Condorcet's principle and the strong no-show paradoxes

Duddy, Conal

National University of Ireland, Galway

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Conal Duddy†

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Abstract

We consider two no-show paradoxes, in which a voter obtains a preferable outcome by abstaining from a vote. One arises when the casting of a ballot that ranks a candidate in first place causes that candidate to lose the election. The other arises when a ballot that ranks a candidate in last place causes that candidate to win. We show that when there are at least four candidates and when voters may express indifference, every voting rule satisfying Condorcet's principle must generate both of these paradoxes.

1 Introduction

A Condorcet winner is a candidate for election who is preferred by a majority in all pairwise comparisons with the other candidates. Condorcet's principle says that a Condorcet winner must be elected whenever there is one (Condorcet 1785). This principle may have undesirable consequences.

Moulin's (1988) no-show paradox arises when the addition of a ballot that ranks candidate x above candidate y may take victory away from x and give it to y. A voting rule that is free from this paradox is said to satisfy the participation principle. Moulin shows that a voting rule cannot satisfy both

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†J.E. Cairnes School of Business and Economics, National University of Ireland, Galway, Ireland. Email: conal.duddy@gmail.com.

Condorcet's principle and the participation principle when there are four or more candidates.[1]

If we insist on satisfying Condorcet's principle then we must tolerate the no-show paradox. However, we may consider some instances of the paradox to be more severe than others. In that case, we may have reason to prefer some "Condorcet-consistent" voting rules over others, since some of them might at least be free from these severe cases of the paradox.

In this paper we consider two special cases of the no-show paradox. These would seem to be especially bizarre instances. One arises when the casting of a ballot that ranks a candidate in first place causes that candidate to lose the election. The other arises when a ballot that ranks a candidate in last place causes that candidate to win. We call these the strong no-show paradoxes after Pérez (2001) and Nurmi (2002). One or both of these special cases of the paradox are also considered by Smith (1973), Richelson (1978), Brams and Fishburn (1983), Saari (1995) and Lepelley and Merlin (2001).

Pérez (2001) demonstrates that Condorcet-consistent voting rules do exist that are free from one or even both of the strong no-show paradoxes, no matter the number of candidates. One of these is the Simpson-Kramer Min-Max rule. That rule is free from both of the strong no-show paradoxes. Young's rule is free from one of the two paradoxes; a ballot that ranks a candidate in last place can never cause that candidate to win under Young's rule. See Pérez (2001) for definitions of both of these rules, and several other Condorcet-consistent rules.

Moulin (1988) and Pérez both consider the aggregation of linear orderings. We consider instead the aggregation of weak orderings. In other words, we allow voters to express indifference. An inspection of Moulin's proof is

[1] The phrase "no-show paradox" was coined by Brams and Fishburn (1983). They note that paradoxes of this kind were described by the Royal Commission Appointed to Enquire into Electoral Systems (1910) and by Meredith (1913). The Royal Commission notes that under the Single Transferable Vote system a party may win two seats instead of one if some of its supporters abstain from the vote. Meredith, also discussing that voting system, describes a paradoxical election:

> Suppose that D (Nationalist), M ([Independent] Unionist) and Z (Unionist) are three [candidates] ... D has 410 votes, M 400, and Z 500. Then M is eliminated, and his votes may be supposed to be transferred to Z, who is elected. But if D were eliminated before M, we may easily suppose that his votes would go to M, who would be elected. The injustice of the result appears even more striking when we reflect that, if D had had 11 votes less, his supporters would have succeeded in returning M instead of Z, as they desired to do.

Those 11 ballot papers rank candidate Z in last place, below M, and yet their inclusion changes the winner from M to Z.

sufficient to confirm that his impossibility result continues to hold true in this case. It is unsurprising that expanding the domain of voting rules by permitting voter indifference does not lead to a possibility result. Typically, in social choice theory, the emergence of possibility is associated with the contraction of a domain rather than the expansion of one (see Gaertner (2001)).[2]

It is less obvious what happens to Pérez's *possibility* result when we allow voters to express indifference. Both the Min-Max rule and Young's rule can be adapted in natural ways to aggregate weak orderings. However, an implication of our theorem is that every possible adaption of either rule results in a rule that violates Condorcet's principle or generates both of the strong no-show paradoxes.

More generally, we show that, when there are four or more candidates and when voters may be indifferent between candidates, every Condorcet-consistent rule generates both of the strong no-show paradoxes.

2 Notation

Let A be a finite set of candidates, and let N_∞ be a finite or countably infinite set of potential voters. Every finite subset of N_∞ is called an electorate. Let $W(A)$ be the set of all weak orderings on A. By a weak ordering we mean a binary relation on A that is transitive and complete.

A *profile* assigns a weak ordering to each voter in an electorate. For every electorate N there is a set of possible profiles $W(A)^N$. We write u_{-i} to denote the profile obtained by removing individual i from profile u.

A voting rule is a function S that assigns a candidate to every possible pair of electorate and profile. So, given an electorate N and a profile u in $W(A)^N$, $S(N, u)$ is the winning candidate.[3]

Given electorate N and profile u in $W(A)^N$, let n_{ab} be the number of voters who prefer a to b less the number of voters who prefer b to a. Let m_a be the greatest value taken by n_{ba} over all b in $A \setminus \{a\}$. If $m_a > 0$ then m_a is the margin of a's greatest pairwise defeat. Candidate a is a Condorcet winner if and only if $m_a < 0$. And m_a is zero if a does not suffer any pairwise defeat but does tie with another candidate in a pairwise comparison.

[2]Nevertheless, Barberà (2007) shows that the introduction of indifferences can sometimes complicate the statement of both positive and negative results in social choice theory.

[3]A voting rule as defined here always takes a single candidate as its value. Ties are not permitted in the outcome. This is also the case in Moulin (1988). Moulin's theorem is extended to the case in which ties are permitted in the outcome by Jimeno, Pérez and García (2009). Their results are similar to some results found in the corresponding literature on extensions of the Gibbard-Satterthwaite theorem (see Taylor (2005) for an overview of that literature).

3 Theorem

We first define *Condorcet-consistency*.

Condorcet-consistent. For all candidates a, all electorates N and all profiles u in $W(A)^N$, $m_a < 0$ implies $S(N, u) = a$.

Next we define *Positive Involvement* and *Negative Involvement*. A voting rule that is free from one of the two strong no-show paradoxes will have one of these two properties. And a voting rule free from both paradoxes will have both properties.

Positive Involvement. For all electorates N that contain at least two voters, all voters i in N and all profiles u in $W(A)^N$, if i weakly prefers $S(N \setminus \{i\}, u_{-i})$ to all other candidates then i weakly prefers $S(N, u)$ to all other candidates.

The Positive Involvement criterion requires that if a candidate ranked in first place by individual i is elected when i does not participate, then a candidate ranked in first place by i should also be elected when i does participate.

Negative Involvement. For all electorates N that contain at least two voters, all voters i in N and all profiles u in $W(A)^N$, if i weakly prefers every candidate to $S(N, u)$ then i weakly prefers every candidate to $S(N \setminus \{i\}, u_{-i})$.

The Negative Involvement criterion requires that if a candidate ranked in last place by individual i is elected when i participates, then a candidate ranked in last place by i should also be elected when i does not participate.

We conclude by stating and proving the theorem.

Theorem. *(i) If there are three candidates or fewer then there are Condorcet-consistent voting rules that satisfy both Positive Involvement and Negative Involvement.*

(ii) If there are at least four candidates and at least 37 (34) potential voters then there is no Condorcet-consistent voting rule that satisfies Positive (Negative) Involvement.

Proof. Following Moulin (1988), we prove part (i) by giving the example of a voting rule that always elects a candidate from the Kramer set (see Kramer (1977)).

Given (N, u), let K, the Kramer set, be the set of all candidates a that minimise m_a. That is, $K = \{a \in A \mid m_a \leq m_b \text{ for all } b \in A\}$. If K is a singleton set then voting rule S elects that candidate in K. And if K

contains more than one candidate then S elects the candidate in K whose name comes first by lexicographic order.

If there is a Condorcet winner a then it follows that $m_a < 0$ and $m_b > 0$ for all other candidates b. Then $K = \{a\}$ and so a will be elected by S. So S is Condorcet-consistent. This is true no matter how many candidates there are.

We now establish that S satisfies Positive Involvement and Negative Involvement when there are three candidates. Let us label the candidates a, b and c. These labels mask the names of the candidates so that their lexicographic ordering by name is unknown to us. Suppose that S elects a at some profile. It follows that $m_a \leq m_b$ and $m_a \leq m_c$. Let us add a voter to this profile to form a new profile.

We consider three cases. The first case is that the new voter strictly prefers a to b and c. The second case is that the new voter weakly prefers b to a, and strictly prefers a to c. The final case is that the new voter weakly prefers both b and c to a.

In the first case m_a falls by one. Since we are adding just one new voter, all margins of pairwise defeat/victory can change by at most one. So m_b and m_c cannot fall by more than one. In other words, m_a remains equal to or falls below (or further below) each of m_b and m_c as a result of the additional voter. Therefore, the new Kramer set is a subset of the original Kramer set, and must contain a. Hence, S elects a at the new profile, consistent with both Positive Involvement and Negative Involvement.

In the second case m_c increases by one, and m_a cannot increase by more than one. In other words, m_c remains equal to or rises above (or further above) m_a. So c can only be in the new Kramer set if that set also contains a, and c was in the original Kramer set (implying that candidate a's name comes before c's by lexicographic order). Hence, S elects a or b at the new profile, as required by both Positive Involvement and Negative Involvement.

In the final case both Positive Involvement and Negative Involvement permit any outcome at the resulting profile, so no contradiction can arise.

When there are just two candidates then it is clear that simple majority rule with lexicographic tie-breaking will satisfy Positive Involvement and Negative Involvement.

We proceed to statement (ii) of the theorem.

Let us now assume that A contains at least four candidates and that S is Condorcet-consistent. We first prove that (iia) if N_∞ contains at least 37 potential voters then S does not satisfy Positive Involvement, and then we prove that (iib) if N_∞ contains at least 34 potential voters then S does not satisfy Negative Involvement. The proof of (iia) is based on the proof of statement (ii) in Moulin (1988).

By way of contradiction, assume that N_∞ contains at least 37 potential voters and S satisfies Positive Involvement. We make the following claim. For all distinct candidates a and b, every electorate N and every profile u in $W(A)^N$,

$$m_a + 1 \leq 37 - |N| \text{ and } m_a + 2 \leq n_{ab} \text{ implies } S(N, u) \neq b. \tag{1}$$

To prove (1), take any electorate N, a profile u in $W(A)^N$ and candidates a and b such that $m_a + 1 \leq 37 - |N|$ and $m_a + 2 \leq n_{ab}$ and assume that $S(N, u) = b$. Since S is Condorcet-consistent, a cannot be a Condorcet winner. Therefore, $m_a \geq 0$. Let M and v be an electorate and profile obtained by adding $m_a + 1$ voters to (N, u), all of whom rank b alone in first place and a alone in second place.

At (N, u) the greatest margin of defeat suffered by a is m_a, and a defeats b by a margin of at least $m_a + 2$. So the addition of $m_a + 1$ voters who all rank b alone in first place and a alone in second place results in all of a's pairwise defeats being reversed, while a continues to pairwise defeat b. So candidate a is a Condorcet winner at (M, v). However, since b is elected at (N, u), Positive Involvement requires that b is elected at (M, v). This contradiction establishes (1).

To complete the proof of (iia) we construct two profiles. Take any four candidates a, b, c and d. The first of the two profiles is described in Table 1. Each number above the horizontal line indicates the number of voters who

6	3	8	7
a	a	d	b
d	d	b	c
c	b	c	a
b	c	a	d

Table 1: A profile

have submitted the ranking below that number. All other candidates (if there are any) are ranked below a, b, c and d by the voters.

Figure 1 is a directed, weighted graph that indicates the margins of pairwise victory and defeat amongst the top four candidates. An edge is directed from b to a and carries a weight of 6 to indicate that b pairwise defeats a by a margin of 6, and so on.

There are 24 participating voters, and we have $m_a = 6$ and $n_{ad} = 8$ so, by (1), S cannot elect d. We have $m_d = 8$ and $m_{db} = 10$ so S cannot elect b. For every candidate x in $A \setminus \{a, b, c, d\}$ we have $m_a = 6$ and $n_{ax} = 24$ so S cannot elect x. Hence the winner must be a or c.

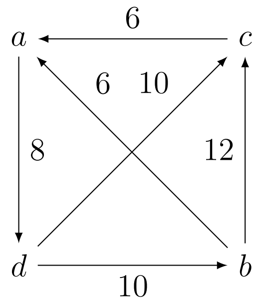

Figure 1: Margins of pairwise victory

Now let us add eight voters to that first profile to create a second profile. All eight of these new voters are indifferent between a and c, and rank those two candidates in joint first place. Their next most preferred candidate is b, followed by d, and they rank all other candidates (if there are any) below d.

Now the graph is as shown in Figure 2.

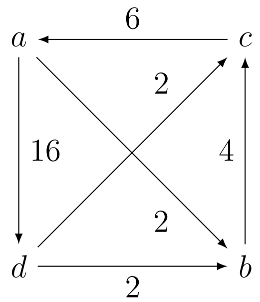

Figure 2: Eight new ballots

There are 32 participating voters, and we find that $m_c = 4$ and $n_{ca} = 6$, so, again by (1), S cannot elect a. We also have $m_b = 2$ and $n_{bc} = 4$, so S cannot elect c. However, Positive Involvement implies that S must elect a or c. This contradiction completes the proof of statement (iia).

Finally, we prove statement (iib).

Let us now assume that N_∞ contains at least 34 potential voters and that S satisfies Negative Involvement (and not necessarily Positive Involvement). Take any four candidates a, b, c and d. A profile is described in Table 2. Let us call this profile u. In this table we use Greek letters to label the six weak orderings that appear in the profile. We say that there are four α voters, five β voters and so on. The x in each column marks the position of all candidates x in $A \setminus \{a, b, c, d\}$ (if there are any), and it is written next to another letter

α	β	γ	δ	ϵ	ζ
4	5	1	6	4	6
b	a	d	a	d	b
c	c	a	d	b	d
a	d	b	b	c	c
dx	bx	cx	cx	ax	ax

Table 2: Another profile

to indicate indifference. For example, the four α voters are indifferent among the candidates in $A \setminus \{a, b, c\}$, and rank all of those candidates in last place.

Figure 3 indicates the margins of pairwise victory and defeat amongst the top four candidates.

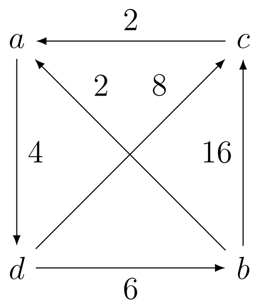

Figure 3: Graph corresponding to Table 2

There are 26 participating voters. If three of the four α voters are deleted from profile u then a's pairwise defeats to b and c are reversed and a becomes a Condorcet winner. So, by Negative Involvement, a candidate in $\{a, b, c\}$ must be elected at profile u. If instead we delete the five β voters then d becomes a Condorcet winner. So b (ranked in last by those voters) cannot be elected at profile u. The winner must be a or c.

Now let us add eight voters to create a final profile v. All eight of these new voters rank c alone in first place, a alone in second place, and rank all other candidates in joint last place.

Now the graph is as shown in Figure 4 (there is a pairwise tie between c and d).

There are 34 participating voters. If we delete the four ϵ voters and five of the six ζ voters then c becomes a Condorcet winner. So a cannot be elected at profile v. If instead we delete the single γ voter and the six δ voters then b becomes a Condorcet winner. So c cannot be elected at profile v. However, Negative Involvement implies that S must elect a or c.

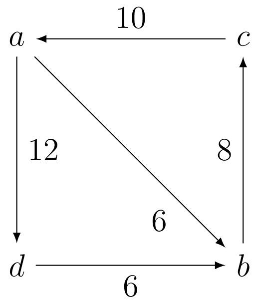

Figure 4: The final profile

This contradiction completes the proof of the theorem. □

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