Closed-Form Solution of General Intertemporal Consumption Maximization Models

Chipman, John S. and Tian, Guoqiang

1993

l Game between Workers and

, Review of Economic Studies

ebliche Ertragsbeteiligung der

bei Ergebnisbeteiligung der
), Struktur und Dynamik der

italism", Journal of Political

ridge University Press.

e Dynamic Inefficiency of

Differential Game Model of

Working Paper, University of

Working Paper, University of

t Modelled as a Differential

radox in Neo-Classical and

, and Economic Growth",

Closed-Form Solutions of General Intertemporal Consumption-Maximization Models[*]

JOHN S. CHIPMAN[†] and GUOQIANG TIAN[‡]

This paper considers explicit representations for very general (discrete and continuous-time) intertemporal consumption-maximization models which allow the instantaneous preferences of the consumer and the time-preference factors to vary over time and for the the non-existence of utility functions, more than one generation of consumers with a given probability of death, many commodities, and, further, a wide class of preferences which do not necessarily satisfy the so-called "regularity conditions" (such as differentiability, strict convexity, boundedness, or continuity) and include most of the well-known preferences in the literature.

1 Introduction

The dynamic consumption-optimization model has been widely used in many fields of economics, such as dynamic macroeconomics, optimal-growth models, life-cycle models, the balance of payments in international trade, and dynamic consumer demand systems.[1] Because of the importance of this model, there has been continued interest in setting forth conditions for obtaining the existence and/or some properties of the solutions of the policy functions. Owing, however, to the considerable mathematical difficulties involved, various restrictive conditions such as the so-called regularity conditions (e.g. differentiability, strict convexity, boundedness, or continuity), the stationarity assumption (e.g., preferences of consumers and/or time-preference factors are invariant over time periods), and the one-commodity assumption, have usually been imposed on instantaneous preferences. Also, closed-form solutions have been obtained only under restrictive assumptions on instantaneous preferences, time-preference factors, and agents' horizons. No results are available for "irregular" assumptions. These restrictions on preferences, the time-preference factors, and agents' horizons obviously limit our understanding of dynamic consumption-maximization problems under more general situations. They also limit the relevance of these problems to practical applications. For these reasons, economists continually strive to weaken the conditions for the intertemporal utility-maximizing model.

In this paper, we consider systematically explicit representations for very general (discrete and continuous time) intertemporal consumption maximization models, which allow for preferences of the consumer and the time-preference factors to vary over time;

[*]We thank M. R. Baye and G. Lozada for useful comments and suggestions.

[†]University of Minnesota

[‡]Texas A&M University.

[1]For references to some of these see Tian and Chipman (1989) and Chipman and Tian (1992).

the non-existence of utility functions; more than one generation of consumers facing a constant probability of death; many commodities; and a wide class of preferences which do not necessarily satisfy the regularity conditions. These preferences include most of the well-known preferences in the economic literature such as the Leontief-Kantorovich, linear, lexicographic (preferences that cannot even be represented by a utility function), Cobb-Douglas and its extensions (such as Klein-Rubin-Samuelson-Geary preferences), CES, Brown-Heien S-branch-tree, homothetic, and parallel preferences. They also include those preferences represented by indirect utility functions of the "Gorman polar form" and "generalized Gorman form" (cf. Chipman and Moore (1990)). Thus our models include most of the existing models (say, Levhari and Srinivasan (1969), Lluch (1973), Cooper and Mclaren (1983), Blanchard (1985), Tian and Chipman (1989)) from which closed-form solutions can be obtained as special cases. We also derive the aggregate consumption-savings and demand functions.

We also note that the essence of monotonic transformations in intertemporal models is that they only change allocations for the consumption-savings profiles over time periods, but within each time period preferences of consumers do not change. That is, monotonic transformations affect the instantaneous demand only through the total consumption function $c(t)$ and thus the functional form of the demand function $h(\cdot)$ is invariant with respect to any monotonic transformation. Note that this interpretation includes the invariance of monotonic transformations of static models as a special case. Indeed, for the static model, which is a special case of the intertemporal model—since wealth (income) is exogenously given and preferences are invariant under monotonic transformations—the demand is invariant under monotonic transformations. Even though monotonic transformations affect the path of consumptions over time, it is still necessary to make suitable monotonic transformations for some instantaneous utility functions to obtain interior and/or explicit solutions for the consumption-savings functions (i.e., it is for guaranteeing that consumption in each period is positive (to smooth the consumption path over time) and/or for obtaining the closed-form solutions). For instance, when the Cobb-Douglas utility function is used as an instantaneous utility function, its logarithmic transformation has been used (see, e.g., Lluch (1973), Ashenfelter and Ham (1979), Tian and Chipman (1989)). Also, a concave monotonic transformation maps a concave monotonic (indirect) utility function into itself so that it does not change the essence of the consumer's problem and thus is innocuous.[2] Further, what kind of monotonic transformation should be used depends on the problem being dealt with and assumptions on the behavior of the consumer over time periods. If we know the forms of the transformations, the coefficients of the monotonic transformation functions can be determined by empirical estimation. MaCurdy (1983) and Blundell and Walker (1986) have studied the problem of monotonic transformations with general forms and derived the particular functional forms, so that the parameters of the transformations can be estimated. In this paper, we will give explicit solutions for a class of instantaneous preferences whose instantaneous indirect utility functions before or after transformation have the same functional form as one resulting from the Box-Cox transformation.[3] Since closed-form solutions can be obtained, the coefficients may easily be estimated.

[2] This is because the intertemporal optimization literature usually only assumes the intertemporal allocations are characterized by the monotonic and concave (indirect) utility functions under which many results on the behavior of individual can be preserved.

[3] A mapping $F : \mathbb{R} \to \mathbb{R}$ is called a Box-Cox transformation if $F(v) = [v^\alpha - 1]/\alpha$.

In dealing with the consumption-maximization problem, we use the approach of maximizing a discounted indirect utility function (defined on the budget space) instead of a discounted (direct) utility function (defined on the commodity space) over a time horizon. After obtaining the solutions of the total consumption-expenditure and savings functions, we can derive the demand functions by, say, the Antonelli-Allen-Roy partial differential equation. This approach has a number of advantages. First, it reduces the dimensionality of the consumption-maximization problem so that the problem becomes simpler and avoids the one-commodity assumption. Second, it enables us to deal with the consumer's optimum problem with non-stationary preferences. Since preferences or tastes of individuals are not invariant over time periods, the assumption of stationarity of preferences is obviously neither realistic nor necessary because explicit solutions can be obtained and the time-inconsistency problem does not arise in our model even if preferences change over time. Third, it permits us to study consumer behavior under more general preferences. Common assumptions on instantaneous preferences are that the (direct) utility functions representing preferences exist, are differentiable, and guarantee interior solutions (that is, the individual consumes all commodities in strictly positive amounts). These severely limit our analyzing the consumer's behavior: for, for some preferences, (direct) utility functions do not exist (e.g., lexicographic preferences), or are not differentiable (e.g., the Leontief-Kantorovich utility function), or do not guarantee interior solutions (e.g., linear utility functions). On the other hand, not only does this approach allow for corner solutions in the commodity space; it does not require assuming the utility function to exist and be differentiable in the commodity space, since kinks and corners in the commodity space do not preclude differentiability in the budget space. In fact, we do not even require preferences in the commodity space to be continuous (e.g., lexicographic preferences).

This paper also extends Blanchard's (1985) overlapping-generations model, in which a finite horizon is introduced through the assumption that at each time period agents face a constant probability of death, to more general assumptions. The main advantage of this approach is its flexibility, since the agent's horizon is a parameter that can be chosen arbitrarily. In particular, when the instantaneous probability of death goes to zero, we obtain the infinite-horizon case as a limiting case.

2 The Model and Assumptions

In the basic model we will study, individuals, as in Blanchard (1985), are assumed to face a given probability of death μ, which is independent of age. The density functions for the constant probability of death are $f(\mu) = (1+\mu)^{-t}\frac{\mu}{1+\mu}$ for the discrete-time case and $f(\mu) = \mu e^{-\mu t}$ for the continuous-time case. The expected remaining lifetime for an individual of any age is given by $\sum_{t=0}^{\infty} t(1+\mu)^{-1}\frac{\mu}{1+\mu} = 1/\mu = \int_0^\infty t\mu e^{-\mu t}dt$. Thus as μ goes to zero life expectancy goes to infinity and reduces to the standard infinite-horizon case. The size of a cohort born at time s, at time t, is $(1+\mu)^{-(t-s)}\frac{\mu}{1+\mu}$ for the discrete-time case and $\mu e^{-\mu(t-s)}$ for the continuous-time case. Thus the size of the population at any time t is $\sum_{s=-\infty}^{t}\left[\frac{\mu}{1+\mu}\right]\left[\frac{1}{1+\mu}\right]^{t-s} = 1 = \int_{-\infty}^{t}\mu e^{-\mu(t-s)}ds$.

Because of uncertain lifetime, all loans require—in addition to regular interest payments—the purchase of life insurance, as in Yaari (1965) and Blanchard (1985). In case of death, the estate is transferred to the life insurance company. It is assumed that there is a large number of agents, and, therefore, competition among insurance companies implies that the percentage purchased is equal to the probability of death. Thus, if an agent's nonhuman wealth is b, he will receive μb if he does not die and pay w if he dies. Suppose that at time t agents consume m consumption goods $x(t)$—a vector in the commodity space $X(t) \subseteq R^m_+$. Denote by $p(t) \in I\!R^m_{++}$, $c(t) \in I\!R_+$, and $Q(t)$ the price vector, the total consumption expenditure, and the budget space whose typical element is $(p(t), c(t)) \in R^{m+1}_+$. Denote by $\succeq_t$ and $\succeq^*_t$ the instantaneous preference relation defined on $X(t)$ and the instantaneous indirect preference relation induced by $\succeq_t$ and defined on $Q(t)$. Let $u(\cdot)$ be the instantaneous (direct) utility function (if it exists) generated by $\succeq_t$, let $v(\cdot)$ be the instantaneous indirect utility function generated by $\succeq^*_t$, and let $x(t) = h(p(t), c(t))$ be the demand function generated by $\succeq_t$.

Since we do not presume the existence of a utility function, we cannot define the indirect utility function in the conventional way, by substituting the demand function into the utility function. Here we will adopt a definition of the indirect utility function given by Chipman and Moore (1976, 1980, 1990):

$$v_t(p'(t), c'(t)) \geq v_t(p''(t), c''(t)) \text{ iff } (p'(t), c'(t)) \succeq^*_t (p''(t), c''(t)), \tag{1}$$

where the indirect preference $\succeq^*_t$ induced by $\succeq_t$ is defined by the condition:

$$(p'(t), c'(t)) \succeq^*_t (p''(t), c''(t)) \text{ iff } h(p'(t), c'(t)) \succeq_t h(p''(t), c''(t)). \tag{2}$$

It is obvious that the above definition of indirect utility function is a more general definition since an indirect utility function under the usual definition is an indirect utility function under the definition of Chipman and Moore. Also Chipman and Moore (1976) showed that $v_t(p(t), c(t))$ is positively homogeneous of degree 0 in $(p(t), c(t))$ and nondecreasing in $c(t)$, and satisfies the Antonelli-Allen-Roy partial differential equations [4] if it is differentiable:

$$x_i(t) = -\frac{\partial v_t(p(t), c(t))/\partial p_i(t)}{\partial v_t(p(t), c(t))/\partial c(t)}. \tag{3}$$

In order to solve the decision functions explicitly, we need to specify a class of preferences and concave monotonic transformations $F_t(\cdot)$. The preferences considered in this paper are characterized by

$$E(t) = \left\{ \succeq_t \,\middle|\, \exists v_t(\cdot) \& F_t(\cdot) \ni F_t(v_t(p(t), c(t))) = \left\{ \left[\frac{c(t) - \psi_t(p(t))}{g_t(p(t))} \right]^\alpha - 1 \right\} \Big/ \alpha \right\}, \tag{4}$$

where $-\infty < \alpha \leq 1$, $c(t) - \psi_t(p(t)) > 0$, $g_t(p(t)) > 0$, and $\psi_t(p(t))$ and $g_t(p(t))$ are homogeneous of degree one in $p(t)$ (cf. Chipman and Moore (1990)). Here $\psi_t(p(t))$ may be interpreted as the consumer's subsistence or minimum-required expenditure. Thus, the indirect utility functions of preferences after some transformations have the same functional form as one resulting from the Box-Cox transformation.[5] Note that when $\alpha = 0$, the logarithmic form of $F_t(v_t(p(t), c(t)))$ can be obtained as a special case:

$$F_t(v_t(p(t), c(t))) = \log[c(t) - \psi_t(p(t))] - \log g_t(p(t)), \tag{5}$$

[4]Those three authors all assumed that $\succeq_t$ is representable by a differentiable direct utility function defined on the interior of the commodity space R^m_+ (for references see Chipman and Moore (1980)). However, Chipman and Moore (1976) gave a simple proof not requiring differentiability (or even existence) of the utility function.

[5]Note that this does not mean the transformation F is the same as the Box-Cox transformation (see Example (e) and Remark 2 below).

which is widely used in the

Denote by $c(s, t+v)$ an

agent born at time s, who m

labor income and the mini

time s.[6] We will assume th

so that the closed-form sol

be obtained for this class c

we also assume that $\sum_{t=0}^{\infty}$

and $\int_0^\infty \Phi(t)\psi_{s,t+v}(p(t+v)$

$\Phi(t) =$

where $r(t) =$ the interest ra

by uppercase letters. Sin

(or integral) of consumpti

aggregate variable $Z(t+v)$

$Z(t+v) = \{$

Thus $C(t+v)$, $B(t+v)$, Y

labor income, and the min

3 Examples of

Before discussing the main

$E(t)$, in fact, are very rich

$E(t)$ specified by (4). The

(a). The preference re

function (cf. Leontief (195

$u_t(x(t)) = \min\{$

Its indirect utility functio

$v_t($

(b). The preference re

u_t

[6]Here, for generality, we all

ages.

which is widely used in the literature.

Denote by $c(s,t+v)$ and $b(s,t+v)$ the consumption and savings for time $t+v$ of an agent born at time s, who makes his plan at time t. Denote by $y(s,t+v)$ and $\psi_{s,t+v}(p(t+v))$ labor income and the minimum-required expenditure at time $t+v$ of an agent born at time s.[6] We will assume that the instantaneous preferences are specified by the set $E(t)$ so that the closed-form solutions for the consumption-savings and demand functions can be obtained for this class of preferences. In order to guarantee that the solutions exist, we also assume that $\sum_{t=0}^{\infty}\Phi(t)y(s,t+v)$, $\sum_{t=0}^{\infty}\Phi(t)\psi_{s,t+v}(p(t+v))$, $\int_0^{\infty}\Phi(t)y(s,t+v)dt$, and $\int_0^{\infty}\Phi(t)\psi_{s,t+v}(p(t+v))dt$ are bounded, where

$$\Phi(t) = \begin{cases} \prod_{\tau=0}^{t-1}[R(\tau)+\mu]^{-1} & \text{if } t \text{ is discrete} \\ e^{-\int_0^t [r(\tau)+\mu]d\tau} & \text{if } t \text{ is continuous,} \end{cases} \tag{6}$$

where $r(t)$ = the interest rate on savings and $R(t) = 1 + r(t)$. Denote aggregate variables by uppercase letters. Since aggregate consumption and savings at $t+v$ are the sum (or integral) of consumption and savings from all cohorts. Thus relation between any aggregate variable $Z(t+v)$ and an individual counterpart $z(s,t+u)$ is

$$Z(t+v) = \begin{cases} \sum_{s=-\infty}^{t+v}\left[\frac{\mu}{1+\mu}\right]\left[\frac{1}{1+\mu}\right]^{t+v-s} z(s,t+v) & \text{if } t \text{ is discrete} \\ \int_{-\infty}^{t+v}\mu e^{-\mu(t+v-s)}z(s,t+v)ds & \text{if } t \text{ is continuous.} \end{cases} \tag{7}$$

Thus $C(t+v)$, $B(t+v)$, $Y(t+v)$, and $\Psi_{t+v}(p(t+v))$ are aggregate consumption, savings, labor income, and the minimum-required expenditure at time $t+v$, respectively.

3 Examples of Preferences

Before discussing the main results of this paper, we show that preferences represented by $E(t)$, in fact, are very rich by providing a number of examples of preferences which are in $E(t)$ specified by (4). These preferences are well-known in the literature.

(a). The preference relation $\succeq_t$ generated by the Leontief-Kantorovich direct utility function (cf. Leontief (1956), Kantorovich (1965)):

$$u_t(x(t)) = \min\{a_1(t)x_1(t), a_2(t)x_2(t), \ldots, a_m(t)x_m(t)\} \quad a_i(t) > 0 \quad \forall i. \tag{8}$$

Its indirect utility function is given by

$$v_t(p(t),c(t)) = \frac{c(t)}{\frac{p_1(t)}{a_1(t)} + \frac{p_2(t)}{a_2(t)} + \cdots + \frac{p_m(t)}{a_m(t)}}. \tag{9}$$

(b). The preference relation generated by the linear utility function

$$u_t(x(t)) = \sum_{i=1}^{m} a_i(t)x_i(t) \quad a_i(t) > 0 \quad \forall i, \tag{10}$$

[6] Here, for generality, we allow the minimum-required expenditure to be different for people of different ages.

whose indirect utility function is then given by

$$v_t(p(t), c(t)) = \frac{a_s(t)c(t)}{p_s(t)}, \tag{11}$$

where $s \in S$ with

$$S = \left\{ s \;\middle|\; \frac{a_s(t)}{p_s(t)} \geq \frac{a_j(t)}{p_j(t)} \quad \forall j \right\}.$$

(c). The preference generated by the lexicographic ordering:

$$x(t) \succ_t x'(t) \text{ iff } x_1(t) > x_1'(t) \text{ or } x_i(t) = x_i'(t) \;\&\; x_j(t) > x_j'(t), \tag{12}$$

for all $i < j$ $(j = 2, 3, \ldots, m)$. The indirect function is given by[7]

$$v_t(p(t), c(t)) = \frac{c(t)}{p_1(t)}. \tag{13}$$

(d). The preference generated by the Klein-Rubin-Samuelson-Geary utility function (cf. Tian and Chipman (1989) for references):

$$u_t(x(t)) = \prod_{i=1}^{m} [x_i - \gamma_i(t)]^{a_i(t)}, \quad a_i(t) > 0 \;\&\; \sum_{i=1}^{m} a_i(t) = 1, \tag{14}$$

whose indirect utility function is then

$$v_t(p(t), c(t)) = \prod_{i=1}^{m} \frac{a_i(t)^{a_i(t)}(c(t) - p(t) \cdot \gamma(t))}{p_i(t)^{a_i(t)}}. \tag{15}$$

(e). The preference generated by the utility function

$$u_t(x(t)) = -\sum (d_i(t) e^{-a_i(t) x_i(t)}) \quad a_i(t) > 0, d_i(t) > 0 \quad \forall i \tag{16}$$

(cf. Arrow and Hurwicz (1958)) for which the indirect utility function is

$$v_t(p(t), c(t)) = \exp\left\{\frac{c(t)}{P(t)}\right\} \sum_{i=1}^{n} d_i(t) \exp\left\{\left(-\sum_{k \neq i}^{n} \frac{p_k(t)}{a_k(t)} \log \frac{d_i(t) a_i(t) p_k(t)}{d_k(t) a_k(t) p_i(t)}\right) \Big/ P(t)\right\} \tag{17}$$

where

$$P(t) = \frac{p_1(t)}{a_1(t)} + \frac{p_2(t)}{a_2(t)} + \cdots + \frac{p_m(t)}{a_m(t)}.$$

(f). The preference generated by the utility function:

$$u_t(x(t)) = -\sum_{i=1}^{m} \frac{a_i(t)}{x_i(t)} \quad a_i(t) > 0 \quad \forall i \tag{18}$$

[7] For the detailed arguments, see Chipman and Moore (1976).

whose indirect utility function is

$$v_t(p(t), c(t)) = -\frac{\left[\sum_{j=1}^{m} (p_j(t) a_j(t))^{1/2}\right]^2}{c(t)}. \tag{19}$$

(g). The preference generated by a homothetic ordering:

$$x(t) \succ_t x'(t) \text{ if and only if } \lambda x_1(t) \succ_t \lambda x_1'(t) \text{ for } \lambda > 0. \tag{20}$$

Then the indirect utility function has the form

$$v_t(p(t), c(t)) = \frac{c(t)}{\chi_t(p(t))}, \tag{21}$$

where $\chi_t(p(t))$ is a function of $p(t)$.

(h). The preference generated by the parallel utility function

$$u_t(x(t)) = x_1(t) + W(x_2(t), \ldots, x_m(t)), \tag{22}$$

where $W(t)$ is a concave function in $(x_2(t), \ldots, x_m(t))$. The indirect utility function then has the form

$$v_t(p(t), c(t)) = \frac{a(t)c(t)}{p_1(t)} + G(p(t)), \tag{23}$$

where $G(t)$ is a function of $p(t)$ only. For detailed discussions about the last two preference orderings and their applications, see Chipman and Moore (1976, 1980). In addition to the above preferences, the Cobb-Douglas utility function, the Modified CES utility function,[8] and the Brown-Heien S-branch-tree utility function are also in $E(t)$ (cf. Brown and Heien (1972)). Tian and Chipman (1989) obtained explicit solutions for logarithmic transformations of those utility functions. Note that example (f) above is simply a monotone transformation $F(v) = -1/v$ of a CES utility function with elasticity of substitution 1/2.

Remark 1 To see that the above preferences are in $E(t)$, we can take the Box-Cox transformation $F_t(v_t) = [v_t^\alpha - 1]/\alpha$ if the preferences are specified by (a)–(d) and (g)–(h), and take the transformation $F_t(v_t) = [(\log v_t)^\alpha - 1]/\alpha$ if the preferences are specified by (e) and $F_t(v_t) = [| v_t |^{-\alpha} - 1]/\alpha$ if they are specified by (f). Note that $\psi_t(p(t)) = 0$ under preferences specified by (a)–(c) and (f)–(g) and $\psi_t(p(t)) = p(t) \cdot \gamma(t)$ under preferences specified by (d).

As a final example, the preferences represented by an indirect utility function of the generalized Gorman polar form specified by Chipman and Moore (1990):

$$v_t(p(t), c(t)) = \frac{[\delta(t)c(t) - \gamma(p(t))]}{[\alpha(p(t)) + \beta(t)c(t)]} \tag{24}$$

are in $E(t)$ since, as shown by Chipman and Moore (1990), they can be equivalently represented by an indirect utility function of the Gorman polar form. Here $\alpha(t)$ and $\gamma(t)$ are both positively homogeneous of degree one, $\alpha(p(t)) + \beta(t)c(t) > 0$ and $\delta(t)\alpha(t) + \beta(t)\gamma(p(t)) > 0$ for $(p(t), c(t)) \in \mathbb{R}_+^{m+1}$, and not both $\beta(t)$ and $\delta(t)$ are equal to zero. We note that all of the above eight examples except (c) are special or limiting cases of (24).

[8] That is, the CES utility function with shifted origin—cf. Tian and Chipman (1989), formula (16) on p. 102.

4 The Discrete-Time Consumer Problem

Let E_t and $\beta(k)$ be the mathematical expectation given information known at t and the pure time-preference factor or subjective rate of discount with $\beta(t) > 0$ and $\sum_{j=0}^{\infty}(1+\mu)^{-j}\beta(t+j) < \infty$.[9] We assume that the agent's intertemporal optimization problem at t is

$$\max E_t\beta(t)^{-1}\sum_{j=0}^{\infty}\beta(t+j)\frac{\left[\frac{c(s,t+j)-\psi(s,t+j)}{g(t+j)}\right]^{\alpha}-1}{\alpha} \quad (-\infty < \alpha \leq 1). \tag{25}$$

Here for ease of notation $v(s,t+j) = v_{t+j}(p(t+j), c(s,t+j))$, $\psi(s,t+j) = \psi_{s,t+j}(p(t+j))$, and $g(t+j) = g_{t+j}(p(t+j))$.

Given the constant probability of death, and if the only source of uncertainty is about the time of death, the above problem is equivalent to the following problem:

$$\max \frac{\mu\beta(t)^{-1}}{(1+\mu)}\sum_{j=0}^{\infty}(1+\mu)^{-j}\beta(t+j)\frac{\left[\frac{c(s,t+j)-\psi(s,t+j)}{g(t+j)}\right]^{\alpha}-1}{\alpha} \quad (-\infty < \alpha \leq 1). \tag{26}$$

If the agent has assets $b(s,t+j-1)$ at time $t+j-1$, he receives $r(t+j-1)b(s,t+j-1)$ in interest and $\mu b(s,t+j-1)$ from the insurance company at time $t+j$. Thus his dynamic budget constraint is

$$c(s,t+j) + b(s,t+j) = [R(t+j-1)+\mu]b(s,t+j-1) + y(s,t+j) \tag{27}$$

with $b(s,t-1)$ given.

In addition to the above budget constraint, the terminal condition (or borrowing constraint):

$$\lim_{j\to\infty}\Phi(t+j)b(s,t+j) = 0 \tag{28}$$

is needed to prevent the consumer from borrowing arbitrarily large amounts.

Solving the budget constraint (27) for $b(s,t+j-1)$ recursively forward and using the terminal condition (28), we obtain the lifetime budget constraint:[10]

$$\Phi^{-1}(t+j)\sum_{k=0}^{\infty}\Phi(t+j+k)c(s,t+j+k) = w(s,t+j), \tag{29}$$

where

$$w(s,t+j) = [R(t+j-1)+\mu]b(s,t+j-1) + \Phi^{-1}(t+j)\sum_{k=0}^{\infty}\Phi(t+j+k)y(s,t+j+k)$$

which consists of nonproperty (earned) income plus property income and can be considered as the permanent income of the consumer.

We first consider the case $\alpha < 1$. The Euler equation (the first-order necessary condition) is given by:

$$c(s,t+j+1) - \psi(s,t+j+1) = \rho(t+j)[c(s,t+j) - \psi(s,t+j)], \tag{30}$$

[9] It is usually assumed that $\beta(k) = \beta^k$, which is a special case here.

[10] Note that $\sum_{k=0}^{\infty}\Phi(t+j+k)c(s,t+j+k)$ must be bounded since $\sum_{k=0}^{\infty}\Phi(t+j+k)y(s,t+j+k)$ is assumed to be convergent.

where

$\rho(t+j)$

Thus

$c(s,t+j+k) = \prod_{\tau=0}^{k-1}$

$= (c(s,t+j) - \psi($

$+\psi(s,t+j+$

Substituting the above eq

$(c(s,t+j) - \psi(s,$

$w(s,t+j) -$

where

$\bar{\psi}(s,t+j$

The $\bar{\psi}(s,t+j)$ may be inte

expenditure; it can be con

Suppose $\sum_{k=0}^{\infty}$

Then we obtain the consu

$c(s,t+j) =$

which is linear in lifetime

and a mixture of preferen

be obtained from the bud

$b(s,t+j) =$

As the indirect utility

$\alpha = 0$), the solution $c(s,t$

$c(s,t+j) =$

Note that if we further

to the result obtained in

$c(s,t+j)$

where

$$\rho(t+j) = \left[\frac{(1+\mu)\beta(t+j)g(t+j+1)^{\alpha}}{[R(t+j)+\mu]\beta(t+j+1)g(t+j)^{\alpha}}\right]^{\frac{1}{\alpha-1}}.$$

Thus

$$\begin{aligned} c(s,t+j+k) &= \prod_{\tau=0}^{k-1}\rho(t+j+\tau)(c(s,t+j)-\psi(s,t+j))+\psi(s,t+j+k) \\ &= (c(s,t+j)-\psi(s,t+j))\left[\frac{\Phi(t+j+k)(1+\mu)^{k}\beta(t+j)g(t+j+k)^{\alpha}}{\Phi(t+j)\beta(t+j+k)g(t+j)^{\alpha}}\right]^{\frac{1}{\alpha-1}} \\ &\quad +\psi(s,t+j+k). \end{aligned}$$

Substituting the above equation into (29) and then rearranging it, we have

$$(c(s,t+j)-\psi(s,t+j))\sum_{k=0}^{\infty}\left[\frac{(1+\mu)^{k}\Phi(t+j+k)^{\alpha}\beta(t+j)g(t+j+k)^{\alpha}}{\Phi(t+j)^{\alpha}\beta(t+j+k)g(t+j)^{\alpha}}\right]^{\frac{1}{\alpha-1}} = w(s,t+j)-\bar{\psi}(s,t+j),$$

where

$$\bar{\psi}(s,t+j) = \Phi^{-1}(t+j)\sum_{k=0}^{\infty}\Phi(t+j+k)\psi(s,t+j+k).$$

The $\bar{\psi}(s,t+j)$ may be interpreted as the present value of minimum-required consumption expenditure; it can be considered as the permanent necessary expenditure.

Suppose $\sum_{k=0}^{\infty}\left[\frac{(1+\mu)^{k}\Phi(t+j+k)^{\alpha}\beta(t+j)g(t+j+k)^{\alpha}}{\Phi(t+j)^{\alpha}\beta(t+j+k)g(t+j)^{\alpha}}\right]^{\frac{1}{\alpha-1}}$ is bounded under the choice of α. Then we obtain the consumption decision function:

$$c(s,t+j) = \frac{w(s,t+j)-\bar{\psi}(s,t+j)}{\sum_{k=0}^{\infty}\left[\frac{(1+\mu)^{k}\Phi(t+j+k)^{\alpha}\beta(t+j)g(t+j+k)^{\alpha}}{\Phi(t+j)^{\alpha}\beta(t+j+k)g(t+j)^{\alpha}}\right]^{\frac{1}{\alpha-1}}}+\psi(s,t+j), \tag{31}$$

which is linear in lifetime discounted wealth with the necessary expenditures as intercepts and a mixture of preference and price terms as slope. The savings function at time t can be obtained from the budget constraint:

$$\begin{aligned} b(s,t+j) &= -\frac{w(s,t+j)-\bar{\psi}(s,t+j)}{\sum_{k=0}^{\infty}\left[\frac{(1+\mu)^{k}\Phi(t+j+k)^{\alpha}\beta(t+j)g(t+j+k)^{\alpha}}{\Phi(t+j)^{\alpha}\beta(t+j+k)g(t+j)^{\alpha}}\right]^{\frac{1}{\alpha-1}}} \\ &\quad +[R(t+j-1)+\mu]b(s,t+j-1)+y(s,t+j)-\psi(s,t+j). \end{aligned} \tag{32}$$

As the indirect utility functions (after transformation) are log-linear in $c(s,t+j)$ (i.e., $\alpha=0$), the solution $c(s,t+j)$ is simply

$$c(s,t+j) = \frac{w(s,t+j)-\bar{\psi}(s,t+j)}{\beta(t+j)^{-1}\sum_{k=0}^{\infty}\beta(t+j+k)(1+\mu)^{-k}}+\psi(s,t+j). \tag{33}$$

Note that if we further assume that $\beta(t+j)=\beta^{t+j}$ with $\beta<1$ and $\mu=0$, (33) reduces to the result obtained in Tian and Chipman (1989):

$$c(s,t+j) = (1-\beta)[w(s,t+j)-\bar{\psi}(s,t+j)]+\psi(s,t+j). \tag{34}$$

Remark 2 When $\psi(s,k) = 0$ for all $k \geq s$,[11] the consumption decisions specified by (33) are the same even if preferences are different.

The intertemporal demand functions can be obtained from the Antonelli-Allen-Roy p.d.e. In particular, those of the instantaneous demand functions from the preferences specified by (a)–(g) can be obtained by substituting the solution $c(s,t)$ into the corresponding static demand functions. For example, when the preference is given by the Leontief-Kantorovich utility function at time $t+j$, the instantaneous demand function is

$$x_i(s,t+j) = \frac{[w(s,t+j) - \bar{\psi}(s,t+j)] \sum_{k=0}^{\infty} \left[\frac{(1+\mu)^k \Phi(t+j+k)^\alpha \beta(t+j) g(t+j+k)^\alpha}{\Phi(t+j)^\alpha \beta(t+j+k) g(t+j)^\alpha}\right]^{\frac{1}{1-\alpha}}}{a_i(s,t+j)\left(\frac{p_1(t+j)}{a_1(s,t+j)} + \frac{p_2(t+j)}{a_2(s,t+j)} + \cdots + \frac{p_m(t+j)}{a_m(s,t+j)}\right)}; \tag{35}$$

when the preference at time $t+j$ is given by the Klein-Rubin-Samuelson-Geary utility function,

$$x_i(s,t+j) = \frac{a_i(s,t+j)[w(s,t+j) - \bar{\psi}(s,t+j)]}{p_i(t+j) \sum_{k=0}^{\infty} \left[\frac{(1+\mu)^k \Phi(t+j+k)^\alpha \beta(t+j) g(t+j+k)^\alpha}{\Phi(t+j)^\alpha \beta(t+j+k) g(t+j)^\alpha}\right]^{\frac{1}{\alpha-1}}} + \gamma_i(s,t+j). \tag{36}$$

We can specialize the above solutions to the case in which $\beta(k) = \beta^k$, $R(k) = R$, $y(s,k) = \phi y(s,k-1)$, $\psi(s,k) = 0$ for all $k \geq s$, $g(k) = g(p(k))$[12], and $p_i(k+1)/p_i(k) = \theta > 0$ (i.e., the inflation rates for all goods are the same over time periods) for all i and k. Then $g(p(t+j+1))/g(p(t+j)) = g(\theta p(t+j))/g(p(t+j)) = \theta$, and $y(s,t+k) = \phi^k y(s,t)$ with $\phi/(R+\mu) < 1$. Thus, $\rho(t+j)$ becomes

$$\rho(t+j) = \left[\frac{\theta^\alpha(1+\mu)}{\beta(R+\mu)^\alpha}\right]^{\frac{1}{\alpha-1}},$$

and equation (31) becomes

$$c(s,t+j)\sum_{k=0}^{\infty}\left[\frac{(1+\mu)\theta^\alpha}{\beta(R+\mu)^\alpha}\right]^{\frac{k}{\alpha-1}} = (R+\mu)b(s,t+j-1) + \sum_{k=0}^{\infty}\left(\frac{\phi}{R+\mu}\right)^k y(s,t+j).$$

Choose α such that $\left[\frac{(1+\mu)\theta^\alpha}{\beta(R+\mu)^\alpha}\right]^{\frac{1}{\alpha-1}} < 1$.[13] Then the consumption-savings functions become

$$c(s,t+j) = \left(1 - \left(\frac{(1+\mu)\theta^\alpha}{\beta(R+\mu)^\alpha}\right)^{\frac{1}{\alpha-1}}\right)\left[(R+\mu)b(s,t+j-1) + \frac{R+\mu}{R+\mu-\phi}y(s,t+j)\right] \tag{37}$$

and

$$\begin{aligned} b(s,t+j) &= \left(\frac{(1+\mu)\theta^\alpha}{\beta(R+\mu)^\alpha}\right)^{\frac{1}{\alpha-1}}(R+\mu)b(s,t+j-1) \\ &\quad + \frac{\left(\frac{(1+\mu)\theta^\alpha}{\beta(R+\mu)^\alpha}\right)^{\frac{1}{\alpha-1}}(R+\mu) - \phi}{R+\mu-\phi}y(s,t+j). \end{aligned} \tag{38}$$

[11] This is true if the instantaneous preferences are specified by, e.g., those in (a)–(c) and (f)–(g).

[12] A sufficient condition for $g(k) = g(p(k))$ is that the instantaneous preferences of the consumer are stationary.

[13] The choice of α makes it possible to obtain the solutions with various values of R, θ, μ, and β.

In particular, when $\alpha = 0$, we have

$$c(s, t+j) = \frac{(1+\mu-\beta)}{(1+\mu)}\left[(R+\mu)b(s, t+j-1) + \frac{R+\mu}{R+\mu-\phi}y(s, t+j)\right] \tag{39}$$

and

$$b(s, t+j) = \beta(R+\mu)b(s, t+j-1) + \frac{\beta(R+\mu)-\phi(1+\mu)}{(R+\mu-\phi)(1+\mu)}y(s, t+j). \tag{40}$$

Now we consider the special case $\alpha = 1$. That is, the indirect utility function before or after transformation is linear in $c(s, t+j)$. For simplicity, we assume $R(k) = R$, $\beta(k) = \beta^k$, $\psi(s, k) = 0$, $g(k) = g(p(k))$, $p_i(k+1)/p_i(k) = \theta$ for all i and $k \geq s$. The problem under a more general situation can be similarly analyzed. Since $g(p(k))$ is homogeneous of degree 1, $g(p(t+j))/g(p(t)) = \theta^j$. Thus the consumer's intertemporal optimization problem (26) under this specification can be written as

$$\max \frac{\mu}{(1+\mu)g(p(t))}\sum_{j=0}^{\infty}\left[\frac{\beta}{(1+\mu)\theta}\right]^j [c(s, t+j) - g(p(t))] \tag{41}$$

subject to the lifetime budget constraint

$$\sum_{k=0}^{\infty}[R+\mu]^{-k}c(s, t+k) = \sum_{k=0}^{\infty}[R+\mu]^{-k}y(s, t+k) + b(s, t-1). \tag{42}$$

We want to find the solutions for the above problem.

Case 1. Suppose $\beta/(1+\mu)\theta = (R+\mu)^{-1}$. Then $\sum_{k=0}^{\infty}(\beta/(1+\mu)\theta)^k c(s, t+k) = \sum_{k=0}^{\infty}(R+\mu)^{-k}c(s, t+k) = \sum_{k=0}^{\infty}(R+\mu)^{-k}y(s, t+k) + b(t-1)$. Therefore, any sequence $\{c(s, t+k)\ \}$ which satisfies the budget constraint is an optimal solution. Note that $c(s, t+k)$ is not a single-valued function but a correspondence.

Case 2. Suppose $\beta/(1+\mu)\theta < (R+\mu)^{-1}$. Since $(R+\mu)\beta/(1+\mu)\theta < 1$, $((R+\mu)\beta/(1+\mu)\theta)^{t+j} < ((R+\mu)\beta/(1+\mu)\theta)^t$ for all $j > 0$, the consumer maximizes by spending all his income at time t but nothing in the following time periods. That is

$$c(s, t+j) = \begin{cases} \sum_{k=0}^{\infty}(R+\mu)^{-k}y(s, t+k) + b(t-1) & \text{if } j = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{43}$$

Case 3. Suppose $\beta/(1+\mu)\theta > (R+\mu)^{-1}$. Since $(R+\mu)\beta/(1+\mu)\theta > 1$, $((R+\mu)\beta/(1+\mu)\theta)^j > (\beta(R+\mu)/(1+\mu)\theta)^k$ for all $j > k$, the consumer can always increase his utility by delaying consuming his income. Thus the optimal solution does not exist.

Remark 3 For a linear indirect utility function in $c(s, t+j)$, Case 1 imposes very strong restrictions on the interest rate, time preference, and inflation. Also, the consumption function is not single-valued. The behavior of the consumer in Cases 2 and 3 does not seem to be realistic. Since the preferences specified by (a)–(g) except (e) are linear in $c(s, t+j)$, monotonic transformations are, sometimes, indeed necessary.

The aggregate consumption-savings functions can be obtained from (7), (31), and (32)

$$C(t+j) = \frac{W(s, t+j) - \bar{\Psi}(t+j)}{\sum_{k=0}^{\infty}\left[\frac{(1+\mu)^k\Phi(t+j+k)^\alpha\beta(t+j)g(t+j+k)^\alpha}{\Phi(t+j)^\alpha\beta(t+j+k)g(t+j)^\alpha}\right]^{\frac{1}{\alpha-1}}} + \Psi(t+j) \tag{44}$$

and

$$B(t+j) = -\frac{W(t+j) - \bar{\Psi}(t+j)}{\sum_{k=0}^{\infty}\left[\frac{(1+\mu)^k\Phi(t+j+k)^{\alpha}\beta(t+j)g(t+j+k)^{\alpha}}{\Phi(t+j)^{\alpha}\beta(t+j+k)g(t+j)^{\alpha}}\right]^{\frac{1}{\alpha-1}}} + [R(t+j-1)+\mu]B(t+j-1) + Y(t+j) - \Psi(t+j). \tag{45}$$

5 The Continuous-Time Consumer Problem

Instead of assuming that time is discrete, we will take the point of view—both more realistic and analytically more convenient—of continuous time. The results in continuous time parallel very closely those in discrete time.

As in Section 3, we assume that the source of uncertainty is only about the time of death. Thus the consumer's intertemporal optimization problem at t is

$$\max \int_0^{\infty} e^{-\int_t^{t+v}[\delta(\tau)+\mu]d\tau}\frac{(c(s,t+v) - \psi(s,t+v))/g(t+v)^{\alpha} - 1}{\alpha}dv \tag{46}$$

subject to the dynamic budget constraint:

$$c(s,t+v) + \dot{b}(s,t+v) = y(s,t+v) + [r(t+v)+\mu]b(t+v) \tag{47}$$

with $b(t)$ given. Here $\delta(v)$ is the pure time-preference factor or subjective rate of discount with $\delta(v) > 0$ and $\dot{b}(s,t+v)$ is the derivative of $b(s,t+v)$ with respect to time. We assume that $\int_0^{\infty} e^{-\int_t^{t+v}[\delta(\tau)+\mu]d\tau}dv$ converges. In addition, the terminal condition (or borrowing constraint):

$$\lim_{v\to\infty} \Phi(t+v)b(s,t+v) = 0 \tag{48}$$

is needed to prevent the consumer from borrowing arbitrarily large amounts.

The Euler equations (the first-order necessary conditions) are

$$\alpha e^{-\int_t^{t+v}[\delta(\tau)+\mu]d\tau}\frac{(c(s,t+v) - \psi(s,t+v)^{\alpha-1})}{g(t+v)^{\alpha}} = \lambda(t+v); \tag{49}$$

$$r(t+v) + \mu = -\frac{\dot{\lambda}(t+v)}{\lambda(t+v)}. \tag{50}$$

Solving the differential equation forward, we get

$$c(s,t+v+u) = (c(s,t+v) - \psi(s,t+v))\left[\frac{\Phi(t+v+u)g(t+v+u)^{\alpha}}{\Phi(t+v)g(t+v)^{\alpha}}\right]^{\frac{1}{\alpha-1}} \times e^{\frac{1}{\alpha-1}\int_{t+v}^{t+v+u}[\delta(\tau)+\mu]d\tau} + \psi(s,t+v+u)). \tag{51}$$

Solving the dynamic budget constraint (47) for $b(s,t+v)$ forward and using the terminal condition (48) , we obtain the lifetime budget constraint:

$$\Phi^{-1}(t+v)\int_0^{\infty}\Phi(t+v+u)c(t+v+u)du = w(s,t+v), \tag{52}$$

where

$w(s,t+v) = b(s,t$

The above budget constrai
that of the earned income
Substituting (51) into t

$c(s,t+v) =$ ... $\int_0^{\infty}\left[\frac{\Phi(t}{\Phi}\right.$

where $\bar{\psi}(s,t+v) = \Phi^{-1}(t$
The demand functions
particular, those demand f
be obtained by substitutin
functions. As $F_{t+v}(v_{t+v}(p(t$
(i.e., $\alpha = 0$), the solution i

$c(s,t+$

Remark 4 The consumpt
utility is logarithmic, $\delta(u)$
If $\mu = 0$, $\delta(u) = \delta$, $\psi(s,$
Polar form and the Box-C
ren (1983), and further, if
preference ordering and ψ

The aggregate consum

$C(t+v) =$ ... $\int_0^{\infty}\left[\right.$

6 Problem Wi

This section discusses the
not only about the time o
cannot obtain closed-form
or after transformation ar
for the consumption with
to be zero (see, e.g., Levh
form solutions can be obt
by (a)–(c) and (f)–(g) in
results can be obtained si

$$\Big]^{\frac{1}{\alpha-1}}$$

$$(t+j) - \Psi(t+j). \qquad (45)$$

er Problem

ne point of view—both more me. The results in continuous

nty is only about the time of blem at t is

$$)/g(t+v)^{\alpha} - 1 \, dv \qquad (46)$$

$$) + \mu]b(t+v) \qquad (47)$$

or subjective rate of discount h respect to time. We assume inal condition (or borrowing

$$(48)$$

ly large amounts. s) are

$$= \lambda(t+v); \qquad (49)$$

$$= -\frac{\dot{\lambda}(t+v)}{\lambda(t+v)}. \qquad (50)$$

$$+u)g(t+v+u)^{\alpha} \Big/ +v)g(t+v)^{\alpha} \Big]^{\frac{1}{\alpha-1}}$$

$$v+u)). \qquad (51)$$

) forward and using the ter- int:

$$w(s,t+v), \qquad (52)$$

where

$$w(s,t+v) = b(s,t+v) + \Phi^{-1}(t+v)\int_0^\infty \Phi(t+v+u)y(s,t+v+u)du.$$

The above budget constraint states that the present value of the expenditure is equal to that of the earned income plus property income.

Substituting (51) into the above equation and solving it for $c(s,t+v)$, we have

$$c(s,t+v) = \frac{w(s,t+v) - \bar{\psi}(s,t+v)}{\int_0^\infty \left[\frac{\Phi(t+v+u)g(t+v+u)}{\Phi(t+v)g(t+v)}\right]^{\frac{\alpha}{\alpha-1}} e^{\frac{1}{\alpha-1}\int_{t+v}^{t+v+u}[\delta(\tau)+\mu]d\tau} du} + \psi(s,t+v), \qquad (53)$$

where $\bar{\psi}(s,t+v) = \Phi^{-1}(t+v)\int_0^\infty \Phi(t+v+u)\psi(s,t+v+u)du$.

The demand functions can then be obtained from the Antonelli-Allen-Roy p.d.e. In particular, those demand functions generated by the preferences specified by (a)–(g) can be obtained by substituting the solution $c(s,t+v)$ into the corresponding static demand functions. As $F_{t+v}(v_{t+v}(p(t+v), c(s,t+v))) = \log(c(s,t+v) - \psi(s,t+v)) - \log g(t+v)$ (i.e., $\alpha = 0$), the solution is simply

$$c(s,t+v) = \frac{w(s,t+v)}{\int_0^\infty e^{-\int_{t+v}^{t+v+u}[\delta(\tau)+\mu]d\tau} du} + \psi(s,t+v). \qquad (54)$$

Remark 4 The consumption function given by (53) is very general. If the instantaneous utility is logarithmic, $\delta(u) = \delta$, $\psi(s,u) = 0$, we obtain the results of Blanchard (1983). If $\mu = 0$, $\delta(u) = \delta$, $\psi(s,u) = \psi(p(u))$, and preferences are specified by the Gorman Polar form and the Box-Cox transformation, we obtain the results of Cooper and Mclaren (1983), and further, if preferences are specified by the Klein-Rubin-Samuelson-Geary preference ordering and $\psi(s,u) = \gamma(u) \cdot p(u)$, we obtain the results of Lluch (1973).

The aggregate consumption can be obtained from (7) and (53):

$$C(t+v) = \frac{W(t+v) - \bar{\Psi}(t+v)}{\int_0^\infty \left[\frac{\Phi(t+v+u)g(t+v+u)}{\Phi(t+v)g(t+v)}\right]^{\frac{\alpha}{\alpha-1}} e^{\frac{1}{\alpha-1}\int_{t+v}^{t+v+u}[\delta(\tau)+\mu]d\tau} du} + \Psi(t+v). \qquad (55)$$

6 Problem With More General Uncertainty

This section discusses the consumer's problem in which the sources of uncertainty are not only about the time of death but also about prices and interest rates. In general, we cannot obtain closed-form solutions for this case unless the indirect utility functions before or after transformation are quadratic (i.e., unless certainty equivalence holds). However, for the consumption with random-return models in which wage income $y(s,t)$ is assumed to be zero (see, e.g., Levhari and Srinivasan (1969) and Sargent (1987, p. 31)), closed-form solutions can be obtained for some preferences (say, for those preferences specified by (a)–(c) and (f)–(g) in Section 2). We only consider the case of discrete time. The results can be obtained similarly for continuous time.

We now assume that the uncertainty about the time of death is independent of the uncertainty about the other sources. Then consumer's intertemporal optimization problem at t can be written as

$$\max \frac{\mu}{1+\mu} E_t \sum_{j=0}^{\infty} \frac{\beta^j}{1+\mu} F_{t+j}\Big(v_{t+j}\big(p(t+j), c(s,t+j)\big)\Big), \quad \beta < 1 \tag{56}$$

subject to the budget constraint

$$c(s,t+j) + b(s,t+j) = [R(t+j-1) + \mu] b(s,t+j-1). \tag{57}$$

Here $R(t+j-1)b(s,t+j-1)$ can be interpreted as the consumer's wealth or assets. The consumer has the option of either consuming all his wealth or "investing" part of it. We assume that $p(t+j)$ and $R(t+j)$ are the sequence of independently and identically distributed random variables. E_{t+j} denotes the conditional mathematical expectation on prices and interest rate at time $t+j$. In addition to the above budget constraint, the terminal condition (or borrowing constraint) is

$$\lim_{j\to\infty} \Phi(t+j) b(s,t+j) = 0. \tag{58}$$

Consider those preferences and monotonic transformations $F_{t+j}(\cdot)$ such that

$$F_{t+j}(v_{t+j}(p(t+j), c(t+j))) = \log c(s,t+j) - \log g(t+j). \tag{59}$$

The Euler equation for this specification then is

$$\frac{1}{c(s,t+j)} = \frac{\beta}{1+\mu} E_{t+j} \frac{R(t+j)+\mu}{c(s,t+j+1)}. \tag{60}$$

It is easily verified that $c(s,t+j) = (1 - \frac{\beta}{1+\mu})[R(t+j-1)+\mu]b(s,t+j-1)$ satisfies the Euler equation. Since $\log c(s,t+j)$ is strictly concave, we know that the optimal policy functions for consumption and savings are $c(s,t+j) = (1-\frac{\beta}{1+\mu})[R(t+j-1)+\mu]b(s,t+j-1)$ and $b(s,t+j) = \frac{\beta}{1+\mu}[R(t+j-1)+\mu]b(s,t+j-1)$.

Similarly, the demand functions can then be obtained from the Antonelli-Allen-Roy p.d.e.

References

ARROW, K. J. and HURWICZ, L. (1958), "On the Stability of the Competitive Equilibrium, I," *Econometrica* **26**, 522–552.

BLANCHARD, O. J. (1985), "Debt, Deficits, and Finite Horizons," *Journal of Political Economy* **93**, 223–247.

BLUNDELL, R. and WALKER, I. (1986), "A Life-Cycle Consistent Empirical Model of Family Labor Supply Using Cross-Section Data," *Review of Economic Studies* **53**, 539–558.

BROWN, M. and HEIEN, D. M. (1972), "The S-Branch Utility Tree: A Generalization of the Linear Expenditure System," *Econometrica* **40**, 737–747.

CHIPMAN, J. S. and MOORE, J. C. (1976), "The Scope of Consumer's Surplus Arguments," in A. M. Tang et al., eds., *Evolution, Welfare and Time in Economics*, D. C. Heath, Lexington, Mass., 69-123.

CHIPMAN, J. S. and MOORE, J. C. (1980), "Compensating Variation, Consumer's Surplus, and Welfare," *American Economic Review* **70**, 933–949.

CHIPMAN, J. S. and MOORE, J. C. (1990), "Acceptable Indicators of Welfare Change, Consumer's Surplus Analysis, and the Gorman Polar Form," in J. S. Chipman, D. McFadden, and M. K. Richter, eds., *Preferences, Uncertainty, and Optimality*, Westview Press, Boulder, Colorado, 68–120.

CHIPMAN, J. S. and TIAN, G. (1992), "A General-Equilibrium Intertemporal Model of an Open Economy," *Economic Theory* **2**, 215–246.

COOPER, R. J. and MCLAREN, K. R. (1983), "Modelling Price Expectations in Intertemporal Consumer Demand Systems: Theory and Application, " *Review of Economics and Statistics* **65**, 282–288.

GORMAN, W. M. (1976), "Tricks with Utility Functions," in M. Artis and R. Nobay, eds., *Essays in Economic Analysis*, Cambridge University Press, Cambridge, 211–243.

KANTOROVICH, L. V. (1965), *The Best Uses of Economic Resources.* Harvard University Press, Cambridge, Mass.

LEONTIEF, W. W. (1956), "Factor Proportions and the Structure of American Trade: Further Theoretical and Empirical Analysis," *Review of Economics and Statistics* **38**, 386–407.

LEVHARI, D. and SRINIVASAN, T. N. (1969), "Optimal Savings Under Uncertainty," *Review of Economic Studies* **36**, 153–163.

LLUCH, C. (1973), "The Extended Linear Expenditure System," *European Economic Review* **15**, 21–32.

MACURDY, T. E. (1983), "A Simple Scheme for Estimating an Intertemporal Model of Labor Supply and Consumption in the Presence of Taxes and Uncertainty," *International Economic Review* **24**, 265–289.

SARGENT, T. J. (1987), *Dynamic Macroeconomic Theory*, Harvard University Press, Cambridge, Mass., 1987.

TIAN, G. and CHIPMAN, J. S. (1989), "A Class of Dynamic Demand Systems," in B. Raj, ed., *Advances in Econometrics and Modeling*, D. Reidel Publishing Company, Dordrecht, Holland, 93–116.

YAARI, M. E. (1965), "Uncertain Lifetime, Life Insurance, and Theory of the Consumer," *Review of Economic Studies* **32**, 137–50.