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Stability analysis in economic dynamics: A computational approach

George E. Halkos and Kyriaki D. Tsilika

Abstract

Modern microeconomics and macroeconomics study dynamic phenomena. Dynamics could predict future states of an economy based on its structural characteristics. Economic dynamics are modeled in discrete and continuous time context, mainly via autonomous difference and differential equations. In this study, we use Xcas and Mathematica as software tools, in order to generate results concerning the dynamic properties of the solutions of the difference and differential equation(s) models and determine whether an economic equilibrium exists. Our computational approach does not require solving the difference or differential equation(s) and makes no assumptions for initial conditions. The results provide quantitative information based on the qualitative properties of the mathematical solutions. The computer codes are fully presented and can be reproduced as they are in computational-based research practice and education. The relevant output of CAS software is created in a way as to be interpreted without the knowledge of advanced mathematics.

Keywords: Stability conditions; software tools; economic equilibrium.

JEL Classification Codes: C02; C62; C63; C88.

1. Introduction

Our work aims to predict the dynamic behavior of economic systems, examining whether they follow a convergent time path or not. The results cover the classes of problems with behavioral equations linear autonomous n -th order difference and differential equations, linear autonomous systems of differential and difference equations and nonlinear autonomous systems of differential equations. The answers come from conditions depending on characteristic roots and/or determinantal expressions and not by the initial values of some elements. Schur's theorem, Routhian analysis, Jacobean matrices and matrix theory are the fundamental mathematics behind our computer codes.

Mathematical software can solve difference or differential equation(s), the prediction though of the dynamic behavior of the solutions requires setting certain initial conditions, which vary from one case study to another. Computer packages provide solutions of the characteristic equations, letting the user evaluate their properties and result to stability conclusions. In our study, computer codes that test necessary and sufficient stability conditions in a black box function are proposed. The input needs no information about the initial state of the system; it requires some basic elements of the analytical model as the characteristic polynomial, the coefficient or the functional matrix or the variable vector.

We create various functions in Mathematica and in Xcas to define whether the process of economic adjustment leads to equilibrium. When equilibrium can be accomplished as time passes, the output of our functions is «stable» (or «asymptotically stable») while when the model follows a divergent time path, the output of our functions is «unstable». Programmed functions in Mathematica are aimed at the wide community of Mathematica users. For readers with no access to

Mathematica or other commercial software of this kind, computer algebra system Xcas is free to single users and institutions.

Using our computational approach the reader can reproduce the results avoiding the serious theoretical and computational difficulty underlying in stability analysis. In addition, the computer codes presented can make an open source material to enrich stability analysis with results like the list of the real or complex characteristic roots in absolute value, the determinants of Schur's and Routh's theorems and the future states of economic systems (in case of homogeneous systems of difference equations).

The structure of the paper is the following. Section 2 introduces the general notion of stability. Sections 3 and 4 present the theoretical framework and the computational techniques of the discrete and continuous time dynamic economic models respectively. Section 5 presents computational performance of the programmed functions in several dynamic economic models, while the last section concludes the paper.

2. General Notions

When the solution of a dynamic economic system follows a restrictive non-explosive path, this is what characterizes economic equilibrium. The structure of the mathematical model that ensures existence of equilibrium has been studied among others by Jury (1974), Folsom et al. (1976), Blanchard and Kahn (1980), Gu et al. (2003), Batra (2006). In economic applications, stability appears to be a property with diachronic interest as shown in studies like Arrow and Hurwitz (1958), Arrow et al. (1959), Champsaur et al. (1977), up to recent studies of Zhang (2005), Ratto (2008), Gomes (2010), Halkos and Papageorgiou (2008, 2010, 2011).

The solution that follows a convergent time path is considered stable. Figure 1 (a-d) depicts the motion of several quantities during a given period. Specifically, figures 1.a and 1.b depict unstable solutions that belong to models with explosive behaviors. Figure 1.c depicts a solution that follows a convergent time path with dampened oscillations; figure 1.d shows solutions with a non-oscillatory convergent time path.

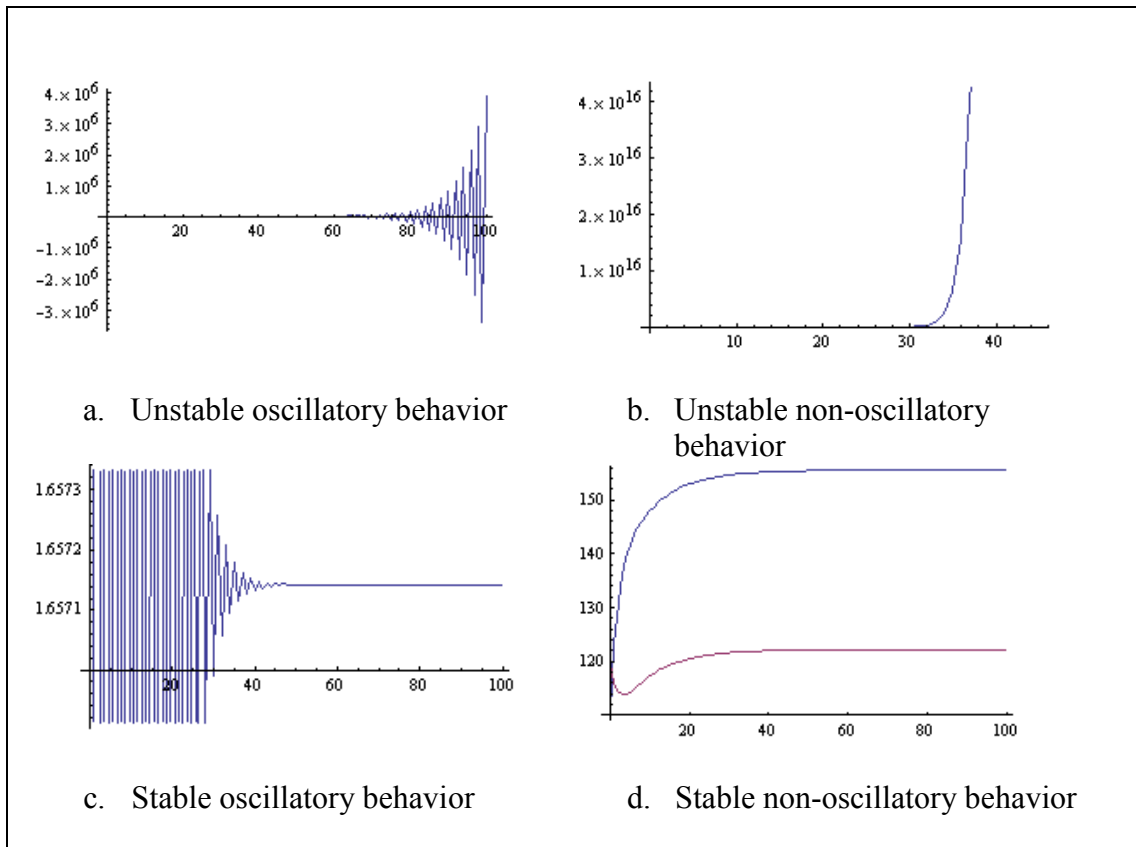


Figure 1: Dynamic evolution of the function under consideration

3. Discrete Time Dynamic Economic Models

In the present section we examine economic models when time is considered a discrete variable. As time passes from one period t to the next $t+1$, $t+2$, etc., the economic system under consideration undergoes a discrete number of states. The mathematical formulation in this context, results in one single or, a simultaneous set of linear difference equations. Linearity is not restrictive in economic applications

since it may be imposed on a model through means of a first order Taylor approximation. Our interest focuses in whether the system reaches a steady state after a number of fluctuations. A stability result is useful information to an economist, as it gives a sense of security and certainty to program business activities.

Classic discrete time discrete state economic models are the Cobweb model with memory of several periods, the Samuelson multiplier-acceleration interaction model, the inflation - unemployment model in discrete time, the dynamic market models, and common macroeconomic and macro-econometric models.

In multi equation models, difference equations are often combined into a single fundamental dynamic equation. A simple second or third order difference equation usually does not suffice to explain the complicated economic mechanisms which occur in practice. As models become larger their dynamic behavior becomes more difficult and less straightforward. Therefore, a proper usage of computer software off-loads manual solving procedures that require heavy mathematical background. Studies towards this direction are found in textbooks like Amman et al. (1996), Varian (1996), Huang and Crooke (1997), Miranda and Fackler (2002), Judd and Tesfatsion (2006) as well as in research papers like Kendrick and Amman (1999).

3.1 Theoretical Framework

Theoretically, stability results come from several kinds of symbolic calculations.

Stability Conditions for Linear Autonomous Difference Equations

Stability Condition I. For a n-th order linear difference equation $a_0 y_{t+n} + a_1 y_{t+n-1} + a_2 y_{t+n-2} + \dots + a_n y_t = b$, a necessary and sufficient condition for the general solution to be stable is

$$|r_i| < 1, i = 1, \dots, n, \quad (1)$$

where r_i are real or complex roots of the characteristic polynomial

$$a_0 r^n + a_1 r^{n-1} + a_2 r^{n-2} + \dots + a_n = 0.$$

Stability Conditions II. Next we present a determinental expression of necessary and sufficient stability conditions.

Schur Theorem. The real polynomial $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n = 0$ is called Schur stable if its roots x_i are $|x_i| < 1$. The condition $|x_i| < 1$ holds if and only if the n determinants Δ_i ($i=1, \dots, n$) are positive. Given

$$\Delta_n = \begin{vmatrix} a_0 & 0 & \dots & 0 & a_n & a_{n-1} & \dots & a_1 \\ a_1 & a_0 & \dots & 0 & 0 & a_n & \dots & a_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n-1} & a_{n-2} & \dots & a_0 & 0 & 0 & \dots & a_n \\ a_n & 0 & \dots & 0 & a_0 & a_1 & \dots & a_{n-1} \\ a_{n-1} & a_n & \dots & 0 & 0 & a_0 & \dots & a_{n-2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_1 & a_2 & \dots & a_n & 0 & 0 & \dots & a_0 \end{vmatrix}$$

the determinants Δ_i are¹:

$$\Delta_1 = \begin{vmatrix} a_0 & a_n \\ a_n & a_0 \end{vmatrix}, \Delta_2 = \begin{vmatrix} a_0 & 0 & a_n & a_{n-1} \\ a_1 & a_0 & 0 & a_n \\ a_n & 0 & a_0 & a_1 \\ a_{n-1} & a_n & 0 & a_0 \end{vmatrix}, \dots, \Delta_n \quad (2)$$

¹ For a detailed analysis see Jury (1974), Neumann (1979), Chiang (1984, chapter 17).

Stability Conditions for Linear Autonomous Systems of Difference Equations

Stability Conditions III. Consider a system of linear constant coefficient first order difference equation in u_k

$$u_{k+1} = Au_k + b \quad (3)$$

where u_k is a vector of date k economic variables, A is constant square matrix and b is a constant vector. It turns out that the eigenvalues of A help determining whether an equilibrium exists. After k steps there are k multiplications of the transformation $A = PDP^{-1}$ and the complementary solution of (3) is the solution of

$$u_{k+1} = A^k u_1 = PD^k P^{-1} u_1 \quad (4)$$

where P is a matrix with columns the characteristic vectors of the matrix A and D is the Jordan matrix of A verifying $D = P^{-1}AP$ (Strang, 1988, 5G p. 264).

The necessary stability conditions for system (3) are

$$\begin{aligned} -n < \text{tr}A < n \\ -1 < |D| < 1 \end{aligned} \quad (5)$$

where $\text{tr}A$ is the trace of the coefficient matrix A , n is the number of variables and $|D|$ is the determinant of D (Strang 1988 5J).

In cases of higher order difference equation systems, by extending the vector u_k and relabeling some lagged variables we can transform a higher order difference equation system into a first order difference equation (Chiang 1984 p. 606).

3.2 Computational Techniques²

3.2.1 Computations in Xcas³

In Xcas programming environment we create **schurseries** function, with arguments the characteristic polynomial (poly) of the difference equation and its variable (var). **schurseries** function calculates the determinant series (2). According to stability conditions II, if all determinants have positive signs, the corresponding difference equation is stable. For a direct answer to this stability test, we define **stabilitytest1** function in Xcas, with arguments the characteristic polynomial (poly) of the difference equation and its variable (var). **stabilitytest1** function returns «stable» for systems with equilibrium state(s) in case where all determinants (2) have positive signs and «unstable» for systems that have explosive behavior otherwise. The computer codes are:

```
A11(poly,var):=matrix(degree(poly,var),degree(poly,var),(j,k)->if(j<k) 0 ; else  
coeff(poly,var)[[j-k+1]]); ;
```

```
A12(poly,var):=matrix(degree(poly,var),degree(poly,var),(j,k)->if(j>k) 0 ; else  
coeff(poly,var)[[degree(poly,var)+j-k+1]]); ;
```

```
A21(poly,var):=matrix(degree(poly,var),degree(poly,var),(j,k)->if(j<k) 0 ; else  
coeff(poly,var)[[degree(poly,var)+k-j+1]]); ;
```

```
A22(poly,var):=matrix(degree(poly,var),degree(poly,var),(j,k)->if(j>k) 0 ; else  
coeff(poly,var)[[k-j+1]]); ;
```

```
schurseries(poly,var):=seq(det(blockmatrix(2,2,[subMat(A11(poly,var),0,0,k,k),s  
ubMat(A12(poly,var),0,0,k,k),subMat(A21(poly,var),0,0,k,k),subMat(A22(poly,v  
ar),0,0,k,k)])),k=0..degree(poly,var)-1); ;
```

² All electronic files are available on request.

³ Xcas is a Computer Algebra System available free in <http://www-fourier.ujf-grenoble.fr/~parisse/giac.html>.

```
stabilitytest1(poly,var):=if([seq(sign(schurseries(poly,var)[[k]]),k=1..degree(poly,
var))]==makelist(1,1,degree(poly,var))) stable; else unstable; ;
```

The codes of **schurseries** and **stabilitytest1** functions are saved in schurtheorem.cxx program file. Working in any session, by writing in a commandline read("schurtheorem.cxx") we can use schurseries and stabilitytest1 functions.

In Xcas environment stability conditions (5) are examined using **stabilitytest2** function, taking system's coefficient matrix (x) as argument. **stabilitytest2** function returns «stable» for systems with equilibrium state(s) in case where conditions (5) are satisfied and «unstable» for systems that have explosive behavior otherwise:

```
stabilitytest2(x):=if (abs(trace(x))<length(x) and abs(det(jordan(x)[[2]]))<1)
"stable"; else "unstable";
```

In case equilibrium exists, we create **steadystate** function with arguments system's coefficient matrix (a) and the system's initial state (initialstate) in a column matrix form. **steadystate** function calculates the asymptotic state of the system in a column matrix form. **steadystate** function uses Xcas' built-in function matpow which calculates the $k=100000$ power of a matrix by jordanization, as shown in (4):

```
steadystate(a,initialstate):=approx(matpow(a,100000))*initialstate
```

3.2.2 Computations in Mathematica

In Mathematica's environment we can easily create the list of the absolute values of the characteristic roots. **charvalues** function takes as arguments the characteristic polynomial (poly) of the difference equation and its variable (var). **charvalues** function generates the list of the absolute values of the real or complex characteristic roots of the difference equation.

```
charvalues[poly_,var_]:=Table[Abs[Root[poly,i]],{i,1,Length[Solve[poly==0,var
]],1}]
```

For a direct answer to the stability test (1), we define **stabilitytest** function in Mathematica, with arguments the characteristic polynomial (poly) of the difference equation and its variable (var). **stabilitytest** function returns «stable» for systems with equilibrium state(s) in case where the maximum element of the list of the absolute values of the characteristic roots is less than 1 and «unstable» for systems that have explosive behavior otherwise:

```
stabilitytest[poly_,var_]:=If[Max[Table[Abs[Root[poly,i]],{i,1,Length[Solve[pol
y==0,var]],1}]]<1,stable,unstable]
```

In the previous section we defined **steadystate** function in Xcas, which calculates the asymptotic behavior of the system $u_{k+1} = A^k u_1 = PD^k P^{-1} u_1$ after $k=100000$ steps. In the present section we define **steadystate** function in Mathematica. Mathematica's **steadystate** function provides greater calculative precision, since the asymptotic behavior of the system is estimated for power $k \rightarrow \infty$.

The first argument of **steadystate** function in Mathematica is system's coefficient matrix (a) and the second argument is system's initial state (initial) in a column matrix form. **steadystate** function calculates the asymptotic behavior of the system in a column matrix form.

```
steadystate[a_,initial_]:=Simplify[JordanDecomposition[a][[1]].Limit[MatrixPow
er[JordanDecomposition[a][[2]],k],k->∞].Inverse[JordanDecomposition[a][[1]]]
.initial//MatrixForm
```

Generalizing previous output, we create **distributionk** function, with arguments system's coefficient matrix (a) and system's initial state (initial). **distributionk** function calculates the system state in step / time k, in a column matrix form.

steadystate and **distributionk** functions in Mathematica are based on the equality

$A^k = PD^kP^{-1}$ and use Mathematica's built-in function `JordanDecomposition` which calculates a similarity matrix and the Jordan canonical form of a square matrix:

```
distributionk[a_,initial_]:=Simplify[JordanDecomposition[a][[1]].MatrixPower[JordanDecomposition[a][[2]],k].Inverse[JordanDecomposition[a][[1]]].initial//MatrixForm
```

4. Continuous Time Dynamic Economic Models⁴

In this section we examine the property of asymptotic stability in several dynamic economic systems, modeled in autonomous differential equation formulations of time parameter t . Asymptotic stability ensures intertemporal equilibrium for the economic quantity the solution stands for, regardless of what the initial conditions happen to be. Frequently applied continuous time economic models, are the Harrod-Domar model, the Solow model, the inflation and unemployment model, the Cagan monetary model, the simplified Keynesian Business Cycle model, the IS-LM model, dynamic market models, competitive equilibrium models, natural resources dynamic models, the Walrasian-Marshalian adjustment process, the Tobin-Blanchard model and the Ramsey model.

Existence of economic equilibrium in continuous time models is checked via a Symbolic language, the Xcas program editor. We generate stability results of ordinary linear and nonlinear differential equations using functional programming in free computer algebra system Xcas.

⁴ This section was presented in the 9th International Conference of Numerical Analysis and Applied Mathematics (ICNAAM), Halkidiki, Greece, September 25 2011 and was published at AIP Conf. Proc. 1389, 1769-1772, DOI:10.1063/1.3636951.

4.1 Theoretical Framework

Definitions

Following Zhang (2005, pp. 59-60) let us consider the system of the form

$$\dot{x}(t) = f(x(t), t)$$

The solution $\phi(t, x_0)$ defined in $[t_0, +\infty)$ is *stable* if, for any given $\varepsilon > 0$, there exists $\delta > 0$ such that if x_0^* is any given vector that satisfies $|x_0 - x_0^*| < \delta$, then the solution $\phi(t, x_0^*)$ with the initial conditions x_0^* exists in $[t_0, +\infty)$ and satisfies

$$|\phi(t, x_0) - \phi(t, x_0^*)| < \varepsilon,$$

for all $t \geq t_0$ (Zhang, 2005).

A solution $\phi(t, x_0)$ in $[t_0, +\infty)$ will be *asymptotically stable* if it is stable and there exists $\mu > 0$ such that if x_0^* is any given vector satisfying $|x_0 - x_0^*| < \mu$, then

$$|\phi(t, x_0) - \phi(t, x_0^*)| \rightarrow 0, \text{ as } t \rightarrow +\infty.$$

The following theorem provides a necessary and sufficient condition for stability in a linear system (Zhang, 2005).

Theorem 4.1 Let A be a $n \times n$ constant matrix in the homogeneous system

$\dot{x}(t) = A_{n \times n} x(t)$ with eigenvalues $\rho_i, i = 1, \dots, n$. The system is asymptotically stable if and only if $\text{Re}\{\rho_i\} < 0, i = 1, \dots, n$. If $\text{Re}\{\rho_i\} > 0$ for any i , the solution is unstable. The same Liapunov stability property holds for the solutions of the linear system $\dot{x}(t) = A_{n \times n} x(t) + b$, where b is a constant vector.

Higher-Order Differential Equations⁵

The n-th order homogeneous linear differential equation $a_n x^{(n)} + a_{n-1} x^{(n-1)} + a_{n-2} x^{(n-2)} + \dots + a_0 x = 0, a_n \neq 0$, is equivalent to the system of n homogeneous linear differential equation $\dot{x}(t) = Ax(t)$,

$$\text{where } x \equiv \begin{pmatrix} x_0 \\ x_1 \\ \dots \\ x_{n-1} \end{pmatrix}, A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & \dots & \dots & -\frac{a_{n-1}}{a_n} \end{pmatrix},$$

and where $x_0 = x, x_i = \dot{x}_{i-1}, i = 1, \dots, n-1$. (Zhang, 2005, p. 319)

The eigenvalues ρ of A are the roots of the following n-th degree polynomial equation $a_n \rho^n + a_{n-1} \rho^{n-1} + a_{n-2} \rho^{n-2} + \dots + a_0 = 0$ with real coefficients. The characteristic equation of the n-th order constant coefficient homogeneous linear differential equation and the characteristic equation of the matrix A are the same.

Routh-Hurwitz Theorem. For the real polynomial

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n = 0$$

the real parts of all its roots x_i are negative if and only if the n determinants Δ_i are positive.

⁵ For more details see Strang (1988, par. 5.4) and Zhang (2005, theorem 8.2.4)..

$$\Delta_1 = a_1, \Delta_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}, \Delta_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix}, \dots, \Delta_n = \begin{vmatrix} a_1 & a_0 & 0 & 0 & \dots & 0 \\ a_3 & a_2 & a_1 & a_0 & \dots & 0 \\ a_5 & a_4 & a_3 & a_2 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_n & a_{n-1} \\ 0 & 0 & 0 & 0 & \dots & a_n \end{vmatrix}$$

Here a_0 is taken to be positive (if $a_0 < 0$, then multiply the equation by -1) (see Samuelson, 1947, pp. 429-435).

Routh -Hurwitz Theorem provides a necessary and sufficient condition for stability in a linear system of differential equations with constant coefficients and in a higher order linear differential equation with constant coefficients and constant term.

Nonlinear Differential Equations

The following theorem provides a necessary and sufficient condition for stability of equilibrium points in a nonlinear system.

Theorem 4.2. (Zhang, 2005, p. 332) We consider a general autonomous system of the

form $\dot{x}(t) = f(x)$ where $x(t)$, is a vector valued function of t and f is a vector

valued function of x . Suppose that x^* is an equilibrium point of $\dot{x}(t) = f(x)$. Let f be

a C^1 function. If all the eigenvalues of the Jacobian matrix $A = \left(\frac{\partial f_i}{\partial x_j} \right)_{n \times n}$ have

negative real parts, then the equilibrium point x^* is asymptotically stable.

4.2 Computations in Xcas

Computer Codes I.

The computer codes presented here are based on Routhian Analysis of sign of real parts of characteristic roots. Programmed functions generate stability results of n-order linear differential equations with constant coefficients and constant term.

In order to surpass the complexity of the low of formation of Routh's determinants we create **routhseries** function in Xcas with arguments the characteristic polynomial (poly) and its variable (var). routhseries generates the first n minors of Δ_n . To calculate the first n minors of Δ_n we use **detrouthseries** function with arguments the characteristic polynomial (poly) and its variable (var).

```
routhseries(poly,var):=seq(subMat(blockmatrix(degree(poly,var),1,[seq(list2mat([seq(if(j<=degree(poly,var))coeff(poly,var)[[j+1]];else0;,j=k..0)],2*degree(poly,var)),k=1..2*degree(poly,var),2)]),0,0,t,t),t=0..degree(poly,var)-1);;
```

```
detrouthseries(poly,var):=seq(det(subMat(blockmatrix(degree(poly,var),1,[seq(list2mat([seq(if(j<=degree(poly,var))coeff(poly,var)[[j+1]];else0;,j=k..0)],2*degree(poly,var)),k=1..2*degree(poly,var),2)]),0,0,t,t)),t=0..degree(poly,var)-1);;
```

Although detrouthseries function generates the results of the stability conditions, we also create a direct stability test in Xcas based on Routhian analysis with the following function:

```
stabtest(poly,var):=if([sign(detrouthseries(poly,var))]==makelist(1,1,degree(poly,var)))stable; else unstable; ;;
```

The codes of functions **routhseries** **detrouthseries** and **stabtest** are saved in rouththeorem.cxx program file. Working in any session, by writing in a commandline read("rouththeorem.cxx") we can use routhseries detrouthseries and stabtest functions.

Computer Codes II

The computer codes presented in this section are based on the sign of the real part of eigenvalues of the coefficient matrix or the jacobian matrix of the system (according to theorems 4.1 and 4.2). Programmed functions provide stability tests of linear constant coefficient first order systems and non-linear first order systems.

In Xcas environment, stability conditions defined in theorem 4.1 are examined using **linearsystemstability** function. **linearsystemstability** function takes system's coefficient matrix as argument and returns «asymptotically stable» for systems with equilibrium state(s) in case where theorem 4.1 holds and «unstable» for systems that have explosive behavior otherwise:

```
linearsystemstability(a):=if([sign(re(eigenvals(a)))]==makelist(-1,1,length(a)))
"asymptotically stable"; else "unstable"; ;;
```

In Xcas environment, stability conditions defined in theorem 4.2 are examined using **nonlinearsystemstability**, which takes as arguments the list of functions (listf) of the second part of the d.e. system $\dot{x} = f(x)$, $f = (f_1, \dots, f_n)^T$, the variable vector (vars) and the equilibrium point (equilibrium). **nonlinearsystemstability** function returns «asymptotically stable» for systems with equilibrium state(s) in case where theorem 4.2 holds and «unstable» otherwise:

```
jacobian(listf,vars):=transpose(seq(seq(diff(listf[[k]],vars[[m]]),k=1..length(listf))
,m=1..length(vars)));;
nonlinearsystemstability(listf,vars,equilibrium):=if([sign(re(eigenvals(subst(jacobi
an(listf,vars),vars=equilibrium)))))]==makelist(-1,1,length(jacobian(listf,vars)))
"asymptotically stable"; else "unstable"; ;;
```

linearsystemstability and **nonlinearstabilitytest** functions are saved in eigentest.cxx program file and can be used in any session by writing read("eigentest.cxx").

5. Testing Stability Conditions for Classic Dynamic Economic Models

The dynamic properties of several models are examined in this section.

5.1 The Samuelson Multiplier-Acceleration Interaction model

The Samuelson multiplier-acceleration interaction model is condensed into a single second order difference equation (Chiang, 1984, p. 585):

$$Y_{t+2} - \gamma(1 + \alpha)Y_{t+1} + a\gamma Y_t = G_0,$$

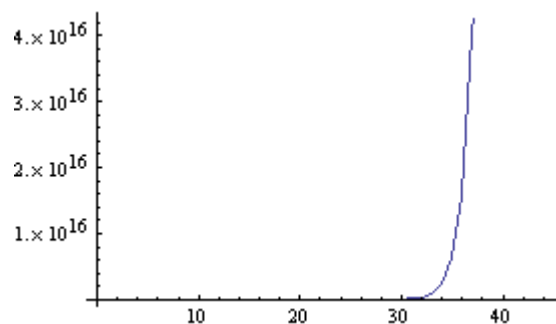
where γ represents the marginal propensity to consume and α stands for the accelerator.

Assuming that $\gamma=0.9$, $\alpha=3$, stabilitytest1 function in Xcas gives

stabilitytest1(x²-0.9*(1+3)*x+0.9*3,x)

unstable

As proven graphically, $y_t \rightarrow +\infty$ as $t \rightarrow +\infty$ (under arbitrarily chosen initial conditions):



5.2 Two macroeconomic models

Example 1

$$\begin{aligned}
 \text{Consumption : } C_t &= 0.7Y_{t-1} \\
 \text{Investment : } I_t &= 3(C_t - C_{t-1}) \\
 G_t &= 0.3Y_{t-1} \\
 \text{Import : } M_t &= 0.3Y_{t-1} \\
 \text{Export : } X_t &= 500 \\
 Y_t &= C_t + I_t + G_t + X_t - M_t
 \end{aligned}$$

which results in

$$Y_{t+2} - 2.8Y_{t+1} + 2.1Y_t = 500$$

Testing stability in Xcas we get

`schurseries(x^2-2.8*x+2.1,x)`

`-3.4,2.1`

Or

`stabilitytest1(x^2-2.8*x+2.1,x)`

unstable

Example 2

Consider two economies:

$$\begin{array}{ll}
 C_{1t} = 0.55Y_{1,t-1} & C_{2t} = 0.7Y_{2,t-1} \\
 I_{1t} = 25 + 0.4Y_{1,t-1} & I_{2t} = 30 + 0.1Y_{2,t-1} \\
 M_{1t} = 12 + 0.3Y_{1,t-1} & M_{2t} = 20 + 0.1Y_{2,t-1} \\
 X_{1t} = M_{2t} & X_{2t} = M_{1t} \\
 Y_{1t} = C_{1t} + I_{1t} + X_{1t} - M_{1t} & Y_{2t} = C_{2t} + I_{2t} + X_{2t} - M_{2t}
 \end{array}$$

In the present case, exports of one economy are imports of the other. After the substitutions, the two macroeconomic models result in the system of difference equations

$$\begin{aligned}
 Y_{1,t+1} - 0.65Y_{1t} - 0.1Y_{2t} &= 33 \\
 Y_{2,t+1} - 0.3Y_{1t} - 0.7Y_{2t} &= 22
 \end{aligned} \quad (2)$$

An initial solution of the system is the solution of the homogeneous system

$$\begin{aligned} Y_{1,t+1} &= 0.65Y_{1t} + 0.1Y_{2t} \\ Y_{2,t+1} &= 0.3Y_{1t} + 0.7Y_{2t} \end{aligned}$$

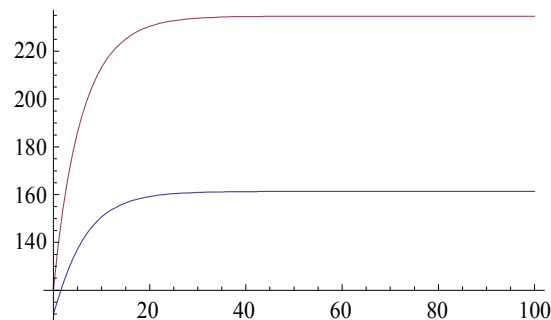
which gives the complementary solution y_c , part of the total solution $y_t = y_c + y_p$ of the nonhomogeneous system.

Testing the stability of the complementary solution in Xcas:

```
stabilitytest2([[0.65,0.1],[0.3,0.7]])
```

"stable"

we conclude that income vector Y_t for both economies reaches a steady state as $t \rightarrow \infty$. The following graph presents the analytical solution of system (2) with initial conditions $Y_{1,0} = 110, Y_{2,0} = 120$ and verifies Xcas' stability result:



5.3 Inflation and Unemployment in Discrete Time

In the discrete time context, three difference equations describe the interaction of inflation and unemployment:

$$\begin{aligned} p_t &= a - T - \beta U_t + h\pi_t \\ \pi_{t+1} - \pi_t &= j(p_t - \pi_t) \\ U_{t+1} - U_t &= -k(m - p_{t+1}) \end{aligned}$$

where p is the actual rate of inflation, π is the expected rate of inflation, U is the rate of unemployment and m is the rate of growth of nominal money. After

elimination of the U_t term, the inflation-unemployment model formulates a second order difference equation in terms of function $p(t)$ (Chiang 1984, p. 592):

$$p_{t+2} - \frac{1 + hj + (1 - j)(1 + \beta k)}{1 + \beta k} p_{t+1} + \frac{1 - j(1 - h)}{1 + \beta k} p_t = \frac{j b \beta k m}{1 + \beta k}$$

Eliminating p , we can rewrite the model as the difference equation system (Chiang 1984, p. 627):

$$\begin{pmatrix} 1 & 0 \\ -kh & 1 + bk \end{pmatrix} \begin{pmatrix} \pi_{t+1} \\ U_{t+1} \end{pmatrix} + \begin{pmatrix} -(1 - j + jh) & jb \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \pi_t \\ U_t \end{pmatrix} = \begin{pmatrix} j(a - T) \\ k(a - T - m) \end{pmatrix}$$

Assuming certain values for the parameters h, a, β, j, k , we test the stability conditions to the inflation and unemployment difference-equation model in Xcas:

`h:=1/2;a:=5/6;b:=1/5;j:=1/3;k:=10`

`stabilitytest1(x^2-(1+h*j+(1-j)*(1+b*k))/(1+b*k))*x+(1-j*(1-h)/(1+b*k)),x)`

`stable`

The same result is achieved for the difference equation system formulation with

`stabilitytest2(inverse([[1,0],[-k*h,1+b*k]])*[[1-j+j*h,-j*b],[0,1]])`

`"stable"`

5.4 Tinbergen Analysis of Economic Fluctuations

Difference equations or mixed difference and differential equations formulate the possible endogenous movements of a schematized economic system in the field of business-cycle analysis. Algebraic manipulations of such multi-equation multi-variable models lead to one final equation, from which the possible movements of the system under consideration are studied (Koopmans, 1940).

A well known numerical equation determined statistically in the analysis of economic fluctuations of U.S. (Tinbergen, 1939, p. 140), is the fourth order Tinbergen's difference equation:

$$Z_t - 0.398Z_{t-1} + 0.220Z_{t-2} - 0.013Z_{t-3} - 0.027Z_{t-4} = 0$$

with the associated characteristic equation

$$x^4 - 0.398x^3 + 0.220x^2 - 0.013x - 0.027 = 0.$$

Checking for stability over time in Xcas

schurseries(x^4-0.398*x^3+0.22*x^2-0.013*x-0.027,x)

0.99927,0.99798,0.94986,0.80951

stabilitytest1(x^4-0.398*x^3+0.22*x^2-0.013*x-0.027,x)

stable

5.5 The Cobweb Model

Extending the Cob-Web model of Hoy et al. (2001, p. 826), we assume the dynamic model with time-lagged supply function:

$$Q_t^S = f + gE_{t-1}(p_t)$$

$$Q_t^D = a + bp_t$$

$$Q_t^S = Q_t^D$$

where $E_{t-1}(p_t)$ is the price expectation for period t estimated at period t-1. We

assume that $E_{t-1}(p_t) = p_{t-1} - r\Delta p_{t-1} - k\Delta p_{t-2}$, $\Delta p_{t-1} = p_{t-1} - p_{t-2}$, $\Delta p_{t-2} = p_{t-2} - p_{t-3}$.

For parameter values $0 \leq r, k \leq 1$, suppliers expect the next price change, to be in the opposite direction of the previous price change. If $-1 \leq r, k \leq 0$, suppliers expect the next price change to be in the same direction as the previous price change. Equilibrium is then expressed via a third order difference equation, relating the price in the current period with the prices in 1, 2 and 3 periods ago:

$$p_{t+3} - \frac{g(1-r)}{b} p_{t+2} + \frac{g(r-k)}{b} p_{t+1} - \frac{gk}{b} p_t = \frac{(f-a)}{b}$$

We will test the stability for several values of g, b, r, k.

In Mathematica we get a list of the absolute values of the characteristic roots by writing:

$g = 13; b = -16; r = -0.3; k = 0.5; a = 60; f = 2;$

$\text{charvalues}[x^3 - g*(1 - r)/b*x^2 - g*(r - k)/b*x - g*k/b, x]$

{1.61464, 0.501601, 0.501601}

Alternatively we decide stability in Mathematica using

$\text{stabilitytest}[x^3 - g*(1 - r)/b*x^2 - g*(r - k)/b*x - g*k/b, x]$

unstable

Testing stability in Xcas, we apply Xcas' programmed function schurseries

$\text{schurseries}(x^3 - g*(1-r)/b*x^2 - g*(r-k)/b*(-16)*x - g*k/b,x)$

0.83496,-0.4673,-7.3874

Xcas' stability test gives

$\text{stabilitytest1}(x^3 - g*(1-r)/b*x^2 - g*(r-k)/b*(-16)*x - g*k/b,x)$

unstable

5.6 A Macroeconometric Model

The following macroeconomic model is a quarterly model constructed by Kmenta and Smith. The equations were estimated with quarterly data over the period 1954 to 1963.

$$\begin{aligned}\hat{C}_t &= -1.7951 + 0.1731Y_t + 0.421(L_t - 0.7275L_{t-1}) + 0.7275C_{t-1} \\ \hat{I}_t^d &= 2.5624 - 0.4411r_t + 0.1381(S_{t-1} - S_{t-2}) + 0.0237t + 0.8917\hat{I}_{t-1}^d \\ \hat{I}_t^r &= 3.6083 - 0.5127r_t + 0.1267(S_{t-1} - S_{t-2}) + 0.0218t + 0.6483\hat{I}_{t-1}^r \\ \hat{I}_t^i &= 3.0782 - 0.8934r_t + 0.3713(S_{t-1} - S_{t-2}) + 0.450t + 0.3178\hat{I}_{t-1}^i \\ \hat{r}_t &= 13.8928 + 0.261Y_t - 0.1501M_t + 0.0588M_{t-1} \\ Y_t &= C_t + I_t^d + I_t^r + I_t^i + G_t \\ S_t &= Y_t - I_t^i \\ L_t &= M_t + TD_t\end{aligned}$$

where Y =gross national product, C =consumption expenditures, I^d =producer's outlays on durable plant and equipment, I^r =residential construction, I^i =increase in inventories, G =government purchases of goods and services plus net foreign investment, S =final sales of goods and services, T =time in quarters (first quarter of 1954=0), r =yield on all corporate bonds, percent per annum, M =money supply, i.e. demand deposits plus currency outside banks, TD =time deposits in commercial banks, L =money supply plus time deposits in commercial banks (representing liquid wealth).

The original model by substituting C_{t-1} , S_{t-1} , S_{t-2} , I^d_{t-1} , I^r_{t-1} , I^i_{t-1} , results in a single dynamic equation:

$$\begin{aligned}
 Y_t = & 3.0716Y_{t-1} - 3.6561Y_{t-2} + 2.0850Y_{t-3} - 0.5585Y_{t-4} + 0.0535Y_{t-5} \\
 & + 1.1427G_t - 2.5300G_{t-1} + 1.3779G_{t-2} + 0.5853G_{t-3} - 0.7463G_{t-4} \\
 & + 0.1784G_{t-5} + 0.3168M_t - 0.7499M_{t-1} + 0.6253M_{t-2} - 0.2000M_{t-3} \\
 & + 0.0082M_{t-4} + 0.0046M_{t-5} + 0.048L_t - 0.1065L_{t-1} + 0.0580L_{t-2} \\
 & + 0.0246L_{t-3} - 0.0314L_{t-4} + 0.075L_{t-5} + 0.1034t - 0.2050(t-1) \\
 & + 0.1267(t-2) - 0.0192(t-3) - 0.0032(t-4) - 0.5113
 \end{aligned}$$

The example is taken from (Pindyck and Rubinfeld, 1998).

In Mathematica we get a list of the absolute values of the characteristic roots by writing:

```
charvalues[x^5-3.0716*x^4+3.6561*x^3-2.085*x^2+0.5585*x-0.0535,x]
{0.207946, 0.593134, 0.593134, 0.855162, 0.855162}
```

Alternatively we check stability conditions using

```
stabilitytest[x^5-3.0716*x^4+3.6561*x^3-2.085*x^2+0.5585*x-0.0535,x]
stable
```

Testing stability in Xcas we get

```
schurseries(x^5-3.0716*x^4+3.6561*x^3-2.085*x^2+0.5585*x-0.0535,x)
```


1, 0.84, 0.24, 0.005, 2e-06

or

stabilitytest1(x^5-3.0716*x^4+3.6561*x^3-2.085*x^2+0.5585*x-0.0535,x)

stable

5.7 Price Trend and Price Expectations

In continuous time dynamic market models, buyers' and sellers' expectations for future price of goods are parameterized through coefficients of derivatives of the price function $p(t)$. The derivative $p'(t)$ stands for the rate of change of the price and $p''(t)$ stands for the rate of change of the rate $p'(t)$ (Chiang 1984, p. 529). In the following examples, we propose some simplified demand and supply functions, modeled by linear differential expressions:

$$\begin{aligned} Q_d &= 60 - 6P(t) + 3P'(t) + 2P''(t) & Q_d &= 60 - 6P(t) - 3P'(t) - 2P''(t) \\ Q_s &= -15 + 10P(t) + 1.5P'(t) + 0.3P''(t) & Q_s &= -15 + 10P(t) - 1.5P'(t) - 0.3P''(t) \end{aligned} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$

Equation of Q_d, Q_s in both examples results in second order differential equations. Then, equilibrium is checked for stability via stabtest function in Xcas:

$$-1.7P''(t) - 1.5P'(t) + 16P(t) = 75$$

$$\text{stabtest}(-1.7*r^2-1.5*r+16,r)$$

unstable

$$1.7P''(t) + 1.5P'(t) + 16P(t) = 75$$

$$\text{stabtest}(1.7*r^2+1.5*r+16,r)$$

stable

In problem (1) price follows a divergent time path and in problem (2) price follows a convergent time path.

5.8 An input-output Leontief model

Consider an economy with two production sectors with input-output Leontief model:

$$\begin{aligned}x_1 &= ax_1 + bx_2 + cx_2' + D_1, \\x_2 &= dx_1 + fx_2 + gx_1' + D_2\end{aligned}$$

where x_1, x_2 are the current production quantities from the two sectors and D_1, D_2 the final demands for x_1, x_2 . Let us check the stability of the equilibrium solutions.

By transforming the initial system of differential equations in the form $x_i' = f_i(x_1, x_2)$, we can use `linearsystemstability` function in Xcas. We present a numerical example:

```
linearsystemstability([[0.2,-0.85],[0.8/3,-0.7/3]])
```

```
"asymptotically stable"
```

5.9 An unstable competitive equilibrium

Let the following aggregate excess demand functions of three goods for three consumers

$$\begin{aligned}z_1(p_1, p_2, p_3) &= \frac{-p_2}{p_1 + p_2} + \frac{p_3}{p_1 + p_3}, \\z_2(p_1, p_2, p_3) &= \frac{-p_3}{p_2 + p_3} + \frac{p_1}{p_1 + p_2}, \\z_3(p_1, p_2, p_3) &= \frac{-p_1}{p_1 + p_3} + \frac{p_2}{p_2 + p_3}.\end{aligned}$$

The equilibrium situation is when $z_i = 0$ for all i , $p_1 = p_2 = p_3$. The equilibrium point is (1,1,1). We consider the following Walrasian tâtonnement

$$\dot{p}_i = z_i(p_1, p_2, p_3)$$

The example is taken from Zhang (2005, p. 387).

We will check if that equilibrium is asymptotically stable. In Xcas' environment

```

nonlinearsystemstability([-p2/(p1+p2)+p3/(p1+p3),-p3/(p2+p3)+p1/(p1+p2),-
p1/(p1+p3)+p2/(p2+p3)], [p1,p2,p3], [1,1,1])
"unstable"

```

6. Conclusions

In this work we considered applications of economic dynamics when time is modeled either in a discrete or a continuous variable. In the discrete time framework, dynamic economic systems give formulations with linear autonomous difference equations. In the continuous time framework, mathematical models constitute of linear or nonlinear autonomous differential equations.

Our study began by gathering theorems and conditions that test whether the system reaches equilibrium, no matter what the initial conditions happen to be. We have presented conditions that ensure convergence based on methodologies from matrix algebra and differential calculus. Our contribution consists in programming functions in Mathematica and Xcas that produce direct stability results. As such, our proposed functions make the necessary symbolic and arithmetic calculations to foresee the possibility of convergence in the long run period. The output generated indicates «stable» for systems that reach equilibrium and «unstable» otherwise. The computational approach adopted in this study, follows functional programming style, similar to Xcas' and Mathematica's built-in functions.

In the discrete time context, we created *stabilitytest* in Mathematica and *stabilitytest1*, *stabilitytest2* functions in Xcas. By *stabilitytest* function, the stability result comes from checking the magnitude of the characteristic roots of a difference

equation. In *stabilitytest1* function, stability test is based on Schur's Theorem, and the result appears without solving the characteristic equation. *stabilitytest2* function was programmed using theory of n-dimensional linear algebra to generate stability conclusions for systems of difference equations in matrix form.

Computer codes in Xcas are written in a separate program level, saved and exported in *schurtheorem.cxx* program file. For stable systems expressed via simultaneous linear homogeneous difference equations, Xcas' *steadystate* and Mathematica' s *steadystate* functions give the stationary value at which the system comes to rest. *distributionk* function in Mathematica evaluates the future state of the system after k steps.

In the study of continuous time economic models, Routh's theorem constitutes a necessary and sufficient condition in determinantal form for asymptotic stability of higher-order differential equations. In the case of simultaneous equations, matrix theory has a part to play. For systems of linear first order differential equations the eigenvalues of the coefficient matrix and for systems of nonlinear first order differential equations the eigenvalues of the jacobian matrix are examined.

Our computer codes use two methodologies, routhian analysis and eigenvalues' calculation, in order to define the long run dynamic evolution of economic systems in continuous time. The property checked is asymptotic stability which is related to the existence of economic equilibrium. All functions are programmed in Xcas in *rouththeorem.cxx* and *eigentest.cxx* program files.

With *routhseries* and *detrouthseries* functions of *rouththeorem.cxx* program file, the user can reproduce routh's determinants symbolically and numerically. By *stabtest* function, stability conditions for higher dimensional differential equations are tested

using Routhian analysis. In this approach, there is no need for solving the corresponding characteristic equations.

In `eigentest.cxx` program file, *linearsystemstability* function tests stability conditions for simultaneous first order linear differential equations with constant coefficients by checking the sign of the real part of the eigenvalues of the coefficient matrix. *nonlinearsystemstability* function tests stability of equilibrium points for simultaneous nonlinear differential equations, by checking the sign of the real part of the eigenvalues of the jacobian matrix. All calculations are made in one entry, with minimum input required.

Numerical experiments on traditional applications of economic dynamics exhibit the simplicity clarity and brevity of input and output of our computer codes.

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