{\gamma}}\left(\frac{\frac{\underline{\theta}p_{12}-p_{22}\Delta\theta}{\rho+p_{12}} + \overline{\theta}\frac{p_{12}\left(\frac{\underline{\theta}}{\overline{\theta}}\right)^{\frac{1-\gamma}{\gamma}}+p_{22}\left(\frac{\overline{\theta}^{\frac{1}{\gamma}}+\underline{\theta}^{\frac{1}{\gamma}}}{2\underline{\theta}^{\frac{1}{\gamma}}}\right)^{1-\gamma}}{\rho+p_{12}} + \frac{\rho\left(\frac{\underline{\theta}^{\frac{1}{\gamma}}+\overline{\theta}^{\frac{1}{\gamma}}}{\underline{\theta}^{\frac{1-\gamma}{\gamma}}}\right)}{2(\rho+p_{12})p_{12}}}{c}\right)^{\frac{1}{\gamma}} \tag{93}$$

$$q_{22}^C = \left(\frac{\overline{\theta}^{\frac{1}{\gamma}}+\underline{\theta}^{\frac{1}{\gamma}}}{2\underline{\theta}^{\frac{1}{\gamma}}}\right) \times q_{12}^C = \left(\frac{\overline{\theta}^{\frac{1}{\gamma}}+\underline{\theta}^{\frac{1}{\gamma}}}{2\underline{\theta}^{\frac{1}{\gamma}}}\right)\left(\frac{\frac{\underline{\theta}p_{12}-p_{22}\Delta\theta}{\rho+p_{12}} + \overline{\theta}\frac{p_{12}\left(\frac{\underline{\theta}}{\overline{\theta}}\right)^{\frac{1-\gamma}{\gamma}}+p_{22}\left(\frac{\overline{\theta}^{\frac{1}{\gamma}}+\underline{\theta}^{\frac{1}{\gamma}}}{2\underline{\theta}^{\frac{1}{\gamma}}}\right)^{1-\gamma}}{\rho+p_{12}} + \frac{\rho\left(\frac{\underline{\theta}^{\frac{1}{\gamma}}+\overline{\theta}^{\frac{1}{\gamma}}}{\underline{\theta}^{\frac{1-\gamma}{\gamma}}}\right)}{2(\rho+p_{12})p_{12}}}{c}\right)^{\frac{1}{\gamma}} \tag{94}$$

Appendix D: Proof of proposition 5

If $\rho < 0$, we guess that the downward individual incentive compatible constraints (2); the $\underline{\theta}$ type's participation constraints (3); the upward local coalition constraints (15) and (18) are binding. Then the constraint equations system can be written as:

$$\begin{pmatrix} p_{11} & p_{12} & 0 & 0 \\ p_{12} & p_{22} & -p_{12} & -p_{22} \\ 2 & -1 & -1 & 0 \\ 0 & 1 & 1 & -2 \end{pmatrix} \times \begin{pmatrix} t_{11} \\ t_{12} \\ t_{21} \\ t_{22} \end{pmatrix} = \begin{pmatrix} \beta_3 - \varepsilon_3 \\ \beta_2 + \varepsilon_2 \\ \beta_{15} - \varepsilon_{15} \\ \beta_{18} - \varepsilon_{18} \end{pmatrix} \tag{95}$$

$$\begin{pmatrix} \beta_3 \\ \beta_2 \\ \beta_{15} \\ \beta_{18} \end{pmatrix} \equiv \begin{pmatrix} \frac{\underline{\theta} p_{11} q_{11}^{1-\gamma} + \underline{\theta} p_{12} q_{12}^{1-\gamma}}{1-\gamma} \\ \frac{\overline{\theta} p_{12}\left(q_{11}^{1-\gamma} - q_{21}^{1-\gamma}\right) + \overline{\theta} p_{22}\left(q_{12}^{1-\gamma} - q_{22}^{1-\gamma}\right)}{1-\gamma} \\ 2^{\gamma}\left(\underline{\theta} - \frac{p_{12}^2 \epsilon \Delta\theta}{p_{11}p_{12}+\rho\epsilon}\right) \frac{(2q_{11})^{1-\gamma} - (q_{12}+q_{21})^{1-\gamma}}{1-\gamma} \\ \left[\left(\underline{\theta} - \frac{p_{22}\epsilon\Delta\theta}{p_{12}}\right)^{\frac{1}{\gamma}} + \overline{\theta}^{\frac{1}{\gamma}}\right]^{\gamma} \frac{(q_{12}+q_{21})^{1-\gamma} - (2q_{22})^{1-\gamma}}{1-\gamma} \end{pmatrix},$$

where

$$\sum_{i,j} p_{ij} t_{ij} = \frac{p_{12}+p_{22}}{\rho+p_{12}}(\beta_3 - \varepsilon_3) - \frac{p_{12}}{\rho+p_{12}}(\beta_2+\varepsilon_2) - \frac{\rho(1-p_{11})}{2(\rho+p_{12})}(\beta_{15}-\varepsilon_{15}) - \frac{\rho p_{22}}{2(\rho+p_{12})}(\beta_{18}-\varepsilon_{18}). \tag{96}$$

To maximize the expected transfer $\sum_{i,j} p_{ij}t_{ij}$, the best choices are $\epsilon = 1, \varepsilon_i = 0, i = 2, 3, 15, 18$. After optimizing with respect to q_{ij} we get the following expressions

$$\left\{\left(\frac{\overline{\theta} p_{12}}{\rho+p_{12}}\right) + \left(\frac{\left(\underline{\theta} - \frac{p_{22}\Delta\theta}{p_{12}}\right)^{\frac{1}{\gamma}} + \overline{\theta}^{\frac{1}{\gamma}}}{2}\right)^{\gamma} \frac{\rho}{\rho+p_{12}}\right\} q_{22}^{-\gamma} = c \tag{97}$$

$$\begin{aligned} &\left(\frac{\overline{\theta} p_{12}}{\rho+p_{12}}\right) q_{21}^{-\gamma} + \left\{\frac{1-p_{11}}{2}\left(\underline{\theta} - \frac{p_{12}^2\Delta\theta}{p_{11}p_{12}+\rho}\right) - \frac{p_{22}}{2}\left(\frac{\left(\underline{\theta} - \frac{p_{22}\Delta\theta}{p_{12}}\right)^{\frac{1}{\gamma}} + \overline{\theta}^{\frac{1}{\gamma}}}{2}\right)^{\gamma}\right\} \\ &\times \frac{\rho}{(\rho+p_{12})p_{12}}\left(\frac{q_{12}+q_{21}}{2}\right)^{-\gamma} = c \end{aligned} \tag{98}$$

$$\begin{aligned} &\left(\frac{\underline{\theta} p_{12} - p_{22}\Delta\theta}{\rho+p_{12}}\right) q_{12}^{-\gamma} + \left\{\frac{1-p_{11}}{2}\left(\underline{\theta} - \frac{p_{12}^2\Delta\theta}{p_{11}p_{12}+\rho}\right) - \frac{p_{22}}{2}\left(\frac{\left(\underline{\theta} - \frac{p_{22}\Delta\theta}{p_{12}}\right)^{\frac{1}{\gamma}} + \overline{\theta}^{\frac{1}{\gamma}}}{2}\right)^{\gamma}\right\} \\ &\times \frac{\rho}{(\rho+p_{12})p_{12}}\left(\frac{q_{12}+q_{21}}{2}\right)^{-\gamma} = c \end{aligned} \tag{99}$$

$$\left\{\underline{\theta}\left(\frac{p_{12}+p_{22}}{\rho+p_{12}}\right)-\overline{\theta}\left(\frac{p_{12}^2}{p_{11}\left(\rho+p_{12}\right)}\right)-\frac{\rho(1-p_{11})}{p_{11}(\rho+p_{12})}\left(\underline{\theta}-\frac{p_{12}^2\Delta\theta}{p_{11}p_{12}+\rho}\right)\right\}q_{11}^{-\gamma}=c \tag{100}$$

From (98) and (99), we can verify the no-arbitrage constraint

$$\frac{q_{12}}{q_{21}}=\frac{\left(\underline{\theta}-\frac{p_{22}\Delta\theta}{p_{12}}\right)^{\frac{1}{\gamma}}}{\overline{\theta}^{\frac{1}{\gamma}}}, \tag{101}$$

since $\epsilon=1$.

The only work left is to verify the monotonic implementability conditions. Because

$$\begin{aligned}
\lim_{\rho\to 0^-}q_{11}^C(\rho)&=\left(\frac{\underline{\theta}-\frac{1-v}{v}\Delta\theta}{c}\right)^{\frac{1}{\gamma}}=q_{11}^N(0)=q_{11}^C(0)\\
\lim_{\rho\to 0^-}q_{12}^C(\rho)&=\left(\frac{\underline{\theta}-\frac{1-v}{v}\Delta\theta}{c}\right)^{\frac{1}{\gamma}}=q_{12}^N(0)=q_{12}^C(0)\\
\lim_{\rho\to 0^-}q_{21}^C(\rho)&=\left(\frac{\overline{\theta}}{c}\right)^{\frac{1}{\gamma}}=q_{21}^N(0)=q_{21}^C(0)\\
\lim_{\rho\to 0^-}q_{22}^C(\rho)&=\left(\frac{\overline{\theta}}{c}\right)^{\frac{1}{\gamma}}=q_{22}^N(0)=q_{22}^C(0)
\end{aligned}$$

and

$$q_{11}^N(0)<\frac{q_{12}^N(0)+q_{21}^N(0)}{2}<q_{11}^N(0),$$

it can be verified that when ρ is close enough to zero the implementability monotonic condition is satisfied strictly: $q_{11}^C(\rho)<\frac{q_{12}^C(\rho)+q_{21}^C(\rho)}{2}<q_{22}^C(\rho)$.

■