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Remark. As has been known, all the basic inequalities are equivalent in the sense of studying positivity (e.g., see Wang [10]). So as to broaden the scope of applications of an inequality its suitable rearrangement is desirable (e.g., see Hua [5] and Iwamoto, Tomkins, and Wang [8]). In conclusion, we pose a question: Can a modified continuous DP approach (e.g., cf. Iwamoto and Wang [6, 7]) analogous to the discrete one adopted to establish Theorem 1 above be found so as to establish Theorem 2? Note also that Theorem 1 follows immediately from Theorem 2 by letting $f(x) = \sum_{j=1}^{n} x_j \chi_{(j-1,j]}$, $T = n$ (see Halmos [4, p. 84]).

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The Maximum Theorem and the Existence of Nash Equilibrium of (Generalized) Games without Lower Semicontinuities

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In this paper we generalize Berge's Maximum Theorem to the case where the payoff (utility) functions and the feasible action correspondences are not lower semicontinuous. The condition we introduced is called the *Feasible Path Transfer Lower Semicontinuity* (in short, FPT l.s.c.). By applying our Maximum Theorem to game theory and economics, we are able to prove the existence of equilibrium for the generalized games (the so-called abstract economics) and Nash equilibrium for games where the payoff functions and the feasible strategy correspondences are not lower semicontinuous. Thus the existence theorems given in this paper generalize many existence theorems on Nash equilibrium and equilibrium for the generalized games in the literature.

1. INTRODUCTION

The "Maximum Theorem" is one of the most useful and powerful theorems in economics, optimization, and game theory. It states that the set of solutions to a maximization problem varies upper semicontinuously as the constraint set of the problem varies in a continuous way. The theorem, first stated and proved by Claude Berge [3; 4, p. 116], is essentially as follows:

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LEMMA 1 (Berge [4, p. 116]). *Let E and Y be topological spaces. If*

(i) *$u: E \times Y \to \mathbb{R}$ is a continuous real-valued function;*

(ii) *$F: E \to 2^Y$ is continuous and compact-valued correspondence,*

then the correspondence $M: E \to 2^Y$ defined, for each $e \in E$, as

$$M(e) = \{y \in F(e): u(e, y) \geqslant u(e, x), \forall x \in F(e)\} \tag{1}$$

is upper semicontinuous and compact-valued, and the real-valued function $\psi: E \to \mathbb{R}$ defined, for each $e \in E$, as

$$\psi(e) = \max\{u(e, y): y \in F(e)\} \tag{2}$$

is continuous.

In applications, following Walker's [25] terminologies, E is generally interpreted to be a set of environments, Y a set of actions that might be undertaken, u a payoff (utility) function, $F(e)$ the set of feasible actions available in environment e, and $M(e)$ the set of optimal actions in environment e. Then the Maximum Theorem gives conditions under which the function $\psi(e)$ (called marginal (or performance, value) function) is continuous and the "maximizing correspondence" $M(e)$ is upper semicontinuous and thus the limit of optimal actions is also an optimal action.

Besides its own importance in the decision situation, the Maximum Theorem can also be used together with other theorems to prove the existence of Nash equilibrium and equilibrium for the generalized game which was first defined by Debreu [8].[1] The hypotheses in the classical existence theorems of Nash equilibrium and equilibrium for the generalized games of these types (e.g., in [17, 18, 8, 12, 11, 19]) typically assumed *continuity* for both the payoff functions and the feasible strategy correspondences, and *quasiconcavity* for the payoff functions (in addition to the usual convexity and compactness assumptions on the values of the feasible strategy correspondences). It is known that in many important economic games the payoff functions and the feasible strategy correspondences are discontinuous (see Dasgupta and Maskin [7] and the references therein). On the other hand, then non-existence of Nash equilibrium in simple economic models was noted a long time ago by Edgeworth [9] in his critique of Bertrand's [5] analysis of the price setting duopolists. To better understand why a model does or does not possess an equilibrium, it is helpful to analyze its behavior and identify the failure—violation of quasiconcavity or of continuity which causes non-existence of equilibrium. In Baye, Tian, and Zhou [2], the authors exposed that the failure of quasiconcavity of the payoff functions is not responsible for the non-existence of equilibrium and gave a condition which is weaker than the quasiconcavity.[2]

In this paper we focus on analyzing the behavior of games where the payoff functions and the feasible strategy correspondences are not lower semicontinuous and on detecting some attributes of the non-existence of Nash equilibrium and equilibrium for the generalized games. We then give weakened conditions that guarantee the existence of Nash equilibrium and equilibrium for (generalized) games. We do so by introducing a weaker condition which is called the *Feasible Path Transfer Lower Semicontinuity* (in short, FPT l.s.c.). With this condition, we generalize Berge's Maximum Theorem to the case where the payoff functions and the feasible action correspondences are not lower semicontinuous. By applying our Maximum Theorem to game theory and economics, we are able to prove the existence of equilibrium for the generalized games and Nash equilibrium for games. Thus the existence theorems given in this paper generalize many existence theorems on Nash equilibrium and equilibrium for the generalized games in the literature.

[1] The generalized game is also called the abstract economy or the social equilibrium in the economics literature.

2. NOTATION AND DEFINITIONS

Let X be a subset of a topological space. For each $x \in X$, denote a neighborhood of $x \in X$ by $\mathcal{N}(x)$.

A function $u: X \to \mathbb{R}$ is said to be *upper semicontinuous* if for each point $x' \in X$, we have

$$\limsup_{x \to x'} u(x) \leqslant u(x'),$$

or equivalently, its epigraph $\operatorname{ep} u \equiv \{(x, a) \in X \times \mathbb{R}: u(x) \geqslant a\}$ is a closed subset of $X \times \mathbb{R}$.

A function $u: X \to \mathbb{R}$ is said to be *lower semicontinuous* if $-u(x)$ is upper semicontinuous.

Let X and Y be two topological spaces, and let 2^Y be the collection of all subsets of Y.

A correspondence $F: X \to 2^Y$ is said to be *upper semicontinuous* (in short, u.s.c.) if the set

$$\{x \in X: F(x) \subset V\}$$

is open for every open set V of Y.

[2] The condition which is called the 0-generalized quasiconcavity is not only necessary but also, under some regular topological conditions, sufficient for the existence of Nash equilibrium.

A correspondence $F: X \to 2^Y$ is said to be *lower semicontinuous* (in short, l.s.c.) if the set

$$\{x \in X: F(x) \cap V \neq \varnothing\}$$

is open for every open set V of Y.

A correspondence $F: X \to 2^Y$ is said to be *continuous* if it is both u.s.c. and l.s.c.

A correspondence $F: X \to 2^Y$ is said to be *closed* (*open*) if its graph is closed (open).

3. The Feasible Path Transfer Lower Semicontinuity and the Maximum Theorem

Let us assume all the conditions in Lemma 1 hold except the lower semicontinuities of u and F, then examine the behavior of the correspondence $M(e)$ defined in Lemma 1. We first state the following lemma.

Lemma 2 (Berge [4, p. 112]). *Let E and Y be two topological spaces, let $G: E \to 2^Y$ and $F: E \to 2^Y$ be correspondences; if*

(i) *G is closed, and*

(ii) *F is u.s.c. and compact-valued,*

then $G \cap F$ is u.s.c. and compact-valued.

Since $M(e) = M(e) \cap F(e)$ and F is assumed to be u.s.c. and compact-valued, to prove that M is u.s.c., in view of Lemma 2, we only need to show that M is closed.

For easy understanding, we temporarily assume that E and F are normed spaces. Then we want to know what the geometric behavior of $M(e)$ is when $e^n \to e$, $y^n \to y$, and $y^n \in M(e^n)$. Since F is u.s.c. and compact-valued, it is closed. Therefore $e^n \to e$, $y^n \to y$, and $y^n \in F(e^n)$ will imply that $y \in F(e)$. When u is u.s.c. and $y \notin M(e)$, they imply respectively that

(i) $\forall \varepsilon > 0$ as n is sufficiently large

$$u(e, y) \geqslant u(e^n, y^n) - \varepsilon;$$

(ii) $\exists z \in F(e)$, such that

$$u(e, z) > u(e, y).$$

But y^n maximizes $u(e^n, \cdot)$ on $F(e^n)$. Therefore by (i) and (ii), it follows that

(iii) $\exists \varepsilon' > 0$ such that for n sufficiently large

$$u(e, z) \geqslant u(e^n, y') + \varepsilon', \qquad \forall y' \in F(e^n).$$

If we see the point $(e, z, u(e, z))$ through $(e^n, F(e^n), u(e^n, F(e^n)))$—a part of the graph of u, like a tunnel, we can see that u has an upper jump at (e, z). If $F(e)$ is l.s.c., we can find a *path* $\bar{y}^n \in F(e^n)$ such that $(e^n, \bar{y}^n) \to (e, z)$. If u is l.s.c. along this *path*, then the upper jump of u at (e, z) will not happen. If $F(e)$ is not l.s.c., a *path* $\bar{y}^n \in F(e^n)$ with $(e^n, \bar{y}^n) \to (e, z)$ may not exist. However, if there is a *path* $\bar{y}^n \in F(e^n)$ (not necessarily $\bar{y}^n \to z$) and after we transfer (e, z) to $(e^n, \bar{y}^n)$, then

$$u(e, z) \leqslant \liminf_{n \to \infty} u(e^n, \bar{y}^n),$$

and thus the upper jump of u at (e, z) can also be avoided. This geometric observation leads us to

Definition 1. Let E and Y be two topological spaces and $F: E \to 2^Y$ be a correspondence.

A function $u(e, y): E \times Y \to \mathbb{R}$ is said to be *Feasible Path Transfer Lower Semicontinuous* (in short, FPT l.s.c.) in e with respect to F if for each $(e, y) \in E \times Y$ with $y \in F(e)$, there exists some neighborhood $\mathcal{N}(e)$ of e such that $\forall e' \in \mathcal{N}(e)$, $\exists y' \in F(e')$ (a feasible path) satisfying

$$u(e, y) \leqslant \liminf_{e' \to e} u(e', y'),$$

or, equivalently, for each $(e, y) \in E \times Y$ with $y \in F(e)$ and $\forall \varepsilon > 0$, $\exists \mathcal{N}(e)$ such that $\forall e' \in \mathcal{N}(e)$, $\exists$ (a feasible path) $y' \in F(e')$ satisfying

$$u(e, y) < u(e', y') + \varepsilon.$$

Remark 1. Observe that

(i) A feasible path always exists although it may have no limit. In the definition, there is no need to assume that a feasible path has a limit.

(ii) If F is l.s.c., then a feasible path $y' \to y$ is always available.

We can use the FPT l.s.c. to generalize Theorem 6.3.1 in Berge [4, p. 115] or Theorem 2.5.2 in Aubin [1, p. 69] to the following proposition which will be used to prove our Maximum theorem. Let E and Y be two topological spaces, $F: E \to 2^Y$ be a nonempty-valued correspondence, and $u: E \times Y \to \mathbb{R}$ be a real-valued function. When we consider the family of maximization problems

$$\psi(e) = \sup\{u(e, y): y \in F(e)\}$$

that depend on the parameter e, by the stability of this family of optimization problems, we usually mean the study of various continuity properties of the marginal (or performance, value) function $\psi(e)$ and the maximum correspondence

$$M(e)=\{y\in F(e): u(e,y)=\psi(e)\}.$$

The following proposition characterizes the l.s.c. of the marginal function. It has also potential applications in quasi-variational inequalities.

PROPOSITION 1. *Let E and Y be two topological spaces, $F: E\to 2^Y$ be a nonempty-valued correspondence, and $u: E\times Y\to\mathbb{R}$ be a real-valued function. Then the marginal function $\psi: E\to\mathbb{R}$ defined, for each $e\in E$, as*

$$\psi(e)=\sup\{u(e,y): y\in F(e)\}$$

is l.s.c. if and only if $u(e,y)$ is FPT l.s.c. in e w.r.t. F.

Proof. When the marginal function $\psi(e)$ is l.s.c. $\forall(e,y)\in E\times Y$ with $y\in F(e)$, $\forall\varepsilon>0$, $\exists\mathcal{N}(e)$ such that $\forall e'\in\mathcal{N}(e)$

$$\psi(e)<\psi(e')+\varepsilon.$$

Then for every such e', by the definition of $\psi(e)$, $\exists y'\in F(e')$ such that

$$u(e,y)\leqslant\psi(e)<u(e',y')+\varepsilon.$$

Therefore $u(e,y)$ is FPT l.s.c. in e w.r.t. F.

On the other hand, when $u(e,y)$ is FPT l.s.c. in e w.r.t. F, to show that ψ is l.s.c. is equivalent to showing that for each $a\in\mathbb{R}$, the set

$$P=\{e\in E: \sup\{u(e,y): y\in F(e)\}\leqslant a\}$$

is closed.

If $e\notin P$, then $\sup\{u(e,y): y\in F(e)\}>a$ and thus $\exists y\in F(e)$ such that

$$u(e,y)>a.$$

Since $u(e,y)$ is FPT l.s.c. in e w.r.t. F, for $\varepsilon=u(e,y)-a>0$, $\exists\mathcal{N}(e)$ such that $\forall e'\in\mathcal{N}(e)$, $\exists y'\in F(e')$ satisfying

$$u(e,y)<u(e',y')+\varepsilon,$$

or

$$u(e',y')>a.$$

Therefore

$$P\cap\mathcal{N}(e)=\varnothing,$$

and P is closed. Q.E.D.

The following theorem is our main result in this paper which generalizes the Maximum theorem of Berge [3, 4] by relaxing lower semicontinuity.

THEOREM 1. *Let E and Y be two topological spaces. If*

(i) *$u: E\times Y\to\mathbb{R}$ is a u.s.c. real-valued function;*

(ii) *$F: E\to 2^Y$ is a nonempty compact-valued and closed correspondence;*

(iii) *$u(e,y)$ is FPT l.s.c. in e w.r.t. F,*

then the maximum correspondence $M: E\to 2^Y$ defined, for each $e\in E$, as

$$M(e)=\{y\in F(e): u(e,y)\geqslant u(e,x), \forall x\in F(e)\}$$

is nonempty compact-valued and closed and the marginal function $\psi: E\to\mathbb{R}$ defined, for each $e\in E$, as

$$\psi(e)=\max\{u(e,y): y\in F(e)\}$$

is l.s.c. If, in addition, F is u.s.c., then M is u.s.c. and ψ is continuous.

Proof. $M(e)$ is nonempty compact-valued as $F(e)$ is nonempty compact-valued and u is u.s.c.

We show that M is closed, i.e., the graph of M

$$Graph(M)=\{(e,y)\in E\times Y: y\in M(e)\}$$

is closed. Thus, we have to show that, if $(e,y)\notin \mathrm{Graph}(M)$, there exist $\mathcal{N}'(e)$ and $\mathcal{N}'(y)$ such that

$$\mathcal{N}'(e)\times\mathcal{N}'(y)\cap Graph(M)=\varnothing.$$

Indeed, if $(e,y)\notin Graph(M)$, i.e., $y\notin M(e)$, then either $y\notin F(e)$ or $y\in F(e)$ but there exists $z\in F(e)$ such that $u(e,z)>u(e,y)$. In the case of $y\notin F(e)$, since F is closed (i.e., its graph is closed), there exists a neighborhood $\mathcal{N}(e,y)$ of (e,y) such that

$$\mathcal{N}(e,y)\cap\mathrm{Graph}(F)=\varnothing.$$

In the case where $y\in F(e)$ but there exists $z\in F(e)$ such that $u(e,z)>$

$u(e, y)$, since u is FPT l.s.c. in e at (e, z) with $z \in F(e)$, there exist $\varepsilon > 0$ and $\mathcal{N}(e)$ such that $\forall e' \in \mathcal{N}(e)$, $\exists z' \in F(e')$ satisfying

$$u(e, y) < u(e, z) - \varepsilon < u(e', z') \leqslant u(e', y'), \qquad \forall y' \in M(e'). \tag{3}$$

Now we claim that there exist $\mathcal{N}'(e) \subset \mathcal{N}(e)$ and $\mathcal{N}'(y)$ such that

$$\mathcal{N}'(y) \cap M(e') = \varnothing, \qquad \forall e' \in \mathcal{N}'(e).$$

Suppose not. Then, for any $\mathcal{N}'(e) \subset \mathcal{N}(e)$ and $\mathcal{N}'(y)$, $\exists e' \in \mathcal{N}'(e) \subset \mathcal{N}(e)$ and $y' \in \mathcal{N}'(y)$ such that $y' \in \mathcal{N}'(y) \cap M(e')$. Thus by (3),

$$u(e, y) < u(e, z) - \varepsilon < u(e', y') \tag{4}$$

or

$$u(e, y) < u(e, z) - \varepsilon \leqslant \limsup_{(e', y') \to (e, y)} u(e', y'),$$

which implies that u is not u.s.c. at (e, y), a contradiction. Therefore

$$\mathcal{N}'(e) \times \mathcal{N}'(y) \cap Graph(M) = \varnothing,$$

and thus M is closed. By Proposition 1, ψ is l.s.c.

Now if, in addition, F is u.s.c., since $M(e) = M(e) \cap F(e)$ and F is compact-valued, then M is u.s.c. by Lemma 2. Also, since u is u.s.c. and F is u.s.c. and compact-valued, ψ is u.s.c. by Theorem 7.3.2 of Berge [4, p. 116] and thus it is continuous. Q.E.D.

As a further application of Proposition 1, let us prove a proposition which is a generalization of several theorems in quasi-variational inequalities (e.g., Theorem 1 in Shih and Tan [21] and also see Yen [28]) by relaxing the conditions on T. By Proposition 1, we can see that the condition we give on T is a necessary and sufficient condition that the function $\sup_{u \in T(y)} \langle u, x - y \rangle$ is l.s.c. in x.

PROPOSITION 2. *Let X be a compact convex subset of a locally convex topological vector space E with its dual E'. Suppose that*

(i) *$F: X \to 2^X$ is a u.s.c. correspondence with nonempty closed convex values,*

(ii) *$T: X \to 2^{E'}$ is a correspondence with nonempty values such that for each fixed $y \in X$, the function $\langle u, x - y \rangle : E' \times X \to \mathbb{R}$ is FPT l.s.c. in x w.r.t. T,*

(iii) *the set $\{x \in X: \sup_{y \in F(x)} \sup_{u \in T(y)} \langle u, x - y \rangle \leqslant 0\}$ is closed.*

Then there exists $\hat{x} \in X$ such that

$$\hat{x} \in F(\hat{x})$$
$$\sup_{u \in T(\hat{x})} \langle u, \hat{x} - y \rangle \leqslant 0 \qquad \forall y \in F(\hat{x}).$$

Proof. Define a function $\phi: X \times X \to \mathbb{R} \cup \{\pm\infty\}$ by

$$\phi(x, y) = \sup_{u \in T(x)} \langle u, x - y \rangle.$$

Then, by Proposition 1, we know that ϕ is l.s.c. in x. Thus, by Theorem 9.3.1 in Aubin [1] or Theorem 3.1 in Zhou and Chen [29], there exists $\hat{x} \in X$ such that

$$\hat{x} \in F(\hat{x})$$
$$\sup_{u \in T(\hat{x})} \langle u, \hat{x} - y \rangle \leqslant 0 \qquad \forall y \in F(\hat{x}).$$

The proof is completed. Q.E.D.

4. EQUILIBRIUM FOR THE GENERALIZED GAMES

Let I be the set of agents which is any countable or uncountable set. Each agent i chooses a strategy x_i in a set X_i of a locally convex topological vector space. Denote by X the (Cartesian) product $\prod_{j \in I} X_j$ and X_{-i} the product $\prod_{j \in I \setminus \{i\}} X_j$. Denote by x and x_{-i} an element of X and X_{-i}. Each agent i has a payoff (utility) function $u_i: X \to \mathbb{R}$. Given x_{-i} (the strategies of others), the choice of the ith agent is restricted to a nonempty, convex and compact set $F_i(x_{-i}) \subset X_i$, the *feasible strategy set*; the ith agent choose $x_i \in F_i(x_{-i})$ so as to maximize $u_i(x_{-i}, x_i)$ over $F_i(x_{-i})$.

DEFINITION 2. A vector $x^* \in X$ is said to be an *equilibrium of a generalized game* (*an abstract economy*) if $\forall i \in I$

(i) $x_i^* \in F_i(x_{-i}^*)$ and

(ii) x_i^* maximizes $u_i(x_{-i}^*, x_i)$ over $F_i(x_{-i}^*)$.

If $F_i(x^{-i}) = X_i$, $\forall i \in I$, the generalized game reduces to the conventional game and the equilibrium is called a *Nash equilibrium*.

Debreu [8] proved that, for a finite number of players and the Euclidean space, an equilibrium of the generalized game exists if (i) X_i is a contractible polyhedron; (ii) F_i is closed; (iii) u_i is continuous such that the function

$$x_{-i} \to \max\{u_i(x_{-i}, x_i) : x_i \in F_i(x_{-i})\}$$

is continuous; (iv) the set

$$M(x_{-i}) = \{x_i \in F_i(x_i) : u_i(x_{-i}, x_i) \geqslant u_i(x_{-i}, z_i), \forall z_i \in F_i(x_{-i})\}$$

is contractible. Since then this classical result has been extended in many ways. Shafer and Sonnenschein [20] and Borglin and Keiding [6] extended Debreu's result to the generalized games with the finite dimensional strategy space and the finitely many agents and without ordered preferences. For the infinite dimensional strategy space and finitely or infinitely many agents case, the existence results were given by Yannelis and Prabhakar [27], Khan and Vohra [14], Toussaint [24], Khan and Papageorgiou [15], Kim, Prikry, and Yannelis [16], Khan [13], Yannelis [26], Tian [22], and others. In this paper we apply our Theorem 1 to prove an existence theorem of equilibrium in the generalized game where the payoff (utility) functions u_i and the feasible strategy correspondences F_i are not l.s.c.

THEOREM 2. *For $i \in I$, let X_i be a nonempty convex compact subset of a locally convex topological vector space, $X = \prod_{j \in I} X_j$, and $X_{-i} = \prod_{j \in I \setminus \{i\}} X_j$. If for each $i \in I$, $x = (x_{-i}, x_i)$,*

(i) *$u_i : X \to \mathbb{R}$ is quasiconcave in x_i and u.s.c. in x;*

(ii) *$F_i : X_{-i} \to 2^{X_i}$ is u.s.c. and nonempty convex compact-valued correspondence;*

(iii) *u_i is FPT l.s.c. in x_i w.r.t. F_i,*

then there exists an equilibrium in the generalized game.

Proof. For each $i \in I$, define the maximizing correspondence

$$M_i(x_{-i}) = \{x_i \in F_i(x_{-i}) : u_i(x_{-i}, x_i) \geqslant u_i(x_{-i}, z_i), \forall z_i \in F_i(x_{-i})\}.$$

The correspondence $M_i : X_{-i} \to 2^{X_i}$ is nonempty convex-valued, because u_i is u.s.c. in x and concave in x_i and $F_i(x_{-i})$ is nonempty convex compact-valued. For these u_i, F_i, and M_i, Theorem 1 guarantees that M_i is u.s.c. and compact-valued. Therefore the correspondence

$$M(x) = \prod_{i \in I} M_i(x_{-i})$$

is u.s.c. by Lemma 3 in Fan [10, p. 124] and nonempty convex compact-valued from $X \to 2^X$. So, by Theorem 1 in Fan [10, p. 122], there exists $x^* \in X$ such that $x^* \in M(x^*)$ and x^* is an equilibrium in the generalized game. Q.E.D.

5. THE EXISTENCE OF NASH EQUILIBRIUM

When the feasible strategy correspondence F_i is constant set-valued, the generalized game reduces to the conventional game. Nash [18] proved that a (Nash) equilibrium of the game exists if each $X_i \subset \mathbb{R}^{L_i}$ is compact, convex, and nonempty, and if u_i is continuous on X and quasiconcave in x^i. Dasgupta and Maskin [7] extended Nash's resuls to games with discontinuous payoff functions. They proved that an equilibrium of the game exists if each $Z^i \subset \mathbb{R}^{L_i}$ is compact, convex, and nonempty, and if the $u^i(x^i, x^{-i})$ is quasiconcave in x^i, upper semicontinuous in x, and graph-continuous[3]. When the feasible strategy correspondence F_i is constant set-valued, every *path* is feasible. The FPT l.s.c. can be restated as

DEFINITION 3. A function $u(e, y) : E \times Y \to \mathbb{R}$ is said to be *Path Transfer Lower Semicontinuous* in e (in short, PF l.s.c.) if for each $(e, y) \in E \times Y$ there exists an $\mathcal{N}(e)$ of e such that $\forall e' \in \mathcal{N}(e)$, $\exists y' \in Y$ satisfying

$$u(e, y) \leqslant \liminf_{e' \to e} u(e', y'),$$

or, equivalently, for each $(e, y) \in E \times Y$ and $\forall \varepsilon > 0$, $\exists \mathcal{N}(e)$ such that $\forall e' \in \mathcal{N}(e)$, $\exists y' \in Y$ satisfying

$$u(e, y) < u(e', y') + \varepsilon.$$

As a special case of Proposition 1, we have

PROPOSITION 3. *Let E and Y be two topological spaces, $u : E \times Y \to \mathbb{R}$ be a real-valued function. Then the marginal (or performance, value) function $\psi : E \to \mathbb{R}$ defined, for each $e \in E$, as*

$$\psi(e) = \sup\{u(e, y) : y \in Y\}$$

is l.s.c. if and only if $u(e, y)$ is PT l.s.c. in e.

[3] Dasgupta and Maskin [7] defined a payoff function to be graph-continuous if for all $\bar{x} \in X$ there exists a function $F_i : X_{-i} \to X_i$ with $F_i(\bar{x}_{-i}) = \bar{x}_i$ such that $u_i(F_i(x_{-i}), x_{-i})$ is continuous at $x_{-i} = \bar{x}_{-i}$.

Remark 2. Observe that a payoff function u is clearly PT l.s.c. if it is graph-continuous. The following theorem generalizes Theorem 2 of Dasgupta and Maskin [7] by relaxing the graph-continuity.

THEOREM 3. *For $i \in I$, let X_i be a nonempty convex compact set of a locally convex topological vector space, $X = \prod_{j \in I} X_j$, and $X_{-i} = \prod_{j \in I \setminus \{i\}} X_j$. If for each $i \in I$, $x = (x_{-i}, x_i)$, $u_i: X \to \mathbb{R}$ is quasiconcave in x_i and u.s.c. in x and PT l.s.c. in x_i then there exists a Nash equilibrium.*

Proof. For each $i \in N$ and each $x_{-i} \in X_{-i}$, let $F_i(x_{-i}) \equiv X_i$ and apply Theorem 2. Q.E.D.

The PT l.s.c. is, in fact, weaker than graph-continuity. To see this let us consider the following example.

EXAMPLE 1. Consider a function $f(x, y)$ defined on the square $Z = [0, 1] \times [0, 1]$ by

$$f(x, y) = \begin{cases} 1 & \text{if} \quad x = y \\ \frac{1}{2} & \text{if} \quad y = 0, x \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

It can be shown that $f(x, y)$ is u.s.c., quasiconcave in x, and PT l.s.c. in x. But it is not graph continuous in x. For any $(x, y) \in Z$ and any $x' \to x$, we just choose $y' = x'$, a path along the diagonal. Then

$$f(x, y) \leqslant \liminf_{x' \to x} f(x', y').$$

Thus $f(x, y)$ is PT l.s.c. in x. While at each given (x, y) with $y = 0$ and $x \neq 0$, and any $y' \neq y$ we have $f(x, y) = 1/2$ and $f(x', y') =$ either 0 $(x' \neq y')$ or 1 $(x' = y')$. Therefore $f(x, y)$ is not graph continuous at (x, y).

As a matter of fact, for any topological vector space X, any $f(x, y)$ defined on $Z = X \times X$ satisfying

$$f(x, y) = \begin{cases} c & \text{if} \quad x = y \\ \leqslant c & \text{otherwise,} \end{cases} \tag{6}$$

is PT l.s.c. in x.

As a final remark, compactness of strategy spaces of the (generalized) games can be relaxed by using the similar techniques in Baye, Tian, and Zhou [2], Tian [22, 23].

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On Efficiency and Duality for Multiobjective Programs

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For a multiobjective nonlinear program which involved inequality and equality constraints, Wolfe, Mond–Weir, and general Mond–Weir type duals are formulated and the concept of efficiency (Pareto optimum) is used to state some duality results under generalized (F, ρ)-convexity assumptions. © 1992 Academic Press, Inc.

1. INTRODUCTION AND PRELIMINARIES

In this paper, our aim is to use the concept of efficiency (Pareto optimum) to formulate some results of duality under generalized (F, ρ)-convexity assumptions for the following class of multiobjective nonlinear program:

$$\text{(VOP)} \begin{cases} \text{minimize } (f_1(x), f_2(x), ..., f_p(x)) \\ \text{subject to } g(x) \leqslant 0, h(x) = 0, \end{cases}$$

where f_i: $\mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, ..., p$, $g = (g_1, g_2, ..., g_m)$, g_j: $\mathbb{R}^n \to \mathbb{R}$, $j = 1, 2, ..., m$, $h = (h_1, h_2, ..., h_q)$, h_k: $\mathbb{R}^n \to \mathbb{R}$, $k = 1, 2, ..., q$ are assumed to be differentiable.

Following Egudo [3] we consider for problem (VOP) the Wolfe vector dual and Mond–Weir vector dual. Further in the last section a general Mond–Weir vector dual is formulated and some duality results are stated. In the case $h \equiv 0$ and different assumptions of convexity (convexity, generalized convexity, or generalized ρ-convexity), Weir [9, 10] and Egudo [2, 3] have used proper efficiency [4] or efficiency to establish some duality results, where Wolfe and Mond–Weir duals are considered. Proofs of strong duality results use characterizations of proper efficiency and efficiency of Geoffrion [4] and Chankong and Haimes [1], respectively. Also we shall use the characterization of efficiency from [1, Theorem 4.1].

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