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Quasi-Variational Inequalities without the Concavity Assumption

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This paper generalizes a foundational quasi-variational inequality by relaxing the (0-diagonal) concavity condition. The approach adopted in this paper is based on continuous selection-type arguments and hence it is quite different from the approach used in the literature. Thus it enables us to prove the existence of equilibrium of the constrained noncooperative games without assuming the (quasi) convexity of loss functions. © 1993 Academic Press, Inc.

1. INTRODUCTION AND PRILIMINARIES

Several recent results such as those in Aubin [3], Aubin and Ekeland [4], Mosco [8], Shih and Tan [13], Zhou and Chen [18], and Tian and Zhou [15] have studied the existence of equilibrium for quasi-variational inequalities which have wide applications to problems in game theory, impulsive control, and economics [2, 3, 6, 8, 12]. In those problems of (quasi-)variational inequalities, a functional $(x, y) \mapsto \phi(x, y)$ is involved. However, all of the results mentioned above assume that ϕ is (0-diagonally) concave in y, while in the problems of variational (minimax) inequalities only (0-diagonal) quasi-concavity is needed to prove the existence. Indeed (0-diagonal) concavity is a crucial assumption in the approach used, say in [3, p. 281], or [4, p. 349], to prove the existence of equilibrium for quasi-

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variational inequalities since it uses the Hahn–Banach theorem and needs the sum of functionals to satisfy the (quasi-)concavity to apply the Ky-Fan minimax inequality.

In this paper, we use a quite different approach to show the existence of equilibrium for quasi-variational inequalities. The approach we adopt is based on continuous selection-type arguments and thus enables us to generalize the existing results by relaxing the (0-diagonal) concavity condition.

We begin with some notation and definitions.

Let X and Y be two topological spaces, and let 2^Y be the collection of all subsets of Y. A correspondence $F: X \to 2^Y$ is said to be *upper semi-continuous* (in short, u.s.c.) if the set $\{x \in X: F(x) \subset V\}$ is open in X for every open subset V of Y. A correspondence $F: X \to 2^Y$ is said to be *lower semi-continuous* (in short, l.s.c.) if the set $\{x \in X: F(x) \cap V \neq \emptyset\}$ is open in X for every open subset V of Y. A correspondence $F: X \to 2^Y$ is said to be *continuous* if it is both u.s.c. and l.s.c. A correspondence $F: X \to 2^Y$ is said to have *open lower sections* if the set $F^{-1}(y) = \{x \in X: y \in F(x)\}$ is open in X for ever $y \in Y$. A correspondence $F: X \to 2^Y$ is said to have *open upper sections* if, for every $x \in X$, $F(x)$ is open in Y. A correspondence $F: X \to 2^Y$ is said to be *closed* if the correspondence has a closed graph, i.e., the set $\{(x, y) \in X \times Y: y \in F(x)\}$ is closed in $X \times Y$. A correspondence $F: X \to 2^Y$ is said to have an *open graph* if the set $\{(x, y) \in X \times Y: y \in F(x)\}$ is open in $X \times Y$. Denote by $\operatorname{co} B$ and $\bar{B}$ the convex hull and the closure of the set B.

Remark 1. It is known that if a correspondence F has an open graph then F open upper and lower sections, and the converse statement may not be true (cf. [5, pp. 265–266]). Also, Yannelis and Prabhakar [16, p. 237] showed that if F has open lower sections, then it is l.s.c., and the converse statement may not be true.

Let X be a topological space. A function $f: X \to \mathbf{R} \cup \{\pm\infty\}$ is said to be *lower semi-continuous* (in short, l.s.c.) on X if for each point $x' \in X$, we have

$$\liminf_{x \to x'} f(x) \geqslant f(x'),$$

or equivalently, its epigraph epi $f \equiv \{(x, a) \in X \times \mathbf{R}: f(x) \leqslant a\}$ is a closed subset of $X \times \mathbf{R}$.

Let X be a convex set of a topological vector space E and let $\phi: X \times X \to \mathbf{R} \cup \{\pm\infty\}$ be a functional. The functional $(x, y) \mapsto \phi(x, y)$ is said to be 0-*diagonally concave* (in short, 0-DCV) in y (cf. [18]), if for any finite subset $\{y_1, ..., y_m\} \subset X$ and any $y_\lambda \in \operatorname{co}\{y_1, ..., y_m\}$ i.e., $y_\lambda = \sum_{j=1}^m \lambda_j y_j$ for $\lambda_j \geqslant 0$ with $\sum_{j=1}^m \lambda_j = 1$), we have

$$\sum_{j=1}^{m} \lambda_j \phi(y_\lambda, y_j) \leqslant 0. \tag{1}$$

A functional $(x, y) \mapsto \phi(x, y)$ is said to be *0-diagonally quasi-concave* (in short, 0-DQCV) in y, if for any finite subset $\{y_1, ..., y_m\} \subset X$ and any $y_\lambda \in \text{co}\{y_1, ..., y_m\}$,

$$\min_j \phi(y_\lambda, y_j) \leqslant 0.$$

A functional $(x, y) \mapsto \phi(x, y)$ is said to be *0-diagonally (quasi-)convex* (in short, 0-DQCX) in y if $-\phi$ is 0-diagonally (quasi-)concave.

Before proceeding to the main theorems, we state some technical lemmas which are needed in the discussions below. The first two lemmas are due to Yannelis [17, p. 103]. The third lemma is due to Michael [7, Theorem 3.1‴] and the last three lemmas (Lemmas 4–6) are due to Yannelis and Prabhakar [16].

LEMMA 1. *Let X and Y be two topological spaces, and let $G: X \to 2^Y$, $K: X \to 2^Y$ be correspondences such that*

(i) *the graph of G is open,*

(ii) *K is l.s.c.*

Then the correspondence $F: X \to 2^Y$ defined by $F(x) = G(x) \cap K(x)$ is l.s.c.

LEMMA 2. *Let X be a topological space and Y be a convex set of a topological vector space, and let $P: X \to 2^Y$ be a correspondence which has an open graph. Then the correspondence $F: X \to 2^Y$ defined by $F(x) = \text{co}\, G(x)$ has open graph as well.*

LEMMAS 3. *Let X be a perfectly normal T_1-topological space and Y be a separable Banach space. Let $\mathscr{D}(Y)$ be the set of all nonempty and convex subsets of Y which are either finite-dimensional or closed or have an interior point. Suppose $F: X \to \mathscr{D}(Y)$ is a l.s.c. correspondence. Then there exists a continuous function $f: X \to Y$ such that $f(x) \in F(x)$ $\forall x \in X$.*

LEMMA 4. *Let X and Y be two topological spaces, and let $G: X \to 2^Y$ and $K: X \to 2^Y$ be correspondences having open lower sections. Then the correspondence $\theta: X \to 2^Y$ defined by, for all $x \in X$, $\theta(x) = G(x) \cap K(x)$, has open lower sections.*

LEMMA 5. *Let X be a topological space and let Y be a convex set of a topological vector space. Suppose a correspondence $G: X \to 2^Y$ has open lower sections. Then the correspondence $F: X \to 2^Y$ defined by $F(x) =$ co $G(x)$ for all $x \in X$ has open lower sections.*

LEMMA 6. *Let X be a paracompact Hausdorff space and Y be topological vector space. Suppose $F: X \to 2^Y$ is a correspondence with nonempty convex values and has open lower sections. Then there exists a continuous functions $f: X \to Y$ such that $f(x) \in F(x)$ for all $x \in X$.*

2. MAIN RESULTS

We now extend the results of Aubin [3, Theorem 9.3.2] and Aubin and Ekeland [4, Corollary 6.4.22] by relaxing the concavity condition. We also extend a result of Zhou and Chen [18, Theorem 3.1] by relaxing the 0-DCV condition. Their results can be stated in the following theorem.

THEOREM 1. *Let Z be a compact convex set in a locally convex Hausdorff topological vector space. Suppose that*

(i) *$K: Z \to 2^Z$ is a continuous correspondence with nonempty closed convex values,*

(ii) *$\phi(x, y): Z \times Z \to \mathbf{R} \cup \{\pm\infty\}$ is lower semi-continuous and is 0-diagonally concave in y.*

Then there exists $x^* \in K(x^*)$ such that $\sup_{y \in K(x^*)} \phi(x^*, y) \leq 0$.

By relaxing the (0-diagonal) concavity condition, we have the following theorem. Note that the method of the proof in the following is different from those given in the references mentioned above. Also, for simplicity, we state the theorem with the weak topology even though it holds for any Hausdorff vector space topology τ which is between the weak topology and the norm topology.

THEOREM 2. *Let Z_w be a nonempty weakly compact convex set in a separable Banach space X with the weak-topology w and Z denote the same set Z_w with the norm topology. Suppose that*

(i) *$K: Z_w \to 2^Z$ is continuous correspondence with nonempty closed and convex values such that $K(x)$ is either finite dimensional or has an (norm) interior point for each $x \in Z$,*

(ii) *$\phi: Z_w \times Z \to \mathbf{R} \cup \{\pm\infty\}$ is l.s.c. and is 0-diagonally quasi-concave in the second variable.*

Then there exists $x^* \in K(x^*)$ such that $\sup_{y \in K(x^*)} \phi(x^*, y) \leq 0$.

Proof. Define a correspondence $P: Z_w \to 2^Z$ by, for each $x \in Z_w$, $P(x) = \{y \in Z: \phi(x, y) > 0\}$. Thus, to show the conclusion of the theorem, it is equivalent to showing that there exists $x^* \in K(x^*)$ such that $K(x^*) \cap P(x^*) = \emptyset$.

By the 0-diagonal quasi-concavity, $x \notin \operatorname{co} P(x)$ for all $x \in Z$. To see this, suppose, by way of contradiction, that there exists some points $x_\lambda \in Z$ such that $x_\lambda \in \operatorname{co} P(x_\lambda)$. Then there exist finite points, $x_1, ..., x_m$ in Z, and $\lambda_j \geqslant 0$ with $\sum_{j=1}^m \lambda_j = 1$ such that $x_\lambda = \sum_{j=1}^m \lambda_j x_j$ and $x_i \in P(x_\lambda)$ for all $i = 1, ..., m$. That is, $\phi(x_\lambda, x_i) > 0$ for all i, which contradicts the hypothesis that $\phi(x, y)$ is 0-DQCV in y.

Define another correspondence $F: Z_w \to 2^Z$ by $F(x) = K(x) \cap \operatorname{co} P(x)$. Let $U_w = \{x \in Z_w: F(x) \neq \emptyset\}$. If $U_w = \emptyset$, this implies $K(x) \cap P(x) = \emptyset$ for every $x \in Z_w$, we only need to show $K(x)$ has a fixed point. And this is guaranteed by Kakutani's fixed point theorem. So the theorem is proved. Now we assume $U_w \neq \emptyset$. Since $\phi(x, y)$ is l.s.c., the set $\{(x, y) \in Z_w \times Z: y \in P(x)\} = \{(x, y) \in Z_w \times Z: \phi(x, y) > 0\}$ is open and thus P has a relative open graph in Z_w. Then it follows from Lemma 2 that the correspondence $\operatorname{co} P(x)$ also has a relative open graph in Z_w. Therefore, by Lemma 1 and the lower semi-continuity of K, F is l.s.c. and thus the correspondence $F|U_w: U_w \to 2^Z$ is l.s.c. in U_w and for all $x \in U_w$, $F(x)$ is nonempty and convex. Now we claim that $F(x)$ either contains an interior point or is finite dimensional. This is clearly true if $K(x)$ is finite dimensional. So we only need to show that $F(x)$ has an interior point if $K(x)$ contains an interior point y_0. To see this, let $x \in U_w$ and $y \in F(x) = K(x) \cap \operatorname{co} P(x)$. Since $K(x)$ is convex, $y_\lambda = y + \lambda(y_0 - y)$ is an interior point for any $0 < \lambda < 1$. Thus any neighborhood $\mathcal{N}_1(y)$ of y contains an interior point of $K(x)$. Since $\operatorname{co} P(x)$ has a relative open graph, $\operatorname{co} P(x)$ is open relative to Z that contains $K(x)$ and $P(x)$. There should be a neighborhood $\mathcal{N}_2(y)$ of y such that $\mathcal{N}_2(y) \cap Z \subset \operatorname{co} P(x)$. So $\mathcal{N}_2(y)$ contains an interior point of $F(x) = K(x) \cap \operatorname{co} P(x)$.

Next we show that Z_w is a perfectly normal T_1-topological vector space. It is clear that Z_w is a normal T_1-topological space, since the dual X^* of a Banach space X separates points in X and Z_w is weakly compact. To show Z_w is perfectly normal, we have to show that any closed set C of Z_w can be written as an intersection of countable open sets. Thanks to the assumption that X is separable. It means X_w is also separable, since the norm-convergence implies the weak-convergence. Let Ω be a countable dense set in X_w. For each closed set C in Z_w, the set $\Omega \backslash C$ is also countable and dense in $X_w \backslash C$. For each $x \in (\Omega \backslash C)$, there are neighborhoods $\mathcal{N}(x)$ and $\mathcal{N}_x(C)$ such that $\mathcal{N}(x) \cap \mathcal{N}_x(C) = \emptyset$. It is clear that $C = \bigcap_{x \in (\Omega \backslash C)} \mathcal{N}_x(C)$, since $(\bigcap_{x \in (\Omega \backslash C)} \mathcal{N}_x(C)) \cap (\bigcup_{x \in (\Omega \backslash C)} \mathcal{N}(x)) = \emptyset$ and $(X_w \backslash C) \subset (\bigcup_{x \in (\Omega \backslash C)} \mathcal{N}(x))$.

Hence, we can apply Lemma 3 to assure that there exists a continuous function $f: U_w \to Z$ such that $f(x) \in F(x)$ for all $x \in U_w$. Note that U_w is open relative to Z_w since F is l.s.c. Define the correspondece $G: Z_w \to 2^Z$ by

$$G(x) = \begin{cases} \{f(x)\} & \text{if } x \in U \\ K(x) & \text{otherwise.} \end{cases} \tag{2}$$

Then G is u.s.c. as for each open set $V \subset Z$, the set

$$\{x \in Z: G(x) \subset V\} = \{x \in Z: K(x) \subset V\} \cup \{x \in U: f(x) \in V\}$$

is w-open in Z_w, because $K: Z_w \to 2^Z$ is u.s.c. Since each w-open set is also an open set, the above set is w-open for each w-open set $V \subset Z_w$. For each $x \in Z_w$, $G(x)$ is a closed convex set, and therefore $G(x)$ is also a w-closed convex set. Now Z_w is w-compact and convex, $G: Z_w \to 2^Z$ is a w-u.s.c. correspondence with nonempty, w-closed, and convex values. By Kakutani's fixed point theorem, there exists a point $x^* \in Z$ such that $x^* \in G(x^*)$. Note that, if $x^* \in U_w$, then $x^* = f(x^*) \in F(x^*) \subset \text{co } P(x^*)$, a contradiction to $x^* \notin \text{co } P(x^*)$. Hence, $x^* \notin U_w$ and thus $x^* \in K(x^*)$ and $K(x^*) \cap \text{co } P(x^*) = \varnothing$ which implies $K(x^*) \cap P(x^*) = \varnothing$. ∎

In the above theorem, compactness of Z_w can be relaxed. We first give a theorem for the finite dimensional topological space. A result for an infinite dimensional topological space will be given after we give Theorems 4 and 5.

THEOREM 3. *Let X be a nonempty convex subset of R^l. Suppose that*

(i) *K is a continuous correspondence with nonempty, compact, and convex values,*

(ii) *$\phi: X \times X \to \mathbf{R} \cup \{\pm\infty\}$ is lower semi-continuous and is 0-diagonally quasi-concave in y,*

(iii) *there exists a nonempty compact set $C \subset X$ such that*

(iii.a) *$K(x) \cap Z \neq \varnothing$ for all $x \in Z$, where $Z = \text{co}\{K(C) \cup C\}$;*

(iii.b) *for each $x \in Z \setminus C$ there exists $y \in K(x) \cap Z$ such that $\phi(x, y) > 0$.*

Then there exists $x^* \in K(x^*)$ such that $\sup_{y \in K(x^*)} \phi(x^*, y) \leqslant 0$.

Proof. Since C is compact and K is a u.s.c. correspondence with compact values by Proposition 3.11 in Aubin and Ekeland [4,p. 113], $K(C)$ is compact and thus Z is compact convex.

Define a correspondence $G: Z \to 2^Z$ by, for each $x \in Z$,

$$G(x) = K(x) \cap Z. \tag{3}$$

Then, by Conditions (ii) and (iii.a), $G(x)$ is nonempty and convex for all $x \in Z$. Since Z is compact and K is closed by Proposition 3.7 in Aubin and Ekeland [4, p. 111], then G is closed and therefore is u.s.c. on Z by Theorem 3.8 in Aubin and Ekeland [4, p. 111]. Also, note that

$$G(x) = \begin{cases} K(x) & \text{if } x \in C \\ K(x) \cap Z & \text{otherwise.} \end{cases} \tag{4}$$

Then, by Theorem 2 there is $x^* \in Z$ such that $x^* \in G(x^*)$ and $\sup_{y \in G(x^*)} \phi(x^*, y) \leqslant 0$. Now $x^* \in C$, for otherwise Hypothesis (ii.b) would be violated, and hence $G(x^*) = K(x^*)$. Therefore, we have $x^* \in K(x^*)$ and $\sup_{y \in K(x^*)} \phi(x^*, y) \leqslant 0$. ∎

Theorem 3 extends the results of Tian and Zhou [15] be relaxing the 0-diagonal concavity condition. Observe that in the case of a compact set X, Assumptions (iii.a)–(iii.b) in Theorem 3 are satisfied by taking $C = X$ and thus Theorem 3 reduces to Theorem 2. Assumption (iii.a) is the necessary and sufficient condition for the correspondence K to have a fixed point when X is not compact (cf. Tian [14]). Assumption (iii.b) is similar to the condition imposed by Allen [1] for variational inequalities with a non-compact set.

We now extend the above theorems to the infinite dimensional topological space. It may be remarked that in Theorems 4 and 5 below the conditions on ϕ are weaker than those in Theorem 2 but we need to strengthen K to have open lower sections.

THEOREM 4. *Let Z be a nonempty, compact, convex, and metrizable set in a locally convex Hausdorff topological vector space. Suppose that*

(a) *$K: Z \to 2^Z$ is correspondence with nonempty convex values and has open lower sections such that $\bar{K}: Z \to 2^Z$ is u.s.c.,*

(b) *$\phi: Z \times Z \to \mathbf{R} \cup \{\pm\infty\}$ is lower semi-continuous in x and is 0-diagonally quasi-concave in y.*

Then there exists $x^* \in \bar{K}(x^*)$ such that $\sup_{y \in K(x^*)} \phi(x^*, y) \leqslant 0$.

Proof. The proof of this theorem is very similar to that of Theorem 2. Define the correspondence $P: Z \to 2^Z$ as before. Again we only need to show that there exists $x^* \in \bar{K}(x^*)$ such that $K(x^*) \cap P(x^*) = \varnothing$.

Since ϕ is l.s.c. in x, then for each $x \in Z$, $P^{-1}(y) = \{x \in Z : \phi(x, y) > 0\}$ is open. Thus P has open lower sections. Also, $x \notin \operatorname{co} P(x)$ for all $x \in Z$ by the 0-DQCV condition.

Also define the correspondence $F: Z \to 2^Z$ and U as before. Since K and P have open lower sections in Z, so do they in U. Then, by Lemma 5, co P has open lower sections in U. Hence, by Lemma 4, the correspondence

$F|U: U \to 2^Z$ has lower open sections in U and for all $x \in U$, $F(x)$ is non-empty and convex. Also, since X is a metrizable space, it is paracompact (cf. Michael [7, p. 831]). Hence, by Lemma 6, there exists a continuous function $f: U \to Z$ such that $f(x) \in F(x)$ for all $x \in U$. Since F has open lower sections and thus is l.s.c. (cf. Remark 1), U is open. Define the correspondence $G: Z \to 2^Z$ by

$$G(x) = \begin{cases} \{f(x)\} & \text{if } x \in U \\ \bar{K}(x) & \text{otherwise.} \end{cases} \tag{5}$$

Then the remaining arguments are the same as those in the proof of Theorem 2. ▮

As a special case of Theorem 4, if the correspondence K is closed-valued on Z, we have the following result.

THEOREM 5. *Let Z be a nonempty, compact, convex, and metrizable set in a locally convex Hausdorff topological vector space. Suppose that*

(a) *$K: Z \to 2^Z$ is an upper semi-continuous correspondence with non-empty closed convex values and has open lower sections,*

(b) *$\phi: Z \times Z \to \mathbf{R} \cup \{\pm\infty\}$ is lower semi-continuous in x and is 0-diagonally quasi-concave in y.*

Then there exists $x^* \in K(x^*)$ such that $\sup_{y \in K(x^*)} \phi(x^*, y) \leqslant 0$.

When $K(x) = Z$ for all $x \in Z$, the quasi-variational inequality reduces to the conventional (minimax) inequality, and consequently Theorems 4 and 5 conclude Theorem 2.11 of Zhou and Chen [18] as a special case which is stated here as a corollary.

COROLLARY 1. *Let Z be a nonempty compact convex metrizable set in a locally convex Hausdorff topological vector space E. Suppose that $\phi: Z \times Z \to \mathbf{R} \cup \{\pm\infty\}$ is lower semi-continuous in x and is 0-diagonally quasi-convex in y. Then there exists $x^* \in Z$ such that $\phi(x^*, y) \leqslant 0$ for all $y \in Z$.*

Remark 2. Even if topological spaces in Theorems 2, 4, and 5 are infinite dimensional, the compactness condition of X can be relaxed if we make the following additional assumptions:

(c) there exist a non-empty compact convex set $Z \subset X$ and a non-empty subset $C \subset Z$ such that

(c.1) $K(C) \subset Z$;

(c.2) $K(x) \cap Z \neq \varnothing$ for all $x \in Z$;

(c.3) for each $x \in Z \backslash C$ there exists $y \in K(x) \cap Z$ with $\phi(x, y) > 0$.

The proof of this generalization is very similar to that of Theorem 3 and thus omitted here.

3. An Application to the Existence of Constrained Games

Aubin [3, pp. 282–283] and Aubin and Ekeland [4, pp. 350–351] used Theorem 1 to prove the existence of equilibrium for the constrained games. However, they need to assume the aggregate loss functions defined in (6) below is concave in y which is equivalent to saying that individual loss functions are convex in their own strategies. In this section, we use our results on quasi-variational inequalities to generalize the results in [3, Theorem 9.3.3] by relaxing the convexity.

Let I be the set of agents which is any (finite or infinite) countable set. Each agent has a choice set X_i, a constraint correspondence $S_i: X_{-i} \to 2^{X_i}$, and a loss function $u_i: \prod_{j \in I} X_j \to \mathbf{R} \cup \{\pm\infty\}$, where $X = \prod_{i \in I} X_i$ and $X_j = \prod_{j \in I \setminus \{i\}} X_j$. Denote by S the product $\prod_{i \in I} S_i$. Denote by x and x_{-i} an element of X and an element of X_{-i}, respectively.

A *constrained game* $\Gamma = (X_i, S_i, u_i)_{i \in I}$ is defined as a family of ordered triples (X_i, S_i, u_i). An *equilibrium* for Γ is an $x^* \in X$ such that $x^* \in S(x^*)$ and $u_i(x^*) \leqslant u(x^*_{-i}, x_i)$ for all $x_i \in S(x^*_{-i})$ and all $i \in I$.

If $S_i(x_{-i}) \equiv X_i$, $\forall i \in I$, the constrained game reduces to the conventional game $\Gamma = (X_i, u_i)$ and the equilibrium is called a *Nash equilibrium*.

Accordingly, we introduce an aggregrate loss function $U: X \times X \to \mathbf{R} \cup \{\pm\infty\}$ defined by

$$U(x, y) = \sum_{i=I} \frac{1}{2^i} [u_i(x) - u_i(x_{-i}, y_i)]. \tag{6}$$

Theorem 6. *Let Z_w be a nonempty weakly compact convex set in a separable Banach space X with the weak-topology w and Z denote the same set Z_w with the norm topology. Suppose that*

(i) *$A: Z_w \to 2^Z$ is a continuous correspondence with nonempty closed and convex values such that $A(x)$ is either finite dimensional or has an (norm) interior point for each $x \in Z$,*

(ii) *$U: Z_w \times Z \to \mathbf{R} \cup \{\pm\infty\}$ is l.s.c. and in 0-diagonally quasi-concave in y.*

Then Γ has an equilibrium.

Proof. By Theorem 2, we know there is $x^* \in S(x^*)$ such that $\sup_{y \in S(x^*)} U(x^*, y) \leqslant 0$. Now let $y = (x^*_{-i}, y_i)$. We then have

$$\frac{1}{2^i} [u_i(x^*) - u_i(x^*_{-i}, y_i)] \leqslant 0$$

for any $y_i \in S_i(x^*_{\hat{i}})$ and all $i \in I$. Hence x^* is an equilibrium of the constrained game. ∎

Remark 3. A sufficient condition for $U(x, y)$ to be lower semi-continuous is that u_i is continuous. Note that by applying Theorem 4 or Theorem 5, we can have similar existence theorems on equilibria of constrained games.

When $S \equiv Z$, the constrained game reduces to the conventional game, as a corollary of the above theorem, we have an existence theorem on Nash equilibrium for games which generalizes the results of Nash [9, 10] and Nikaido and Isoda [11] by relaxing the convexity condition.

COROLLARY 2. *Let Z be a nonempty compact convex metrizable set in a locally convex Hausdorff topological vector space E. Suppose that $U: Z \times Z \to \mathbf{R} \cup \{\pm\infty\}$ is upper semi-continuous in x and is 0-diagonally quasi-concave in y. Then Γ has an Nash equilibrium.*

EXAMPLE 1. In order to see that our theorems indeed are generalizations of the existing theorems in the literature, consider a two person noncooperative game with some "nice" constraints and the loss functions of player 1 and player 2 given by

$$u_1(x_1, x_2) = x_1^2 x_2^2$$

and

$$u_2(x_1, x_2) = -x_1^2 x_2^2,$$

respectively. One can see that the loss function u_2 of player 2 is not convex in x_2. Furthermore it can be shown that the aggregate loss function

$$\begin{aligned} U(x, y) &= u_1(x) - u_1(y_1, x_2) + u_2(x) - u_2(x_1, y_2) \\ &= -y_1^2 x_2^2 - x_1^2 y_2^2 \end{aligned}$$

is neither quasi-concave in x nor quasi-concave in y, but 0-diagonally quasi-concave, so with some "nice" constraints we can still show the existence of equilibrium.

As a final remark, by using Theorem 3, we can generalize Theorem 6 by relaxing the compactness of the strategy space.

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