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## ABSTRACT

Data envelopment analysis (DEA) is a linear programming methodology to evaluate the relative technical efficiency for each member of a set of peer decision making units (DMUs) with multiple inputs and multiple outputs. It has been widely used to measure performance in many areas. A weakness of the traditional DEA model is that it cannot deal with negative input or output values. There have been many studies exploring this issue, and various approaches have been proposed.

In this paper, we develop a variant of the traditional radial model whereby original values are replaced with absolute values as the basement to quantify the proportion of improvements to reach the frontier. The new radial measure is units invariant and can deal with all cases of the presence of negative data. In addition, the VRM model preserves the property of proportionate improvement of a traditional radial model, and provides the exact same results in the cases that the traditional radial model can deal with. Examples show the advantages of the new approach.

**Keywords**: (D) Data Envelopment Analysis; Negative data in DEA; Variant of radial measure; Unit invariance

## 1. Introduction

Data envelopment analysis (DEA), originally developed by Charnes et al. (1978), is a linear programming methodology for evaluating the relative technical efficiency for each member of a set of peer decision making units (DMUs) with multiple inputs and multiple outputs (Charnes et al., 1978). It has been widely used to measure performance in many areas. A weakness of a traditional DEA model is that it requires the assumption that all the inputs and outputs have non-negative values, while in many situations negative values, especially those of outputs, such as profit, could exist.

Among the various approaches proposed for dealing with negative data, the simplest one is to exchange the role between inputs and outputs. If inputs are all negative or non-positive, they can be treated as positive outputs so that their absolute values can be increased, which means a decrease of negative inputs. And vice versa if outputs are all negative or non-positive, they can be treated as positive inputs so that their absolute values can be reduced, which means an increase of negative outputs (Scheel, 2001; Zhu, 2009). However this method cannot be applied to variables with both positive and negative values. In addition, treating negative outputs as positive inputs may not reflect the true production process.

The commonly used approach to dealing with negative data is based on the property of "translation invariance". A DEA model is expected to be translation invariant if translating the original input and/or output data values results in a new model that has the same optimal solution as the old one (Ali and Seiford, 1990; Cooper et al., 2007). For DEA models with translation invariance, negative values can be turned into positive values by imposing a big enough positive value to the variable with negative values so that all the values of the variable become positive (Charnes et al., 1983; Lovell and Pastor, 1995; Seiford and Zhu, 2002). The additive model under variable returns to scale (VRS) is translation invariant, which means that the results will not change after a positive scalar is added to any input or output in the additive model ( Charnes et al., 1985; Ali and Seiford, 1990; Lovell and Pastor, 1995; Pastor, 1996). Lovell et al. (1995) developed a normalized weighted additive model that is both translation invariant and units invariant. However, both the traditional additive model and the normalized additive model have some drawbacks. Firstly, they yield the 'furthest' targets on the production frontier for inefficient units; and secondly they cannot provide an efficiency score for an inefficient unit (Portela et al., 2004). Another translation-invariant model is the radial model under VRS technology (BCC model), but it only has restricted translation invariance. Input-translation invariance exists only in output-oriented models and output-translation invariance exists only in input-oriented models. The efficiency status (efficient or inefficient) of the evaluated DMU is translation invariant in both input and output-oriented BCC models, though, its efficiency score is not (classification invariance only) (Ali and Seiford, 1990; Lovell and Pastor, 1995; Pastor, 1996; Cooper et al., 2007).

Portela et al. (2004) put forth a directional distance model (range directional measure, RDM) using range values (absolute gap between the initial evaluated value and the best observed value of a variable) as the direction vector, which can deal with inputs or outputs with positive and/or negative values (Portela et al., 2004). The advantage of the RDM over the generic directional distance model, which can natively deal with negative data, is that it is units invariant and it yields efficiency scores between 0 and 1 for inefficient units. The efficiency measurement process applied is similar to but not the same as that of radial model. The results obtained from the RDM model are generally different than those delivered by the conventional radial model in the cases that the latter model can deal with, such as non-negative value data or negative value free DMUs.

On the basis of the idea proposed by Portela et al. (2004), Sharp et al. (2006) introduced a modified slack-based measure (MSBM) which can deal with data with positive and/or negative values (Sharp et al., 2007). Similar to the RDM model, MSBM model is broadly unable to yield same results in the cases the traditional slack-based measure (SBM) model can deal with.

Emrouznejad et al. (2010) propose a semi-oriented radial measure (SORM), which is applicable to datasets include variables which can take both negative and positive values (Emrouznejad et al., 2010a; Emrouznejad et al., 2010b). The essence of SORM is that it breaks down each variable into two variables, one assigned the negative values and the other the positive values of the original variable. Similarly to RDM and MSBM, SORM is able to deal with negative data without changes of origin (data translation). The preservation of the origin means a form of radial pursuit of targets to a certain extent, although it is in terms of the positive or negative part of a variable but not in terms of the variable as a whole. SORM yields the same results as those determined by the traditional radial model in case no negative data exist for the evaluated DMUs. The disadvantage of the SORM is the artificial increase of the number of variables (inputs and/or outputs). As a result, the method may not necessarily determine Pareto efficient targets. In addition, SORM may lead to targets worse than the observed values.

In this paper, we propose a variant of radial measure (VRM) that yields a measure of efficiency and also is able to handle variables consisting of positive values for some and negative values for other sample DMUs. This paper unfolds as follows. Section 2 introduces the VRM model. Section 3 provides a brief comparison between VRM and two more approaches focused on negative data handling in DEA. Moreover, in the next section, numerical examples are quoted to demonstrate the differences between the related methods, as well as advantages and defects of the VRM method are revealed. Conclusive remarks are presented in the last section.

## 2. Variant of the radial measure (VRM)

The traditional input-oriented radial model can be formulated as

$$\begin{aligned} &\min \ \theta \\ &st \ \ X\lambda \le \theta x_0 \\ &\quad Y\lambda \ge y_0 \\ &\quad [\sum_j \lambda_j = 1] \end{aligned}$$

$$\lambda \geq 0 \qquad (1)$$

The traditional output-oriented radial model can be formulated as

$$\max \ \phi$$

$$st \ \ X\lambda \leq x_0$$

$$Y\lambda \geq \phi y_0$$

$$[\sum_j \lambda_j = 1]$$

$$\lambda \geq 0 \qquad (2)$$

The constraint Σλ=1 is kept under variable returns to scale technology (BCC) and dropped under constant returns to scale technology (CCR) (Banker et al., 1984). The efficiency of the evaluated $DMU_0$ is defined as the optimal value $\theta^*$ in the input-oriented model and $1/\varphi^*$ in the output-oriented model.

The above radial models can be equivalently transformed into the following formulations by replacing θ with 1-β in the input-oriented model and φ with 1+β in the output-oriented model, respectively.

As a result, the input-oriented model is re-written as

$$\max \ \beta$$

$$st \ \ X\lambda + \beta x_0 \leq x_0$$

$$Y\lambda \geq y_0$$

$$[\sum_j \lambda_j = 1]$$

$$\lambda \geq 0 \qquad (3)$$

and the output-oriented model becomes

$$\max \ \beta$$

$$st \ \ X\lambda \leq x_0$$

$$Y\lambda - \beta y_0 \geq y_0$$

$$[\sum_j \lambda_j = 1]$$

$$\lambda \geq 0 \qquad (4)$$

After transformation, the efficiency of the evaluated $DMU_0$ is equal to 1-$\beta^*$ in input-oriented model and 1/(1+$\beta^*$) in output-oriented model. Here β expresses the measurement of inefficiency. To be more precise, β measures the degree of improvements for the evaluated DMU to reach the frontier by applying the ratio of proportionate input decrease to the observed input value (input-oriented) or the ratio of proportionate output increase to the observed output value (output-oriented).

In input- (output-) oriented radial models, existence of negative inputs (outputs) of the evaluated DMUs will lead the ‘improvements’ produced to a wrong direction. In other words, the increased inputs or decreased outputs may be regarded as targets for inefficient DMUs to reach a ‘faulty’ frontier. To avoid such a flaw, we propose a variant of the traditional radial model by applying the absolute values of inputs (outputs) instead of their original values in the left hand side of the constraint. The variant of radial measure (VRM) is identical to the traditional radial model when negative data are absent or negative data do not lead to a wrong direction of input (output) improvements (e.g., negative input data in an output-oriented model or negative output data in an input-oriented model). The VRM model rectifies the wrong direction and, at the same time, preserves the radial property when negative data do exist.

After replacement, the input-oriented VRM under VRS becomes

$$\max \ \beta$$

$$st \ \ X\lambda + \beta\left|x_0\right| \leq x_0$$

$$Y\lambda \geq y_0$$

$$\sum_j \lambda_j = 1$$

$$\lambda \geq 0 \qquad (5)$$

and the output-oriented VRM under VRS results

$$\begin{aligned}
&\max \quad \beta \\
&st \quad X\lambda \le x_0 \\
&\quad Y\lambda - \beta|y_0| \ge y_0 \\
&\quad \sum_j \lambda_j = 1 \\
&\quad \lambda \ge 0
\end{aligned} \tag{6}$$

In case positive and negative input (output) data appear in a particular variable, the developed VRM model assures that the targets assigned to the sample DMUs for performance improvement will respect the traditional DEA concept expressed either by input shrinkage or by output expansion.

## 3. Related Methods

In this section, other two approaches, RDM proposed by Portela et al. (2004) and SORM introduced by Emrouznejad et al. (2010b), is discussed.

### Model 1: Range directional measure (RDM)

The general directional distance model under VRS is expressed as (Chambers et al., 1996; Chung et al., 1997; Chambers et al., 1998)

$$\begin{aligned}
&\max \quad \beta \\
&st \quad X\lambda + \beta g_x \le x_0 \\
&\quad Y\lambda - \beta g_y \ge y_0 \\
&\quad \sum_j \lambda_j = 1 \\
&\quad \lambda \ge 0
\end{aligned} \tag{7}$$

where $g_x$ and $g_y$ are direction vectors for inputs and outputs respectively.

Directional distance function model can deal with negative data in itself, and under VRS it is translation invariant. The drawback of this particular function is that it is not unit invariant and cannot provide an efficiency measure between 0 and 1. The

RDM model put forth by Portela et al. (2004) overcomes the aforementioned shortcoming. The key feature of RDM is that it uses the subtraction between the input value of the evaluated DMU and the smallest value of the input as the input direction vector.

$$g_x = R_x = x_0 - Min(x_j), \quad j = 1, 2, \dots n,$$

Simultaneously, it uses the subtraction between the biggest value of the output and the output value of the evaluated DMU as the output direction vector,

$$g_y = R_y = Max(y_j) - y_0, \quad j = 1, 2, \dots n.$$

Additionally, the non-oriented RDM model can be formulated as

$$\begin{aligned} & \max \ \beta \\ & st \ \ X\lambda + \beta R_{x0} \le x_0 \\ & \quad\ \ Y\lambda - \beta R_{y0} \ge y_0 \\ & \quad\ \ \sum_j \lambda_j = 1 \\ & \quad\ \ \lambda \ge 0 \end{aligned} \qquad (8)$$

The RDM model can deal with negative data as well as the general directional model, and also, under VRS, it determines an efficiency measure between 0 and 1 by the definition of $(1-\beta^*)$.

Here we used an output-oriented RDM model by setting $\beta=0$ for input-related constraints. In order for the output-oriented model to be comparable with the efficiency measure defined in the output radial model, we introduce another efficiency measure $1/(1+\beta^*)$, in addition to $(1-\beta^*)$ used by Portela et al. (2004) to calculate the efficiency score.

**Model 2: SORM (semi-oriented radial measure)**

To tackle the problem of the traditional radial model related to possible faulty direction specification for input (output) improvements in case negative data exist, Emrouznejad et al. (2010) proposed an alternative. This particular approach, treats each variable (v) recording positive and negative values for the sample DMUs as consisting of the sum of two variables ($v = v^1 - v^2$) as follows.

$$v^1_{ij} = \begin{cases} v_{ij} & if\ v_{ij} \geq 0, \\ 0 & if\ v_{ij} < 0, \end{cases} \quad \text{and} \quad v^2_{ij} = \begin{cases} 0 & if\ v_{ij} \geq 0, \\ -v_{ij} & if\ v_{ij} < 0. \end{cases}$$

To be convenient to make comparisons between SORM and VRM, we use the equivalent formulations for SORM by replacing "h" used by Emrouznejad et al. (2010b) with (1-β) in input-oriented model and (1+β) in output-oriented model . Subsequent to the transformation of the variables with negative data described, the traditional input-oriented radial model (3) is turned into input-oriented SORM.

$$\begin{aligned} & \max\ \beta \\ & st\ \ X\lambda + \beta x_0 + s^- = x_0 \\ & \quad V^1\lambda + \beta v^1_0 + s^{1-} = v^1_0 \\ & \quad (-V^2)\lambda + \beta(-v^2_0) + s^{2-} = (-v^2_0) \\ & \quad Y\lambda \geq y_0 \\ & \quad \sum_j \lambda_j = 1 \\ & \quad \lambda, s^-, s^+ \geq 0 \qquad \qquad (9) \end{aligned}$$

By applying a reverse replacement process, in which (1-β) in (9) is substituted by "h", the formulas will be turned back into the formulas used by Emrouznejad et al (2010b). Namely, the input-oriented SORM model

$$\begin{aligned} & \min\ h \\ & st\ \ X\lambda \leq hx_0 \\ & \quad V^1\lambda \leq hv^1_0 \\ & \quad V^2\lambda \geq hv^2_0 \\ & \quad \sum_j \lambda_j = 1 \\ & \quad \lambda \geq 0 \qquad \qquad (10) \end{aligned}$$

By adopting the precedent practice, the output-oriented SORM model

$$
\begin{aligned}
& \max \quad \beta \\
& st \quad X\lambda + s^{-} = x_0 \\
& \quad Y\lambda - \beta y_0 - s^{+} = y_0 \\
& \quad V^1\lambda - \beta v_0^1 - s^{1+} = v_0^1 \\
& \quad (-V^2)\lambda - \beta(-v_0^2) - s^{2+} = (-v_0^2) \\
& \quad \sum_j \lambda_j = 1 \\
& \quad \lambda, s^{-}, s^{+} \ge 0 \qquad (11)
\end{aligned}
$$

is transformed into the original one introduced by Emrouznejad et al. (2010b)

$$
\begin{aligned}
& \max \quad h \\
& st \quad X\lambda \le x_0 \\
& \quad Y\lambda \ge hy_0 \\
& \quad Y^1\lambda \ge hy_0^1 \\
& \quad Y^2\lambda \le hy_0^2 \\
& \quad \sum_j \lambda_j = 1 \\
& \quad \lambda \ge 0 \qquad (12)
\end{aligned}
$$

Particularly in the input-oriented SORM model (9), regarding inefficient DMUs, the targets of $x_0$ and $v_0^1$ are properly directed towards input decline. Nevertheless, the $-v_0^2$ side, of $v_0$ is mis-specified. In other words, considering the target value of $-v_0^2$ that is $(-v_0^2) - \beta(-v_0^2) - s^{2-}$, unlike the slack movement ($-s^{2-}$) that is in the right direction, the radial movement $\beta v_0^2$ is misdirected. Consequently, the $-v_0^2$ target results, either their improvement or deterioration, are the predominance product of the absolute values between the radial movement and the slack movement. For instance, if $\beta v_0^2 \geq s^{2-}$, the direction of the total movement is correct, else it is wrong.

Moreover, the improvement or the worsening of the target of (v) depends on the sign of the combined movement of $v_0^1$ and $-v_0^2$. As a result, it is possible for the input targets to become worsened for the optimal solutions of the input-oriented SORM model, and similarly, the output targets may be deteriorated by applying the output-oriented SORM model. An example following in the next section proves this position.

## 4. Numerical Example

### 4.1 Data Description

Two examples are quoted in order to reveal the differences between the two dominant negative data handing methods in DEA: RDM and SORM, and the proposed VRM approach. The first example, based on the same dataset as the work of Emrouznejad et al. (2010a), consists of 13 DMUs, two input and three output variables. One input variable is positive (cost) and the other non-positive (effluent). One of the output variables is positive (saleable) and the remaining two are non-positive (methane and CO2). The second case that confirms the discrepancy between the efficiency results obtained by the three methods applies a single positive input (X) and twin, one positive (Y1) and one negative (Y2), output dataset for four operational units.

For both examples, the output-oriented VRS model is adopted.

### 4.2 Variant of Radial Model (VRM) application

**Example 1.** Despite the inconsistency of the efficiency scores calculated after introducing the twin input – twin output dataset to the RDM, SORM and VRM models (Table 1), there is unitary identification of the efficient DMUs (Table 2). Particularly, SORM tends to underestimate the efficiency scores of the inefficient units compared to those yielded by the RDM and VRM approaches. Since h* is always greater than unity for inefficient DMUs, the constraint referred to the negative part of the output $Y^2\lambda \le hy_0^2$ is much looser than it should be. In case h* is much greater than unity, the efficiency scores imputed will be equal to those yielded from the traditional output-oriented radial model that solely processes the non-negative output data.

Table 1 Input-Output data

| DMU | (I1)Cost | (I2)Effluent | (O1)Saleable | (O2)CO2 | (O3)Methane |
|---|---|---|---|---|---|
| D01 | 1.03 | -0.05 | 0.56 | -0.09 | -0.44 |
| D02 | 1.75 | -0.17 | 0.74 | -0.24 | -0.31 |
| D03 | 1.44 | -0.56 | 1.37 | -0.35 | -0.21 |
| D04 | 10.80 | -0.22 | 5.61 | -0.98 | -3.79 |
| D05 | 1.30 | -0.07 | 0.49 | -1.08 | -0.34 |
| D06 | 1.98 | -0.10 | 1.61 | -0.44 | -0.34 |
| D07 | 0.97 | -0.17 | 0.82 | -0.08 | -0.43 |
| D08 | 9.82 | -2.32 | 5.61 | -1.42 | -1.94 |
| D09 | 1.59 | 0.00 | 0.52 | 0.00 | -0.37 |
| D10 | 5.96 | -0.15 | 2.14 | -0.52 | -0.18 |
| D11 | 1.29 | -0.11 | 0.57 | 0.00 | -0.24 |
| D12 | 2.38 | -0.25 | 0.57 | -0.67 | -0.43 |
| D13 | 10.30 | -0.16 | 9.56 | -0.58 | 0.00 |

Table 2 Efficiencies scores (RDM, SORM and VRM applications)

| DMU | RDM $(1-\beta^*)$ | RDM $1/(1+\beta^*)$ | SORM | VRM |
|---|---|---|---|---|
| D01 | 0.97 | 0.97 | 0.629 | 0.906 |
| D02 | 0.91 | 0.92 | 0.447 | 0.770 |
| D03 | 1.00 | 1.00 | 1.000 | 1.000 |
| D04 | 0.50 | 0.67 | 0.594 | 0.684 |
| D05 | 0.92 | 0.93 | 0.406 | 0.771 |
| D06 | 0.97 | 0.97 | 0.861 | 0.861 |
| D07 | 1.00 | 1.00 | 1.000 | 1.000 |
| D08 | 1.00 | 1.00 | 1.000 | 1.000 |
| D09 | 0.99 | 0.99 | 0.912 | 0.912 |
| D10 | 0.63 | 0.73 | 0.386 | 0.730 |
| D11 | 1.00 | 1.00 | 1.000 | 1.000 |
| D12 | 0.81 | 0.84 | 0.255 | 0.645 |
| D13 | 1.00 | 1.00 | 1.000 | 1.000 |

By applying SORM and the traditional radial model to a reduced dataset, excluding the two non-positive outputs (methane and $CO_2$), the efficiency results are equal except two cases: D09 and D11 (Table 3)

Table 3 Efficiencies scores (SORM and traditional BCC applications)

| DMU | SORM | BCC(1 output) |
|---|---|---|
| D01 | 0.629 | 0.629 |
| D02 | 0.447 | 0.447 |
| D03 | 1.000 | 1.000 |
| D04 | 0.594 | 0.594 |
| D05 | 0.406 | 0.406 |
| D06 | 0.861 | 0.861 |
| D07 | 1.000 | 1.000 |
| D08 | 1.000 | 1.000 |
| D09 | **0.912** | **0.345** |
| D10 | 0.386 | 0.386 |
| D11 | **1.000** | **0.477** |
| D12 | 0.255 | 0.255 |
| D13 | 1.000 | 1.000 |

The subsequent table reports the benchmarks and intensity vector λ for inefficient DMUs. Particularly, the direction of improvement for the inefficient units uncovered by the three models differ entailing significant differences in the benchmarks and the intensity vector. The reason of this inconsistency is sourced in the measurement of direction. Namely, the direction is defined by the range vector ($R_o$) in RDM, is partially radial in SORM and completely radial in VRM.

Table 4 Benchmarks and intensity vector from RDM, SORM and VRM

| DMU | RDM | SORM | VRM |
|---|---|---|---|
| D01 | D03(0.015); D07(0.980); D13(0.006) | D03(0.128); D07(0.872) | D07(0.813); D11(0.187) |
| D02 | D03(0.447); D07(0.365); D11(0.131); D13(0.057) | D03(0.965); D13(0.035) | D03(0.407); D11(0.549); D13(0.044) |
| D04 | D08(0.033); D11(0.205); D13(0.763) | D08(0.028); D13(0.972) | D08(0.031); D11(0.137); D13(0.832) |
| D05 | D03(0.702); D07(0.298) | D03(0.702); D07(0.298) | D03(0.080); D07(0.006); D11(0.913) |
| D06 | D03(0.939); D13(0.061) | D03(0.939); D13(0.061) | D03(0.939); D13(0.061) |
| D09 | D11(1.000) | D11(1.000) | D11(1.000) |
| D10 | D03(0.080); D11(0.403); D13(0.517) | D03(0.490); D13(0.510) | D03(0.080); D11(0.403); D13(0.517) |
| D12 | D03(0.894); D13(0.106) | D03(0.894); D13(0.106) | D03(0.679); D11(0.211); D13(0.110) |

RDM and VRM yield improved target values than the observed levels. By applying SORM, worsened target values may be obtained as illustrated in Table 5. For instance, the target levels of $CO_2$ for units D01 and D02 are moved to the opposite direction than was expected, expressing deterioration rather than improvement. To be more precise, for D01 and D02, the target $CO_2$ values decrease from -0.09 to -0.11 and from -0.24 to -0.36, respectively.

Table 5 Targets from RDM, SORM and VRM

| DMU | (O1)Saleable | | | | (O2)$CO_2$ | | | | (O3)Methane | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Observed | Target | | | Observed | Target | | | Observed | Target | | |
| | | RDM | SORM | VRM | | RDM | SORM | VRM | | RDM | SORM | VRM |
| D01 | 0.56 | 0.88 | 0.89 | 0.77 | -0.09 | -0.09 | **-0.11** | -0.07 | -0.44 | -0.42 | -0.40 | -0.39 |
| D02 | 0.74 | 1.53 | 1.66 | 1.29 | -0.24 | -0.22 | **-0.36** | -0.17 | -0.31 | -0.28 | -0.20 | -0.22 |
| D04 | 5.61 | 7.59 | 9.45 | 8.21 | -0.98 | -0.49 | -0.60 | -0.53 | -3.79 | -0.11 | -0.05 | -0.09 |
| D05 | 0.49 | 1.21 | 1.21 | 0.64 | -1.08 | -0.27 | -0.27 | -0.03 | -0.34 | -0.28 | -0.28 | -0.24 |
| D06 | 1.61 | 1.87 | 1.87 | 1.87 | -0.44 | -0.36 | -0.36 | -0.36 | -0.34 | -0.20 | -0.20 | -0.20 |
| D09 | 0.52 | 0.57 | 0.57 | 0.57 | 0.00 | 0.00 | 0.00 | 0.00 | -0.37 | -0.24 | -0.24 | -0.24 |
| D10 | 2.14 | 5.28 | 5.55 | 5.28 | -0.52 | -0.33 | -0.47 | -0.33 | -0.18 | -0.11 | -0.10 | -0.11 |
| D12 | 0.57 | 2.24 | 2.24 | 2.10 | -0.67 | -0.37 | -0.37 | -0.30 | -0.43 | -0.19 | -0.19 | -0.19 |

**Example 2.** We use another simple demonstrative dataset to identify the differences between RDM, SORM and the newly developed VRM in detecting efficient units (Table 6). Based on this particular dataset, units deemed inefficient by RDM and SORM are regarded as "weak" efficient by VRM. Considering unit A, which records

the largest value for the output Y2, within the four sample DMUs, is expected to be "weak" efficient in case an output-oriented BCC model is applied. However, unit A is evaluated inefficient by SORM due to the peculiarity of this particular model to identify, in some cases, worsened target values for negative outputs. For example, the target value, determined by SORM, for Y2 negative output of unit A is deteriorated while value -4 is recommended instead of the actual -2.

According to the RDM model, units A and D are regarded as inefficient due to the zero value of $R_o$ for the output of the evaluated unit when the unit reaches its maximum level. The zero coefficient for β in the constraint results in an optimal non-zero value for β. Under the same circumstances, when the traditional output-oriented radial model is applied, the optimal β value is always equal to null. Just in cases all the outputs of a unit under evaluation obtain maximal values; β is unbounded relying on the RDM model.

Table 6 Second example data

| DMU | X | Y1 | Y2 |
|---|---|---|---|
| A | 1 | 2 | -2 |
| B | 1 | 3 | -2 |
| C | 1 | 4 | -3 |
| D | 1 | 4 | -4 |

Table 7 Efficiencies for second example data

| DMU | RDM ($1-\beta^*$) | RDM $1/(1+\beta^*)$ | SORM | VRM |
|---|---|---|---|---|
| A | 0.50 | 0.67 | 0.50 | 1.00* |
| B | 1.00 | 1.00 | 1.00 | 1.00 |
| C | 1.00 | 1.00 | 1.00 | 1.00 |
| D | 0.50 | 0.67 | 1.00 | 1.00* |

*: Weak efficient.

Table 8 Targets for second example data

| DMU | Y1 | | | | Y2 | | | |
|---|---|---|---|---|---|---|---|---|
| | Observed | Target | | | Observed | Target | | |
| | | RDM | SORM | VRM | | RDM | SORM | VRM |
| A | **2** | **3** | **4** | **3** | **-2** | **-2** | **-4** | **-2** |
| B | 3 | 3 | 3 | 3 | -2 | -2 | -2 | -2 |
| C | 4 | 4 | 4 | 4 | -3 | -3 | -3 | -3 |
| D | 4 | 4 | 4 | 4 | -4 | -3 | -3 | -3 |

### 4.3 Advantages and drawbacks of VRM

The VRM model provides a simple method to evaluate DMUs incorporating input and/or output variables with solely negative values or a mixture of positive and negative values without requiring any data transformation. The essence of VRM that makes it capable of dealing with negative data is the use of absolute values instead of the original negative ones to determine the necessary proportional improvements for the inefficient DMUs to reach the best-practice frontier adopting the radial projection models. The power of the developed model derives from its flexibility and catholic philosophy in handling negative data introduced to DEA; the VRM model is applicable not only in cases in which part of the dataset consists of negative values but also when fully-dominated datasets by negative values appear.

The newly introduced model goes beyond the RDM method which yields the same results as those of the traditional radial model when the traditional approach is free of negative data. The VRM model preserves the radial property towards the inefficient units' projection to their best-practice frontier dismissing the need for actual data transformation or modification of the axis of origin. As a result, this particular model never leads to worsened target values than the original ones. It could be said that VRM is a generalization of the traditional radial DEA model that extends its applicability from solely non-negative data handling to partially or fully negative datasets assessment.

The drawback of VRM is summarized in the weakness of monotonicity engagement for inefficient DMUs while improving their inputs or outputs in the whole set of real numbers (ℝ). Though, monotonicity is preserved in the intervals $[-\infty, 0]$ and $[0, +\infty]$, this issue is raised just in case a variable contains both positive input and output values (mixed-sign variable). A process proposed for tackling such a problem is the division of the mixed-sign variable into two single-sign variables as follows.

$$v^1_{ij} = \begin{cases} v_{ij} & if\ v_{ij} \geq 0, \\ 0 & if\ v_{ij} < 0, \end{cases} \text{ and } v^2_{ij} = \begin{cases} 0 & if\ v_{ij} \geq 0, \\ v_{ij} & if\ v_{ij} < 0. \end{cases}$$

This transformation is similar but not common to that introduced by SORM. Particularly, according to VRM concept, the mixed-sign variable is the summation result of two artificial variables ($v=v^1 + v^2$) that one takes positive (or non-negative) values and the other negative (or non-positive) values. Unlike VRM transformation process, in SORM, both artificial variables take positive (or non-negative) values. Additionally, in SORM, the mixed-sign variable (v) is the difference between the two artificial variables ($v=v^1 - v^2$).

## 5. Conclusion

In the presence of negative input or output data, the traditional input or output-oriented radial models, respectively, lead to flawed results due to their weakness to identify the magnitude of the negative-signed data to the optimization process. In this context, the target values obtained towards efficiency attainment express worsened input or output levels compared with the original values introduced.

The (normalized) additive model, the range directional measures (RDM), the modified slack-based model (MSBM) and the semi-oriented radial model (SORM) could be applied in order to tackle the disorientation problem raised in case negative values exist within the dataset under assessment, though without lacking weaknesses. Namely, the (normalized) additive model does not provide efficiency measure. The RDM model may be unbounded when the evaluated DMU performs the maximum values for every output variable or engages the minimum levels for all the inputs.

Following, the SORM model may get deteriorated targets and suffer from disorientation for the sample units that either all their input or all their output values are negative.

In this paper, we developed a variant of the traditional radial model, in which the original values are replaced by their absolute values to quantify the proportion of improvements in order to reach the best-practice frontier. The VRM model is units invariant and can deal omnipotently with every case of negative values presence in the dataset under assessment. In addition, the VRM model preserves the proportionate improvement property of the traditional radial model, and also yields the same results with the traditional model in the cases in which the latter one is applicable.

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