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Generalized KKM Theorems, Minimax Inequalities, and Their Applications[1]

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Abstract. This paper extends the well-known KKM theorem and variational inequalities by relaxing the closedness of values of a correspondence and lower semicontinuity of a function. The approach adopted is based on Michael's continuous selection theorem. As applications, we provide theorems for the existence of maximum elements of a binary relation, a price equilibrium, and the complementarity problem. Thus our theorems, which do not require the openness of lower sections of the preference correspondences and the lower semicontinuity of the excess demand functions, generalize many of the existence theorems such as those in Sonnenschein (Ref. 1), Yannelis and Prabhakar (Ref. 2), and Border (Ref. 3).

Key Words. KKM theorem, variational inequalities, complementarity problem, price equilibrium, maximal elements, binary relations.

1. Introduction

The classical Knaster–Kuratowski–Mazurkiewicz (KKM) theorem in Ref. 4 is of fundamental importance in nonlinear convex analysis, game theory, economics, and optimization theory; it is equivalent to many basic results such as the Sperner lemma, the Brouwer fixed-point theorem, and the Ky Fan minimax inequality. Many generalizations of the KKM theorem and the Ky Fan minimax inequality have been given, such as those in Ky Fan (Refs. 5–7), Aubin (Ref. 8), Yen (Ref. 9), Aubin and Ekeland (Ref. 10), Takahashi (Ref. 11), Zhou and Chen (Ref. 12), Bardaro and Ceppitelli (Refs. 13–14), Shih and Tan (Refs. 15–16), Tian (Refs. 17–19),

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and Tian and Zhou (Ref. 20) among many others. However, all the results obtained are based on the assumptions that correspondences for generalized KKM theorems are transfer closed-valued and that functions for variational inequalities are lower semicontinuous.

The purpose of this paper is to generalize the KKM theorem and the Ky Fan inequality by relaxing the closedness of values of a correspondence for the KKM theorem and the lower semicontinuity of a function for variational inequality when the topological vector spaces under consideration are separable Banach spaces. Since we do not assume that a correspondence is transfer closed-valued, the finite intersection property cannot be used to prove the generalized KKM theorem. The approach adopted in this paper is based on a selection theorem (Theorem 3.1‴ of Michael, Ref. 21). As noted, our results also relax the compactness and convexity of sets, and the quasiconcavity of functions of the Ky Fan minimax inequality. Since the Ky Fan minimax inequality is a fundamental variational inequality, many existence theorems for variational inequalities and convex analysis can also be generalized by our minimax inequality.

As applications of these results to economics and optimization theory, we generalize a class of existence theorems on the maximal elements of a binary relation, price equilibrium, and the complementarity problem by relaxing the compactness and convexity of choice sets, the openness of lower sections of preference correspondences, and the continuity of excess demand functions. Further, we prove below that our generalized KKM theorem, minimax inequality, and existence theorems on maximal elements are equivalent to one another.

The paper is organized as follows. Notation and definitions are given in Section 2. The generalized versions of the KKM theorem and their proofs are given in Section 3. In Section 4, we give the generalized Ky Fan minimax inequality by using our generalized KKM theorem. The existence of maximal elements is presented in Section 5. In Section 6, we consider price equilibrium and the complementarity problem.

2. Notation and Definitions

Let X and Y be two topological spaces, and let 2^Y be the collection of all subsets of Y. A correspondence $F: X \to 2^Y$ is said to be upper semicontinuous (in short, u.s.c.) if the set $\{x \in X: F(x) \subset V\}$ is open in X for every open subset V of Y. A correspondence $F: X \to 2^Y$ is said to be lower semicontinuous (in short, l.s.c.) if the set $\{x \in X: F(x) \cap V \neq \emptyset\}$ is open in X for every open subset V of Y, or equivalently if, for any $y \in Y$, $x \in F(y)$,

and any $\mathcal{N}(x)$ neighborhood of x, there exists a neighborhood $\mathcal{N}(y)$ of y such that

$$\mathcal{N}(x) \cap F(y') \neq \emptyset, \qquad \text{for all } y' \in \mathcal{N}(y).$$

A correspondence $F: X \to 2^Y$ is said to be continuous if it is both u.s.c. and l.s.c. A correspondence $F: X \to 2^Y$ is said to have open lower sections if the set

$$F^{-1}(y) = \{x \in X: y \in F(x)\}$$

is open in X for every $y \in Y$. A correspondence $F: X \to 2^Y$ is said to have open upper sections if, for every $x \in X$, $F(x)$ is open in Y. Denote by cl B and con B the closure and convex hull of the set B, respectively. A correspondence $F: X \to 2^Y$ is said to be transfer closed-valued on X if, for every $x \in X$, $y \notin F(x)$ implies that there exists a point $x' \in X$ such that $y \notin \text{cl } F(x')$. A set B is said to be σ-compact if there is a sequence $\{B_n\}$ of compact sets of B satisfying

$$B = \bigcup_{n=1}^{\infty} B_n.$$

Definition 2.1. FS-Convexity. Let Y be a convex subset of a topological vector space E, and let $\emptyset \neq X \subset Y$. A correspondence $F: X \to 2^Y$ is said to be FS-convex[3] on X if, for every finite subset $\{x_1, x_2, \ldots, x_m\}$ of X,

$$\text{con}\{x_1, x_2, \ldots, x_m\} \subset \bigcup_{j=1}^{m} F(x_j).$$

Remark 2.1. Note that the FS-convexity of F implies that every point $x \in X$ is a fixed point of $F(x)$, i.e., $x \in F(x)$.

Definition 2.2. SS-Convexity. Let Y be a convex subset of a topological vector space E. A correspondence $U: Y \to 2^Y$ is said to be SS-convex[4] if

$$x \notin \text{con } U(x), \qquad \text{for all } x \in Y.$$

Definition 2.3. Generalized SS-Convexity. Let Y be a convex subset of a topological vector space E, and let $\emptyset \neq X \subset Y$. A correspondence $U: Y \to 2^X$ is said to be generalized SS-convex on X if, for every finite subset $\{x_1, x_2, \ldots, x_m\}$ of X and $x_0 \in \text{con}\{x_1, x_2, \ldots, x_m\}$,

$$x_j \notin U(x_0), \qquad \text{for some } 1 \leq j \leq m.$$

[3]FS stands for Fan (Ref. 6) and Sonnenschein (Ref. 1).
[4]SS stands for Shafer and Sonnenschein (Ref. 22).

Remark 2.2. Note that the SS-convexity of U implies the generalized SS-convexity. The converse statement may not be true unless $X = Y$.

Let X be a topological space. A function $f: X \to \mathbb{R} \cup \{\pm\infty\}$ is said to be lower semicontinuous if its set $\{x \in X: f(x) \leq a\}$ is a closed subset of X for all $a \in \mathbb{R}$.

Definition 2.4. γ-Diagonal Quasi-Concavity. Let Y be a convex subset of a topological vector space E, and let $\varnothing \neq X \subset Y$. A function $\phi(x, y): X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ is said to be γ-diagonally quasi-concave (γ-DQCV) in x, if, for every finite subset $\{x_1, \ldots, x_m\} \subset X$ and any $x_0 \in \operatorname{con}\{x_1, \ldots, x_m\}$, we have

$$\inf_{1 \leq j \leq m} \phi(x_j, x_0) \leq \gamma.$$

Remark 2.3. The above definition on γ-DQCV is more general than that of Zhou and Chen (Ref. 12). Here, we do not require that $X = Y$ and that X be convex.

Remark 2.4. It is easily shown that a function $\phi: X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ is γ-DQCV in x if and only if the correspondence $F: X \to 2^Y$, defined by

$$F(x) = \{y \in Y: \phi(x, y) \leq \gamma\}, \qquad \text{for all } x \in X,$$

is FS-convex.

Let Y be a topological space, and let $\succ$ be a (strict) preference relation defined on Y which is a subset of $Y \times Y$. Here, Y may be considered as a consumption space. The expression $(x, y) \in \succ$ is written as $x \succ y$ and reads "x is (strictly) preferred to y." For each x, let $U_s(x)$ be the upper-contour set of x whose elements are strictly preferred to x. We call the correspondence U_s the preference correspondence.

In some cases, not all points in Y can be chosen; so, let $B \subset Y$ be a choice set, which may be considered as, say, the budget set or the feasible set.

Definition 2.5. Maximal Element. A binary relation $\succ$ is said to have a maximal element on the subset B of Y if there exists a point $x^* \in B$ such that $\neg x \succ x^*$ for all $x \in B$, or equivalently $U_s(x^*) = \varnothing$ on B, where $\neg$ stands for "it is not the case that."

3. Generalized KKM Theorems

In this section we give two generalized KKM theorems. Before proceeding to the main theorems, we state some technical lemmas which are due to Michael (Ref. 21, Proposition 2.6 and Theorem 3.1‴).

Lemma 3.1. Let X be a topological space; let Y be a convex set of a topological vector space; and let $\phi: X \to 2^Y$ be l.s.c. Then, the correspondence $\psi: X \to 2^Y$ defined by $\psi(x) = \text{con}\, \phi(x)$ is l.s.c.

Lemma 3.2. Let X be a perfectly normal T_1-topological space, and let Y be a separable Banach space. Let $\mathscr{D}(Y)$ be the collection of all nonempty and convex subsets of Y which are either finite-dimensional or closed or have an interior point. Suppose that $F: X \to \mathscr{D}(Y)$ is an l.s.c. correspondence such that $F(x)$ is nonempty and convex for all $x \in X$. Then, there exists a continuous function $f: X \to Y$ such that $f(x) \in F(x)$ for all $x \in X$.

We begin by stating the KKM theorem whose proof can be found in Fan (Ref. 7).

Theorem 3.1. KKM Theorem. In a topological vector space, let Y be a compact convex set, and let $\varnothing \neq X \subset Y$. Suppose that $F: X \to 2^Y$ is a correspondence such that:

(a) it is closed-valued on Y;
(b) it is FS-convex on X.

Then, $\bigcap_{x \in X} F(x) \neq \varnothing$.

Here, by relaxing the closedness condition, we extend Theorem 3.1 to the following result.

Theorem 3.2. In a separable Banach space, let Y be a compact convex set, and let $\varnothing \neq X \subset Y$. Let $\mathscr{D}(Y)$ be the collection of all nonempty subsets of Y which are either finite-dimensional or closed or have an interior point. Suppose that $F: X \to \mathscr{D}(Y)$ is a correspondence such that:

(a) the correspondence $U: Y \to 2^X$, defined by

$$U(y) = \{x \in X: y \notin F(x)\} = X \backslash F^{-1}(y),$$

is l.s.c. on Y;
(b) F is FS-convex on X.

Then, $\bigcap_{x \in X} F(x)$ is nonempty and compact.

Proof. Define a correspondence $G: Y \to 2^Y$ by

$$G(x) = \begin{cases} F(x), & \text{if } x \in X, \\ Y, & \text{otherwise.} \end{cases} \tag{1}$$

Then,

$$\bigcap_{x \in Y} G(x) = \bigcap_{x \in X} F(x) \neq \varnothing.$$

Thus, we only need to prove that

$$\bigcap_{x \in Y} G(x) \neq \varnothing$$

and is compact. We first note that G is clearly FS-convex. Consider a correspondence $P: Y \to 2^Y$ defined by

$$P(y) = \{x \in Y: y \notin G(x)\} = Y \backslash G^{-1}(y).$$

Then, $P(y) = U(y)$ for all $y \in Y$. Indeed, it is clear that $U(y) \subset P(y)$ for all $y \in Y$. So we only need to show $P(y) \subset U(y)$ for all $y \in Y$. Suppose that $x \in P(y)$. Then, $x \in Y$ and $y \notin G(x)$. Thus, by the definition of $G(x)$, we must have $x \in X$, and therefore $y \notin F(x)$. So $y \in U(y)$. Hence,

$$P(y) = U(y), \qquad \text{for all } y \in Y,$$

and thus P is l.s.c.

Since

$$\{y \in Y: P(y) = \varnothing\} = \bigcap_{x \in Y} G(x),$$

then proving that

$$\bigcap_{x \in Y} G(x) \neq \varnothing$$

is equivalent to proving that

$$\{x: P(x) = \varnothing\} \neq \varnothing.$$

Now, we show that P is SS-convex on Y, i.e., $y \notin \operatorname{con} P(y)$ for all $y \in Y$. Indeed, suppose, by way of contradiction, that $y_0 \in \operatorname{con} P(y_0)$. Then, there exist $\{y_1, \ldots, y_m\}$ and $\lambda_i \geq 0$, $i = 1, \ldots, m$, with $\sum_{i=1}^m \lambda_i = 1$, such that $y_0 = \sum_{i=1}^m \lambda_i y_i$ and $y_i \in P(y_0)$ for all i. Then, $y_i \notin G^{-1}(y_0)$, and thus

$$y_0 \notin G(y_i), \qquad \text{for all } i,$$

which contradicts the fact that G is FS-convex.

Now, suppose that, for all $y \in Y$, $P(y) \neq \varnothing$. Then the correspondence $\psi: Y \to 2^Y$, defined by $\psi(x) = \operatorname{con} P(x)$ for all $y \in Y$, is nonempty and convex valued. Since P is l.s.c. on Y, by Lemma 3.1, $\operatorname{con} P$ is l.s.c. on Y. Hence, by Lemma 3.2, there exists a continuous function $f: Y \to Y$ such that $f(x) \in \psi(x)$ for all $x \in Y$. Hence, by the Brouwer fixed-point theorem, there exists a point $x^* \in Y$ such that $x^* \in f(x^*)$. Then,

$$x^* = f(x^*) \in \psi(x^*) = \operatorname{con} P(x^*),$$

a contradiction. Then, $\operatorname{con} P(x^*) = \varnothing$, and thus

$$\bigcap_{x \in X} G(x) \neq \varnothing.$$

Finally, we show that $\bigcap_{x \in X} G(x)$ is compact. Since

$$\bigcap_{x \in Y} G(x) = \{y \in Y: P(y) = \varnothing\},$$

to show that $\bigcap_{x \in X} G(x)$ is closed, it suffices to show that $\{y \in Y: P(y) \neq \varnothing\}$ is open. But this comes from the fact that

$$\{y \in Y: P(y) \neq \varnothing\} = \{y \in Y: P(y) \cap Y \neq \varnothing\}$$

is open in Y, since P is l.s.c. Thus, $\bigcap_{x \in X} G(x)$ is closed and therefore compact by the compactness of Y. □

Remark 3.1. Observe that, in the case where $F(x)$ is closed in Y, P has open lower sections, and thus P is lower semicontinuous by Proposition 4.1 of Yannelis and Prabhakar (Ref. 2). In fact, they provide an example in their Remark 4.1 to show that the lower semicontinuity condition of P is strictly weaker than the open lower sections condition of P. Thus, Theorem 3.2 extends Theorem 3.1 by using the weaker continuity condition. But on the other hand, we need to require that the topological vector space in Theorem 3.2 be a separable Banach space, which is stronger than that required in Theorem 3.1. This is the cost that one must pay in order to have the weaker topological condition. Nevertheless, for most problems considered in economics and optimization theory, this requirement is reasonable.

Remark 3.2. In Tian (Refs. 17–18), Theorem 3.1 has been generalized by relaxing the closedness of values of F to the transfer closedness of values of F which in general has no relationship with the lower semicontinuity of P.

In the following, we generalize Theorem 3.2 by relaxing the compactness of Y. Since F is not necessarily closed-valued nor transfer closed-

valued, we cannot, as in the approach adopted by Fan (Ref. 7) and Tian (Ref. 18), use the finite intersection property to argue the nonemptiness. Instead, we assume that Y is σ-compact. For simplicity, we also assume that the topological space is finite dimensional.

Theorem 3.3. Let Y be a σ-compact convex set of $\mathbb{R}^m$, and let $\varnothing \neq X \subset Y$. Suppose that $F: X \to 2^Y$ is a correspondence such that

(a) the correspondence $U: Y \to 2^X$, defined by

$$U(y) = \{x \in X: y \notin F(x)\},$$

is l.s.c. on Y;

(b) F is FS-convex on X;

(c) there is a nonempty compact subset X_0 of X such that, for each $y \in Y \backslash X_0$, there is an $x \in X_0$ such that $y \notin F(x)$.

Then, $\bigcap_{x \in X} F(x)$ is nonempty and compact.

Proof. Again, define the correspondence $P: Y \to 2^Y$ by

$$P(y) = \{x \in Y: y \notin G(x)\}.$$

Thus, to show that

$$\bigcap_{x \in X} F(x) \neq \varnothing,$$

it suffices to show that

$$P(y^*) = \varnothing, \qquad \text{for some } y^* \in Y.$$

Since Y is σ-compact, there is a sequence $\{Y_n\}$ of compact sets of Y satisfying

$$Y = \bigcup_{n=1}^{\infty} Y_n.$$

Let

$$K_n = \mathrm{co}\left\{\bigcup_{j=1}^{n} Y_i \cup X_0\right\}.$$

Then, $\{K_n\}$ is an increasing sequence of compact convex sets each containing X_0 with

$$\bigcup_{n=1}^{\infty} K_n = Y;$$

cf. Border (Ref. 3, p. 10). By Theorem 3.2, for each K_n, it follows from Conditions (a) and (b) that there exists a point $y_n \in K_n$ such that

$P(y_n) = \varnothing$. Since $X_0 \subset K_n$, Condition (c) implies that $y_n \in X_0$. Since X_0 is compact, we can extract a convergent subsequence $y_n \to y^* \in X_0$.

Suppose that $P(y^*) \neq \varnothing$. Let $x \in P(y^*)$. Since P is l.s.c. for any neighborhood $\mathcal{N}(x)$, there exists a neighborhood $\mathcal{N}(y^*)$ of y^* such that

$$\mathcal{N}(x) \cap P(y) \neq \varnothing, \qquad \text{for all } y \in \mathcal{N}(y^*).$$

Then,

$$x \in P(y), \qquad \text{for all } y \in \mathcal{N}(y^*).$$

Thus, for large enough n, we have $y_n \in \mathcal{N}(y^*)$ and $x \in K_n$. Hence, $x \in P(y_n) \cap K_n$, contradicting $P(y_n) \cap K_n = \varnothing$. Hence, $P(y^*) = \varnothing$ and thus $\bigcap_{x \in X} F(x) \neq \varnothing$. Similarly, since

$$\{y \in Y: P(y) \neq \varnothing\} = \{y \in Y: P(y) \cap Y \neq \varnothing\}$$

is open in Y by the lower semicontinuity of P,

$$\bigcap_{x \in Y} G(x) = \{y \in Y: P(y) = \varnothing\}$$

is closed and therefore compact, since it is contained in the compact set X_0. □

4. Generalized Ky Fan Minimax Inequality

In this section, we use the results obtained in the last section to extend the minimax inequalities of Fan (Ref. 5), Allen (Ref. 23), and Zhou and Chen (Ref. 12) by relaxing the convexity of sets and the lower semicontinuity of functions.

Theorem 4.1. Let Y be a σ-compact convex set of $\mathbb{R}^m$; let $\varnothing \neq X \subset Y$; let $\gamma \in \mathbb{R}$; and let $\phi: X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ be a function such that:

(i) the correspondence $P: Y \to 2^X$, defined by

$$P(y) = \{x \in X: \phi(x, y) > \gamma\},$$

is l.s.c.;

(ii) it is γ-diagonally quasi-concave in x;

(iii) there exists a nonempty compact set $C \subset X$ such that, for each $y \in Y \backslash C$, there exists $x \in C$ with $\phi(x, y) > \gamma$.

Then, there exists a point $y^* \in X$ such that $\phi(x, y^*) \leq \gamma$ for all $x \in X$.

Proof. For $x \in X$, let

$$F(x) = \{y \in Y: \phi(x, y) \leq \gamma\}.$$

Then, by Conditions (i)–(iii) of Theorem A.1, $F(x)$ satisfies Conditions (a)–(c) of Theorem 3.3. Hence, by Theorem 3.3, $\bigcap_{x \in X} F(x) \neq \varnothing$ on X. Thus, there is a point $y^* \in X$ such that

$$\phi(x, y^*) \leq \gamma, \qquad \text{for all } x \in X. \qquad \square$$

Remark 4.1. Note that Condition (i) of Theorem 4.1 is satisfied if $\phi(x, y)$ is lower semicontinuous in y; Condition (iii) is satisfied if $X = Y$ and Y is compact.

Corollary 4.1. Generalized Ky Fan's Minimax Inequality. Let the hypotheses of Theorem 4.1 hold for $\gamma = \sup_{x \in X} \phi(x, x)$. Then, there exists a point $y^* \in X$ such that

$$\phi(x, y^*) \leq \sup_{x \in X} \phi(x, x), \qquad \text{for all } x \in X.$$

Remark 4.2. Theorem 4.1 or Corollary 4.1 generalizes the minimax inequalities of Fan (Ref. 5) by relaxing the quasiconcavity and lower semicontinuity of ϕ and the convexity and compactness of X; of Allen (Ref. 23) by relaxing the quasiconcavity and lower semicontinuity of ϕ and the convexity of X; and of Zhou and Chen (Ref. 12) by relaxing the lower semicontinuity of ϕ and the convexity of X.

The following claim states that we can also derive Theorem 3.3 from Theorem 4.1.

Claim 4.1. Theorem 4.1 $\Rightarrow$ Theorem 3.3.

Proof. Define

$$G = \{(x, y) \in X \times Y: y \in F(x)\},$$

and define $\phi: X \times Y \to \mathbb{R} \cup \{+\infty\}$ by

$$\phi(x, y) = \begin{cases} \gamma, & \text{if } (x, y) \in G, \\ +\infty, & \text{otherwise,} \end{cases}$$

where $\gamma \in \mathbb{R}$. Then,

$$\{y \in Y: \phi(x, y) \leq \gamma\} = \{y \in Y: y \in F(x)\}.$$

Thus, Conditions (i)–(iii) of Theorem 4.1 are satisfied by Conditions (a)–(c) of Theorem 3.3. Thus, by Theorem 4.1, there exists a point $y^* \in X$ such that $\phi(x, y^*) \leq \gamma$ for all $x \in X$. That is,

$$y^* \in F(x), \qquad \text{for all } x \in X.$$

So,

$$\bigcap_{x \in X} F(x) \neq \varnothing, \qquad \text{on } X.$$

Thus, Theorem 3.3 and Theorem 4.1 are equivalent. □

5. Existence of Maximal Elements

In this section, we use Theorem 3.3 to prove the following theorem, which gives sufficient conditions for the existence of maximal elements of preference relations on noncompact and nonconvex choice sets. It will be noted that the preference correspondence may not have open lower sections. The preference relations may be nontransitive-nontotal.

Theorem 5.1. Let Y be a σ-compact convex set of $\mathbb{R}^m$, let $\varnothing \neq B \subset Y$, and let $\succ$ be a binary relation on Y such that:

(a) $\hat{U}_s: Y \to 2^B$, defined by

$$\hat{U}_s(y) = \{x \in B: x \succ y\}, \qquad \text{for all } y \in Y,$$

is l.s.c. on Y;

(b) $\hat{U}_s$ is generalized SS-convex on B;

(c) there exists a nonempty compact set $C \subset Y$ such that, for each $y \in Y \backslash C$, there exists $x \in C$ with $x \succ y$.

Then, $\succ$ has a maximal element on B; i.e., there exists some $x^* \in B$ such that $\hat{U}_s(x^*) = \varnothing$.

Proof. Let

$$F(x) = \{y \in Y: x \notin \hat{U}_s(y)\}.$$

Then,

$$\{x \in B: \hat{U}_s(x) = \varnothing\} = \bigcap_{x \in B} F(x),$$

and Conditions (a)–(c) of Theorem 5.1 imply Conditions (a)–(c) of Theorem 3.3. So, we only need to show that F is FS-convex on B. By way

of contradiction, suppose that there exists a point $x_0 \in \text{con}\{x_1, x_2, \ldots, x_m\}$ of Y which is not in

$$F(x_j) = \{y \in Y: x \notin \hat{U}_s(y)\} = Y \backslash \hat{U}_s^{-1}(x), \qquad \text{for all } j.$$

Then, $x_j \succ x_0$, so $x_j \in \hat{U}_s(x_0)$ for all j, which contradicts the generalized SS-convexity of $\hat{U}_s(x_0)$. Hence, by Theorem 3.3,

$$\bigcap_{x \in B} F(x) \neq \varnothing.$$

So, there exists some point $x^* \in B$ such that $\hat{U}_s(x^*) = \varnothing$. □

Remark 5.1. Theorem 5.1 generalizes the results of Sonnenschein (Ref. 1) and Yannelis and Prabhakar (Ref. 2) by relaxing the openness of lower sections of preference correspondences and the compactness and convexity of choice sets. Note that Theorem 5.1 and Theorem 2 in Tian (Ref. 18) give two different ways of generalizing the results of Sonnenschein (Ref. 1) and Yannelis and Prabhakar (Ref. 2).

It may be remarked that Theorem 5.1 is clearly equivalent to Theorem 3.3. Thus, our Theorem 3.3, Theorem 4.1, and Theorem 5.1 are equivalent to one another.

6. Price Equilibrium and Complementarity Problem

In the following, we will study the existence of price equilibrium by using Theorem 4.1. The equilibrium price problem is to find a price vector p which clears the markets for all commodities [the excess demands $f(p) \leq 0$ for the free disposal equilibrium price or $f(p) = 0$] under the assumption of the Walras law. Here, we give an existence theorem on price equilibrium by relaxing the lower semicontinuity of the excess demand functions on the Euclidean space $\mathbb{R}^{n+1}$.

Theorem 6.1. Let Δ_n be the closed standard n-simplex, and let $f: \Delta_n \to \mathbb{R}^{n+1}$ be an excess demand function such that:

(i) the correspondence $P: \Delta_n \to 2^{\Delta_n}$, defined by

$$P(q) = \{p \in \Delta_n : p \cdot f(q) > 0\},$$

is lower semicontinuous in q;

(ii) for all $p \in \Delta_n$, $p \cdot f(p) \leq 0$ (the Walras law).

Then, there exists a $q^* \in \Delta_n$ such that $f(q^*) \leq 0$.

Proof. Let $\phi(p, q) = p \cdot f(q)$. Then, ϕ satisfies all conditions of Theorem 4.1 with $\gamma = 0$, by noting that Δ_n is compact, so that Condition (iii) of Theorem 4.1 is satisfied. Thus, there exists some $q^* \in \Delta_n$ such that

$$\phi(p, q^*) = p \cdot f(q^*) \leq 0, \qquad \text{for all } p \in \Delta_n.$$

Hence, $f(q^*) \leq 0$. □

Remark 6.1. Note that Condition (i) of Theorem 6.1 holds if f is lower semicontinuous.

We now consider a mathematically more general problem, which is known as the nonlinear complementarity problem.

Let X be a convex cone of a topological vector space E, let $f: X \to E^*$ (dual of E). The problem is to find a p such that $f(p) \in X^* \subset E^*$ (which is the polar cone) and $\langle p, f(p) \rangle = 0$. In particular, if $X = \mathbb{R}_+^{n+1}$, then the condition that $f(p) \in X^*$ becomes $f(p) \leq 0$. In the following, we give an existence theorem on complementarity problems extending the results of Allen (Ref. 23, Corollary 2). Again for simplicity, we assume that E is the Euclidian space $\mathbb{R}^{n+1}$.

Theorem 6.2. Let X be a σ-compact convex cone in $\mathbb{R}^{n+1}$, and let $f: X \to \mathbb{R}^{n+1}$ be a function such that:

(i) the correspondence $P: X \to 2^X$, defined by

$$P(y) = \{x \in X: \langle x - y, f(y) \rangle > 0\},$$

is lower semicontinuous;

(ii) there exists a nonempty compact set $C \subset X$ such that, for each $y \in X \backslash C$, there exists some $x \in C$ with $\langle x - y, f(y) \rangle > 0$.

Then, there exists $y^* \in X$ such that $f(y^*) \leq 0$ and $\langle y^*, f(y^*) \rangle = 0$.

Proof. For every $(x, y) \in X \times X$, define the function $\phi: X \times X \to \mathbb{R}$ by

$$\phi(x, y) = \langle x - y, f(y) \rangle.$$

Since ϕ is linear in p, it satisfies Condition (ii) of Theorem 4.1. Conditions (i) and (iii) of the same theorem are clearly satisfied. Then, by applying Theorem 4.1 to $\phi(x, y)$ with $\gamma = 0$, we have the existence of some $y^* \in X$ such that

$$\phi(x, y^*) \leq 0, \qquad \text{for all } x \in X.$$

That is,

$$\langle x - y^*, f(y^*) \rangle \leq 0, \qquad \text{for all } x \in X.$$

From Lemma 1 of Allen (Ref. 23), we have that $\langle y^*, f(y^*) \rangle = 0$. So, $f(q^*) \leq 0$. □

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