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Abstract:

In this article we propose a quick, efficient, and easy method to detect whether a time series Y_t possesses any nonlinear feature. The advantage of our proposed nonlinearity test is that it is not required to know the exact nonlinear features and the detailed nonlinear forms of Y_t. Our proposed test could also be used to test whether the model, including linear and nonlinear, hypothesized to be used for the variable is appropriate as long as the residuals of the model being used could be estimated. Our simulation results show that our proposed test is stable and powerful while our illustration on Wolf's sunspots numbers is consistent with the findings from existing literature.

Keywords: **linearity, nonlinearity, U-statistics, Volterra expansion**

1 Introduction

It is well-known that nonlinearity is always observed in many time series like natural data and economic and financial time series, including some benchmark datasets such as the sunspot, Canadian lynx (Tong, 1990, 1995; Tjostheim, 1994), and inflation rate (Engle 1982). In practice, nonlinearity is common in both stationary or non-stationary time series. Nevertheless, detecting nonlinearity in time series is very important because very often academics and practitioners have to know this feature in the data before conducting their analysis. For example, Fourier analysis assumes the time series to be linear and stationary while, on the other hand, the wavelet analysis is raised for linear but nonstationary. Thus, before academics and practitioners apply Fourier analysis and/or wavelet analysis in their work, they have to examine whether there is any nonlinearity in the time series. This is only one of the many reasons why testing for nonlinearity is one of most important issues in time series analysis. There are many works on this area for stationary and nonstationary time series. In this paper we focus on developing a nonlinearity test for a stationary time series, which is often ignored by academics and practitioners especially in applied science such as finance and economics.

It is a growing interest in the testing, estimation, specification, and developing properties for nonlinearity for decades. Nonlinear features include asymmetric cycles, nonlinear relationship among the variables being studied and their lags, time irreversibility, sensitivity to initial conditions, and others. There are infinitely many nonlinear forms to be explored. The early development of nonlinear time series analysis focused on various nonlinear parametric forms (Tong, 1990; Tjotheim, 1994), including the ARCH-model (Engle, 1982; Bollerslev, 1986) and the threshold model (Tong, 1990; Tiao and Tsay, 1994). On the other hand, developments in nonparametric regression techniques provide an alternative to model nonlinear time series (Tjotheim, 1994; Yao and Tong, 1995 a,b; Härdle, Lütkepohl, and Chen 1997; Masry and Fan, 1997). Nonetheless, the most general form of a nonlinear stationary process is the Volterra expansion.

In particular, various tests for linearity have been proposed to illustrate the nonlinear nature of certain well-known processes (Subba Rao & Gabr, 1980; Hinich, 1982; Maravell, 1983; Hinich & Patterson, 1985) and to support the need for nonlinear time series models (Granger & Andersen, 1978). Keenan (1985) adopts the idea of Tukey's (1949) one degree of freedom test for nonadditivity to derive a time-domain statistic, as an alternative of the frequency-domain statistics, for example, bispectrum, for discriminating between nonlinear and linear models. Keenan's test is motivated by the similarity of Volterra expansions to polynomials. Tsay (1986) improves Keenan's test and obtains a more powerful test.

Since the number of parameters of the nonlinearity part could be very large, this could affect the performance of the existing nonlinear tests. In addition, nonlinearity may occur in many and could be infinitely ways. The advantage of our proposed nonlinearity test is that it is not required to know the exact nonlinear features and the detailed nonlinear forms of a time series. Under the null hypothesis of linearity, residuals of an appropriate linear model should be independent, in this paper we use this idea to develop a new nonlinearity test to check whether there is any nonlinearity in a time series. As a nonlinear phenomenon is typically more complex and more difficult to model than a linear one, so it is not reasonable to restrict the form of nonlinearities. The objective in this paper is to circumvent the limitation of Volterra expansion or other similar approaches that result in many parameters in the estimation by developing a new method to test the nonlinearity for a time series that do not involve many parameters.

We note that our test could not only be used to detect any nonlinearity for the variable being examined. Our test could also be used to test whether the hypothesized model, including linear and nonlinear, to the variable being examined is appropriate as long as the residuals of the model being used could be estimated. We will discuss this feature more in the conclusion section.

The result of our simulation shows that Tsay's test is more powerful than our proposed test in a region while our test is more powerful in another region. We note that this finding is not surprised because nonlinearity may occur in many ways and thus there may not exist any single test that could dominate the others in detecting nonlinearity. However, our simulation shows that our proposed test has three desirable features when comparing with Tsay's test: our proposed test is more stable, the power of our proposed test increases while that of Tsay's test could decrease when the magnitude of parameter increases, and the power of our proposed test reaches one quickly while that of Tsay's test could not reach one when the magnitude of parameter increases. Thus, the result of our simulation supports our claim that our proposed test is a more desirable test.

At last, we apply both Tsay's test and the nonlinearity test we developed in this paper to test whether there is any nonlinear feature in the sunspots data, one of the most typical nonlinear cases. Our findings show that both our proposed test and Tsay's test draw the same conclusion that there exists nonlinearity in the Wolf's sunspots numbers.

The remainder of the paper is organized as follows. In Section 2, we first discuss the Volterra expansion and state the nonlinearity test developed by Tsay (1986). Thereafter, we develop our proposed new nonlinearity test to circumvent the limitation of Volterra expansion. In Section 3, we illustrate the superiority of the nonlinearity test we developed in Section 2 by conducting simulation to examine its performance over that of the test developed by Tsay (1986). In Section 4, we illustrate the applicability of our proposed nonlinearity test by applying it to examine whether there is any nonlinear feature in the sunspots data and compare the result with Tsay's test. Section 5 wraps up the paper by providing several well-grounded observations.

2 Theory

A purely stochastic time series model for Y_t is a function of an independent and identically-distributed (iid) sequence containing the current and past shocks; that is, $Y_t = f(\varepsilon_t, \varepsilon_{t-1}, \cdots)$. If $f(\cdot)$ is a linear function of its arguments, the model is linear. On the other hand, if there exists any nonlinearity in $f(\cdot)$, the model is nonlinear. One of the most commonly used linear models is an ARMA process that could be presented as an MA representation or AR equation (Box, Jenkins, and Reinsel, 1994).

A nonlinear phenomenon is typically more complex and more difficult to model than a linear one, and the available tools are not comprehensive and ineffective. There are many approaches, for example, parametric, semi-parametric, and nonparametric approaches, to identify an appropriate form in nonlinear modeling. Also, there are several nonlinearity tests available, which may be divided into two categories: portmanteau tests, which test for departure from linear models without specifying alternative models, and the tests designed for some specific alternatives. Recently, the tests that make use of nonparametric and semiparametric fitting have received considerable attention. For example, Fan and Yao (2003) introduce a likelihood ratio test for a linear model against a TAR alternative with two regimes introduced by Chan and Tong (1990) and Chan (1990b). Although the test is designed for a specified alternative, it may be applied to test for a departure to a general smooth nonlinear function since a piecewise linear function will provide a better approximation than that from a global linear function. In addition, Cox (1981) suggests to use quadratic or cubic regression for testing nonlinearity. The tests could be parametric or nonparametric statistics. The Ljung-Box statistics of squared residuals, the bispectral test, and the Brock, Dechert, and Scheinkman (BDS) test are nonparametric methods. The RESET test (Ramsey, 1969), the F tests of Tsay (1986, 1989), and other Lagrange multiplier and likelihood ratio tests depend on specific parametric functions.

One of the most commonly used approaches is to apply Volterra expansion (Wiener, 1958; Brillinger, 1970) to expand a nonlinear and stationary time series, say, Y_t, to be in terms of the linear, quadratic, cubic, etc. such that

$$Y_t = \mu + \sum_{-\infty}^{\infty} a_u \varepsilon_{t-u} + \sum_{u,v=-\infty}^{\infty} a_{uv} \varepsilon_{t-u} \varepsilon_{t-v} + \sum_{u,v,w=-\infty}^{\infty} a_{uvw} \varepsilon_{t-u} \varepsilon_{t-v} \varepsilon_{t-w} + \cdots \,, \qquad (1)$$

where $\{\varepsilon_t, -\infty < t < \infty\}$ is a strictly stationary process of independent and identically distributed random variables with mean zero.

If the null hypothesis of linearity is true, residuals of an appropriate linear model should be independent. For example, if the model by using the linear and quadratic terms of the Volterra expansion is the right model, any violation of independence in the residuals reveals that the hypothesized model with the linearity assumption is not appropriate. This is the basic idea used in the development of various nonlinearity tests.

2.1 Tsay's F Test

Tsay (1986) develops a nonlinearity test based on the idea of using the Volterra expansion. His test is popular and is well-known to have decent power on detecting nonlinearity in a sequence, say, $\{Y_t\}$. Thus, we first discuss his test in our paper. The null hypothesis is that

$$H_0 \quad : \quad \text{there is no nonlinearity in the time series being examined.} \tag{2}$$

Readers may refer to Tsay (1986) for more details. The test mainly consists three steps:

Step 1: Regress Y_t on $\{1, Y_{t-1}, \cdots, Y_{t-M}\}$ by least squares and obtain the residuals $\{\hat{e}_t\}$, for $t = M+1, \cdots, T$. The regression model is denoted by $Y_t = \mathbf{W}_t\mathbf{\Phi} + e_t$, where $\mathbf{W}_t = (1, Y_{t-1}, \cdots, Y_{t-M})$, M is a pre-specified positive integer, and T is the length of sequence $\{Y_t\}$.

Step 2: Regress the vector $\mathbf{Z}_t$ on $\{1, Y_{t-1}, \cdots, Y_{t-M}\}$ and obtain the residual vector $\{\hat{\mathbf{X}}_t\}$, for $t = M+1, \cdots, T$. Here, the multivariate regression model is $\mathbf{Z}_t = \mathbf{W}_t\mathbf{H} + \mathbf{X}_t$, where $\mathbf{Z}_t$ is an $M^* = \frac{1}{2}M(M+1)$ dimensional vector defined by $\mathbf{Z}_t^T = \text{vech}(\mathbf{V}_t^T\mathbf{V}_t)$ with $\mathbf{V}_t = (Y_{t-1}, \cdots, Y_{t-M})$ and vech denotes the half stacking vector. In other words, $\mathbf{Z}_t^T$ is obtained from the symmetric matrix $\mathbf{V}_t^T\mathbf{V}_t$ by the usual column stacking operator and using only those elements on or below the main diagonal of each column.

Step 3: Regress $\hat{e}_t$ on $\hat{\mathbf{X}}_t$ and let $\hat{F}$ be the F ratio of the mean square of regression to the mean square of error. That is, fit $\hat{e}_t = \hat{\mathbf{X}}_t\boldsymbol{\beta} + \varepsilon_t, (t = M+1, \cdots, T)$ and obtain the Tsay's test by

$$\hat{F} = \{(\sum \hat{\mathbf{X}}_t \hat{e}_t)(\sum \hat{\mathbf{X}}_t^T \hat{\mathbf{X}}_t)^{-1}(\sum \hat{\mathbf{X}}_t^T \hat{e}_t)/M^*\}/\{\sum \hat{\varepsilon}_t^2/(T-M-M^*-1)\} \ , \quad (3)$$

where the summations are over t from $M+1$ to T.

Under the null hypothesis of linearity and for large T, the statistic $\hat{F}$ follows approximately a F-distribution with degrees of freedom $\frac{1}{2}M(M+1)$ and $T - \frac{1}{2}M(M+3) - 1$. Thus, for test level α, one could reject the null hypothesis of linearity if

$$\hat{F} > F_{\left(\frac{1}{2}M(M+1), T-\frac{1}{2}M(M+3)-1\right)}(\alpha) \ . \quad (4)$$

2.2 New Non-Linearity Test

The major drawback of applying the Volterra expansion is that the number of parameters is too large. To circumvent the limitation, one could assume a_u, a_{uv}, and a_{uvw} in equation in (1) to be functions of small numbers of parameters. However, the problem of this approach is that we do not know the forms of "functions" and in fact, such "functions" may not exist. Thus, in this paper we introduce another approach to circumvent the limitation of the Volterra expansion of getting too many parameters. It is not necessary to assume any form of nonlinear function for Y_t for our proposed test. To identify any nonlinearity of time series $\{Y_t\}$, it is common that an AR model is used to remove any serial correlation in the data (Tsay, 1986), and thereafter apply the nonlinearity tests to the residual series of the model. In this paper we follow this suggestion to first fit the following linear AR model like the first step in Tsay's F test to identify its linearity:

$$Y_t = \sum_{i=1}^{p} \phi_i Y_{t-i} + e_t \ , \quad (5)$$

where $\varepsilon_t \sim \mathrm{WN}(0, \sigma^2)$ and WN stands for 'white noise.' After removing the linear components in $\{Y_t\}$ by introducing the linear model in (5), we proceed to examine whether there is any remaining incremental power from time t to the later time $t+h$ in the residuals sequence. If such power is identified, the model is concluded that there exists nonlinear feature in the corresponding residuals, $\{\hat{e}_t\}$. We are using this concept to develop a nonlinearity test to the residual series $\{\hat{e}_t\}$ of the variables being studied to examine whether there is any remaining nonlinearity in the residuals. For simplicity, we denote Y_t to be the corresponding residuals of the variable being examined. We first state the following definition:

Definition 2.1 *For any strictly stationary and weakly dependent series $\{Y_t\}$, the m-length lead vector of Y_t is given by*

$$Y_t^m \equiv \big(Y_t, Y_{t+1}, \cdots, Y_{t+m-1}\big),\ m = 1, 2, \cdots,\ t = 1, 2, \cdots$$

and L_y-length lag vector of Y_t is defined as

$$Y_{t-L_y}^{L_y} \equiv \big(Y_{t-L_y}, Y_{t-L_y+1}, \cdots, Y_{t-1}\big), L_y = 1, 2, \cdots, t = L_y + 1, L_y + 2, \cdots .$$

In addition, we define

$$Y_{t-L_y}^{m+L_y} \equiv \big(Y_{t-L_y}, \cdots, Y_{t-1}, Y_t, Y_{t+1}, \cdots, Y_{t+m-1}\big), L_y = 1, 2, \cdots, t = L_y + 1, L_y + 2, \cdots .$$

Series $\{Y_t\}$ ***does not possess any nonlinearity*** *if and only if*

$$Pr\left(\|Y_t^m - Y_s^m\| < e \,\Big|\, \| Y_{t-L_y}^{L_y} - Y_{s-L_y}^{L_y} \| < e\right) = Pr\left(\|Y_t^m - Y_s^m\| < e\right) ,$$

where $Pr(\cdot\,|\,\cdot)$ denotes conditional probability and $\|\cdot\|$ denotes the maximum norm which is defined as

$$\|X - Y\| = max\big(|x_1 - y_1|, |x_2 - y_2|, \cdots, |x_n - y_n|\big) ,$$

for any two vectors $X = \big(x_1, \cdots, x_n\big)$ and $Y = \big(y_1, \cdots, y_n\big)$.

In addition, we define

$$\begin{aligned} C_1\big(m + L_y, e\big) &\equiv Pr\left(\| Y_{t-L_y}^{m+L_y} - Y_{s-L_y}^{m+L_y} \| < e\right) \\ C_2\big(L_y, e\big) &\equiv Pr\left(\| Y_{t-L_y}^{L_y} - Y_{s-L_y}^{L_y} \| < e\right) \\ C_3\big(m, e\big) &\equiv Pr\left(\|Y_t^m - Y_s^m\| < e\right) . \end{aligned}$$

Because

$$
\begin{aligned}
&Pr\left(\| Y_t^m - Y_s^m \|< e \,\Big|\, \| Y_{t-L_y}^{L_y} - Y_{s-L_y}^{L_y} \|< e\right)\\
=&\frac{Pr\left(\| Y_t^m - Y_s^m \|< e, \| Y_{t-L_y}^{L_y} - Y_{s-L_y}^{L_y} \|< e\right)}{Pr\left(\| Y_{t-L_y}^{L_y} - Y_{s-L_y}^{L_y} \|< e\right)}\\
=&\frac{C_1\big(m+L_y,e\big)}{C_2\big(L_y,e\big)} \quad ,
\end{aligned}
$$

when one tests the existence of the nonlinearity of a sequence $\{Y_t\}$, instead of testing the linearity hypothesis stated in (2), one could test the following hypothesis:

$$
H_0 : \frac{C_1\big(m+L_y,e\big)}{C_2\big(L_y,e\big)} - C_3\big(m,e\big) = 0\,. \tag{6}
$$

$\{Y_t\}$ is said to possess nonlinearity if the hypothesis H_0 in (6) is rejected.

Under Definition 2.1, the nonlinearity test statistic is given by

$$
\sqrt{n}\left(\frac{C_1\big(m+L_y,e,n\big)}{C_2\big(L_y,e,n\big)} - C_3\big(m,e,n\big)\right)\,, \tag{7}
$$

where

$$
\begin{aligned}
C_1\big(m+L_y,e,n\big) &\equiv \frac{2}{n(n-1)}\sum\sum_{t<s} I\left(y_{t-L_y}^{m+L_y}, y_{s-L_y}^{m+L_y}, e\right)\,,\\
C_2\big(L_y,e,n\big) &\equiv \frac{2}{n(n-1)}\sum\sum_{t<s} I\left(y_{t-L_y}^{L_y}, y_{s-L_y}^{L_y}, e\right)\,,\\
C_3\big(m,e,n\big) &\equiv \frac{2}{n(n-1)}\sum\sum_{t<s} I\left(y_t^m, y_s^m, e\right)\,, \text{and}\\
I(x,y,e) &= \begin{cases} 0, & \text{if } \|x-y\| > e \\ 1, & \text{if } \|x-y\| \le e \end{cases}\,.
\end{aligned}
$$

Here, $t, s = L_y + 1, \cdots, T - m + 1$, $n = T + 1 - m - L_y$, T is the length of sequence Y_t. The test statistic possesses the following property:

Theorem 2.1 *For given values of m, L_y, and $e > 0$ defined in Definition 2.1 and under the assumptions that $\{Y_t\}$ is strictly stationary, weakly dependent, and satisfies* *the conditions stated in Denker and Keller [13], if $\{Y_t\}$ does not possess any nonlinear feature, then the test statistic defined in (7) is distributed as $N\big(0, \sigma^2(m, L_y, e)\big)$ asymptotically. When the test statistic in (7) is too far away from zero, we reject the linearity null hypothesis defined in (2) or (6). A consistent estimator of the variance $\sigma^2(m, L_y, e)$ follows:*

$$\hat{\sigma}^2\big(m, L_y, e\big) = \widehat{\nabla f(\theta)}^T \cdot \hat{\Sigma} \cdot \widehat{\nabla f(\theta)}^T ,$$

in which:

$$\widehat{\nabla f(\theta)} = \left[\frac{1}{\hat{\theta}_2}, \frac{-\hat{\theta}_1}{\hat{\theta}_2^2}, -1\right]^T = \left[\frac{1}{C_2\big(L_y, e, n\big)}, -\frac{C_1\big(m + L_y, e, n\big)}{C_2^2\big(L_y, e, n\big)}, -1\right]^T ,$$

each component $\Sigma_{i,j}$ $(i, j = 1, 2, 3)$, of the covariance matrix Σ is given by:

$$\Sigma_{i,j} = 4 \cdot \sum_{k \geq 1} \omega_k E(A_{i,t} \cdot A_{j,t+k-1}),$$

$$\omega_k = \begin{cases} 1 & \textit{if } k = 1 \\ 2, & \textit{otherwise} \end{cases} ,$$

$$\begin{aligned} A_{1,t} &= h_{11}\left(y_{t-L_y}^{m+L_y}, e\right) - C_1(m + L_y, e)\,, \\ A_{2,t} &= h_{12}\left(y_{t-L_y}^{L_y}, e\right) - C_2(L_y, e)\,, \\ A_{3,t} &= h_{13}\left(y_t^m, e\right) - C_3(m, e)\,, \end{aligned}$$

where $\mathbf{z}_t = Y_{t-L_y}^{m+L_y}$, and $h_{1i}(\mathbf{z}_t)$, $i = 1, \cdots, 3$, is the conditional expectation of $h_i(\mathbf{z}_t, \mathbf{z}_s)$ given the value of $\mathbf{z}_t$ as follows:

$$\begin{aligned} h_{11}\left(y_{t-L_y}^{m+L_y}, e\right) &= E\big(h_1 \,\big|\, y_{t-L_y}^{m+L_y}\big)\,, \\ h_{12}\left(y_{t-L_y}^{L_y}, e\right) &= E\big(h_2 \,\big|\, y_{t-L_y}^{L_y}\big)\,, \\ h_{13}\left(y_t^m, e\right) &= E\big(h_3 \,\big|\, y_t^m\big)\,. \end{aligned}$$

Moreover, a consistent estimator of $\Sigma_{i,j}$ elements is given by:

$$\hat{\Sigma}_{i,j} = 4 \cdot \sum_{k=1}^{K(n)} \omega_k(n) \left[\frac{1}{2(n-k+1)} \sum_t \Big(\hat{A}_{i,t}(n) \cdot \hat{A}_{j,t-k+1}(n) + \hat{A}_{i,t-k+1}(n) \cdot \hat{A}_{j,t}(n)\Big)\right] ,$$

in which $t = L_y + k, \cdots, T - m + 1$*,* $K(n) = [n^{1/4}]$*,* $[x]$ *is the integer part of* x*,*

$$\begin{aligned}
\omega_k(n) &= \begin{cases} 1, & \textit{if } k = 1 \\ 2\big(1 - [(k-1)/K(n)]\big), & \textit{otherwise} \end{cases}, \\
\hat{A}_{1,t} &= \frac{1}{n-1}\left(\sum_{s \neq t} I\big(Y_{t-L_y}^{m+L_y}, Y_{s-L_y}^{m+L_y}, e\big)\right) - C_1\big(m + L_y, e, n\big), \\
\hat{A}_{2,t} &= \frac{1}{n-1}\left(\sum_{s \neq t} I\big(Y_{t-L_y}^{L_y}, Y_{s-L_y}^{L_y}, e\big)\right) - C_2\big(L_y, e, n\big), \\
\hat{A}_{3,t} &= \frac{1}{n-1}\left(\sum_{s \neq t} I\big(Y_t^m, Y_s^m, e\big)\right) - C_3\big(L_y, e, n\big), \\
& \quad t, s = L_y + 1, \cdots, T - m + 1 .
\end{aligned}$$

The hypothesis H_0 is rejected at level α if

$$\left| \sqrt{n}\left(\frac{C_1\big(m + L_y, e, n\big)}{C_2\big(L_y, e, n\big)} - C_3\big(m, e, n\big)\right)\right| > N\Big(\frac{\alpha}{2}; 0, \hat{\sigma}^2(m, L_y, e)\Big),$$

and, in this situation, Y_t is concluded to possess nonlinearity.

3 Simulation

In this section, we illustrate the superiority of the nonlinearity test we have developed in Section 2 by conducting simulation to examine its performance over that of the test developed by Tsay (1986). For simplicity, we call the test developed by Tsay (1986) "Tsay test" and the test developed in this paper "BHW test."

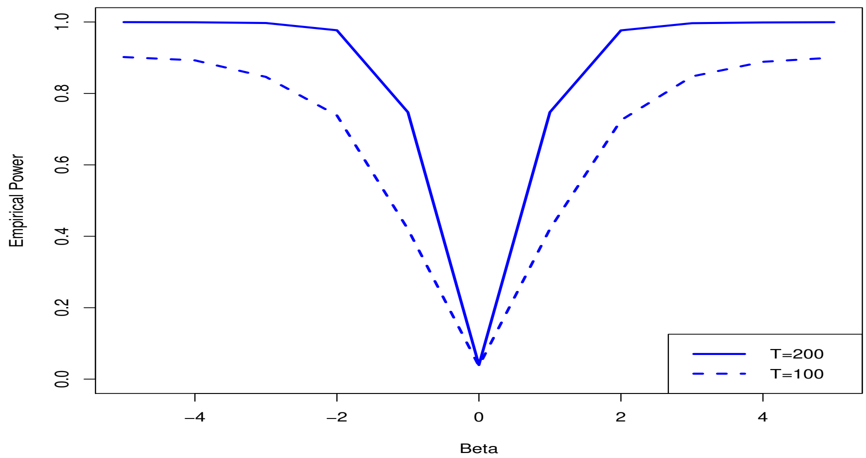
As Volterra expansion in (3) is one of the most commonly used forms for a nonlinear and stationary time series while threshold autoregressive model is another popular method in nonlinear analysis, in this paper we will use the following two models in our simulation:

$$\begin{aligned}
&\text{Model A :} \quad Y_t = \varepsilon_t + \beta \varepsilon_{t-1}\varepsilon_{t-2} \quad , \quad \text{and} \\
&\text{Model B :} \quad Y_t = \begin{cases} -\beta Y_{t-1} + \varepsilon_t & Y_{t-1} \geq 0 \\ \beta Y_{t-1} + \varepsilon_t & Y_{t-1} < 0 \end{cases},
\end{aligned} \tag{8}$$

where $\{\varepsilon_t\}$ is assumed to be iid $N(0,1)$ for both Models A and B and $|\beta| < 1$ for Model B. Readers may refer to Tsay (1986) for more information about Model A and we modify a simple threshold autoregressive model in Fan and Yao (2003) to get Model B. Both models are stationary, weakly dependent, and satisfy the conditions stated in Denker and Keller [13] and thus we can use the test statistic defined in (7) to conduct our simulation. We use 10000 replications to generate different samples in our simulation to examine the performance of our test with Tsay's test.

Let R be the times of rejection the null hypothesis that Y_t does not possess any nonlinearity in the 10000 replications at level 5% and the empirical power is then $\frac{R}{10000}$. To conduct simulation, we let $L_y = m = 1$ and $e = 1.5$ for the BHW test and let $M = 4$ for the Tsay's F test, this is the same M used in Tsay (1986) in his simulations.

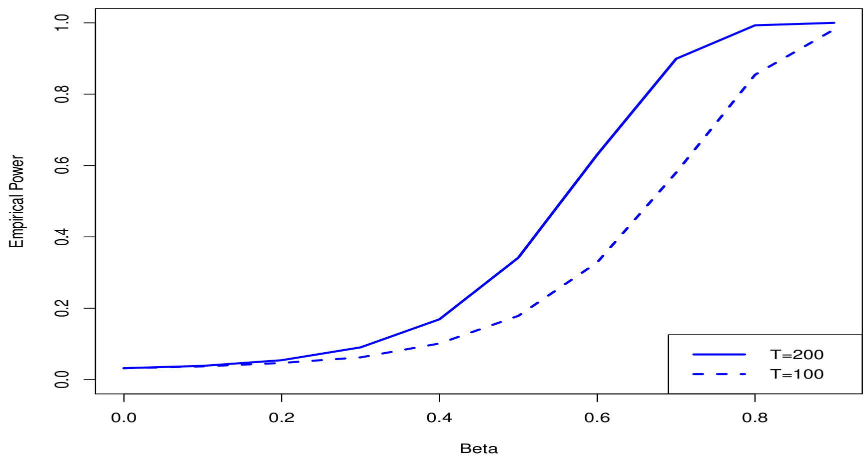
Figure 1: Empirical Power of the BHW Test for different values of β in Model A.



Note: The solid line and dotted line show the power of the BHW Test for different values of β in Model A for the sample size $T = 100$ and 200, respectively. Simulation is conducted with the test level $\alpha = 5\%$ and 10000 replications.

We first conduct simulation for the BHW test for the sample size $T = 100$ and 200 for both Models A and B. The results are plotted in Figures 1 and 2, respectively. For Model B, we only conduct simulation for $\beta \geq 0$ due to the symmetry property of the model. From both Figures 1 and 2, our findings show that (1) for both $T = 100$ and 200, our test gets higher power when nonlinear feature weights more in absolute values, (2) for any β, the empirical power increases as the length T increases, and (3) when $T = 200$, our test's

Figure 2: Empirical Power of the BHW Test for different values of β in Model B.



Note: The solid line and dotted line show the power of the BHW Test for different values of β in Model A for the sample size $T = 100$ and 200, respectively. Simulation is conducted with the test level $\alpha = 5\%$ and 10000 replications.

power quickly reach 1, inferring that our test is powerful and stable.

We turn to compare the power and size of our test with those of Tsay's test for different values of β in Models A and B as shows Figures 3 and 4 and Tables 1 and 2, respectively. For Model A, we observe from Figure 3 and Table 1 that Tsay's test is more powerful than our proposed test for $0 < |\beta| < 1$ whereas our proposed test is much more powerful than Tsay's test when $|\beta| > 1$. However, our simulation shows that (1) the empirical power of Tsay's test decreases sharply when $|\beta| > 1$ and (2) it decreases further when the magnitude of $|\beta|$ increases further after 1 and becomes stabilized at power below 0.4 when $|\beta| > 2$. On the contrary, the empirical power of our proposed test increases steadily as nonlinear weight $|\beta|$ increases, and quickly increases to 1 when the length $T = 200$. This shows that our proposed test is more stable than Tsay's test. For Model B, the conclusion drawn from the results of our simulation are similar to those for Model A: (1) Tsay's test is more powerful than our proposed test when $|\beta| < 0.65$ while our proposed test is more powerful afterward for both $T = 100$ and 200, and (2) the empirical power of Tsay's test decreases sharply when $|\beta| > 0.65$ and decreases further when the magnitude

Table 1: Powers of the BHW and Tsay tests for different values of β in Model A

	$T = 100$		$T = 200$	
β	BHW test	Tsay test	BHW test	Tsay test
-5	0.9018	0.2694	0.9996	0.3054
-4	0.8926	0.2667	0.9992	0.3157
-3	0.8462	0.2854	0.9971	0.3419
-2	0.7384	0.3403	0.9768	0.4467
-1	0.4199	0.5971	0.7473	0.8533
0	0.0367	0.0543	0.0395	0.0495
1	0.4205	0.5958	0.7475	0.8473
2	0.7248	0.3434	0.9765	0.4503
3	0.8475	0.2804	0.9966	0.3412
4	0.8885	0.2541	0.9987	0.312
5	0.9003	0.2641	0.9994	0.303

Simulation is conducted with the test level $\alpha = 5\%$ and 10000 replications, sample size $T = 100$ and 200. Model A is defined in (8).

of β increases further whereas the empirical power of our proposed test increases steadily as β increases, and quickly increases to 1 for both $T = 100$ and 200. Thus, our proposed test is more stable than Tsay's test and is more powerful for large magnitude of β. We note that because nonlinearity may occur in many ways, there may not exist any single test that could dominate the others in detecting nonlinearity. Thus, we are not surprised that Tsay's test is more powerful than ours in a region while our test is more powerful in another region. Nonetheless, to be stable is one of the most important features for a test statistic and since our proposed test more stable than Tsay's. In addition, the power of our proposed test reaches one quickly when the magnitude of β increases is a desirable property while the power of Tsay's test is decreasing when the magnitude of β increases is not a desirable feature. At last, from Tables 1 and 2, we notice that the size of our proposed test is less than 5% while, in general, the size of Tsay's test is more than 5% when the level of significance is 5%. This infers that in general Tsay's test slightly overreject the null hypothesis whereas our proposed test slightly underreject the null hypothesis of linearity when there is no nonlinearity. Underrejection the null hypothesis when the null

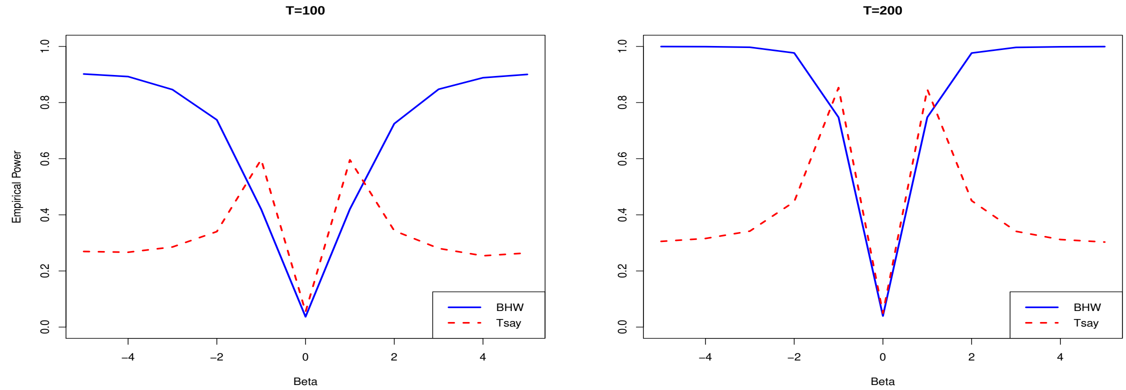
Table 2: Powers of the BHW and Tsay tests for different values of β in Model B

	$T = 100$		$T = 200$	
β	BHW test	Tsay test	BHW test	Tsay test
0	0.0318	0.0528	0.032	0.0525
0.1	0.037	0.0624	0.0385	0.068
0.2	0.0466	0.0908	0.0543	0.1335
0.3	0.0623	0.1401	0.0903	0.2714
0.4	0.1011	0.2251	0.169	0.4728
0.5	0.1786	0.3274	0.3418	0.6685
0.6	0.3285	0.4317	0.6307	0.8168
0.7	0.5802	0.5204	0.8992	0.8839
0.8	0.8544	0.5352	0.993	0.8963
0.9	0.9811	0.4395	0.9999	0.8004

Simulation is conducted with the test level $\alpha = 5\%$ and 10000 replications, sample size $T = 100$ and 200. Model B is defined in (8).

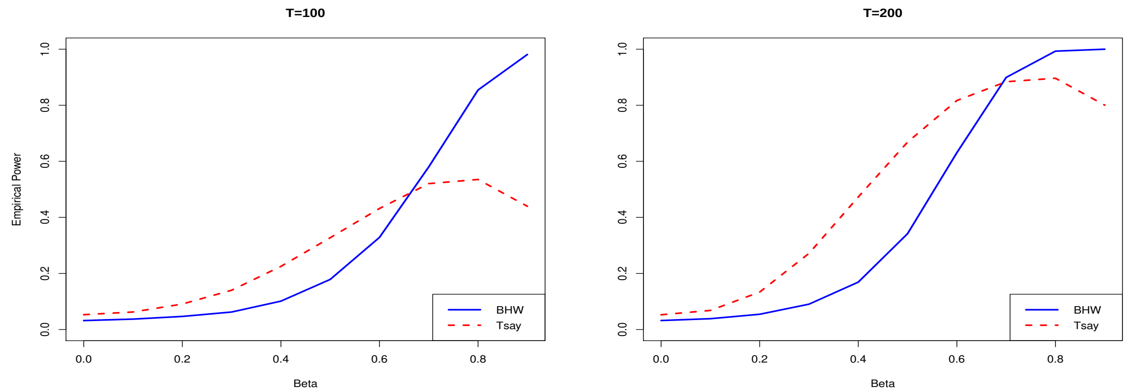
hypothesis is true is not a non-desirable property and thus, together with the features of our proposed test we discussed before, we could claim that our test is a more desirable test.

Figure 3: Empirical Power of Tsay's and BHW's Tests for different values of β in Model A.



Note: The left panel shows the plot for the sample size $T = 100$ and the right panel displays the plot for $T = 200$. The solid line exhibits the BHW's test while the dashed line shows the power of Tsay's Test for different values of β in Model B. Simulation is conducted with the test level $\alpha = 5\%$ and 10000 replications.

Figure 4: Empirical Power of Tsay's and BHW's Tests for different values of β in Model B.

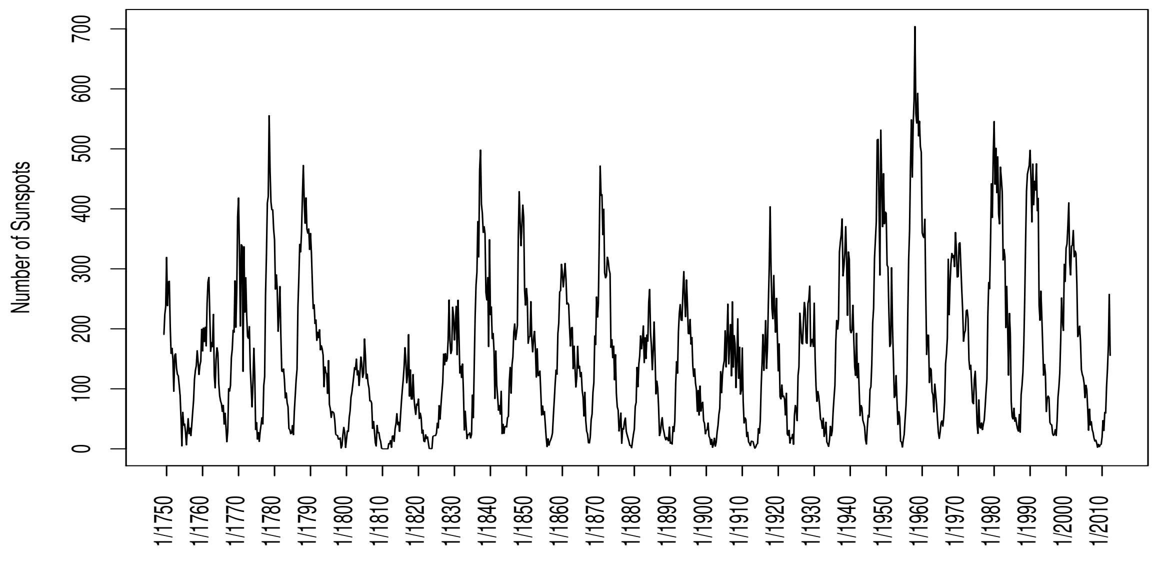


Note: The left panel shows the plot for $T = 100$ and the right panel displays the plot for $T = 200$. The solid line exhibits the BHW's test while the dashed line shows the power of Tsay's Test for different values of β in Model A. Simulation is conducted with the test level $\alpha = 5\%$ and 10000 replications.

4 Illustration

In this section, we illustrate the applicability of the nonlinearity test we have developed in Section 2 by applying both Tsay's test and our proposed nonlinearity test to test whether there exists any nonlinear feature in the sunspots data. Dark spots on the surface of the sun have consequences in the overall evolution of its magnetic oscillation. They also relate to the motion of the solar dynamo. The Zurich series of sunspot relative numbers is one of the most commonly analyzed in the literature. Izenman (1983) attributes the origin and subsequent development of the Zurich series to Johann Rudolf Wolf (1816-1893) who introduced a formula for calculating the sunspots numbers, which is given by $R = k(10g + f)$, where g is the number of groups of sunspots, f is the total number of individual spots and k is a constant for the observations. Thus, to honor the contribution by Johann Rudolf Wolf, it is common to call sunspots number "Wolf's sunspots number."

Figure 5: Wolf's Sunspots Numbers



Note: Quarterly Wolf's sunspots numbers from first quarter of 1749 to first quarter of 2012.

The earliest linear model built for these data is probably done by Yule (1927) who first introduces the class of linear autoregressive models to analyze the data. Since then, the literature of linear time series analysis of these data has been growing almost exponentially. Moran (1954), Schaerf (1964), Craddock (1967), Box, Jenkins, and Reinsel (1994). Bloomfield (1976), and Akaike (1978) are only some among many others. However, some works, see, for example, Tong and Lim (1980), Ghaddar (1980), and Lim (1981), point out that linear model is not adequate for fitting the data and forecasting.

In this paper we illustrate the applicability of our proposed test and Tray's test to examine the nonlinearity in the quarterly Wolf's sunspot numbers from the first quarter of 1749 to the first quarter of 2012. Let Y_t be Wolf's quarterly sunspots numbers from the first quarter of 1749 to the first quarter of 2012, we draw its time series plot in Figure 5. We first discuss how to use our test statistic to examine whether there is any nonlinearity in $\{Y_t\}$. To do so, as we discussed in Section 2, we first fit the data by using the following AR(p) model:

$$Y_t = \sum_{i=1}^{p} \phi_i Y_{t-i} + e_t,\ \ e_t \sim \mathrm{WN}(0, \sigma^2) \tag{9}$$

to the sunspot data. We find that the "best" linear model for the sunspot data is

$$\begin{aligned} Y_t &= 19.88492 - 0.70514 Y_{t-1} - 0.1549 Y_{t-2} - 0.18732 Y_{t-3} - 0.0834 Y_{t-4} \\ &\quad +0.10553 Y_{t-6} + 0.07121 Y_{t-7} + 0.08101 Y_{t-9} + e_t \end{aligned} \tag{10}$$

and the detailed results are exhibited in Table 1. Thereafter, we apply the Ljung-Box test to test whether the autocorrelations up to lag k for the residuals are zero and display the results in Table 2. In addition, we plot the autocorrelations of the residuals in Figure 6. The results from Table 2 and Figure 6 show that the autocorrelations of the residuals are not significantly different from zero for any lag up to 42 and thus one may conclude that the AR model in (10) is adequate and there is no other linear relationship remained in the residuals.

One may believe that the linear model in (10) is appropriate and it could explain the sunspot data well. To check whether this is true, we apply the test we have developed in Section 2 to examine whether there is any nonlinearity in the standardized residuals

Table 3: The Results of the Linear AR Model

Parameter	Estimate	Standard Error	t Value
intercept	19.88492	2.28728	8.694***
Y_{t-1}	0.70293	0.03056	23.004***
Y_{t-2}	0.15452	0.03756	4.114***
Y_{t-3}	0.18726	0.03785	4.948***
Y_{t-4}	0.08835	0.03539	2.497**
Y_{t-6}	-0.10490	0.03538	-2.965***
Y_{t-7}	-0.07221	0.03466	-2.083**
Y_{t-9}	-0.08301	0.02474	-3.355***

Note: This table exhibits the results of the linear AR model as shown in (10).
*, **, and *** mean significant at levels 10%, 5%, and 1%, respectively.

(that is, $\left(\hat{e}_t - \text{mean}(\hat{e}_t)\right)/\sqrt{\text{var}(\hat{e}_t)}$) obtained from fitting the linear models in (9), To do so, we use $L_y = m = 1$ and $e = 1.5$ for our proposed test, as the same values being used our simulation. The p value of the BHW test is $6.98374e^{-7}$, which strongly reveals nonlinearity within the residuals. Thus, applying our test, one could realize that there still exists nonlinearity component in the sunspot data. This result is consistent with the findings by Tong and Lim (1980), Tong (1983), and many others. In addition, we use Tsay's test to detect the nonlinearity in the Wolf's Sunspots numbers. Its p value is $3.541611e^{-14}$, inferring that both our proposed test and Tsay's test draw the same conclusion that there exists nonlinearity in the Wolf's Sunspots numbers.

5 Conclusion

Academics are interested in developing nonlinearity tests that could be divided into two categories: portmanteau tests, which test for departure from linear models without specifying alternative models, and the tests designed for some specific alternatives. More recently, the tests that make use of nonparametric and semiparametric fitting have received considerable attention (Fan and Yao, 2003). They introduce a likelihood ratio test for a linear model against a TAR alternative with two regimes introduced by Chan and Tong (1990) and Chan (1990b). Although the test is designed for a specified alternative,

Table 4: Autocorrelation Check: The Result of Ljung-Box Test

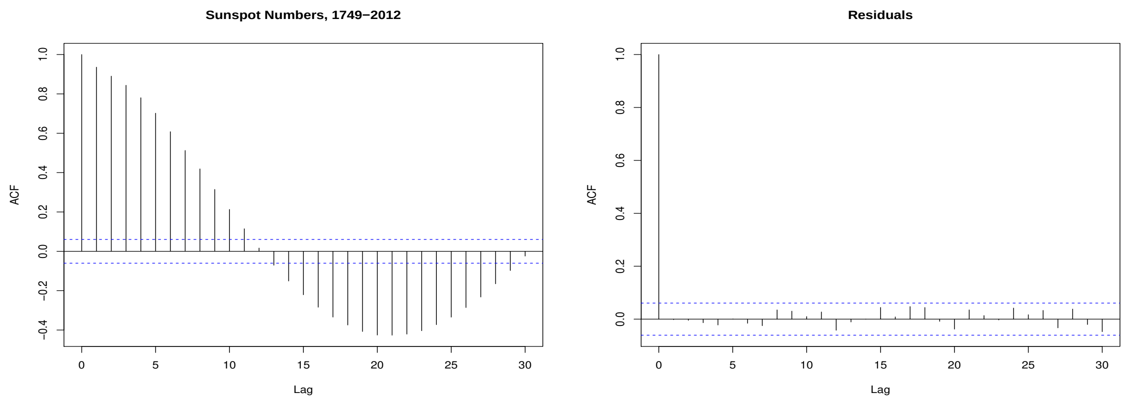
Check for Sunspots Numbers			Check for Residuals		
Lag (k)	df	$\chi^2(k)$	Lag (k)	df	$\chi^2(k)$
6	6	4075.119***	12	5	6.632
12	12	4708.268***	18	11	13.3774
18	18	5146.997***	24	17	18.3663
24	24	6232.194***	30	23	25.4344
30	30	6540.412***	36	29	33.2317
36	36	7060.406***	42	35	46.5823

Note: The null hypothesis of Ljung-Box test is that the autocorrelations up to lag k in the population from which the sample is taken are 0. $\chi^2(k)$ is the test statistic with k degrees of freedom. Readers may refer to Ljung and Box (1978) for more details of the test. The left panel displays the values of $\chi^2(k)$ for the Sunspots numbers while the right panel shows the values for the residuals after fitting the linear AR model as shown in (10).
*, **, and *** mean significant at levels 10%, 5%, and 1%, respectively.

it may be applied to test for a departure to a general smooth nonlinear function since a piecewise linear function will provide a better approximation than that from a global linear function. This is in the same direction as Cox (1981) who suggested the use of quadratic or cubic regression for testing nonlinearity. The tests could be parametric or nonparametric statistics. The Ljung-Box statistics of squared residuals, the bispectral test, and the Brock, Dechert, and Scheinkman (BDS) test are nonparametric methods. The RESET test (Ramsey, 1969), the F tests of Tsay (1986, 1989), and other Lagrange multiplier and likelihood ratio tests depend on specific parametric functions.

A nonlinear phenomenon is typically more complex and more difficult to model than a linear one, and the available tools are much less comprehensive and less effective. The number of parameters of the nonlinearity part could be very large, this could affect the performance of the existing nonlinear tests. Nonlinearity may occur in many and could be infinitely ways, so it is not our intention to develop a single test that dominates the others in detecting nonlinearity. There are many works on this area. In this paper we focus on nonlinearity within a stationary time series, which is often ignored by many people especially in applied science such as finance and economics. We add a reliable,

Figure 6: Plots of the Autocorrelation Functions



Note: The left panel exhibits the ACF for Sunspots numbers whereas the right panel displays the ACF for the residuals after fitting the linear AR model as shown in (10).

user-friendly, desirable, and powerful test to the nonparametric nonlinearity test category in the literature. As a nonlinear phenomenon is typically more complex and more difficult to model than a linear one, so it is not reasonable to restrict the form of the nonlinearities at the stage of detecting them within a sequence. Our test circumvents this type of limitation, including the Volterra expansion.

Our simulation shows that Tsay's test is more powerful than ours in a region while our test is more powerful in another region. We note that this finding is not surprised because nonlinearity may occur in many ways that there may not exist any single test that could dominate the others in detecting nonlinearity. However, our simulation shows that our proposed test has three desirable features than Tsay's test: (1) our proposed test is more stable, (2) the power of our proposed test increases while that of Tsay's test could decrease when the magnitude of parameter increases, and (3) the power of our proposed test reaches one quickly but the power of Tsay's test is decreasing when the magnitude of β increases. Thus, the results of our simulation support our claim that our test is a more desirable test.

Thereafter, we apply both Tsay's test and the nonlinearity test we developed in this paper to test whether there exists any nonlinear feature in the sunspots data, one of the most typical nonlinear cases. Our findings show that both our proposed test and Tsay's test draw the same conclusion that there exists nonlinearity in the Wolf's Sunspots numbers. The illustration reveals that our test is useful.

At last, we note that our test could not only be used to detect any nonlinearity for the variable being examined. If one believes a predetermined model could be fitted to the variable and its residuals could be estimated. Then, the test developed in this paper could also be used to examine whether there is an nonlinearity in the residuals and, in turn, test whether the model being used to fit to the variable is appropriate. For example, if one believes that Model A, which could be linear or nonlinear, is the right model for the data and thus she could fit Model A to the variable, obtain its residuals, and thereafter apply our test to test the residuals and see whether our test rejects the null hypothesis of linearity. If it does not, this infers that Model A is appropriate to be used for the variable being studied. On the other hand, if our test rejects the linearity of the residuals, this infers that the model is not appropriate. However, if one could not find any model to be appropriate for the data but one could find, say two models, Model A and Model B, that could be the best choices for the data and one could estimate the residuals for both Models A and B. Then, one could still apply the our proposed statistic to test for their residuals and the one with smaller p-value will be the more desirable model for the data.

There are many nonlinear time series models, for example, the bilinear models (Granger and Andersen, 1978), the threshold autoregressive (TAR) model, (Tong, 1978), the state-dependent model (Priestley, 1980), the Markov switching model (Hamilton, 1989). the nonlinear state-space model (Carlin, Polson, and Stoffer, 1992), the functional coefficient autoregressive model (Chen and Tsay, 1993a), the nonlinear additive autoregressive model (Chen and Tsay, 1993b), and the multivariate adaptive regression spline model (Lewis and Stevens, 1991). One may not be able to estimate the residuals for some of these nonlinear time series models. However, it is still possible for academics and practitioners to estimate the residuals for some nonlinear time series models, for example, one could choose a few terms such as the linear, quadratic and cubic terms in the Volterra expansion to be the one's desired nonlinear time series models. As long as the residuals of the nonlinear time series models can be estimated, one could apply the test developed in this paper to test whether there is still any nonlinearity in the residuals. If the null hpyothesis of linearity is not rejected, then one could conclude that the chosen nonlinear time series model is appropriate.

References

[1] AKAIKE H. (1978). On the likelihood of a time series model. *J. R. Statist. Soc. D* **27**, 217-235.

[2] BLOOMFIELD, P. (1976). *Fourier Analysis of Time Series: An Introduction.* New York: John Wiley.

[3] BOLLERSLEV, T. (1986). Generalized autoregressive conditional heteroscedasticity. *J. Econometrics* **31**, 307-327.

[4] BOX, G. E. P., JENKINS, G. M. & REINSEL, G. C. (1994). *Time Series Analysis Forecasting and Control.* Prentice-Hall.

[5] BRILLINGER, D. R. (1970). The identification of polynomial systems by means of higher order spectra. *J. Sound. Vib.* **12**, 301-313.

[6] CARLIN, B. P., POLSON, N. G. & STOFFER, D. S. (1992). A Monte Carlo approach to nonnormal and nonlinear state space modeling. *J. Am. Statist. Assoc.* **87**, 493-500.

[7] CHAN, K. S. (1990b). Testing for threshold autoregression. *Ann. Statist.* **18**, 1886-1894.

[8] CHAN, K. S. & TONG, H. (1990). On likelihood ratio tests for threshold autoregression. *J. R. Statist. Soc. B* **52**, 469-476.

[9] CHEN, R. & TSAY, R. S. (1993a). Functional-coefficient autoregressive models. *J. Am. Statist. Assoc.* **88**, 298-308.

[10] CHEN, R. & TSAY, R. S. (1993b). Nonlinear additive ARX models. *J. Am. Statist. Assoc.* **88**, 955-967.

[11] COX, D. R. (1981). Statistical analysis of time series: Some recent developments (with discussion). *Scand. J. Statist.* **8**, 93-115.

[12] CRADDOCK, J. M. (1967). An experiment in the analysis and prediction of time series. *The Statistician* **17**, 257-268.

[13] DENKER, M. & KELLER, G. (1983). On U-statistics and v. Mises' Statistics for weakly Dependent Processes. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **64**, 505-522.

[14] DENKER, M. & KELLER, G. (1986). Rigorous statistical procedures for data from dynamical system. *J. Statist. Phys.* **44**, 67-93.

[15] ENGLE, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of U.K. inflation. *Econometrica* **50**, 987-1008.

[16] FAN, J. Q. & YAO, Q. W. (2003). *Nonlinear time series: nonparametric and parametric methods.* New York: Springer-Verlag.

[17] GHADDAR, D. K. (1980). Some diagnostic checks of non-linear time series models. M.sc. dissertation, University of Manchester, U.K.

[18] GRANGER, C. W. J. & ANDERSEN, A. P. (1978). *An Introduction to Bilinear Time Series Models.* Gottingen: Vandenhoek and Ruprecht.

[19] HAMILTON, J. D. (1989). A new approach to the economic analysis of nonstationary time series and the business cycle. *Econometrica* **57**, 357-384.

[20] HÄRDLE, W., LÜTKEPOHL, H. & CHEN, R. (1997). A review of nonparametric time series analysis. *Int. Statist. Re.* **65**, 49-72.

[21] HINICH, M. J. (1982). Testing for Gaussianity and linearity of a stationary time series. *J. Time Series Anal.* **3**, 169-176.

[22] HINICH, M. J. & PATTERSON, D. M. (1985). Evidence of nonlinearity in daily stock returns. *J. Bus. Econ. Statist.* **3**, 69-77.

[23] IZENMAN, A. J., Wolf, J. R. & Wolfer, H. A. (1983). An historical note on Zurich sunspot relative numbers. *J. R. Statist. Soc. A* **146**, 311-318.

[24] KEENAN, D. M. (1985). A Tukey non-additivity-type test for time series nonlinearity. *Biometrika* **72**, 39-44.

[25] KOWALSKI, J. & Tu, X. M. (2007). *Modern Applied U-Statistics.* New York: John Wiley & Sons.

[26] LEWIS, P. A. W. & STEVENS, J. G. (1991). Nonlinear modeling of time series using multivariate adaptive regression spline (MARS). *J. Am. Statist. Assoc.* **86**, 864-877.

[27] LIM, K.S. (1981). On threshold time series modelling. Unpublished Ph.D. Thesis. University of Manchester, U.K.

[28] MARAVELL, A. (1983). An application of nonlinear time series forecasting. *J. Bus. Econ. Statist.* **1**, 66-74.

[29] MASRY, E. & FAN, J. (1997). Local polynomial estimation of regression functions for mixing processes. *Scand. J. Statist.* **24**, 165-179.

[30] MORAN, P. A. P. (1954). Some experiments on the prediction of sunspot numbers. *J. R. Statist. Soc. B* **116**, 112-117.

[31] NEWEY, W. K. & WEST, K. D. (1987). A simple, positive semi-definite, heteroskedasticity and autocorrelation constistent covariance matrix. *Econometrica* **55**, 703-708.

[32] PRIESTLEY, M. B. (1980). State-dependent models: a general approach to nonlinear time series analysis. *J. Time Series Anal.* **1**, 47-71.

[33] RAMSEY, J. B. (1969). Tests for specification errors in classical linear least squares regression analysis. *J. R. Statist. Soc. B* **31**, 350-371.

[34] SCHAERF, M. C. (1964). Estimation of the covariance and autoregressive structure of a stationary time series. Tech. Rep., Department of Statistics, Stanford University, Stanford, California, U.S.A.

[35] SERFLING, R. (1980). *Approximation Theorems of Mathematical Statistics.* New York: John Wiley & Sons.

[36] SUBBA RAO, T. & GABR, M. M. (1980). A test for linearity of stationary time series. *J. Time Series Anal.* **1**, 145-158.

[37] TIAO, G. C. & TSAY, R. S. (1994). Some advances in nonlinear and adaptive modeling in time series. *J. Forecasting* **13**, 109-131.

[38] TJOTHEIM, D. (1994). Non-linear time series: A selective review. *Scand. J. Statist.* **21**, 97-130.

[39] TUKEY, J. W. (1949). One degree of freedom for non-additivity. *Biometrics* **5**, 232-242.

[40] TONG, H. (1978). *On a threshold model. In C. H. Chen (ed.), Pattern Recognition and Signal Processing.* Amsterdam: Sijhoff & Noordhoff.

[41] TONG, H. (1983). *Threshold Models in Non-linear Time Series Analysis.* New York: Springer-Verlag.

[42] TONG, H. (1990). *Non-linear Time Series: A Dynamical Systems Approach.* Oxford: Oxford University Press.

[43] TONG, H. (1995). A personal overview of non-linear time series analysis from a chaos perspective (with discussion). *Scand. J. Statist.* **22**, 399-445.

[44] TONG, H. & LIM, K. S. (1980). Threshold autoregression, limit cycles and cyclical data (with discussion). *J. R. Statist. Soc. B* **42**, 245-292.

[45] TSAY, R. S. (1986). Nonlinearity tests for time series. *Biometrika* **73**: 461-466.

[46] TSAY, R. S. (1989). Testing and modeling threshold autoregressive processes. *J. Am. Statist. Assoc.* **84**, 231-240.

[47] WIENER, N. (1958). *Non-linear Problems in Random Theory.* Cambridge, Mass: M.I.T. Press.

[48] YAO, Q. & TONG, H. (1995a). On initial-condition sensitivity and prediction in nonlinear stochastic systems. *Bull. Int. Statist. Inst.* **10**, 395-412.

[49] YAO, Q. & TONG, H. (1995b). On prediction and chaos in stochastic systems. In Chaos and Forecasting ed. by H Tong, World Scientific, Singapore, pp. 57-86.

[50] YULE, G. U. (1927). On a method of investigating periodicities in disturbed series with special reference to Wolfer's sunspot numbers. *Philosophical Transactions of the Royal Society (London) A* **226**, 267-298.