# A Characterization of the Top Trading Cycles Mechanism for the School Choice Problem

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**Abstract**

This paper characterizes the top trading cycles mechanism for the school choice problem. Schools may have multiple available seats to be assigned to students. For each school a strict priority ordering of students is determined by the school district. Each student has strict preference over the schools. We first define weaker forms of fairness, consistency and resource monotonicity. We show that the top trading cycles mechanism is the unique Pareto efficient and strategy-proof mechanism that satisfies the weaker forms of fairness, consistency and resource monotonicity. To our knowledge this is the first axiomatic approach to the top trading cycles mechanism in the school choice problem where schools have a capacity greater than one.

Key Words: Top Trading Cycles Mechanism, School Choice Problem

JEL Classification: C78, D61, D78, I20

## 1 Introduction

In their seminal paper, Abdulkadiroğlu and Sönmez [2003] introduce the school choice problem. Before that paper, in some of the major cities students were assigned to public schools via deficient mechanisms which give high incentives to the students to misreport their true preferences in order to get better allocations. To eliminate the gaming, they propose two competing strategy-proof mechanisms: the Top Trading Cycle (TTC) mechanism and the Deferred Acceptance (DA) mechanism.

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The TTC mechanism is not only strategy-proof but also Pareto efficient. However, it fails to be fair[1]. On the other hand, the DA mechanism satisfies fairness but fails to be Pareto efficient. When the policy makers decided to adopt one of the two strategy-proof mechanisms, the DA mechanism was selected due to its better features in terms of respecting school district priorities.[2] However, in 2012 New Orleans Recovery School District became the first school district to adopt TTC.

Adoption of the TTC by New Orleans school district shows us that some school districts may value efficiency over fairness. If Pareto efficiency and strategy-proofness are the main objectives of the school districts then TTC can be considered one of the candidates. However, it is not the unique Pareto efficient and strategy-proof mechanism. For instance, the serial dictatorship mechanism also satisfies these two axioms.[3] In this paper, we try to help the policy makers who are willing to adopt a Pareto efficient and strategy-proof mechanism by providing the full characterization of the TTC mechanism. Our characterization is based on Pareto efficiency, strategy-proofness, mutual best along with two axioms that we introduce: resource monotonicity for top-ranked students and weak consistency. We show that TTC mechanism is the unique mechanism satisfying Pareto efficiency, strategy-proofness, mutual best, weak consistency and resource monotonicity for top-ranked students.

"Mutual best"[4] requires that a student be assigned to the school at the top of his preference whenever he has the highest priority at that school. A mechanism is "resource monotonic for top-ranked students" if the assignment of the top-ranked student for a school is not worsened when the number of available seats in that school increases. A mechanism is said to be "weakly consistent" if the removal of a set of agents with their assignments does not affect the assignments of the remaining agents as long as each agent is the top-ranked student for one of the assignment of the removed agent.

Mutual best, weak consistency and resource monotonicity for top-ranked students are weaker forms of fairness, consistency[5] and resource monotonicity[6], respectively. TTC mechanism does not satisfy fairness, consistency and resource monotonicity. In particular, there does not exist a mechanism that is fair, strategy-proof and consistent.[7] Moreover Pareto efficiency and fairness are in-

[1]Fairness is the natural counterpart of the stability in the school choice context [Balinski and Sönmez, 1999]. An allocation is fair if there does not exist a student who prefers another school to his assignment and that school admitted a student with lower priority.

[2]School districts in Boston, New York City and Denver have adopted versions of the DA mechanism.

[3]Pycia and Ünver [2011b] provide a class of mechanisms satisfying strategy-proofness and Pareto efficiency in the school choice problem.

[4]Morrill [2012] uses the same axiom in the characterization of TTC in a school choice problem where each school has only one available seat.

[5]A mechanism is consistent if whenever a set of agents are removed with their assignments then all the remaining agents will be assigned to their initial assignment when we run the mechanism only considering the remaining agents and remaining copies of the objects.

[6]Resource monotonicity requires that if the number of available objects increases then all agents should be affected in the same direction [Chun and Thomson, 1988].

[7]Alcalde and Barbera [1994] show that DA mechanism is the unique strategy-proof and fair mechanism but it fails to be consistent.

compatible.[8] Therefore, we cannot have a mechanism satisfying all of the axioms.[9] Kesten [2006] shows that TTC satisfies fairness, consistency and resource monotonicity if the priority order satisfies strong acyclicity condition. In this paper, we show that TTC is not totally unsuccessful in these three dimensions and none of the Pareto efficient and strategy-proof mechanisms can perform better than TTC in all the three dimensions.

A mechanism which fails to satisfy mutual best, resource monotonicity for top-ranked students and consistency may not meet the demands of both students (families) and school districts. We consider mutual best as a must fairness requirement in the school choice context. For instance, most school districts give highest priority at a school to a student whose elder sibling is already attending that school and most of the families have preference over keeping their children in the same school [Pathak, 2011]. Therefore, both parents and school districts benefit from the mutually best mechanisms . Similarly, resource monotonicity for top-ranked students is a must resource monotonicity requirement. We modify this requirement in two ways. When public goods are allocated, we should not have a decrease in the welfare of any of the agents. Otherwise, providing less and less public goods will be a clear solution for the policy makers. Therefore, we restrict our attention to the mechanisms under which the welfare of agents weakly increases when the number of available objects increases.[10] We also modify the resource monotonicity axiom by only requiring not to have a reduction in the welfare of the top-ranked student for the school whose number of seats has increased. Therefore any resource monotonic mechanism under which welfare of the agents weakly increase with an increase in the number of available objects satisfies resource monotonicity for top-ranked students. Consistency is a desired property in the school choice context where the assignment process for different types of schools are done separately. For instance, in New York City the assignment of exam and mainstream schools are done separately [Abdulkadiroğlu et al., 2009]. Therefore, running a consistent mechanism will prevent the request of remaining agents for another run when the other agents are removed with their assignments.

Although, mutual best and resource monotonicity for top-ranked students axioms are enough to prove our uniqueness result, the TTC mechanism satisfies stronger forms of these two axioms. TTC respects the priority of student $i$ for school $s$ if the number of students with higher priority for school $s$ is less than the number of available seats in that school. Moreover, if the policy makers and families are only sensitive to priority violation in the upper priority groups then TTC can be considered to have a good performance in terms of respecting priorities. Under TTC mechanism, the students who are ranked at the top $q$ of the priority order of school $s$ cannot be made worse off due to the increase in the number of available seats from $q$ to $q'$.

[8]Balinski and Sönmez [1999] show that there does not exist fair and Pareto efficient mechanism.

[9]Serial dictatorship mechanism satisfies four of them. It fails to be fair.

[10]Kojima and Ünver [2010] define resource monotonicity similarly.

This is the first paper characterizing TTC mechanism in the school choice context where each school may have more than one available seat. Abdulkadiroğlu and Che [2010] and Morrill [2012] provide alternative characterizations of TTC mechanism in the school choice context where each school is restricted to have only one available seat. Abdulkadiroglu and Che show that TTC mechanism is the only mechanism that is Pareto efficient, strategy-proof and recursively respects top priorities.[11] Morill characterizes the TTC mechanism in two different ways. He first shows that TTC is the unique mechanism which is strategy-proof, Pareto efficient, and independent of irrelevant rankings[12] and satisfies mutual best. He also demonstrates that TTC is the unique mechanism satisfying Pareto efficiency, independence of irrelevant rankings, weak Maskin monotonicity and mutual best. Results of these two papers do not hold in the school choice problem where schools may have more than one available seat [Morrill, 2012]. Sönmez and Ünver [2010] provide the characterization of the you request my house-I get your turn (YRMH-IGYT) mechanisms in the house allocation problems with existing tenants [Abdulkadiroğlu and Sönmez, 1999]. They show that YRMH-IGYT mechanism is the unique mechanism satisfying Pareto efficiency, strategy-proofness, individual rationality, weak neutrality[13] and consistency.[14] Pycia and Ünver [2011a] introduce a class of mechanism called trading cycles mechanisms and show that in the house allocation problem a mechanism is individually rational, Pareto efficient, group strategy-proof if an only if it is a trading cycles mechanism.[15] Pycia and Ünver [2011b] also analyze trading cycles mechanism in the school choice environment where each school may have more than one available seat and show that trading cycles mechanisms are Pareto efficient and strategy-proof.

The rest of the paper is organized as follows: In Section 2 we introduce the model and properties of mechanisms. In Section 3 we describe the TTC mechanism. We present our main results in Section 4. In Section 5 we show the independence of axioms used in our main results. A brief conclusion is given in the final section.

## 2 Model

A school choice problem is a list $[I, S, q, P, \succ]$ where

- $I$ is the set of students,

[11]A mechanism respects top priorities if an agent is assigned an object, then the agent that is top-ranked by that object should not be assigned to a worse object than that object.

[12]A mechanism is independent of irrelevant rankings if whenever the ranking of an agent at an object's priority order does not affect the assignment of that agent then it does not affect the assignment of all the other agents.

[13]If a mechanism satisfies weak neutrality then the outcome of that mechanism will not depend on the names of the unoccupied objects.

[14]Sönmez and Ünver [2010] also consider a weaker version of consistency in the house allocation problem with existing tenants.

[15]The TTC mechanism belongs to the class of the trading cycles mechanisms.

- $S$ is the set of schools,
- $q = (q_s)_{s \in S}$ is the quota vector where $q_s$ is the number of available seats in school $s$,
- $P = (P_i)_{i \in I}$ is the preference profile where $P_i$ is the strict preference of student $i$ over the schools including no-school option,
- $\succ = (\succ_s)_{s \in S}$ is the priority profile where $\succ_s$ is the priority relation of school $s$ over $I$.

We denote the no-school option with $s_\emptyset$ and $q_{s_\emptyset} = \infty$. Let $R_i$ be the at-least-as-good-as relation associated with the strict preference order $P_i$ and for all $s,\ s' \in S \cup s_\emptyset$ $sR_is'$ if and only if $s = s'$ or $sP_is'$. We assume that there are no ties in the priority profiles of schools.[16]

A matching is a function $\mu : I \to S \cup s_\emptyset$ such that $\mu(i) = s$ and $\mu(i) = s'$ if only if $s = s'$. If $\mu(i) = s_\emptyset$ then student $i$ is unassigned. In a matching $\mu$, the number of students assigned to a school $s$ cannot exceed the total number of available seats in school $s$. Let $\mathcal{M}$ be the set of all possible matchings.

A mechanism is a procedure which selects a matching for each problem. That is, a mechanism $\varphi$ takes the preference profile of the students, the priority order of students for schools, the quota vector, then selects a matching for every problem. The matching selected by mechanism $\varphi$ in problem $[I, S, q, P, \succ]$ is denoted by $\varphi\,[I, S, q, P, \succ]$. Let $\varphi\,[I, S, q, P, \succ]\,(i)$ denote the assignment of student $i \in I$ by mechanism $\varphi$ for problem $[I, S, q, P, \succ]$.

Student $i$ *strictly prefers* matching $\mu$ to matching $\mu'$ if he strictly prefers $\mu(i)$ to $\mu'(i)$, $\mu(i)P_i\mu'(i)$. A matching $\mu$ is *Pareto efficient* if there does not exist a matching $\mu' \in \mathcal{M}$ in which each student is not worse off and at least one student is strictly better off. More formally, matching $\mu$ is Pareto efficient if there does not exist a matching $\mu' \in \mathcal{M}$ where $\mu'(i)R_i\mu(i)$ for each $i \in I$ and $\mu'(j)P_j\mu(j)$ at least for one $j \in I$. A mechanism $\varphi$ is *Pareto efficient* if for all problems it selects a Pareto efficient matching.

A mechanism $\varphi$ is *strategy-proof* if it is (weakly) dominant strategy for all students to tell their preferences truthfully. Formally, a mechanism $\varphi$ is strategy-proof if for every preference profile $P$ and $P'_i$ $\varphi\,[I, S, q, P, \succ]\,(i)R_i\varphi\,[I, S, q, (P'_i, P_{-i}), \succ]\,(i)$ for all student $i \in I$. Here, $P_{-i}$ represents the true preference profile of students except $i$.

Let $t_i^{\succ}$ be the set of schools ranking $i$ over all other students under priority profile $\succ$. Formally, $t_i^{\succ} = \{s \in S | i \succ_s j\ \forall j \in I \setminus i\}$. A mechanism $\phi$ is *mutually best* if whenever there exists $s \in t_i^{\succ}$ such that $sP_is'$ for all $s' \in S \setminus \{s\}$ then $\phi\,[I,\ S,\ q,\ P,\ \succ]\,(i) = s$ for all $i \in I$.[17]

A mechanism $\phi$ is *resource monotonic* if for all $s \in S$, all $q'_s \leq q_s$ either for all $i \in I$ , $\phi\,[I, S, q, P, \succ]\,(i)R_i\,\phi\,[I, S, (q'_s, q_{-s}), P, \succ]\,(i)$ or for all $i \in I$ $\phi\,[I, S, (q'_s, q_{-s}), P, \succ]\,(i)R_i\phi\,[I, S, q, P, \succ]\,(i)$.[18]

[16] School districts mostly use random tie breaking rules.

[17] Morrill [2012] defines mutual best similarly.

[18] See Thomson [2000], Ehlers and Klaus [2003] and Kesten [2009] for related results.

I use a different version of resource monotonicity. Intuitively, if student $i$ has the highest priority for school $s$ then his welfare should not be worsened when the number of seats in school $s$ increases. I formally define resource monotonicity for top-ranked students as follows: A mechanism $\phi$ is *resource monotonic for top-ranked students* if for all $i \in I$ and all $q'_s \geq q_s > 0$ where $s \in t_i^{\succ}$ $\phi\left[I, S, (q'_s, q_{-s}), P, \succ\right](i) R_i \phi\left[I, S, q, P, \succ\right](i)$.

Before introducing our consistency axiom we need additional notation.

For any school $s \in S$, priority order $\succ_s$, and a set of students $J \subset I$, let $\succ_s^J$ be the restriction of priority order $\succ_s$ to students in $J$. Let $\succ^J = (\succ_s^J)_{s\in S}$ and $\succ^{-J} = (\succ_s^{I\setminus J})_{s\in S}$.

Given a problem $[I, S, q, P, \succ]$, a set of students $J \subset I$, and a quota profile $\tilde{q} \leq q$ we say $\left[J, S, \tilde{q}, P_{-J}, \succ^J\right]$ is the restriction of the problem $[I, S, q, P, \succ]$ to students in $J$ and quota profile $\tilde{q}$.[19]

A mechanism is consistent if whenever a set of students are removed with their assignments then all the remaining students will be assigned to their initial assignment when we run the mechanism only considering the remaining students and objects.[20] Formally, a mechanism $\phi$ is *consistent* if for any problem $[I, S, q, P, \succ]$, when we remove a set of students $J \subset I$ together with their assignments $\phi[I, S, q, P, \succ](J)$, then for any $i \in I \setminus J$

$$\phi[I \setminus J, S, \tilde{q}, P_{-J}, \succ^{-J}](i) = \phi[I, J, q, P, \succ](i)$$

where $\tilde{q}_s$ is the number of available seats remaining in school $s$.

In this paper, we introduce a weaker version of the consistency axiom.[21] A mechanism satisfies weak consistency if whenever we remove a set of students with their assignment such that the student with the highest priority for one of the removed student's assignment is also another removed student then the assignments of the remaining students do not change.

A mechanism $\phi$ is *weakly consistent* if for any problem $[I, S, q, P, \succ]$, when we remove a set of students $J \subset I$ together with their assignments $\phi[I, S, q, P, \succ](J)$ satisfying $|t_j^{\succ} \cap \phi[I, S, q, P, \succ](J)| = 1$ for each $j \in J$ , then for any $i \in I \setminus J$

$$\phi[I \setminus J, S, \tilde{q}, P_{-J}, \succ^{-J}](i) = \phi[I, J, q, P, \succ](i).$$

Our restriction on the set of students and seats removed is simple. It is easy to see that any mechanism which is consistent based on the traditional definition satisfies the weaker form of it that we define here.

[19] Similar notation is used in Sönmez and Ünver [2010].

[20] See Thomson [1990] and Ergin [2000] for related results.

[21] Sönmez and Ünver [2010] also modifies the definition of the consistency axiom. In that paper, they characterize YRMH-IGYT in the house allocation problem with existing tenants. YRMH-IGYT also fails to satisfy the consistency axiom but satisfies the modified version defined in that paper.

## 3 Top Trading Cycles Mechanism

In the school choice context, the TTC mechanism was first introduced by Abdulkadiroğlu and Sönmez [2003]. It was based on the Gale's top trading cycles algorithm [Shapley and Scarf, 1974]. It is a direct mechanism and for any given problem $[I, S, q, P, \succ]$ it works iteratively in a number of steps:

**Top Trading Cycles Mechanism (TTC):**

*Step 1:* Assign a counter to each school and set it to the quota of each school. Each student points to his most preferred school. Each school points to the top-ranked student in its priority order. School $s_\emptyset$ points to all students pointing to it. Due to the finiteness there is at least one cycle.[22] Assign every student in the cycles to the school he points to and remove him. The counter of each school in a cycle is reduced by one and if it reduces to zero, the school is also removed.

In general,

*Step k:* Each student points to his most preferred school among the remaining ones. Each remaining school points to the student with the highest priority among the remaining ones. School $s_\emptyset$ points to all students pointing to it. There is at least one cycle. Assign every student in the cycles to the school he points to and remove him. The counter of each school in a cycle is reduced by one and if it reduces to zero, the school is also removed.

The algorithm terminates when all students are assigned.

We illustrate the dynamics of TTC mechanism in the following example.

**Example 1** *Let* $S = \{s_1, s_2, s_3, s_4\}$ , $I = \{i_1, i_2, i_3, i_4, i_5\}$ *and* $q = (1, 1, 1, 2)$. *The preferences of students and priorities are as follows:*

$$
\begin{array}{ll}
i_1 : s_1 P_{i_1} s_2 P_{i_1} s_3 P_{i_1} s_4 & s_1 : i_5 \succ_{s_1} i_3 \succ_{s_1} i_4 \succ_{s_1} i_2 \succ_{s_1} i_1 \\
i_2 : s_2 P_{i_2} s_1 P_{i_2} s_4 P_{i_2} s_3 & s_2 : i_3 \succ_{s_2} i_1 \succ_{s_2} i_4 \succ_{s_2} i_2 \succ_{s_2} i_1 \\
i_3 : s_1 P_{i_3} s_3 P_{i_3} s_4 P_{i_3} s_2 & s_3 : i_3 \succ_{s_3} i_2 \succ_{s_3} i_4 \succ_{s_3} i_1 \succ_{s_3} i_5 \\
i_4 : s_3 P_{i_4} s_4 P_{i_4} s_1 P_{i_4} s_2 & s_4 : i_1 \succ_{s_4} i_3 \succ_{s_4} i_2 \succ_{s_4} i_5 \succ_{s_4} i_4 \\
i_5 : s_4 P_{i_5} s_1 P_{i_5} s_2 P_{i_5} s_3 &
\end{array}
$$

*Step 1: Each students points to his most preferred school and each schools points to the student with the highest priority. There is only one cycle:* $(s_1, i_5, s_4, i_1)$. *We assign each student in the cycle to the school he points to and remove him:* $\mu(i_1) = s_1$ *and* $\mu(i_5) = s_4$. *We also reduce the counter of each school in the cycle and remove only* $s_1$ *since its counter reduces to zero.*

*Step 2: Each remaining students points to his most preferred remaining school and each remaining schools points to the student with the highest priority among the remaining ones. There is only one cycle:* $(s_3, i_3)$. *We assign the student in the cycle to the school he points to and remove him:*

[22] A cycle is an ordered list of distinct schools and distinct students $(s_1, i_1, s_2, ..., s_k, i_k)$ where $s_1$ points to $i_1$ , $i_1$ points to $s_2$ , ... , $s_k$ points to $i_k$ , $i_k$ points to $s_1$ .

*$\mu(i_3) = s_1$. We also reduce the counter of the school in the cycle and remove it, $s_3$, since its counter reduces to zero.*

*Step 3: Each remaining students points to his most preferred remaining school and each remaining schools points to the student with the highest priority among the remaining ones. There is only one cycle: $(s_2, i_4, s_4, i_2)$. We assign each students in the cycle to the school he points to and remove him: $\mu(i_2) = s_2$ and $\mu(i_4) = s_4$. We also reduce the counter of each school in the cycle and remove only both of them since their counter reduce to zero.*

*The mechanism terminates since all students are assigned.*

# 4 Results

In the following theorem, we show that TTC is Pareto efficient, strategy-proof, weakly consistent, resource monotonic for top-ranked students and mutually best. Moreover, there does not exist another mechanism satisfying all these axioms. We prove it in the Appendix.

**Theorem 1** *In school choice problem TTC is the unique mechanism satisfying*

- *Pareto efficiency*
- *Strategy-proofness*
- *Weak consistency*
- *Resource monotonicity for top-ranked students*
- *Mutual best.*

In the next section, we show that there always exist another mechanism satisfying only four of the five axioms.

Mutual best can be considered as a very weak fairness requirement and satisfying it may not make a mechanism more desirable. In the following proposition, we show that TTC mechanism satisfies much stronger fairness requirement.

**Proposition 1** *Under TTC mechanism, each student weakly prefers his assignment to each school $s$ for which he is ranked at the top $q_s$ portion of that school's priority order.*

**Proof.** Suppose not. Let student $i$'s rank for school $s$ be $r < q_s$ and he be assigned to school $s'$ such that $sP_is'$. School $s$ will start pointing student $i$ after $r-1$ students are assigned to it if $i$ is not assigned in an earlier step. First consider the case that $i$ is not assigned before $s$ points him. School

$s$ will keep pointing $i$ until he is removed. Therefore, $i$ will be assigned to $s$ whenever he points to that school. Now consider the case that $i$ is assigned before $s$ points to him. In this case, $i$ should be assigned to a better school and he never points to $s$.

■

We can also show that TTC mechanism satisfies a general form of resource monotonicity for top-ranked student.

**Proposition 2** *When the number of available seats in school $s$ is increased from $q_s$ to $\tilde{q}_s$, keeping everything else the same, then TTC mechanism assigns top $q_s$ students in school $s$'s priority order to weakly better schools.*

**Proof.** We refer to the proof of Theorem 1. The part that we prove TTC mechanism is resource monotonic for top-ranked students can be extended for top $q_s$ students. It follows from the fact that the first $q \leq q_s$ seats of school $s$ cannot be filled before top $q$ students in school $s$'s priority order are removed.

■

So far, we show that TTC mechanism outperforms other strategy-proof and Pareto efficient mechanisms. Some school districts consider fairness as the most important concern and these districts select DA mechanism instead of the TTC mechanism. In the rest of this section, we focus on the fairness and the performance of the TTC in terms of respecting priorities.

In the most of the school districts, priority structure is determined based on some exogenous rules. For instance, Boston school district gives the highest priority for a school to the students living in the same walk zone and having a sibling attending that school.[23] The second priority is given to students having a sibling attending that school but living outside the walk zone of that school. Students who are only living in the same walk zone have the third priority and the fourth priority is given to the remaining students. Ties between students in the same priority group is broken by random lottery. That is, the priority structure, $\succ$, in any problem is determined based on the priority groups and random draw. Public policy makers and families might give more importance respecting priorities in the upper priority groups [Abdulkadiroğlu, 2011]. In Proposition 3, we show that TTC is successful at respecting priorities in the upper priority groups under some realistic conditions. Before presenting our results we need some notation.

Suppose there are $n$ priority groups and respecting priorities in the first $n^*$ priority group is more important. Let $G_i : S \to \mathcal{N}$ be a function and $G_i(s)$ be the priority group that student $i$ belongs to for school $s$. We say student $i$'s preference $P_i$ is **perfectly correlated with** the priority groups if the following condition holds: if $G_i(s) < n^*$ and $G_i(s) < G_i(s')$ then $sP_is'$. A preference profile

[23] This priority group is known as sibling-walk zone priority.

$P = (P_i)_{i \in I}$ is perfectly correlated with the priority groups if each student's preference is perfectly correlated with the priority groups. As an example, suppose the first priority group (sibling-walk zone) in Boston is given more importance than the others. Then the preference profile of the students is perfectly correlated with the priority groups if each student having sibling-walk zone priority in some school ranks one of the schools for which he has sibling-walk zone priority at the top of his preference list.

Now we are ready to present our result on the performance of the TTC mechanism in terms of respecting priorities.

**Proposition 3** *Let $\pi$ be the outcome of TTC mechanism in problem $[I, S, q, P, \succ]$. There does not exist a student and school pair $(i, s)$ such that $G_i(s) < n^*$, $sP_i\pi(i)$, there exists another student $j$ assigned to $s$ and $i \succ_s j$ if any one of the following conditions holds:*

*(a) The total number of students in the first $n^*$ priority class of each school $s$ is less than or equal to $q_s$.*

*(b) Preference profile $P$ is perfectly correlated with the priority groups.*

# 5 Independence of Axioms

Below we show the independence of axioms mentioned in Theorem 1.

- *Strategy-proof, weakly consistent, resource monotonic for top-ranked students, and mutually best, but not Pareto efficient:* Consider the following problem. Two schools $S = \{a, b\}$ with one available seat and two students $I = \{\mathbf{1}, \mathbf{2}\}$. Let the preference profile $P$ and priority order $\succ$ be

| $P_{\mathbf{1}}$ | $P_{\mathbf{2}}$ |
|---|---|
| $b$ | $a$ |
| $a$ | $b$ |
| $s_\emptyset$ | $s_\emptyset$ |

| $\succ_a$ | $\succ_b$ |
|---|---|
| **1** | **2** |
| **2** | **1** |

  Let mechanism $\psi$ assign **2** to $b$ and **1** to $a$. Let $\psi$ select the same assignment in the above problem independent of preferences. For all other problems, $\psi$ selects the same matching as TTC mechanism. Mechanism $\psi$ fails to be Pareto efficient and satisfies other 4 properties.

- *Strategy-proof, weakly consistent, resource monotonic for top-ranked students, and Pareto efficient, but not mutually best:* Serial dictatorship mechanism is strategy-proof, (weakly) consistent, and Pareto-efficient. Moreover, when the number of available seats in a school is increase all students' welfare weakly improve. That is, it satisfies more generalized version of the resource monotonicity for top-ranked students. However, it fails to be mutually best.

- *Strategy-proof, weakly consistent, Pareto efficient, and mutual best mechanism, but not resource monotonic for top-ranked students:* Consider the following problem: Two schools $S = \{a, b\}$ with one available seat and three students $I = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$. Let the preference profile $P$ and priority order $\succ$ be

| $P_{\mathbf{1}}$ | $P_{\mathbf{2}}$ | $P_{\mathbf{3}}$ | $\succ_a$ | $\succ_b$ |
|---|---|---|---|---|
| $b$ | $b$ | $a$ | $\mathbf{1}$ | $\mathbf{3}$ |
| $a$ | $a$ | $b$ | $\mathbf{2}$ | $\mathbf{1}$ |
| $s_\emptyset$ | $s_\emptyset$ | $s_\emptyset$ | $\mathbf{3}$ | $\mathbf{2}$ |

  Let mechanism $\psi$ assign $\mathbf{3}$ to $a$ and $\mathbf{1}$ to $b$ in this problem. If the number of available seats in school $a$ is increased to 2 then $\psi$ assigns $\mathbf{1}$ and $\mathbf{3}$ to $a$ and $\mathbf{2}$ to $b$. Let $\psi$ select the same assignment in the above problem where $a$ has two available seats and $\mathbf{1}$ ranks $a$ above $s_\emptyset$ and assign $\mathbf{1}$ to $s_\emptyset$ if he ranks $a$ below $s_\emptyset$. For all other problems $\psi$ selects the same matching as TTC mechanism. Mechanism $\psi$ fails to be resource monotonic for top-ranked students and satisfies other 4 properties.

- *Strategy-proof, Pareto efficient, mutually best mechanism, resource monotonic for top-ranked students but not weakly consistent:* Consider the following problem. Three schools $S = \{a, b, c\}$ with one available seat and three students $I = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$. Let the preference profile $P$ and priority order $\succ$ be

| $P_{\mathbf{1}}$ | $P_{\mathbf{2}}$ | $P_{\mathbf{3}}$ | $\succ_a$ | $\succ_b$ | $\succ_c$ |
|---|---|---|---|---|---|
| $c$ | $a$ | $a$ | $\mathbf{1}$ | $\mathbf{1}$ | $\mathbf{1}$ |
| $a$ | $b$ | $b$ | $\mathbf{2}$ | $\mathbf{2}$ | $\mathbf{2}$ |
| $b$ | $c$ | $c$ | $\mathbf{3}$ | $\mathbf{3}$ | $\mathbf{3}$ |

  Let mechanism $\psi$ assign $\mathbf{1}$ to $c$ and $\mathbf{2}$ to $b$ and $\mathbf{3}$ to $a$ in this problem. Let $\psi$ select the same matching as long as $\mathbf{1}$ and $\mathbf{3}$ submit the same preferences and $\mathbf{2}$ ranks $b$ over $s_\emptyset$. If we remove $\mathbf{1}$ with his assignment then $\mathbf{2}$ is assigned to $a$ and $\mathbf{3}$ is assigned to $b$. For all other problems $\psi$ selects the same matching as TTC mechanism. Mechanism $\psi$ fails to be consistent and satisfies other 4 properties.

- *Pareto efficient, mutually best mechanism, resource monotonic for top priority students and consistent but not strategy-proof:* The Boston mechanism is Pareto efficient, resource monotonic and consistent [Kojima and Ünver, 2010]. Moreover, in the first step of the Boston mechanism when a student applies to his most popular school for which he has the highest priority he will be assigned to that school. Therefore it satisfies mutual best. The Boston mechanism fails to be strategy-proof (Abdulkadiroglu and Sonmez, 2003) and satisfies other 4 properties.

# 6 Conclusion

TTC mechanism has been studied extensively in the market design literature. It and its variants have been proposed as one of the best alternatives in many matching markets including public school choice systems, on-campus housing and the kidney exchange programs. However, TTC mechanism has never been characterized for the cases where objects have a capacity greater than one, i.e. school choice problem. In this paper, we provide the first characterization of the TTC mechanism in the school choice problem. Our characterization will help the school districts choose between strategy-proof and Pareto efficient mechanisms. In particular, TTC mechanism is the unique strategy-proof and Pareto efficient mechanism satisfying mutual best, weak consistency and resource monotonicity for top-ranked students.

We also focus on the performance of the TTC mechanism in terms of respecting priorities. We show that TTC mechanism respects priorities in the upper priority classes. If the policy makers and families are only sensitive for the priority violations in the upper priority classes then TTC mechanism will meet their needs.

# Appendix

**Proof of Theorem 1.**

We first show that the TTC mechanism satisfies all of the axioms in the theorem. Then, we show that it is the unique mechanism satisfying all of the axioms. Pareto efficiency and strategy-proofness of TTC follows from Abdulkadiroğlu and Sönmez [2003].

**Mutual Best:** Suppose TTC does not satisfy mutual best. Then, there exists a student school pair, $(i, s)$, such that student $i$ has the highest priority for school $s$ and prefers school $s$ to any other school and $i$ is not assigned to $s$ by TTC. In the first step of the TTC, $s$ will point to $i$ and $i$ will point to $s$. They will form a cycle and $i$ will be assigned to $s$. Therefore, TTC satisfies mutual best.

**Resource Monotonicity:** To show that TTC is resource monotonic for top-ranked students take a student school pair $(i, s)$ such that $s \in t_i^{\succ}$ and $q_s > 0$. Denote the assignment of TTC in problem $[I, S, q, P, \succ]$ with $\mu$. Now consider the problem $[I, S, (\tilde{q}_s, q_{-s}), P, \succ]$ where $\tilde{q}_s > q_s$. We consider a variant of the TTC mechanism in which only one cycle is removed in each step.[24] Fix the cycle selection rule. In particular, let $Cy(k)$ be the cycle that is selected in the $k^{th}$ step of the variant of the TTC mechanism when we consider the problem $[I, S, q, P, \succ]$. Let $s$ be removed in step $k$ of TTC when we consider problem $[I, S, q, P, \succ]$. We will also select $Cy(\tilde{k})$ in step $\tilde{k} < k$ if we observe that cycle when we run the variant of TTC for the problem $[I, S, (\tilde{q}_s, q_{-s}), P, \succ]$.

[24] TTC is independent of the order in which cycles are selected.

School $s$ cannot be removed before student $i$ is assigned to a school in problem $[I, S, q, P, \succ]$. Therefore, $i$ is assigned in step $k' \leq k$ in the problem $[I, S, q, P, \succ]$. To see this recall that in the TTC mechanism, $s$ will point to $i$ until $i$ is removed. Therefore, none of the seats of $s$ will be assigned to any student before $i$ is removed. Also note that all the cycles selected in step $k'' < k'$ in problem $[I, S, q, P, \succ]$ will be observed in step $k''$ of TTC when we consider the problem $[I, S, (\tilde{q}_s, q_{-s}), P, \succ]$ because none of them includes a student pointing to $s$ and an increase in the number of available seats in $s$ will not affect their assignments. As a result the set of remaining schools in step $k'$ of the TTC mechanism in both problem will be the same and we will observe the cycle including $i$ in both problems.

**Weak Consistency:** We again consider the variant of the TTC that is defined above. Let $J$ be the set of students and let $\mu(J)$ be their assignments. Due to the requirement in the definition of the weak consistency we only check the case in which each student in $J$ has the highest priority for one of the schools in $\mu(J)$. Suppose none of the students in $J$ belongs to a $Cy(k)$ where $k < \tilde{k}$. Then, it is clear that the assignment of students in $Cy(k)$ where $k < \tilde{k}$ will not be affected by the removal of students in $J$ with their assignments. Suppose $i \in Cy(\tilde{k})$. Let $\mu(i)$ be his assignment. Therefore, $i_1$ who is the top-ranked student in the priority order of $\mu(i)$ should be in $J$. This is also true for the top-ranked student of the school that $i_1$ is assigned. Due to the finiteness we should have a cycle. That is, $Cy(\tilde{k}) \subseteq J$ and $\mu(Cy(\tilde{k})) \subseteq \mu(J)$. Therefore, removing these students before running the TTC mechanism or removing them within the mechanism will not affect the assignments of the remaining students.

**Uniqueness:** Suppose there exists another mechanism $\phi$ satisfying all these 5 properties and there exists a problem $[I,\ S,\ q,\ P,\ \succ]$ in which $\phi$ and TTC select different matchings. We will consider the version of TTC mechanism in which only one cycle is removed in a step and if there are more than 1 cycle the one which will be removed is selected based on some exogenous rule, i.e. the cycle with the school having the lowest index . Then suppose that each student removed before step $k \geq 1$ of the TTC mechanism is assigned to the same school under $\phi$ and TTC. Denote these students with set $J$. Let $i$ be the student who is removed in the step $k$ of TTC and assigned to a different school by $\phi$. If we remove students assigned in the first step of TTC with their assignments then assignments of the remaining students in the outcome of both mechanisms will not change due to the weak consistency. We can continue removing all students in $J$ with their assignments and still remaining students will be assigned to the same schools.[25] Denote the reduced problem with $\left[\tilde{I}, \tilde{S}, \tilde{q}, \tilde{P}, \tilde{\succ}\right]$. Here, $\tilde{I} = I \setminus J$, $\tilde{S} = S$, $\tilde{q}_s = -\sum\limits_{i \in J} 1(\phi[q, P, \succ](i) = s) + q_s$, $\tilde{P} = P_{\tilde{I}}$ and $\tilde{\succ} = \succ^{\tilde{I}}$. In this reduced problem student $i$ will be removed in the first step of the TTC mechanism. Let $s$ be the school pointing student $i$ in the first step of TTC mechanism in the reduced problem. By the

[25]Here we remove students in the following order: $Cy(2) - Cy(3) - ... - Cy(k-2) - Cy(k-1)$.

definition of the TTC mechanism student $i$ should be the top-ranked student in $\tilde{\succ}_s$. We consider two cases. In the first case student $i$ is assigned to $s$ and in the second case $i$ is assigned to another school by TTC.

**Case 1:** Student $i$ points to the school $s$ in the first step of TTC. School $s$ should be the most preferred school in $P_i$ among the ones having available seats. Suppose $i$ reports $P_i' : sP_i's_\emptyset$. Due to the strategy-proofness TTC will assign $i$ to $s$ and $\phi$ will assign to $s_\emptyset$. Any mutual best mechanism should assign $i$ to $s$ in the reduced problem. Therefore, $\phi$ fails to satisfy mutual best.

**Case 2:** In this case $i$ is assigned to $s' \neq s$ and there is another student $j$ assigned to school $s$. Now suppose student $i$ reports $s'P_i'sP_i's_\emptyset$. TTC will select the same matching. Due to the strategy-proofness $\phi$ will assign $i$ to either $s$ where he is top-ranked or $s_\emptyset$.[26] First consider the latter case in which $i$ is assigned to $s_\emptyset$ by $\phi$ when he submits $s'P_i'sP_i's_\emptyset$. Now consider the case that $i$ submits $sP_i''s_\emptyset$ and keeping everything same. Due to the strategy-proofness he will be assigned to $s_\emptyset$ by $\phi$. However this will violate mutual best. Therefore the latter case is not possible. Therefore, when $i$ submits $P_i'$ he will be assigned to $s$ by $\phi$. Now consider the case where $i$ submits $P_i'$ and $\tilde{q}_s = 1$. Since $\phi$ is resource monotonic for top-ranked students, $i$ cannot be assigned to his top choice $s'$ by $\phi$. Then he will be assigned to $s$ or $s_\emptyset$. Due to the aforementioned reasons he will be assigned to $s$. Therefore student $j$ who is assigned to $s$ by TTC will be assigned to an other school by $\phi$. Given $s$ is the top choice of $j$ among the schools with available seats $j$ prefers his assignment under TTC to $\phi$.

Note that since student $j$ is assigned to a school by TTC in the first step there should be another school $s''$ where $j$ is the top-ranked student. If we repeat the same steps for student $j$ then we will show that when $\tilde{q}_{s''} = 1$ and $j$ submits $sP_j's''P_j's_\emptyset$ he will be assigned to $s''$ by $\phi$. We can keep continue and show that $\phi$ will assign all the students who are assigned in the first step of TTC to one of the schools pointing to them in the first step of TTC in the reduced problem. Therefore they will be assigned to strictly worse school by $\phi$ and no other student will be assigned to those schools since all schools quota will be equalized to 1 when we keep repeating. Therefore a trade between these students will increase the welfare without worsening any other student and $\phi$ fails to be Pareto-efficient.

■

**Proof of Proposition 3.** Part (a) of the proposition is a direct result of Proposition 2. We prove Part (b) by using the definition of the TTC mechanism. In particular, we use the variant of TTC mechanism in which only one cycle is removed in each step (see Proof of Theorem1). Consider the students who are ranked at the top of the priority order of schools. Then among these students

[26] Here, it is possible that $i$ can be also assigned to another school that he doesn't include to his preference list. However, we can prove that this will violate either strategy-proofness of mutual best as a similar way that we follow for showing that $i$ cannot be assigned to $s_\emptyset$.

find the students who are pointed by schools that they belong to the $k^{th}$ priority group and there does not exist a student pointed by a school that he belongs to the $l^{th}$ priority group where $l < k$. If $k > n^*$ then we are done. If $k \leq n^*$ then among these students, select the one who is favored in the random draw and denote him by $i_1$. We claim that in this step, $i_1$ is pointed by his most popular school. Suppose not. Then he is pointed by another school $s$ and his most popular school $s'$ is pointing another student $i'$. Given $s'P_{i_1}s$ then $G_{i_1}(s) \geq G_{i_1}(s')$. Moreover, $G_{i_1}(s') = k > G_{i'}(s')$ since $i_1$ is the most favored student in the random draw and $i' \succ_{s'} i_1$. This contradicts with the fact that there does not exist a student pointed by a school that he belongs to the $l^{th}$ priority group where $l < k$. Then student $i_1$'s priority is not violated in any school because he is assigned to his most popular school.

We show in Theorem 1 that TTC mechanism satisfies weak consistency. That is, when we remove $i_1$ with his assignment the remaining students will be assigned to the same school by TTC mechanism in the updated problem. Therefore, we can consider the reduced problem as a new problem and repeat the steps above and show that there does not exist a student and school pair $(i, s)$ such that $G_i(s) < n^*$, $sP_i\pi(i)$, there exists another student $j$ assigned to $s$ and $i \succ_s j$.

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