Minimax inequality equivalent to the Fan-Knaster-Kuratowski-Mazurkiewicz Theorem

Lin, Y. Joseph and Tian, Guoqiang

1993

Appl Math Optim 28:173–179 (1993)

Applied Mathematics & Optimization

Minimax Inequalities Equivalent to the Fan–Knaster–Kuratowski–Mazurkiewicz Theorems

Y. Joseph Lin[1] and Guoqiang Tian[2]

[1] Graduate School of Management, University of California, Riverside, CA 92521, USA

[2] Department of Economics, Texas A&M University, College Station, TX 77843, USA

Communicated by R. Triggiani

Abstract. The purpose of this note is to give further generalizations of the Ky Fan minimax inequality by relaxing the compactness and convexity of sets and the quasi-concavity of the functional and to show that our minimax inequalities are equivalent to the Fan–Knaster–Kuratowski–Mazurkiewicz (FKKM) theorem and a modified FKKM theorem given in this note.

Key Words. The minimax inequality, Variational inequalities, The FKKM theorem, Noncompact and nonconvex sets, Equivalence.

AMS Classification. 49A29, 90C33, 90C50.

1. Introduction

The Knaster–Kuratowski–Mazurkiewicz (KKM) theorem is a very basic and useful result which is equivalent to many basic theorems such as Sperner's lemma, Brouwer's fixed-point theorem, and Ky Fan's minimax inequality. Since Knaster *et al.* [12] gave this theorem, many generalizations of the KKM theorem have been given. Among these generalizations, an important one is the so-called Fan–Knaster–Kuratowski–Mazurkiewicz (FKKM) theorem which was obtained by Ky Fan [9, Theorem 1] (also see Theorem 4 of [10]) and can be used to prove and/or generalize many existence theorems such as fixed-point theorems and coincidence theorems for noncompact convex sets and intersection theorems for sets with convex sections (cf. [8]). Subsequently, Ben-El-Mechaickh *et al.* [4], [5] gave fixed-point theorems for set-valued mappings without compactness of the domain. These fixed-point theorems in fact can be proved to be equivalent to the FKKM theorem. Tarafdar [15] also gave a fixed-point theorem which is equivalent to the FKKM theorem. In this note we generalize the minimax inequalities of Fan [8], Allen [1], and Zhou and Chen [19] which have wide applications to mathematical programming, partial differential equation theory, game theory, impulsive control, and economics [2], [3], [6], [11], [13], [14], [16]–[18] and show that our minimax inequalities are equivalent to the FKKM theorem and a modified FKKM theorem obtained in this note.

We begin with some definitions. Throughout the paper all topological vector spaces are assumed to be Hausdorff and are denoted by E.

Let X be a subset of E and let $\varphi: X \to \mathbb{R} \cup \{\pm\infty\}$. We say the functional φ is *lower semicontinuous* if, for each point x', we have

$$\liminf_{x \to x'} \varphi(x) \geq \varphi(x').$$

An equivalent definition of the lower semicontinuity of φ is that the set $\{x \in X: \varphi(x) \leq a\}$ is closed for every $a \in \mathbb{R}$.

Let Y be a convex subset of E and let $\emptyset \neq X \subset Y$. A functional $\varphi(x, y)$: $X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ is said to be *γ-diagonally quasi-concave* (*γ-DQCV*) in x [19] if, for any finite subset $\{x_1, \ldots, x_m\} \subset X$ and any $x_\lambda = \sum_{j=1}^m \lambda_j x_j$ with $\lambda_j \geq 0$ and $\sum_{j=1}^m \lambda_j = 1$, we have

$$\min_{1 \leq j \leq m} \varphi(x_j, x_\lambda) \leq \gamma.$$

Remark 1. The above definition on γ-DQCV is slightly more general than that of Zhou and Chen [19]. Here we do not require that $X = Y$ and that X be convex.

It is easily shown that an equivalent definition of the γ-diagonal quasi-convexity is that the convex hull of every finite subset $\{x_1, x_2, \ldots, x_m\}$ of X is contained in the corresponding union $\bigcup_{j=1}^m \{y \in Y: \varphi(x_j, y) \leq \gamma\}$.

Remark 2. Zhou and Chen [19] gave a class of diagonal (quasi-)concavity (convexity) conditions which are weaker than the usual (quasi-)concavity (convexity) conditions and from which many theorems in convex analysis and (quasi-)variational inequalities can be generalized.

Denote the convex hull of the set Z by co Z.

2. Generalizations of the Ky Fan Minimax Inequality

Fan [9], [10] has obtained a further generalization of the classical KKM theorem which is stated in the following theorem.

Theorem 1 (Ky Fan). *In a Hausdorff topoligical vector space, let Y be a convex set and $\varnothing \neq X \subset Y$. For each $x \in X$, let $F(x)$ be a relatively closed subset of Y such that the convex hull of every finite subset $\{x_1, x_2, \ldots, x_m\}$ of X is contained in the corresponding union $\bigcup_{j=1}^{m} F(x_j)$. If there is a nonempty subset X_0 of X such that the intersection $\bigcap_{x \in X_0} F(x)$ is compact and X_0 is contained in a compact convex subset of Y, then $\bigcap_{x \in X} F(x) \neq \varnothing$.*

If we want $X \cap [\bigcap_{x \in X} F(x)] \neq \varnothing$, we can modify Theorem 1 to the following form:

Theorem 2. *In a Hausdorff topological vector space, let Y be a convex set and $\varnothing \neq X \subset Y$. For each $x \in X$, let $F(x)$ be a relatively closed subset of Y such that the convex hull of every finite subset $\{x_1, x_2, \ldots, x_m\}$ of X is contained in the corresponding union $\bigcup_{j=1}^{m} F(x_j)$. If there is a nonempty set X_0 of X such that for each $y \in Y \backslash X_0$ there exists $x \in X_0$ with $y \notin F(x)$, and X_0 is contained in a compact convex subset of Y, then $X \cap (\bigcap_{x \in X} F(x)) \neq \varnothing$.*

Proof. By assumption we know that $\bigcap_{x \in X_0} F(x)$ is a closed subset of X_0. Since X_0 is contained in a compact convex subset of Y, $\bigcap_{x \in X_0} F(x)$ is compact. Thus, by Theorem 1, $\bigcap_{x \in X} F(x) \neq \varnothing$. Now for any $y \in \bigcap_{x \in X} F(x)$ we must have $y \in \bigcap_{x \in X_0} F(x) \subset X_0$, for otherwise $y \notin F(x)$ for some $x \in X_0$. Therefore, $y \in X$. □

We now prove the following minimax inequality independently of Theorem 1 and show the equivalence of Theorem 1 with our minimax inequality in Theorem 3 below.

Theorem 3. *Let Y be a nonempty convex subset of a Hausdorff topological vector space E, let $\varnothing \neq X \subset Y$, and let $\varphi: X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ be a functional such that*

(i) *$(x, y) \mapsto \varphi(x, y)$ is lower semicontinuous in y;*
(ii) *$(x, y) \mapsto \varphi(x, y)$ is γ-DQCV in x;*
(iii) *there exists a nonempty subset C of X such that $\bigcap_{x \in C} \{y \in Y: \varphi(x, y) \leq \gamma\}$ is compact and C is contained in a compact convex subset B of Y.*

Then there exists a point $y^* \in Y$ such that $\varphi(x, y^*) \leq \gamma$ for all $x \in X$.

Proof. If $\gamma = \infty$, the conclusion is clearly true. So we assume that $\gamma \neq \infty$. We now prove the theorem by considering two cases.

Case 1. We first consider the case where Y is compact. Suppose, by way of contradiction, that for every $y \in Y$ there exists some point $x \in X$ such that

$$\varphi(x, y) > \gamma. \tag{1}$$

For each $x \in X$, define

$$N(x) = \{y \in Y: \varphi(x, y) > \gamma\}.$$

Then, for all $x \in X$, $N(x)$ is open in Y ($N(x)$ may be empty for some x) by assumption (i). Thus, by (1), we have

$$Y \subset \bigcup_{x \in X} N(x).$$

Since Y is compact, $\{N(x)\}$ has a finite subcover $N(x_1), \ldots, N(x_m)$. Choose a partition of unity μ_j: $Y \to \mathbb{R}$, subordinate to $\{N(x)\}$. Define a map B: $Y \to Y$ by

$$B(y) = \sum_{j=1}^{m} \mu_j(y)x_j,$$

which is continuous and maps Y into $S \equiv \text{co}\{x_1, \ldots, x_m\}$. In particular, B maps S into itself. By Brouwer's fixed-point theorem, there exists $x_\lambda \in S$ such that $B(x_\lambda) = x_\lambda$.

Let $I = \{j: 1 \leq j \leq m \;\&\; \mu_j(x_\lambda) > 0\}$. Then $x_\lambda = \sum_{j \in I} \mu_j(x_\lambda)x_j$ and, for all $j \in I$, $x_\lambda \in N(x_j)$ and thus $\varphi(x_j, x_\lambda) > \gamma$. However, this contradicts assumption (ii).

Case 2. We now consider the case where Y is not compact. Let $D = \bigcap_{x \in C} \{y \in Y: \varphi(x, y) \leq \gamma\}$. Then D is compact by assumption.

Consider an arbitrary finite subset $\{x_1, \ldots, x_m\}$ of X. Let

$$X_1 = X_0 \cup \{x_1, \ldots, x_m\}$$

and let $A = B \cup \{x_1, \ldots, x_m\}$. Since B is compact convex, co A is compact. Also since $X \cup B$ is a subset of Y, we have co $A \subset Y$. Hence, by the conclusion in Case 1, there exists a vector $y' \in \text{co } A$ such that $\varphi(x, y') \leq \gamma$ for all $x \in X_1$. Thus $y' \in D$.

For each $x \in X_1$, let

$$K(x) = \{y \in Y: \varphi(x, y) \leq \gamma\}.$$

Then $y' \in D \cap [\bigcap_{j=1}^{m} K(x_j)]$. Thus, the collection $\{D \cap K(x): x \in X\}$ has the finite intersection property. Since D is compact and $K(x)$ is closed, $D \cap K(x)$ is compact. Hence $\bigcap_{x \in X} [D \cap K(x)] \neq \varnothing$ and therefore $\bigcap_{x \in X} K(x) \neq \varnothing$. So there exists a vector $y^* \in Y$ such that $y^* \in K(x)$ for all $x \in X$ and thus $\varphi(x, y^*) \leq \gamma$ for all $x \in X$. □

Note that Theorem 3 cannot guarantee $y^* \in X$ even if $y^* \in \bigcap_{x \in X} F(x)$. If we require $y^* \in X$, we need to strengthen condition (iii) of Theorem 3 and have the following theorem.

Theorem 4. *Suppose all the conditions in Theorem* 3 *hold except that assumption* (iii) *is replaced by*

(iii)′ *there exists a nonempty set $C \subset X$ such that for each $y \in Y \backslash C$ there exists $x \in C$ with $\varphi(x, y) > \gamma$ and C is contained in a compact convex subset of Y.*

Then there exists a point $y^* \in X$ such that $\varphi(x, y^*) \leq \gamma$ for all $x \in X$.

Proof. By conditions (i) and (iii)′, $D \equiv \bigcap_{x \in C} \{y \in Y: \varphi(x, y) \leq \gamma\}$ is a closed subset of C. Since C is contained in a compact convex subset of Y, D is compact. Thus by Theorem 3 there exists a point $y^* \in Y$ such that $\varphi(x, y^*) \leq \gamma$ for all $x \in X$. Now y^* must be in C, for otherwise hypothesis (iii)′ would be violated. Therefore, $y^* \in X$. □

Remark 3. Theorem 4 is a generalization of the minimax inequality of Fan [8] by relaxing the quasi-concavity of φ and the convexity and compactness of X; a generalization of Allen [1] by relaxing the quasi-concavity of φ and the convexity of X; and a generalization of Zhou and Chen [19] by relaxing the convexity of X.

As an application of Theorem 4, we give the following theorem which generalizes a theorem in Fan [7], [10].

Theorem 5. *Let X be a set in a normal vector space E, and let $\psi: X \to E$ be a continuous map which can be continuously extended to a convex subset Y of E which contains X. Let C be a nonempty subset of X. Suppose, for every $y \in Y \setminus C$, there exists $x \in C$ such that*

$$\|x - \psi(y)\| < \|y - \psi(y)\|$$

and C is contained in a compact convex subset of Y. Then there is a point $\hat{y} \in X$ satisfying

$$\|\hat{y} - \psi(\hat{y})\| = \min_{x \in X} \|x - \psi(\hat{y})\|.$$

(In particular, if $\psi(\hat{y}) \in X$, then $\hat{y}$ is a fixed point of ψ.)

Proof. Define $\varphi: X \times Y \to \mathbb{R}$ by

$$\varphi(x, y) = \|y - \psi(y)\| - \|x - \psi(y)\|.$$

Then $\varphi(x, y)$ satisfies all the assumptions with $\gamma = 0$ for Theorem 4. Hence the result follows from Theorem 4. □

3. Equivalence of Theorems 1 and 3

As noted, the fixed-point theorems of Ben-El-Mechaickh *et al.* [4], [5] and Tarafdar [15] are equivalent to Theorem 1. Here we prove that our Theorem 3 is also equivalent to Theorem 1.

Theorem 1 ⇒ Theorem 3.

Proof. For $x \in X$, let $F(x) = \{y \in Y: \varphi(x, y) \leq \gamma\}$. By condition (i), $F(x)$ is closed in Y. Also, since φ is γ-DQ-CV in x, the convex hull of every finite subset

$\{x_1, x_2, \ldots, x_m\}$ of X is contained in the corresponding union $\bigcup_{j=1}^{m} F(x_j)$. Condition (iii) says that $\bigcap_{x \in C} F(x)$ is compact and C is contained in a compact convex subset of Y. Hence, by Theorem 1, there is a point y^* in $\bigcap_{x \in X} F(x)$, so $\varphi(x, y^*) \leq \gamma$ for all $x \in X$. □

Theorem 3 ⇒ Theorem 1.

Proof. Define $G = \{(x, y) \in X \times Y : y \in F(x)\}$ and define $\varphi : X \times Y \to \mathbb{R} \cup \{+\infty\}$ by

$$\varphi(x, y) = \begin{cases} \gamma & \text{if } (x, y) \in G, \\ +\infty & \text{otherwise,} \end{cases}$$

where $\gamma \in \mathbb{R}$.

Since $F(x)$ is a relatively closed subset of Y, then, for every $x \in X$, φ is lower semicontinuous in $y \in Y$. Also, since the convex hull of every finite subset $\{x_1, x_2, \ldots, x_m\}$ of X is contained in the corresponding union $\bigcup_{j=1}^{m} F(x_j)$, then for any x_λ in the convex hull we have $x_\lambda \in F(x_j)$ for some j. Hence $\varphi(x_j, x_\lambda) \leq \gamma$ for some j and thus φ is γ-DQCV in $x \in X$ for every $y \in Y$. Also,

$$\bigcap_{x \in X_0} \{y \in Y : \varphi(x, y) \leq \gamma\} = \bigcap_{x \in X_0} F(x)$$

is compact and X_0 is contained in a compact convex subset of Y by the assumptions of Theorem 1.

Thus by Theorem 3 there exists a point $y^* \in Y$ such that $\varphi(x, y^*) \leq \gamma$ for all $x \in X$. That is, $y^* \in F(x)$ for all $x \in X$. So $\bigcap_{x \in X} F(x) \neq \emptyset$. □

Thus the Ben-El-Mechaickh *et al.* fixed-point theorems, Tarafdar's fixed-point theorem, the FKKM theorem (Theorem 1), and Theorem 3 are equivalent to one another. We can similarly show that Theorems 2 and 4 are also equivalent.

Acknowledgment

We wish to thank an anonymous referee for helpful comments and suggestions.

References

1. Allen G (1977) Variational inequalities, complementarity problems, and duality theorems. J Math Anal Appl 58:1–10
2. Aubin JP (1979) Mathematical Methods of Game and Economic Theory. North-Holland, Amsterdam
3. Aubin JP, Ekeland I (1984) Applied Nonlinear Analysis. Wiley, New York
4. Ben-El-Mechaickh H, Deguire P, Granas A (1982) Points fixes et coincidences pour les applications multivoques (applications de Ky Fan). C R Acad Sci Paris Sér I Math 295:337–340
5. Ben-El-Mechaickh H, Deguire P, Granas A (1982) Points fixes et coincidences pour les fonctions multivoques II (applications de type φ et φ^*). C R Acad Sci Paris Sér I Math 295:381–384

6. Border KC (1985) Fixed Point Theorems with Applications to Economics and Game Theory. Cambridge University press, Cambridge
7. Fan K (1969) Extensions of two fixed point theorems of F. E. Browder. Math Z 112:234–240
8. Fan K (1972) A minimax inequality and application. In: Inequalities, vol 3 (Shisha O, ed). Academic Press, New York, pp 103–113
9. Fan K (1979) Fixed-point and related theorems for non-compact sets. In: Game Theory and Related Topics (Moeschlin O, Pallaschke D, eds). North-Holland, Amsterdam, pp 151–156
10. Fan K (1984) Some properties of convex sets related to fixed points theorems. Math Ann 266:519–537
11. Hartman PT, Stampacchia G (1966) On some nonlinear ecliptic differential functional equations. Acta Math 115:153–188
12. Knaster B, Kuratowski C, Mazurkiewicz S (1929) Ein Beweis des Fixpunktsatzes für n-demensionale Simpliexe. Fund Math 14:132–137
13. Lassonde M (1983) On the use of KKM multifunctions in fixed point theory and related topics. J Math Anal Appl 97:151–201
14. Mosco U (1976) Implicit variational problems and quasi-variational inequalities. Lecture Notes in Mathematics, vol 543. Springer-Verlag, Berlin, pp 83–156
15. Tarafdar E (1987) A fixed point theorem equivalent to the Fan–Knaster–Kuratowski–Mazurkiewicz theorem. J Math Anal Appl 128:475–479
16. Tian G (1991) Fixed points theorems for mappings with non-compact and non-convex domains. J Math Anal Appl 158:160–167
17. Tian G (1992) Existence of equilibrium in abstract economies with discontinuous payoffs and non-compact choice spaces. J Math Econom 21:379–388
18. Tian G, Zhou J (1991) Quasi-variational inequalities with non-compact sets. J Math Anal Appl 160:583–595
19. Zhou J, Chen G (1988) Diagonal convexity conditions for problems in convex analysis and quasi-variational inequalities. J Math Anal Appl 132:213–225

Accepted 18 August 1992