Some remarks on lower hemicontinuity of convex multivalued mappings

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SOME REMARKS ON LOWER HEMICONTINUITY OF CONVEX MULTIVALUED MAPPINGS

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Summary. For a multifunction a condition sufficient for lower hemicontinuity is presented. It is shown that under convexity of graph it is possible for a multifunction to be not continuous only when a special representation of points of its domain is not feasible.

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1. Introduction

In the paper we deliver a sufficient condition for lower hemicontinuity of graph-convex multifunctions from a set $X \subset \mathbf{R}^n$ into $Y \subset \mathbf{R}^m$. This class of multifunctions plays an important role in the theory of convex multisectoral growth models (see [6]) and dynamic programming (see [5], p. 66-100) - lower hemicontinuity is a very useful property since it is one of conditions for validity of the famous Berge's Maximum Theorem ([1], p. 116), which allows to conclude about continuity of solutions to optimization problems.

At the same time we also give some analogues and extensions of existing theorems on behavior of concave functions and graph-convex mappings. From [3] it is known that for a closed bounded subset X of $\mathbf{R}^n$ to be a polytope is equivalent to following fact: every closed concave function defined on X is continuous. In our paper we state that if every graph-convex non-constant multifunction is lower hemicontinuous on a compact set X, then X is a polytope. Moreover there is an equivalence: if X is a polytope, then every graph-convex non-constant mapping is lower hemicontinuous on X (corollary 2). Further, from theorem 10.2 in [7] we know that if X is locally simplicial,[1] then every closed concave function is continuous - we proved an analogue of this result in terms of lower hemicontinuous graph-convex mappings (see theorem 2, lemma 1 and remark 1).

[1] For definition of locally simplicial sets see [7], p.84.

Assumption of local simplicity on technological set was used (among others 'standard' assumptions) in [2] to show that reduced-model utility function is continuous. This assumption is a rationale for continuity of utility functions in reduced models of growth. Theorem 4 allows to make weaker assumptions on technological set but they 'leave' continuity of reduced-model utility function intact.

Last but not least we amend a theorem from [5] which asserts that every graph-convex mapping from a locally compact set is lower hemicontinuous[2] and we give counterexamples in which mappings are not lower hemicontinuous at a boundary point of domain (see examples 1 and 3). In theorem 4 we give equivalence of lower hemicontinuity of all graph-convex mappings on X and some property of X. This theorem gives an extension to theorem 5.9 b from [8].

The next part of the paper gives us notation. Section 3 contains counterexamples mentioned above. Section 4 includes main results of the paper.

2. Notation

In what follows $intA$, clA, $bndA$, $extA$, $convA$ denote interior of A, closure of A, boundary of A, set of extreme points of A and convex hull of A, where $A \subset \mathbf{R}^n$, respectively. For $x \in \mathbf{R}^n$ $\|x\|$ denotes Euclid norm of x. $B(x, \epsilon)$ denotes closed ball centered at $x \in \mathbf{R}^n$ of radius $\epsilon > 0$.

3. Preliminaries

Recall the definition of lower hemicontinuity ([5], p. 56):

Definition 1. *Let $\emptyset \neq X \subset \mathbf{R}^n$, $\emptyset \neq Y \subset \mathbf{R}^m$, and $\Gamma : X \to Y$ be a multifunction s.t. $\forall x \in X\ \Gamma(x) \neq \emptyset$. Γ is lower hemicontinuous at $x \in X$ (l.h.c. at x), if $\forall y \in \Gamma(x)\ \forall \{x_n\}_{n=1}^{+\infty} \subset X,\ x_n \to x\ \exists \{y_n\}_{n=1}^{+\infty} \in \prod_{n=1}^{+\infty} \Gamma(x_n) \quad y_n \to y$. Γ is called lower hemicontinuous (l.h.c.) if it is l.h.c. at every $x \in X$.*

For the further part of the paper we state

Assumption 1. *$X \subset \mathbf{R}^n$, $intX \neq \emptyset$, $\emptyset \neq Y \subset \mathbf{R}^m$ and X is convex. $\Gamma : X \to Y$ is a multifunction s.t. $\forall x \in X\ \Gamma(x) \neq \emptyset$ and if $X_1 \subset X$ is a bounded set in Euclid norm, then there exists a bounded set $Y_1 \subset Y$ s.t. $\forall x \in X_1\ \Gamma(x) \cap Y_1 \neq \emptyset$.*

[2]It is true that such a mapping is lower hemicontinuous on interior of domain - [8], p. 155, theorem 5.9 b.

In a very often referred book [5] the following theorem was presented ([5], p.61):

Theorem 1. *Let assumption 1 hold and suppose* $\forall x \in X\ \exists \epsilon > 0 : B(x, \epsilon) \cap X$ *is closed. Assume further that the graph of* Γ *is convex.* Γ *is l.h.c.*

However it turns out that the above theorem is not true in general which is shown by

Example 1. *Let* $X = \{x \in \mathbf{R}^2 : \|x\| \leq 1\}$, $Y = [0, 1]$ *(unit sector of real line). Define* Γ *as follows*

$$\forall x \in X \qquad \Gamma(x) := \begin{cases} [0,1] & , \textit{if } \|x\| < 1 \vee x = (1,0); \\ \{0\} & , \textit{if } \|x\| = 1. \end{cases}$$

It is easy to check that all assumptions of theorem 3 hold. Γ *is not l.h.c. at* $(1, 0)$*: take* $x = (1, 0), 1 \in \Gamma(x)$ *and sequence* $x \notin \{x_n\}_{n=1}^{+\infty}, \forall n \|x_n\| = 1, x_n \to x$*; it is seen that* $\forall n\, \Gamma(x_n) = \{0\}$*, so that if* $y_n \in \Gamma(x_n)$*, then* $y_n = 0$ *- we can not approximate* $y = 1 \in \Gamma(1, 0)$ *by any sequence contained in* $\{\Gamma(x_n)\}_{n=1}^{+\infty}$.

Remark that the multifunction from example 1 is not l.h.c. at a boundary point of its domain. At the first glance it appears that strengthening of assumptions of theorem 1 by adding closedness of graph will fix the error (graph of Γ from example 1 is not closed). But this is not the point. Consider

Example 2. *Let* $X = \{x \in \mathbf{R}^2 : \|x\| \leq 1\}$, $Y = [0, 1]$. *Define* Γ*:*

$$\forall x \in X \qquad \Gamma(x) := \begin{cases} \left[0, \frac{1-x_1^2-x_2^2}{2(1-x_1)}\right], & \textit{if } x \neq (1,0); \\ [0,1], & \textit{elsewhere.} \end{cases}$$

If $\|x\| = 1$*, then* $\Gamma(x) = 0$*. Moreover one can show that* $\forall x \in X\ \Gamma(x) \subseteq [0, 1]$ *and graph of* Γ *is convex and closed. But* Γ *is not l.h.c. at* $(1, 0)$ *(take sequences as in example 1).*

4. Results

A 'correct' version of theorem 1 is as follows:

Theorem 2. *Let assumptions of theorem 1 hold and suppose that* $\forall x \in X \exists \epsilon > 0 \forall y \in X\ 0 < \|y - x\| < \epsilon \Rightarrow \exists t \in [0, 1]\ \exists d \in X,\ \|d - x\| = \epsilon :\ y = tx + (1 - t)d$*.* Γ *is l.h.c.*

Proof. Fix $x \in X$. And let $\epsilon' > 0$ be s.t. $B(x, \epsilon')$ is closed. It is obvious that if hypothesis of the theorem holds for some ϵ at x, then by convexity of X it holds for all numbers strictly less than ϵ so w.l.o.g. assume that it holds for $0 < \epsilon < \epsilon'$. The next part of the proof is as in [5], p. 61:
Let $X_1 := B(x, \epsilon) \cap X$ - it is a compact set. We shall show that Γ is l.h.c. at x. Let $y \in \Gamma(x)$ and $\{x_n\}_{n=1}^{+\infty} \subset X_1,\ x_n \to x$ and $Y_1 \subset Y$ be a bounded set such that $\forall x \in X_1\, \Gamma(x) \cap Y_1 \neq \emptyset$. W.l.o.g. assume $x \notin \{x_n\}_{n=1}^{+\infty}$. Now fix N so that $\forall n \geq N\, \|x_n - x\| < \epsilon$ and consider further only such n-s. It holds that $\forall n\ \exists d_n \in X\ \|d_n - x\| = \epsilon\ \exists t_n \in (0,1):\ x_n = (1-t_n)x + t_n d_n$. For every d_n choose an $y_n \in \Gamma(d_n) \cap Y_1$. Convexity of graph of Γ implies that $\forall n\ (1-t_n)y + t_n y_n \in \Gamma((1-t_n)x + t_n d_n) = \Gamma(x_n)$. Since $(1-t_n)x + t_n d_n \to x, \|x - d_n\| = \epsilon,\ t_n \in (0,1)$ then $t_n \to 0$ which means $(1-t_n)y + t_n y_n \to y$, since $\{y_n\}_{n=1}^{+\infty}$ is bounded - l.h.c. at x follows. Since x is arbitrary - the thesis follows. Q.E.D. $\square$

The below lemma shows that 'representation' in hypothesis of theorem 2 is equivalent to finiteness of $extX$ if X is compact.

Lemma 1. *Let $X \subset \mathbf{R}^n$ be a compact convex set, $intX \neq \emptyset$. The following formulations are equivalent*

(1) $\forall x \in X\ \exists \epsilon > 0 \forall y \in X\ 0 < \|y - x\| < \epsilon \Rightarrow y \notin extX$;
(2) $\forall x \in X \exists \epsilon > 0 \forall y \in X\ 0 < \|y - x\| < \epsilon \Rightarrow \exists t \in [0,1]\ \exists d \in X,\ \|d - x\| = \epsilon:\ y = tx + (1-t)d$;
(3) *X is a polytope.*

Proof. '1 $\Rightarrow$ 2' Since X is compact and convex, as closure of a convex set, it follows by the Krein-Milman theorem that $extX \neq \emptyset$ ([4], p. 38). If $extX$ is not a finite set, then by the Bolzano-Weierstrass theorem exists a cluster point for which formulation 1 is violated. Further, since $conv(extX) = X$ ([4], p. 39) and $extX$ is finite we get that X is a polytope. There exists a finite number m of halfspaces $H_i^+ := \{x \in \mathbf{R}^n : a^i x \geq \alpha^i\}$ ([4], p. 40), where a^i, α^i are respectively a vector from $\mathbf{R}^n$ and a real number, $i = 1, \ldots, m$, such that

$$X = \bigcap_{i=1}^{m} H_i^+.$$

Fix a point $x \in bndX$ and define a number ϵ:

$$\epsilon := 2^{-1} \min\{\rho(x, bndH_i^+) : \; i \in I_1\},$$

where $I_1 := \{i \in \{1, \ldots, n\} : a^i x > \alpha^i\}$, $\rho(x, A) := \inf\{\|x - y\| : y \in A\}$. Since, by assumption X is a compact set with nonempty interior it follows that $\epsilon > 0$. Now take any $y \in X$ s.t. $0 < \|x - y\| < \epsilon$. By the choice of ϵ $\forall i \in I_1 : a^i y > \alpha^i$ and $a^i y \geq \alpha^i$ for the rest of indices. Define $d := x + \frac{\epsilon}{\|y-x\|}(y - x)$. It holds that $\|d - x\| = \epsilon$, $\forall i \in I_1$ $a^i d > \alpha^i$ (by definition of ϵ) and since $\forall t > 0 \forall i \notin I_1 : a^i d = a^i x + ta^i(y - x) = \alpha^i + t(a^i y - \alpha^i) \geq \alpha^i$, then $d \in X$. Finally $y = (1 - t')x + t'd$, where $t' := \frac{\|y-x\|}{\epsilon} \in (0, 1)$, $d \in X$, $\|x - d\| = \epsilon$.
'2 $\Rightarrow$ 1' Let $x \in X$. Choose $\epsilon > 0$ as in the second formulation and fix $y \in X, 0 < \|x-y\| < \epsilon$. There exists $d \in X$, $\|d - x\| = \epsilon$, and $t \in (0, 1) : y = tx + (1 - t)d$ and we get that $y \notin extX$ which proves the thesis.
'3 $\Leftrightarrow$ 1' This follows immediately from the proof of part '1 $\Rightarrow$ 2' and definition of polytope ([4], p. 39). Q.E.D. □

Now we are ready to prove that every graph-convex mapping defined on a polytope is l.h.c.

Theorem 3. *Let assumption 1 hold and suppose* Γ *is graph-convex. If* X *is a polytope, then* Γ *is l.h.c.*

Proof. By lemma 1 assumptions of theorem 2 are met and the thesis follows. Q.E.D. □

Remark 1. *By lemma 1 we could equivalently assume in the hypothesis that the second condition of lemma 1 holds. It should be obvious, by the proof and lemma 1, that the thesis would hold true if we assumed that* X *is locally simplicial (and even omitted assumption on boundedness of* X*), since if set* X *is locally simplicial then - by the very definition of local simplicity ([7], p. 84) - it meets condition 1 of lemma 1, and therefore condition 2 is also met, so that we can apply theorem 2.*

The following example shows that 'representation' hypothesis of theorem 2 is crucial for its validity in general case i.e. if X is any subset of $\mathbf{R}^n$.[3]

[3]In theorem 2 it is assumed on X only that it is a convex set with non-empty interior.

Example 3. *Let* $X = \{x \in \mathbf{R}^2 : \|x\| \leq 1\} \times \mathbf{R}$*. X is not compact and has no extreme point but in spite of this the thesis of theorem 2 does not hold. Let* $\Gamma : X \to \mathbf{R}$ *be defined as follows*

$$\forall x \in X \qquad \Gamma(x) := \begin{cases} \left[0, \frac{1-x_1^2-x_2^2}{2(1-x_1)}\right] & , \textit{if } x \neq (1,0,x_3), x_3 \in \mathbf{R}; \\ [0,1] & , \textit{if } x = (1,0,x_3), x_3 \in \mathbf{R}. \end{cases}$$

Γ *is not l.h.c. at* $x = (1,0,x_3), x_3 \in \mathbf{R}$ *(see example 3).*

The main result of the paper is theorem 4.

Theorem 4. *Assume* $X \subset \mathbf{R}^n$*,* $intX \neq \emptyset$ *and X is convex. Fix some* $\overline{x} \in X$*. The following formulations are equivalent:*

1. *Every non-empty-valued graph-convex multifunction* $\Gamma : X \to Y$*, where Y is an arbitrary non-empty subset of* $\mathbf{R}^m$*,* $m \in \{1,2,\ldots\}$*, s.t. there exists a bounded convex neighbourhood* X_1 *of* $\overline{x}$ *s.t.* $Y_1 = \bigcup_{x \in X_1} \Gamma(x)$ *is bounded, is l.h.c. at* $\overline{x}$*.*
2. $\exists \epsilon > 0 \forall y \in X \; 0 < \|y - \overline{x}\| < \epsilon \Rightarrow \exists t \in [0,1] \; \exists d \in X, \; \|d - \overline{x}\| = \epsilon : \; y = t\overline{x} + (1-t)d.$

Proof. '2 $\Rightarrow$ 1' This is a consequence of proof of theorem 2.
'1 $\Rightarrow$ 2' Assume that $\overline{x} \in X$ and for no number $\epsilon > 0$ formulation 2 holds. For all $k = 1,2,\ldots$ choose $x_k \in X$ s.t. $0 < \|x_k - \overline{x}\| < 1/k$ and for all $t \in [0,1] \, \forall d \in X \; \|d - \overline{x}\| = 1/k : \; x_k \neq t\overline{x} + (1-t)d$. Let

$$t_k := \sup\{t \in [0,1] : \; x_k = t\overline{x} + (1-t)y, \; t \in [0,1], y \in X\} \qquad k = 1,2,\ldots.$$

Since $x_k \neq \overline{x}$, then $t_k < 1$ for all k and $x_k' := (1-t_k)^{-1}(x_k - t_k\overline{x}) \in clX$, $k = 1,2,\ldots$ are well-defined points having following properties that stem from definition of t_k: $0 < \|x_k' - \overline{x}\| \leq 1/k$, $\forall t \in (0,1] : \; t\overline{x} + (1-t)x_k' \in X$, $\forall t < 0 : \; t\overline{x} + (1-t)x_k' \notin X$. It also holds that $x_k' \xrightarrow{k} \overline{x}$.
Let a function $g : X \to [0,1]$ be given by

$$g(x) := \begin{cases} 1 & , \text{ if } x = \overline{x}, \\ 0 & , \text{ if } x \neq \overline{x}. \end{cases}$$

Denote graph of g by $Gr(g)$ i.e. $Gr(g) := \{(x, g(x)) \in X \times [0,1] : x \in X\}$. Define $G := conv(Gr(g))$ and another function $p : X \to [0,1]$

$$p(x) := \sup\{\lambda \in [0,1] : \; (x,\lambda) \in G\}.$$

Since $(\overline{x},1) \in G$, then $p(\overline{x}) = 1$. Let $\forall k \, \forall q > 1 : x_k^q = q^{-1}\overline{x} + (1-q^{-1})x'_k$. It follows that $\forall k \, \forall q \, \|x_k^q - \overline{x}\| \leq k^{-1}$ and $\limsup_q p(x_k^q) = 0$. We shall substantiate the latter. By the Caratheodory's theorem ([4], p. 2)

$$G = \left\{ \sum_{i=1}^{n+2} \alpha^i (x^i, g(x^i)) : \sum_{i=1}^{n+2} \alpha^i = 1, \ \alpha^i \geq 0, x^i \in X, \, i = 1, \ldots, n+2 \right\}.$$

Value $p(x)$ is strictly greater than zero only if there exists $(x,\lambda) \in G : \lambda > 0$. Since $G \ni (x,\lambda) = \sum_{i=1}^{n+2} \alpha^i (x^i, g(x^i))$ for some $\alpha^i \geq 0, x^i \in X$ and $g(x) > 0$ only if $x = \overline{x}$, then every x for which holds $p(x) > 0$ is representable as $x = t\overline{x} + (1-t)y$ for a number $t \in (0,1]$ and some $y \in X$. If for some k $\limsup_q p(x_k^q) > \epsilon > 0$ then there is a subsequence $\{x_k^{q_j}\}_{j=1}^{+\infty}, 1 < q_j < q_{j+1} \, \forall j$ s.t. $p(x_k^{q_j}) > \epsilon$ and $x_k^{q_j} = \lambda_j \overline{x} + (1-\lambda_j)y_j$, $\lambda_j > \epsilon$, $y_j \in X$. But at the same time $x_k^{q_j} = {q_j}^{-1}\overline{x} + (1-{q_j}^{-1})x'_k$ and we get for all j

$${q_j}^{-1}\overline{x} + (1-{q_j}^{-1})x'_k = \lambda_j \overline{x} + (1-\lambda_j)y_j,$$

and therefore

$$x'_k = (1-{q_j}^{-1})^{-1}(\lambda_j - {q_j}^{-1})\overline{x} + (1-{q_j}^{-1})^{-1}(1-\lambda_j)y_j.$$

It is easy to see that x'_k is a convex combination of $\overline{x}$, $y_j \in X$. By definition of x'_k and the above equation we get

$$x_k = \alpha_j \overline{x} + (1-\alpha_j)y_j,$$

where $\alpha_j = t_k + (1-t_k)(1-{q_j}^{-1})^{-1}(\lambda_j - {q_j}^{-1})$. But it contradicts definition of t_k since $1 \geq \alpha_j > t_k$ and $y_j \in X$. So that fixing $0 < \epsilon < 1$ for all $k = 1, 2, \ldots$ we can find q_k s.t. $x_k^{q_k} \in X : p(x_k^{q_k}) < \epsilon$.

Define a multifunction $\Gamma : X \to [0,1]$:

$$\forall x \in X \ \Gamma(x) := [0, p(x)].$$

We have that $\Gamma(\overline{x}) = [0,1]$ and $\forall k : \Gamma(x_k^{q_k}) = [0, p(x_k^{q_k})] \subset [0,\epsilon)$, where $x_k^{q_k} \in X$ are constructed and chosen as above. Since by construction $x_k^{q_k} \to \overline{x}$ it is sufficient to show that Γ has convex graph - this will contradict formulation 1. To this end we shall show that $p(\cdot)$ is a concave function on X. Let $x', x'' \in X$. For any integer $m \geq 1$ there exist $\lambda'_m, \lambda''_m : \ p(x') - 1/m < \lambda'_m, p(x'') - 1/m < \lambda''_m, (x', \lambda'_m), (x'', \lambda''_m) \in G$. We get $\forall t \in [0,1]$: $(tx' + (1-t)x'', t\lambda'_m + (1-t)\lambda''_m) \in G$ by convexity of G and it follows $\forall t \in [0,1]$ $p(tx' + (1-$

$t)x'') \geq t\lambda'_m + (1-t)\lambda''_m$. So it holds that $\forall m = 1, 2, \ldots$ $\forall t \in [0,1]$ $tp(x') + (1-t)p(x'') - 1/m < t\lambda'_m + (1-t)\lambda''_m \leq p(tx' + (1-t)x'')$ and taking limit $m \to +\infty$ concavity of $p(\cdot)$ follows. Q.E.D. □

Corollary 1. *Every graph-convex mapping meeting assumption 1 is l.h.c. iff for each $x \in X$ condition 2 of theorem 4 holds.*

Finally we get as a corollary of theorems 3 and 4:

Corollary 2. *Suppose $\emptyset \neq intX \subset \mathbf{R}^n$ and let X be compact and convex. Every non-empty-valued graph-convex and bounded mapping Γ is l.h.c. on X iff X is a polytope.*

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