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TRANSFER METHOD FOR CHARACTERIZING THE EXISTENCE OF MAXIMAL ELEMENTS OF BINARY RELATIONS ON COMPACT OR NONCOMPACT SETS*

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Abstract. This paper systematically studies the existence of maximal elements for unordered binary relations on compact or noncompact sets in a general topological space. This is done by developing a method, called *transfer method*, to derive various necessary and sufficient conditions that characterize the existence of maximal elements for a binary relation in terms of: (1) (generalized) transitivity conditions under certain topological assumptions; (2) topological conditions under certain (generalized) transitivity assumptions; and (3) (generalized) convexity conditions under certain topological assumptions. There are two basic approaches in the literature to prove the existence by providing sufficient conditions. One assumes certain convexity and continuity conditions for a topological vector space and the other assumes certain weakened transitivity and continuity conditions for a general topological space. The results unify those two approaches and generalize almost all of the existing results in the literature.

Key words. binary relations, maximal elements, transfer continuities, transfer transitivities, transfer convexities

AMS(MOS) subject classifications. 49A27, 90C48, 90C31, 90B50

1. Introduction. Let Y be a topological space and $u: Y \to \mathbf{R}$ be a function. The classical Weierstrass theorem states that u attains its maximum on any nonempty compact set $X \subset Y$ if u is upper semicontinuous. As generalizations of the Weierstrass theorem, Tian and Zhou [18] proved two theorems that give necessary and sufficient conditions for u to attain its maximum on a nonempty compact set by introducing the notion of transfer continuities. The idea behind this is quite simple. To characterize the existence of maximal points for a function u, for given $u(x) > u(y)$, we really do not have to know the topological relations between x and a neighborhood $\mathcal{N}(y)$ of y. All we need to know is the topological relations between a neighborhood of y and a point x' in the upper part of $u(y)$, i.e., whether x can be transferred to x', a point in the upper part of $u(y)$ such that $u(x') > (\geqq) u(y')$: $y' \in \mathcal{N}(y)$, and if so, u is said to be *transfer (weakly) upper continuous* on X. However, in many cases in economics, decision analysis, optimization, and game theory, *a binary relation is not representable by a function even for an ordering.* Thus many results are given in the literature to prove the existence of maximal elements of a binary relation for this case. See, e.g., Yu [22], Borwein [4], Luc [10], and others who study the existence of maximal elements for a partial ordering induced by a convex cone in a topological vector space; and Fan [7],[1] Schmeidler [13], Sonnenschein [14], Shafer [11], Shafer and Sonnenschein [12], Bergstrom [3], Walker [19], Yannelis and Prabhakar [21], Campbell and Walker [5], Tian [16], [17], and others who study the existence of maximal elements for unordered binary relations by assuming either certain convexities or certain transitivities (at least acyclicity). Most of the results provide only sufficient conditions.

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[1] Fan [7, Lem. 4] does not phrase his results in terms of maximizing binary relations, but his results can be interpreted that way.

In this paper we systematically study maximization of binary relations. Note that there are two basic approaches to unordered binary relations in the literature: one through "weak" (i.e., reflexive) binary relations, and the other through "strict" (i.e., irreflexive) binary relations. Kim and Richter [9] made the connection between these two approaches and proved that *these two approaches are equally valid: definitions and theorems in one approach correspond in parallel to definitions and theorems in the other approach.* So in this paper we, without loss of generality, deal with only "strict" binary relations.

Let Y be a set and $X \subset Y$ be a subset. Denote by "$\succ$" an "irreflexive (strict) binary relation" on Y and "$\succeq$" the completion of "$\succ$," i.e., $x \succeq y$ means that $y \succ x$ does not hold, and thus "$\succeq$" is a "reflexive (weak) and complete binary relation." Here $y \succ x$ is read "y is *strictly preferred* (*or dominated*) to x" and y is said to be a dominator to x. Let A be a subset of Y and $y \in Y$. Denote by $y \succ (\succeq) A$ if $y \succ (\succeq) x$ for all $x \in A$ and y is said to be a dominator (maximizer) to A.

An element $x^* \in X$ is said to be a *maximal element* of the binary relation "$\succ$" on X if $x \succeq X$, i.e., x^* has no dominator in X.

Our objective in this paper is to study the existence of maximal elements for a binary relation "$\succ$" on a nonempty compact or noncompact set. We characterize the existence in terms of: (1) certain topological conditions, (2) certain (generalized) transitivity conditions, and (3) certain generalized convex (geometric) conditions. We extend the notion of transfer continuities further to *transfer transitivities* and *transfer convexities.* We call this notion the *transfer method.* The basic idea behind it is as follows. For topology, given $x \succ y$, the conventional continuity conditions describe topological behavior or relations between x and a neighborhood of y. For transitivity, given a finite subset $X_0 = \{x_1, x_2, \cdots, x_n\}$, conventional transitivities describe "relations" within the finite set X_0, i.e., the "internal relations." For geometry and algebra, given a finite subset $X_0 = \{x_1, x_2, \cdots, x_n\}$, conventional convexity conditions describe "relations" between this finite set and its convex hull. To characterize the existence of maximal elements for "$\succ$," when $x \succ y$, we do not have to know the topological relations between x and a neighborhood of y, the internal relations of the finite subset X_0, and the geometric and algebraic relations between the finite set and its convex hull. We only need to know, for topology, the topological behavior or relations between a neighborhood of y and an element x' in its "upper" part (so x can be transferred to a certain element x' in the "upper" part of a neighborhood of y); for transitivity, the relations between the finite subset X_0 and an element x' in the "upper" part of the finite subset X_0, i.e., the "external relation"; for geometry, the relations between the finite set X_0 and the convex hull of a corresponding finite subset in the part not "below" X_0. Conditions describing the topological relations between a neighborhood of y and an element in its "upper" part are called *transfer continuities*; conditions describing the relations between the finite subset X_0 and an element in its "upper" part are called *transfer transitivities*; and conditions describing the geometric relations between the finite subset X_0 and the convex hull of a corresponding finite set in the part not "below" X_0 are called *transfer convexities.*

This paper consists of four sections. In § 1, we introduce various transfer conditions and we explore their connections with conventional conditions and some of their properties as preliminaries for further development. In § 2, we characterize the existence of maximum elements for binary relations on nonempty compact sets by giving necessary and/or sufficient conditions in terms of: (1) transfer transitivity conditions under certain transfer continuity assumptions, (2) transfer continuity conditions under certain transfer transitivity assumptions, and (3) transfer convexity conditions under certain transfer continuity assumptions. In § 3, we first discuss some properties of the definitions in § 1, and then provide several theorems to characterize the existence of maximum elements for binary relations on nonempty noncompact sets by also giving necessary and/or sufficient conditions in terms of various transfer conditions. In § 4, as concluding remarks, we first indicate that our results can be used to give conditions under which the maximum correspondence in Walker's Maximum Theorem is nonempty valued, which is required for many applications in decision analysis and game theory and serves as part of our motivation for this work. Then we show how a maximization problem, with respect to a (weak) binary relation, can be converted to a maximization problem, with respect to a (strict) binary relation, so our approach can be applied.

1.1. Transfer transitivities. In the following definition, whenever $K = X$, "to K" will be replaced by "on X" or omitted.

Definition 1. Let K be a subset of a set X. A binary relation "$\succ$" defined on X is said to be:

(1) *Transfer n-maximal* to K, if for each finite subset $\{x_1, x_2, \cdots, x_n\} \subset X$ there exists $x' \in K$ such that $x' \geqq \{x_1, x_2, \cdots, x_n\}$;

(2) *transfer finitely maximal* to K, if it is transfer n-maximal to K for all $n = 1, 2, \cdots$;

(3) *n-acyclic* on X, if $x_1 \succ x_2 \succ \cdots \succ x_k$ implies $x_1 \geqq x_k$ for all $k = 1, 2, \cdots, n$ (1-acyclic just means $x \geqq x$ for all $x \in X$);

(4) *acyclic* on X, if it is n-acyclic for all $n = 1, 2, \cdots$;

(5) *transfer n-strict maximal* to K, if for all y_i, x_i in X with $y_i \succ x_i$, $i = 1, 2, \cdots, n$ there exists $x' \in K$ such that $x' \succ \{x_1, x_2, \cdots, x_n\}$;

(6) *transfer finitely strict maximal* to K, if it is transfer n-strict maximal to K for all $n = 1, 2, \cdots$;

(7) *n-link transitive* on X, if $y \succ x_0 \geqq x_1 \geqq \cdots \geqq x_n \succ z$ implies $y \succ z$;

(8) *link transitive* on X, if it is n-link transitive on X for all $n = 0, 1, 2, \cdots$;

(9) *fully transitive* on X, if its completion "$\geqq$" is transitive on X, i.e., $x \geqq y \geqq z$ imply $x \geqq z$.

Remark 1. Here we can see that many definitions in the literature have been unified. The way we define those transitivities makes it easier for us to save terminologies and to see implications among different transitivities. For instance, in Definitions 1(1), 1(3), 1(5), and 1(7), the case for $n+1$ implies the same case for n. Conventionally:

(1) a 1-acyclic "$\succ$" is said to be irreflexive, i.e., not $x \succ x$ or $x \geqq x$ for all $x \in X$;

(2) a 2-acyclic "$\succ$" is said to be asymmetric, i.e., $x \succ y$ and not $y \succ x$, which implies $x \geqq y$, and is also said to be a "preference" relation;

(3) a 0-link transitive "$\succ$" is said to be (weakly) transitive in [5]. Therefore, a 0-link transitive "$\succ$" induces a partial ordering;

(4) a 1-link transitive "$\succ$" is said to be extratransitive in [5].

Remark 2. Cone preference relations have been adopted very often in (multiple-criteria decision making) vector optimization (cf. Yu [22], Borwein [4], Tanaka [15], Ferro [8], and Luc [10]). Therein (weak) cone preferences are defined to induce partial orderings. Here we show that a cone preference is just a very special case of our approach.

Let X be a subset of a real topological vector space Y and let C be a convex cone in Y. Let $C^- = -C$. Define (see, e.g., [10]) the (weak) cone preference "$\geqq_c$" in Y by $y \geqq_c x$ if and only if $y - x \in C$. Thus its asymmetric part of $\geqq_c$, denoted by $\succ_c$, i.e., $y \succ_c x$ whenever $y \geqq_c x$ and not $x \geqq_c y$, defines a strict preference relation. Then a point $x^* \in X$ is said to be an *efficient* (*or minimal*) point of X with respect to C if no $x \in X$ such that $x^* >_c x$, i.e., either $x^* - x \notin C$ or $x^* - x \in C \cap C^-$ for all $x \in X$. For such a (weak) cone preference, we can define a (strict) cone preference relation "$\succ$" in Y by $y \succ x$ if and only if $x - y \in C \setminus (C \cap C^-)$, and write its completion "$\geqq$" by $x \geqq y$ whenever $y \succ x$ does not hold, i.e., $x \geqq y$ if either $x - y \notin C$ or $x - y \in C \cap C^-$. Now following our definition, a maximal element of "$\succ$" on X is an element $x^* \in X$ such that $x^* \geqq x$, for all $x \in X$, i.e., in this case $x^* - x \notin C$ or $x^* - x \in C \cap C^-$ for all $x \in X$. Thus x^* is an efficient point of "$\geqq_c$" if and only if it is a maximal point of "$\succ$."

It may be remarked that the above-defined (strict) cone binary relation "$\succ$" is 0-link transitive on X, i.e., $z \succ y \succ x$ implies $z \succ x$. To see this we only need to show that $x' \in C \setminus (C \cap C^-)$ and $y' \in C \setminus (C \cap C^-)$ imply $x' + y' \in C \setminus (C \cap C^-)$. Since C is a convex cone, $x' + y' \notin C \setminus (C \cap C^-)$ implies $x' + y' \in (C \cap C^-)$ and $y' \in C$ implies $-y' \in C^-$. Then $x' = (x' + y') + (-y') \in C^-$and leads to a contradiction.

Thus our approach is very general and includes the cone preference as a special case. It then frees us, in considering vector optimization, from using linear structures and from restricting a binary relation to being defined by a cone. We believe our *transfer method* can be applied to cone preference to both derive and characterize the existence of maximal elements.

Remark 3. Campbell and Walker [5] overlooked the fact that the pseudotransitivity in [5], defined by "$x_1 \succ x_2 \geqq x_3 \succ x_4$ implies $x_1 \succ x_4$ when $x_2 \neq x_3$," is weaker than the 1-link transitivity when $\succ$ is asymmetric. The pseudotransitivity and 1-link transitivity are equivalent by noting that the pseudotransitivity implies the 0-link transitivity (since the pseudotransitivity and the 0-link transitivity together clearly imply the 1-link transitivity). To see this, suppose that $x \succ y \succ z$ (which implies $x \neq z$ by the asymmetricity), but $z \geqq x$. Then we have $y \succ z \geqq x \succ y$. The pseudotransitivity implies $y \succ y$ but this is impossible.

Since our objective is to characterize the existence of *maximal* elements for a binary relation, to better understand those transitivities stated in Definition 1, it is beneficial for us to restate some of these transitivities in terms of maximization terminologies.

LEMMA 1. *Let "$\succ$" be a binary relation on a set X.*

(1) *For any fixed integer $n = 1, 2, 3, \cdots$, the binary relation is n-acyclic on X if and only if any n elements $\{x_1, x_2, \cdots, x_n\} \subset X$ have an internal maximal element, i.e., there exists $x_i \in \{x_1, x_2, \cdots, x_n\}$ such that $x_i \geqq \{x_1, x_2, \cdots, x_n\}$. Consequently, the binary relation is acyclic on X if and only if for any integer n, any finite subset $\{x_1, x_2, \cdots, x_n\} \subset X$ has an internal maximal element.*

(2) *The binary relation is acyclic if it is 0-link transitive.*

Proof. (1) The second part of (1) follows from the first part, so we only need to prove the first part. The cases $n = 1, 2$ are obvious. For $n > 2$, to prove the "only if" part, we assume that the binary relation is n-acyclic and that there exists n elements $\{x_1, x_2, \cdots, x_n\} \subset X$ without an internal maximal element. These elements therefore form a $k+1$ cycle for some integer k with $3 \leqq k \leqq n$. Without loss of generality, we assume that the $k+1$ cycle is of the form $x_1 \succ x_2 \succ \cdots \succ x_k \succ x_1$. Since the binary relation is k-acyclic, we have $x_1 \geqq x_k \succ x_1$, and this is impossible.

To prove the "if" part, we assume that for each fixed integer $n > 2$ the binary relation has an internal maximal element for any n elements in X. This implies that the binary relation has an internal maximal element for any k elements in X with $3 \leqq k \leqq n$. Let $\{x_1, x_2, \cdots, x_k\}$ be k elements in X with $x_1 \succ x_2 \succ \cdots \succ x_k$. Since $x_k \succ x_1$ will force a $k+1$ cycle to form, i.e., these k elements have no internal maximal elements, and reduce to a contradiction, we must have $x_1 \geqq x_k$ and thus the binary relation is k-acyclic for all $1 \leqq k \leqq n$.

(2) Without loss of generality, if there is a cycle of the form $x_1 \succ x_2 \cdots \succ x_k \succ x_1$ for some $1 \leqq k \leqq n$, 0-link transitivity will lead to $x_1 \succ x_1$—a contradiction. □

LEMMA 2. *The binary relation is* 1*-link transitive on* X *if and only if for any integer* n *and any* x_i *and* y_i *with* $y_i \succ x_i$, $i = 1, 2, \cdots, n$, *there exists* $1 \leqq k \leqq n$ *such that* $y_k \succ \{x_1, x_2, \cdots, x_n\}$.

Proof. The "if" part is obvious. For if $y \succ x_1 \succ x_2 \succ z$, then either $y \succ \{x_1, z\}$ or $x_2 \succ \{x_1, z\}$. But $x_1 \geqq x_2$, so $y \succ \{x_1, z\}$ and thus "$\succ$" is 1-link transitive. Now we prove the "only if" part by mathematical induction. When $n = 1$ it is obviously true. Suppose it is true for all $n \leqq m$. Now for $n = m+1$, if $y_i \succ x_i$, $1 \leqq i \leqq m+1$, then, according to the assumption on $n \leqq m$, there exists y_k, $1 \leqq k \leqq m$, such that $y_k \succ \{x_1, x_2, \cdots, x_m\}$. If $y_k \succ x_{m+1}$, it is done. Otherwise we have $y_{m+1} \succ x_{m+1} \geqq y_k \succ \{x_1, x_2, \cdots, x_m\}$. By the 1-link transitivity, we obtain $y_{m+1} \succ \{x_1, x_2, \cdots, x_m\}$. Then $y_{m+1} \succ \{x_1, x_2, \cdots, x_m, x_{m+1}\}$. □

Remark 4. Definitions 1(3), 1(4), 1(7), and 1(8) are of conventional types and Definitions 1(1), 1(2), 1(5), and 1(6) are of transfer types. By consulting Lemmas 1 and 2 we can see how we applied the transfer method to the conventional Definitions 1(3), 1(4), 1(7), and 1(8) (we simply allow the dominator or maximal element to n elements to exist inside or outside the n elements) to obtain, respectively, Definitions 1(1), 1(2), 1(5), and 1(6). Therefore, they are very natural generalizations of the conventional assumptions. It is these transfer conditions that enable us to avoid the asymmetric assumption. When $K = X$ we have the following implications among various transitivities stated in Definition 1, while none of their converses hold ($\oplus$ means that the binary relation is asymmetric):

$$
\begin{tikzcd}
 & (9) \arrow[d, Rightarrow, "\oplus"'] & \\
 & (8) \arrow[d, Rightarrow] & \\
 & (7) \arrow[dl, Rightarrow] \arrow[dr, Rightarrow, "(n>0)"] & \\
(4) \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & & (6) \arrow[dl, Rightarrow, "\oplus"'] \arrow[d, Rightarrow] \\
(3) \arrow[dr, Rightarrow, "(\text{same } n)"'] & (2) \arrow[d, Rightarrow] & (5) \arrow[dl, Rightarrow, "\oplus(\text{same } n)"] \\
 & (1) &
\end{tikzcd}
$$

where

(3)$\Rightarrow$(1) follows from Lemma 1(1);

(7)$\Rightarrow$(6) follows from Lemma 2;

(7)$\Rightarrow$(4) follows from Lemma 1(2);

(5)$\Rightarrow$(1) because if n elements have no maximal element, each one of them has a dominator; then (5) guarantees the existence of an outside dominator (a maximal element under asymmetry) to all these n elements;

(4)$\Rightarrow$(2) follows from Lemma 1(1).

Next we provide two examples to demonstrate that for a binary relation the acyclic condition strictly implies the transfer finitely maximal condition, while the acyclic condition is independent of the transfer finitely strict maximal condition.

Example 1. Let $Y = \mathbf{C}$, the complex plane. Define a binary relation "$>$" for any $z_1, z_2 \in Y$ by

$$(1) \quad z_1 > z_2 \quad \text{iff} \begin{cases} \text{either} & |z_1| < |z_2| \text{ and } z_1, z_2 \text{ are on the same ray from the origin} \\ \text{or} & |z_1| = |z_2| \text{ but } z_1 = e^{i\theta} z_2, \text{ for } 0 < \theta \leqq \pi/2. \end{cases}$$

Let X be the unit disk on the complex plane $\mathbf{C}$. Then for each r, $0 < r \leqq 1$, we have a cycle

$$(r, 0) > (0, -r) > (-r, 0) > (0, r) > (r, 0).$$

However, the origin is the unique maximal point on X, which is strictly preferred to any other point. So "$>$" is transfer finitely strict maximal (of course, transfer finitely maximal) on X.

Example 2. Let $Y = \mathbf{C}$, the complex plane. Define a binary relation "$>$" for any $z_1, z_2 \in Y$ by

$$(2) \quad z_1 > z_2 \quad \text{iff} \begin{cases} \text{either} & |z_1| < |z_2| \text{ and } \arg(z_1) = \arg(z_2) \\ \text{or} & |z_1| = |z_2| \text{ but } z_1 = e^{i\theta} z_2, \text{ for } 0 < \theta \leqq \pi/2. \end{cases}$$

Here the argument of the origin, arg (0), is regarded as zero. Let X be the unit disk on the complex plane $\mathbf{C}$. Then for each r, $0 < r \leqq 1$, we have a cycle

$$(r, 0) > (0, -r) > (-r, 0) > (0, r) > (r, 0).$$

However, the origin is the unique maximal point on X, thus "$>$" is transfer finitely maximal on X. Indeed, we have $(0, 0) > (0, r)$ for all $0 < r \leqq 1$, and $(0, 0) \geqq$ any other points (where $>$ does not hold). If we let X be the upper half of the unit disk, including the bottom line, then it is easy to see that "$>$" is acyclic, but is not transfer finitely strict maximal on X. So we can see that the acyclic condition and the transfer finitely strict maximal condition are two independent conditions. We point out here that the 0-link transitive condition and the transfer finitely strict maximal condition are also independent.

1.2. Transfer continuities and convexities.

DEFINITION 2. Let X be a subset of a topological space Y and let z be any point in Y; denote $\mathcal{N}(z)$ a neighborhood of z. The binary relation "$>$" defined on Y is said to be:

(1) *upper continuous* on X, if for any $x \in X$ and $y \in Y$, $x > y$ implies that there exists $\mathcal{N}(y)$ such that $x > \mathcal{N}(y)$;

(2) *weakly upper continuous* on X, if for any $x \in X$ and $y \in Y$, $x > y$ implies that there exists $\mathcal{N}(y)$ such that $x \geqq \mathcal{N}(y)$.

For convenience, in further developments we define the weakly upper contour correspondence $U_w : X \to 2^Y$ by $U_w(x) = \{y \in Y : y \geqq x\}$ for each $x \in X$, and we define the strictly upper contour correspondence $U_s : Y \to 2^X$ by $U_s(x) = \{y \in X : y > x\}$ for each $x \in Y$.

Remark 5. Note that "$>$" is upper continuous on X if and only if U_s has (relatively) open lower sections on X, i.e., if and only if $U_s^{-1}(x)$ is open for all $x \in X$. The upper continuity is called the lower continuity in [3] and [20] and the weakly upper continuity in [5] is called the weakly lower continuity. The reason we call them "upper" is that when the binary relation "$>$" is represented by a real-valued function on Y our definitions coincide with the usual upper semicontinuities.

Let Z be a convex subset in a topological vector space E. A correspondence $P : Z \to 2^Z$ is said to be SS-*convex* (refer to Shafer and Sonnenschein) if $x \notin \operatorname{co} P(x)$ for all $x \in Z$. Here we used co A to denote the convex hull of a set A.

DEFINITION 3. Let Z be a convex subset of a topological vector space E and let $\emptyset \neq X \subset Z$. A correspondence $P: Z \to 2^X$ is said to be *generalized* SS-*convex* on X (cf. [17]) if for every finite subset $\{x_1, x_2, \cdots, x_m\}$ of X and $x_0 \in \operatorname{co}\{x_1, x_2, \cdots, x_m\}$, $x_j \notin P(x_0)$ for some $1 \leqq j \leqq m$.

Remark 6. Note that the SS-convexity implies the generalized SS-convexity. The converse statement may not be true unless $X = Z$.

Let Z be a convex subset in a topological vector space and let $\emptyset \neq X \subset Z$. A correspondence $G: X \to 2^Z$ is said to be FS-*convex* (refer to Fan [17] and Sonnenschein [14]) if for any finite set $\{x_1, x_2, \cdots, x_n\} \in X$, $\operatorname{co}\{x_1, x_2, \cdots, x_n\} \subset \bigcup_{i=1}^n G(x_i)$. Next we apply the transfer method to generalize the above conventional continuities and convexities. Once the definitions are compared, the ideas behind the transfer method become clear.

In the following definition, whenever $K = X$, "to K" will be replaced by "on X" or omitted.

DEFINITION 4. Let X be a set of a topological space Y and $K \subset X$ be a subset. The binary relation "$\succ$" on Y is said to be

(1) *transfer upper continuous* to K, if for any $x \in X$ and $y \in Y$, $x \succ y$ implies that there exist $x' \in K$ and $\mathcal{N}(y)$ such that $x' \succ \mathcal{N}(y)$;

(2) *transfer pseudoupper continuous* to K, if for any $x \in X$ and $y \in Y$, $x \succ y$ implies that there exist $x' \in K$ and $\mathcal{N}(y)$ such that $x' \succ y$ and $x' \succeq \mathcal{N}(y)$;

(3) *transfer weakly upper continuous* to K, if for any $x \in X$ and $y \in Y$, $x \succ y$ implies that there exist $x' \in K$ and $\mathcal{N}(y)$ such that $x' \succeq \mathcal{N}(y)$.

DEFINITION 5. Let X be a topological space and let Z be a convex subset in a topological vector space. A correspondence $G: X \to 2^Z$ is said to be *transfer* FS-*convex* on X if for any finite set $\{x_1, x_2, \cdots, x_n\} \subset X$ there exists a corresponding finite set $\{y_1, y_2, \cdots, y_n\} \subset Z$ such that for any subset $\{y_{i1}, y_{i2}, \cdots, y_{is}\}$ $(1 \leqq s \leqq n)$ of $\{y_1, y_2, \cdots, y_n\}$ we have

$$\operatorname{co}\{y_{i1}, y_{i2}, \cdots, y_{is}\} \subset \bigcup_{r=1}^{s} G(x_{ir}),$$

where $\{x_{i1}, x_{i2}, \cdots, x_{is}\}$ is a corresponding subset of $\{x_1, x_2, \cdots, x_n\}$.

DEFINITION 6. Let X be a topological space and let Z be a convex subset in a topological vector space. A correspondence $P: Z \to 2^X$ is said to be *transfer* SS-*convex* on X if for any finite set $\{x_1, x_2, \cdots, x_n\} \subset X$ there exists a corresponding finite set $\{y_1, y_2, \cdots, y_n\} \subset Z$ such that for any subset $\{y_{i1}, y_{i2}, \cdots, y_{is}\}$ $(1 \leqq s \leqq n)$ of $\{y_1, y_2, \cdots, y_n\}$ and $y_{i0} \in \operatorname{co}\{y_{i1}, y_{i2}, \cdots, y_{is}\}$ we have $x_{ir} \notin P(y_{i0})$.

DEFINITION 7. Let Z be a convex subset in a topological vector space and let $\emptyset \neq X \subset Z$. The binary relation $\succ (\succeq)$ is said to be *transfer* SS-*convex* (*transfer* FS-*convex*) on X if $U_s: Z \to 2^X$ ($U_w: X \to 2^Z$) is transfer SS-convex (transfer FS-convex) on X.

Remark 7. Conventional convexity conditions give relations between a finite set $\{x_1, x_2, \cdots, x_n\}$ and its convex hull $\operatorname{co}\{x_1, x_2, \cdots, x_n\}$. Transfer convexity conditions give relations between a finite set $\{x_1, x_2, \cdots, x_n\}$ and the convex hull of a corresponding finite set $\{y_1, y_2, \cdots, y_n\}$, which may differ from $\{x_1, x_2, \cdots, x_n\}$.

DEFINITION 8. Let X and Y be two topological spaces. A correspondence $G: X \to 2^Y$ is said to be *transfer closed-valued* on X if for every $x \in X$, $y \notin G(x)$ implies that there exists $x' \in X$ such that $y \notin \operatorname{cl} G(x')$, i.e., $y \notin$ the closure of $G(x')$.

In the remainder of this section we prove several lemmas that give the interconnections between different definitions and that will be useful in later proofs.

LEMMA 3. (1) *Let Y be a topological space and let $\emptyset \neq X \subset Y$. Then the correspondence $U_w: X \to 2^Y$ is closed-valued on X if and only if "$>$" is upper continuous on X; the correspondence $U_w: X \to 2^Y$ is transfer closed-valued on X if and only if "$>$" is transfer upper continuous to X.*

(2) *Let Z be a convex subset in a topological vector space and let $\emptyset \neq X \subset Z$. Then the correspondence $U_w: X \to 2^Z$ is* FS-*convex on X if and only if $U_s: Z \to 2^X$ is generalized* SS-*convex on X, and the binary relation "$>$" is transfer* SS-*convex on X if and only if "$\geqq$" is transfer* FS-*convex on X.*

Proof. The proof follows immediately from the definitions. □

LEMMA 4. *Let Z be a nonempty convex subset of a topological vector space and let $\emptyset \neq X \subset Z$. Suppose "$>$" is a binary relation on Z such that $U_w: X \to 2^Z$ is finitely closed for each $x \in X$ (i.e., the intersection of $U_w(x)$ with any finite-dimensional subspace of Z is closed). Then "$>$" is transfer finitely maximal on X if and only if $>$ ($\geqq$) is transfer* SS-*convex (transfer* FS-*convex) on X.*

Proof. By [6], U_w has the finite intersection property if and only if U_w is transfer FS-convex on X and therefore if and only if U_s is transfer SS-convex on X. It is clear that U_w has the finite intersection property if and only if "$>$" is transfer finitely maximal on X. □

LEMMA 5. *Let Y be a topological space and let $\emptyset \neq X \subset Y$ and let "$>$" be a binary relation on Y. Then $\bigcap_{x \in X} \operatorname{cl} U_w(x) = \bigcap_{x \in X} U_w(x)$ if and only if U_w is transfer closed-valued or equivalently if and only if "$>$" is transfer upper continuous on X.*

Proof. Sufficiency. It is clear that $\bigcap_{x \in X} U_w(x) \subset \bigcap_{x \in X} \operatorname{cl} U_w(x)$. So we only need to show $\bigcap_{x \in X} \operatorname{cl} U_w(x) \subset \bigcap_{x \in X} U_w(x)$. Suppose $y \notin \bigcap_{x \in X} U_w(x)$. Then $y \notin U_w(z)$ for some $z \in X$. Since U_w is transfer closed-valued on X, there exists some $z' \in X$ such that $y \notin \operatorname{cl} U_w(z')$ and then $y \notin \bigcap_{x \in X} \operatorname{cl} U_w(x)$.

Necessity. Assume $\bigcap_{x \in X} \operatorname{cl} U_w(x) = \bigcap_{x \in X} U_w(x)$. If $y \notin U_w(x)$, then $y \notin \bigcap_{x \in X} \operatorname{cl} U_w(x) = \bigcap_{x \in X} U_w(x)$ and thus $y \notin \operatorname{cl} U_w(x')$ for some $x' \in X$. Thus U_w is transfer closed-valued on X. □

2. Maximization of binary relations on compact sets. There are two basic approaches in the literature to showing nonemptiness of the set of maximal elements on a nonempty compact set without assuming transitivity of the binary relation. One approach, under some convexity and continuity conditions, was developed by Fan [7], Sonnenschein [14], Shafer [11], Shafer and Sonnenschein [12], Yannelis and Prabhakar [21], and Tian [16], [17], among others. The other approach may be found in Bergstrom [3], Walker [19] (under acyclic and upper continuity assumptions), and Campbell and Walker [5] (under the 1-link transitivity, compactness for the space, and weakly upper continuity for the binary relation). In this section we generalize and unify the two approaches by giving several theorems that characterize the existence of maximal elements for a binary relation on a compact set. Theorem 1 characterizes the existence of maximal elements of a binary relation on a compact set in terms of transfer continuity (topological condition) for a given weakened transitivity condition. Theorem 2 characterizes the existence of maximal elements of a binary relation in terms of transfer transitivity for a given weakened topological condition (transfer continuity) and Theorem 3 characterizes the existence of maximal elements of a binary relation in terms of geometric conditions (transfer convexities) for a given weakened topological condition (transfer continuities).

LEMMA 6. *Let X be any subset of a topological space Y and let "$>$" be any binary relation on Y.*

(1) *If "$>$" has a maximal element on X, then "$>$" is transfer finitely maximal on X.*

(2) *If "$>$" has a maximal element on X, then "$>$" is transfer weakly upper continuous on X.*

(3) *If "$>$" is transfer upper continuous on X, then the set of all maximal elements on X is closed (possibly empty) in X. If "$>$" is fully transitive and the set of all maximal elements on X is nonempty and closed, then "$>$" is transfer upper continuous.*

Proof. The proof follows immediately from the definitions. □

THEOREM 1. *Let X be a nonempty compact topological space and let the binary relation "$>$" on X be transfer finitely strict maximal on X. Then "$>$" has a maximal element on X if and only if "$>$" is transfer weakly upper continuous on X.*

Proof. Sufficiency. Suppose, by way of contradiction, that "$>$" does not have a maximal element. Then for each $y \in X$, there exists $x \in X$ such that $x > y$. By the transfer weakly upper continuity of "$>$," there exist $x' \in X$ and a neighborhood $\mathcal{N}(y)$ such that $x' \geqq y'$ for all $y' \in \mathcal{N}(y)$. It follows that $X = \bigcup_{y \in X} \mathcal{N}(y)$. Since X is compact, there exist finite points $\{y_1, y_2, \cdots, y_n\}$ such that $X = \bigcup_{i=1}^{n} \mathcal{N}(y_i)$. Let x_i' be the associated point such that $x_i' \geqq y'$ for all $y' \in \mathcal{N}(y_i)$. Since we assume that there is no maximal element, for the finite subset $\{x_1', x_2', \cdots, x_n'\}$, by the transfer finitely strict maximal property there exists $x' \in X$ such that $x' > x_i'$, for all $i = 1, 2, \cdots, n$. However, $x' \in \mathcal{N}(y_j)$ for some $j = 1, 2, \cdots, n$. We have $x_j' \geqq x'$. It leads to a contradiction. So X has a maximal element.

Necessity. This follows from Lemma 6(1). □

THEOREM 2. *Let X be a nonempty compact topological space.*

(1) *Assume that the binary relation "$>$" is transfer upper continuous on X. Then the set of all maximal elements on X is nonempty and compact if and only if "$>$" is transfer finitely maximal on X.*

(2) *Assume that the binary relation "$>$" on X is asymmetric (i.e., 2-acyclic) and fully transitive. Then the set of all maximal elements on X is nonempty and compact on X if and only if "$>$" is transfer upper continuous on X.*

Proof of (1). The necessity follows from Lemma 6(1). We prove the sufficiency. Since "$>$" is transfer finitely maximal on X, for every finite subset $\{x_1, \cdots, x_n\}$, there is $x' \in X$ such that for each $i = 1, 2, \cdots, n$, $x' > x_i$ or $x' \geqq x_i$. Define $U_w(x) = \{y \in X: y \geqq x\}$. Thus U_w and then cl U_w have the finite intersection property. By Lemma 5, $\bigcap_{x \in X} U_w(x) = \bigcap_{x \in X} \text{cl } U_w(x) \neq \emptyset$ on the compact set X. So the set of all maximal elements on X, which is $\bigcap_{x \in X} U_w(x) = \bigcap_{x \in X} \text{cl } U_w(x)$, is nonempty and compact.

Proof of (2). The sufficiency follows from part (1) and we only need to prove the necessity. Notice that under the full transitivity, for any nonmaximal element y and any maximal element x we have $x > y$. Since the set of all maximal elements is closed, any nonmaximal element y has an $\mathcal{N}(y)$ that contains no maximal element. Therefore, for any maximal element x we have $x > y'$ for all $y' \in \mathcal{N}(y)$. That is, "$>$" is transfer upper continuous. □

Thus Theorem 1 generalizes the results of Campbell and Walker [5] by relaxing the weakly upper continuity and the 1-link transitivity (pseudotransitivity) of "$>$." It also generalizes the results of Tian and Zhou [18] by relaxing the full transitivity of "$>$." Theorem 2(1) generalizes the results of Fan [7], Sonnenschein [14], Shafer [11], Shafer and Sonnenschein [12], Yannelis and Prabhakar [21], and Tian [17] by relaxing the upper continuity and (generalized) SS-convexity of "$>$" and the convexity of X. Theorem 2(1) also generalizes the results of Bergstrom [3] and Walker [19] by relaxing the upper continuity and acyclicity of $>$. Thus our results unify two basic approaches to the existence of maximal elements by giving necessary and sufficient conditions.

Remark 8. If we compare the conditions of Theorems 1 and 2(1), we can find that there is a trade-off between the transfer transitivities and the transfer continuities (a trade-off between transitivity conditions and topological conditions): If one condition is weakened, then the other must be strengthened and vice versa. Many theorems we give below will also have this trade-off relation.

Theorem 3 below, which is obtained in Tian [16], is a special case of Theorem 2 (which needs to assume that X is a subset of a topological vector space). We state it here as an alternative.

THEOREM 3. *Let Z be a nonempty convex compact subset of a topological vector space and let $\emptyset \neq X \subset Z$. Let "$>$" be a transfer upper continuous binary relation on X. Then the set of all maximal elements of "$>$" on X is nonempty and compact if and only if "$>$" is transfer* SS-*convex on X.*

Lemma 3(2) and Lemma 4 give partial interconnections between Theorem 2(1) and Theorem 3.

Remark 9. At this point, it is quite natural to conjecture that the transfer finitely strict maximal condition in Theorem 1 might be further weakened, or to ask, for a binary relation on a compact set: What is the weakest possible transitivity condition under which the existence of maximal elements is equivalent to the transfer weakly upper continuity? This question is related to our understanding of the fundamental structures of mathematics, namely, topology, transitivity, and their interconnections. So far it is still an open question. However, Campbell and Walker [5] provide a clue. They construct an example [5] in which a binary relation is weakly upper continuous (and thus is transfer weakly upper continuous) and 0-link transitive but fails to have a maximal element on a nonempty compact set. Therefore, under the transfer weakly upper continuity, any transitivity condition proposed, other than the transfer finitely strict maximal condition, must be weaker than the 1-link transitive condition and independent of or stronger than the 0-link transitive condition.

For any function u, we can define a fully transitive binary relation "$>$" as follows: $x > y$ if and only if $u(x) > u(y)$. Thus the transfer continuities of a function u can be similarly defined. As direct consequences of the above theorems, we provide two corollaries that are generalizations of the classical Weierstrass theorem and are obtained in Tian and Zhou [18].

COROLLARY 1 [18]. *Let X be a nonempty compact topological space and let u $X \to \mathbf{R} \cup \{-\infty\}$ be a function. Then u attains its maximum on X if and only if u is transfer weakly upper continuous on X.*

COROLLARY 2 [18]. *Let X be a nonempty compact topological space and let $u: X \to \mathbf{R} \cup \{-\infty\}$ be a function. Then the set of maximum points of u on X is nonempty and compact if and only if u is transfer upper continuous on X.*

The following examples show that the above corollaries are very useful for us to see whether or not the maximum points of functions exist—even though these functions are very discontinuous.

Example 3. Consider a function u defined on the interval $X = [0, 1]$ by

$$u(x) = \begin{cases} 1+x & \text{if } x \text{ is a rational number,} \\ x & \text{otherwise.} \end{cases} \tag{3}$$

We can easily see that u is not upper semicontinuous. In order to see that u is transfer upper continuous, for any neighborhood $\mathcal{N} \subset [0, 1]$, we may choose any rational number x' such that $\sup\{x \mid x \in \mathcal{N}\} \leqq x' \leqq 1$. In addition, by Corollary 2, we know the set of all maximal points is nonempty and compact. In fact, $x = 1$ is a unique maximum point of u on $[0, 1]$.

Example 4. Consider the so-called Dirichlet function u defined on the interval $X=[0, 1]$ by

$$u(x)=\begin{cases}1 & \text{if } x \text{ is an irrational number,}\\ 0 & \text{if } x \text{ is a rational number.}\end{cases} \tag{4}$$

Note that u defined by (4) is clearly not transfer upper continuous. However, it is transfer weakly upper continuous by choosing x' as any irrational number. Thus, by Corollary 1, u has a maximum point. We can also easily see that the set of maximum points of u on $[0, 1]$ is a set consisting of all irrational numbers and thus is not compact.

Example 5. Now if a function u is defined on the interval $X=[0, 1]$ by

$$u(x)=\begin{cases}x & \text{if } 0 \leqq x < 1,\\ 0 & \text{if } x=1,\end{cases} \tag{5}$$

then u is not transfer weakly upper continuous on X. This is because for $y=1$ and $x \in (0, 1)$, we cannot find any $x' \in X$ and neighborhood $\mathcal{N}(y)$ of y such that $u(z) \leqq u(x')$ for all $z \in \mathcal{N}(y)$. Thus, by Corollary 1, we know that u does not have any maximum point. In fact, we can easily see that u does not have a maximum point on $[0, 1]$.

3. Maximization of binary relations on noncompact sets. For application purposes the compactness assumption of a set is sometimes too restrictive, especially when solving problems with data in infinite dimension. In this section we prove several theorems that give necessary and/or sufficient conditions for the existence of maximal elements of a binary relation on noncompact sets in terms of topological conditions (transfer continuities) for given transitivities, or in terms of transfer transitivities for given topological conditions (transfer continuities), or in terms of transfer convexities for given topological conditions. Thus, by applying our transfer method, we generalize almost all of the results in the literature and all results in the last section. Furthermore, by using the "transfer" feature of our transfer method, we are able to provide an approach with potential applications in constrained maximization.

Recall that the function u in Example 4 is not transfer upper continuous, but it is easy to see that it is transfer pseudo-upper continuous. So the transfer upper continuity strictly implies the transfer pseudo-upper continuity. To show that the transfer pseudo-upper continuity strictly implies the transfer weakly upper continuity, we set up the following example.

Example 6. Let $X=K$ be the unit disk in the complex plane $\mathbf{C}$. Define a binary relation "$\succ$" on X by

$$z_1 \succ z_2 \quad \text{if } \arg(z_1) > \arg(z_2),$$

for all $z_1, z_2 \in Z$. Since for the origin 0, its argument arg (0) is not defined, $0 \succeqq z$ for all $z \in Z$. So "$\succ$" is transfer weakly upper continuous. But when we observe the behavior of "$\succ$" around the point $z=(x, 0)$ for $x>0$, we can see that "$\succ$" is not transfer pseudo-upper continuous. Obviously, "$\succ$" is 0-link transitive. As a matter of fact, "$\succ$" is also 1-link transitive. For if $z_1 \succ z_2 \succeqq z_3 \succ z_4$, then we have $\arg(z_1) > \arg(z_2)$ and $\arg(z_3) > \arg(z_4)$, which also implies that $z_3 \neq 0$. It follows that $\arg(z_2) \geqq \arg(z_3)$. Therefore, $\arg(z_1) > \arg(z_4)$ or $z_1 \succ z_4$. Thus "$\succ$" is 1-link transitive.

The above example shows that for a binary relation "$\succ$" under the 1-link transitivity, the transfer pseudo-upper continuity strictly implies the transfer weakly upper continuity. However, when a preference relation "$\succ$" is fully transitive, it is easy to see that the transfer pseudo-upper continuity is equivalent to the transfer weakly upper continuity. So the question may be asked: What is the weakest possible transitivity condition under which the transfer pseudo-upper continuity is equivalent to the transfer weakly upper continuity. We need to consider another question before we can answer this one.

For a binary relation "$\succ$" defined on a set X, if $x \in X$ is not a maximal element of "$\succ$" on X, then there exists an element $y \in X$ such that $y \succ x$. We may be concerned with the question of whether or not there exists a maximal element $x^* \in X$ such that $x^* \succ x$. In general, the answer is no. But under certain transitivity conditions, the answer is yes.

LEMMA 7. *If the binary relation "$\succ$" is 2-link transitive on X and $\succ$ has a maximal element, then for any nonmaximal element $x \in X$ there exists a maximal element $x^* \in X$ such that $x^* \succ x$.*

Proof. Suppose, by way of contradiction, that there is a nonmaximal element $y \in X$ such that $y \geqq x^*$ for every maximal element x^* on X. Then there is an element $z \in X$ such that $z \succ y$. Note that z must be a nonmaximal element since $y \geqq x^*$ for all maximal elements x^* on X. So there is an element $x \in X$ such that $x \succ z \succ y$. Let x^* be any maximal element on X. Then we have $z \succ y \geqq x^* \geqq x \succ z$, which implies that $z \succ z$ by the 2-link transitivity—a contradiction. □

It can be seen in the above proof that 2-link transitivity can be replaced by a weaker condition "$x_1 \geqq x_2 \geqq x_3 \succ x_4 \Rightarrow x_1 \geqq x_4$."

LEMMA 8. *Let "$\succ$" be a 2-link transitive binary relation on a topological space X. Then "$\succ$" is transfer weakly upper continuous if and only if it is transfer pseudo-upper continuous.*

Proof. We only need to show that under the assumption, the transfer weakly upper continuity implies the transfer pseudo-upper continuity. When "$\succ$" is transfer weakly upper continuous and $x \succ y$, there exists $x' \in X$ and $\mathcal{N}(y)$ of y such that $x' \geqq \mathcal{N}(y)$. If "$\succ$" has a maximal element on X, by Lemma 7, there exists a maximal element $x^* \in X$ such that $x^* \succ y$ and $x^* \geqq \mathcal{N}(y)$. If "$\succ$" does not have a maximal element on X, notice that the 2-link transitivity implies the transfer finitely strict maximal condition; then there exists $x_0 \in X$ such that $x_0 \succ \{x, x'\}$, or $x_0 \succ x \succ y$ and $x_0 \succ x' \geqq \mathcal{N}(y)$. Then it follows that $x_0 \succ y$ and $x_0 \geqq \mathcal{N}(y)$, by noting that the 2-link transitivity implies the 0-link transitivity. Therefore, "$\succ$" is transfer pseudo-upper continuous. □

In the following, we provide various necessary and sufficient conditions to characterize the existence of binary relations on noncompact sets.

THEOREM 4. *Let X be a topological space.*

(1) *The set of all maximal elements of the binary relation "$\succ$" on X is nonempty and closed if there exists a nonempty compact set $K \subset X$ such that "$\succ$" is transfer finitely maximal and transfer upper continuous to K.*

(2) *Assume that the binary relation "$\succ$" on X is 2-acyclic and fully transitive. Then the set of all maximal elements on X is nonempty and closed if and only if there exists a nonempty compact set $K \subset X$ such that "$\succ$" is transfer upper continuous to K.*

(3) *Assume that the binary relation "$\succ$" on X is 2-acyclic and fully transitive. Then the set of all maximal elements on X is nonempty and compact if and only if there exists a nonempty compact set $K \subset X$ such that "$\succ$" is transfer upper continuous to K and for each $y \in X \setminus K$, there exists $x \in K$ such that $x \succ y$.*

Proof of (1). By the existence result in Theorem 2(1), we can see that "$\succ$" has a maximal element x^* on K. Suppose "$\succ$" has no maximal element on X. By the transfer upper continuity to K, there exists $x \in K$ such that $x \succ x^*$. This leads to a contradiction. The closedness of the set of all maximal elements follows Lemma 6(3).

Proof of (2). The sufficiency follows from (1) and the necessity is similar to that of Theorem 2(2). Just let $K = \{x^*\}$, where x^* is any maximal element on X.

Proof of (3). The sufficiency is similar to that of (2) and we only need to note that all the maximal elements must be in K. The proof of the necessity is similar to that of Theorem 2(2) by letting K be the set of maximal elements on X. □

COROLLARY 3. *Let X be a topological space and $u: X \to \mathbf{R}$ be a function. Then*

(1) *the set of all maximal elements of u on X is nonempty and closed if and only if there exists a nonempty compact set $K \subset X$ such that u is transfer upper continuous to K;*

(2) *the set of all maximal elements of u on X is nonempty and compact if and only if there exists a nonempty compact set $K \subset X$ such that u is transfer upper continuous to K and for each $y \in X \backslash K$ there is $x \in K$ with $u(x) > u(y)$.*

THEOREM 5. *Let X be a topological space. Suppose that the binary relation "$\succ$" on X is 1-link transitive. Then X has a maximal element if there exists a nonempty compact set $K \subset X$ such that "$\succ$" is transfer pseudo-upper continuous to K.*

Proof. By Theorem 1, "$\succ$" has a maximal element x^* on K. Suppose that "$\succ$" has no maximal element on X. Then there exist $y \in X$ and $x \in K$ such that $x \succ y \geqq y \succ x^*$ by the transfer pseudo-upper continuity to K. But this implies $x \succ x^*$ by the 1-link transitivity, which contradicts the fact that x^* is a maximal element of "$\succ$" on K. So "$\succ$" has a maximal element on X. □

Remark 10. Note that in the example provided by Campbell and Walker [5] the binary relation is not only 0-link transitive and weakly upper continuous but also transfer pseudo-upper continuous on a compact set. However, it fails to have a maximal element. So the 1-link transitive assumption in Theorem 5 cannot be replaced by the 0-link transitivity. □

THEOREM 6. *Let X be a topological space and let the binary relation "$\succ$" on X be such that there exists a nonempty compact set $K_1 \subset X$ such that "$\succ$" is transfer finitely strict maximal to K_1. Then X has a maximal element if and only if there exists a nonempty compact subset $K_2 \subset X$ such that "$\succ$" is transfer weakly upper continuous to K_2.*

Proof. The necessity is trivial. Just let $K_2 = \{x^*\}$, where x^* is any maximal element on X. We only need to prove the sufficiency.

When there exists a nonempty compact subset $K_1 \subset X$ such that "$\succ$" is transfer finitely strict maximal to K_1, let $K = K_1 \cup K_2$. Then "$\succ$" is transfer finitely strict maximal and transfer weakly upper continuous to K. By Theorem 1, there is a maximal element x^* on K. Suppose, by way of contradiction, that "$\succ$" has no maximal element on X. Then there exists an element $y \in X$ such that $y \succ x^*$. But "$\succ$" is transfer finitely strict maximal to K; for the nonmaximal element $x^* \in X$ there exists $x \in K$ such that $x \succ x^*$, a contradiction. So X has a maximal element in K_1. □

The following corollary is a complete characterization for a function to attain its maximum values.

COROLLARY 4. *Let X be a topological space and let $u: X \to \mathbf{R}$ be a function. Then the set of all maximal elements of u on X is nonempty if and only if there exists a nonempty compact set $K \subset X$ such that u is transfer weakly upper continuous to K.*

THEOREM 7. *Let X be a topological space and let "$\succ$" be a binary relation on X. Assume that*

(1) *there is an element $x_0 \in X$ such that* cl $U_w(x_0)$ *is compact in X;*

(2) *U_w is transfer upper continuous on X.*

Then the set of all maximal elements of "$\succ$" on X is nonempty and compact if and only if "$\succ$" is transfer finitely maximal on X.

Proof. The necessity follows from Lemma 6(1). We only need to show the sufficiency. Since "$\succ$" is transfer finitely maximal on X, U_w has a finite intersection property on X and so does cl U_w. Now cl $U_w(x) \cap$ cl $U_w(x_0)$ is compact and has a finite intersection property as well. So $\bigcap_{x \in X}$ cl $U_w(x) \neq \emptyset$ and is compact. Since condition (2) is equivalent to $\bigcap_{x\in X} U_w(x)=\bigcap_{x\in X} \mathrm{cl}\, U_w(x)$, $\bigcap_{x\in X} U_w(x)$ is nonempty and compact. □

Similarly, we can extend Theorem 3 to cover binary relations on sets that are not convex or compact.

THEOREM 8 [16]. *Let Z be a nonempty convex subset of a topological vector space, X a nonempty subset of Z, and "$>$" a binary relation on Z. Assume that*

(1) *there is a vector $x_0 \in X$ such that* cl $U_w(x_0)$ *is compact in Z;*

(2) *U_w is transfer upper continuous on X;*

(3) *for each $y\in Z\backslash X$, there exists $x\in X$ such that $x>y$.*

Then the set of all maximal elements of "$>$" on X is nonempty and compact if and only if "$>$" is transfer SS-*convex on X.*

THEOREM 9. *Let X be a topological space and "$>$" be a transfer upper continuous binary relation on X. Then the set of all maximal elements on X is nonempty and closed if and only if there exists a nonempty compact subset $K\subset X$ such that "$>$" is transfer finitely maximal to K.*

Proof. The necessity is trivial. Just let $K=\{x^*\}$, where x^* is any maximal element on X. We only need to prove the sufficiency. First we show that

$$\bigcap_{x\in X} \mathrm{cl}\, U_w(x)\cap K\neq\emptyset. \tag{6}$$

In fact, since "$>$" is transfer finitely maximal to K, for any finite subset $\{x_1, x_2, \cdots, x_n\}\subset X$, there exists $y\in K$ such that $x \geqq x_i$, $i=1,2,\cdots,n$. That is,

$$\bigcap_{i=1}^{n} U_w(x_i)\cap K\neq\emptyset.$$

It follows that

$$\bigcap_{i=1}^{n} \mathrm{cl}\, U_w(x_i)\cap K\neq\emptyset.$$

However, for each $x\in X$, the set cl $U_w(x)\cap K$ is compact and therefore

$$\bigcap_{x\in X} \mathrm{cl}\, U_w(x)\cap K\neq\emptyset.$$

Due to the assumption that "$>$" is transfer upper continuous, Lemma 5 reads

$$\bigcap_{x\in X} U_w(x)=\bigcap_{x\in X} \mathrm{cl}\, U_w(x),$$

which is a nonempty closed subset in X. This completes the proof. □

Remark 11. Theorems 7 and 9 are generalizations of Theorem 2(1). They coincide if X is compact. Note that there is a trade-off between Theorem 7 and Theorem 9. Assumption (1) in Theorem 7 has been removed in Theorem 9, but the condition that "$>$" is transfer finitely maximal on X in Theorem 7 has been strengthened to the condition that "$>$" is transfer finitely maximal to a compact subset $K\subset X$. As a result, the conclusion in Theorem 7 that the set of all maximal elements is nonempty and compact becomes weaker in Theorem 9. □

4. Concluding remarks. In this section we give some further remarks.

Let E (environment space) and Y (action space) be two topological spaces; let $F: E\to 2^Y$ be a nonempty-valued correspondence; and let "$>_e$" be a family of the binary relations on Y that depends on the parameter $e \in E$. Define a binary correspondence $P: E \times Y \to 2^Y$ by

$$P(e, y) = \{x \in Y: x \succ_e y\}$$

for $(e, y) \in E \times Y$. To study a family of maximization problems with respect to the parameterized binary relation "$\succ_e$," we define the maximum (marginal) correspondence $M: E \to 2^Y$, for each $e \in E$, as

$$M(e) = \{y \in F(e): P(e, y) \cap F(e) = \emptyset\}.$$

Berge [1], [2, p. 116] first studied various continuity properties of the maximum correspondence $M(e)$ for a simple case where

$$M(e) = \{y \in F(e): u(e, y) \geqq u(e, x), \forall x \in F(e)\}$$

for some function $u: E \times Y \to \mathbf{R}$. He proved that if u is a continuous function and F is a nonempty compact-valued continuous correspondence, then the maximum correspondence M is nonempty compact-valued and upper semicontinuous. Since then, this theorem, called Berge's Maximum Theorem, has become one of the most useful and powerful theorems in economics, optimization, and game theory. Walker [20] extended Berge's Maximum Theorem to maximization with respect to binary relations. He gave conditions under which M is an upper semicontinuous correspondence with compact (but possibly empty) values. In [18], a further generalization is obtained by giving necessary and sufficient conditions, but M is still possibly empty-valued. Just as Berge's Maximum Theorem can be used to prove the existence of Nash equilibrium and equilibrium for the generalized game with payoff functions, Walker's Maximum Theorem can be used to prove the existence of Nash equilibrium and equilibrium for the generalized game without ordered binary relations if the nonemptiness of the maximum correspondence $M(e)$ can be guaranteed. It is worth indicating that our work in this paper is partially motivated by this problem and the results established here can be applied to giving various conditions under which $M(e)$ are nonempty valued.

Let Y be a topological space and $X \subset Y$ be a subset. For a given (weak) binary relation "$\succ^*$" on Y, if a maximal element on X with respect to "$\geqq^*$" is defined as an element $x^* \in X$ such that for each $x \in X$, either $x^* \geqq^* x$ or x^* and x cannot be compared, then we can define a (strict) binary relation "$\succ$" as the *asymmetric part* of "$\geqq^*$," i.e., $y \succ x$ whenever $y \geqq^* x$ and not $x \geqq^* y$ and write the completion "$\geqq$" of "$\succ$" by $y \geqq x$ whenever $x \succ y$ does not hold. Then follow our definition that a maximal element of "$\succ$" on X is an element $x^* \in X$ such that $x^* \geqq x$ for all $x \in X$, which reads: For each $x \in X$ either $x^* \geqq x$ or x^* and x cannot compare. So these two definitions for maximal elements on X coincide and a maximization problem with respect to the (weak) binary relation can be converted to a maximization problem with respect to the (strict) binary relation. Note that the above-defined (strict) binary relation "$\succ$" is always asymmetric (2-acyclic).

Finally, we would like to mention that the results stated in the above sections can also be used to prove the existence of greatest elements for a weak (reflexive) binary relation "$\geqq^*$." Let Y be a topological space and $X \subset Y$ be a subset. For a weak binary relation "$\geqq^*$" on Y, a point $x^* \in X$ is said to be a *greatest element* of $\geqq^*$ on X if $x^* \geqq^* x$ for all $x \in X$. For this weak binary relation "$\geqq^*$," we can define a strict binary relation "$\succ^*$" as follows. $x \succ^* y$ if and only if not $y \geqq^* x$. Then we can easily see that $x^* \in X$ is a greatest element of $\geqq^*$ on X if and only if x^* is a maximal element of "$\succ^*$" on X. Thus proving the existence of a greatest element of a weak binary relation is equivalent to proving the existence of a maximal element of the reduced strict binary relation.

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