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Abstract. Based on a dynamic general equilibrium model we study how the composition of technical progress, along three dimensions, affects transitional dynamics, with an emphasis on the speed of convergence. The three dimensions are, first, the degree to which technical change is embodied, second, the extent to which an endogenous source, learning, drives productivity advances, and, third, the extent to which the vehicle of learning is gross investment rather than net investment. The analysis shows that the speed of convergence, both ultimately and in a finite distance from the steady state, depends strongly and negatively on the importance of learning in the growth engine and on gross investment being the vehicle of learning rather than net investment. In contrast to a presumption implied by "old growth theory", a rising degree of embodiment in the wake of the computer revolution is *not* likely to raise the speed of convergence when learning by investing is the driving force of productivity increases.

Keywords and Phrases: Transitional dynamics, speed of convergence, learning by investing, embodied technological progress, decomposable dynamics.

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1 Introduction

This paper analyzes, within the context of a dynamic general equilibrium model, the effects of changes in the composition of technical progress on transitional dynamics — with an emphasis on the speed of convergence.

The speed of convergence is important because it indicates what weight should be placed on transitional dynamics of a growth model relative to the steady-state behavior. Whether the speed of convergence is likely to go up or down in the future matters for the evaluation of growth-promoting policies. In growth models with diminishing returns successful growth-promoting policies have transitory growth effects and permanent level effects. Slower convergence implies that the full benefits are slower to arrive.

There is a substantial literature attempting to empirically estimate the speed of convergence and theoretically assess what factors affect it. One of the first econometric studies of "conditional convergence" was accomplished by Barro and Sala-i-Martin (1992). They found a speed of convergence of around 2% a year, implying that the time it takes to recover half the initial distance from steady state is around 35 years (assuming no further disturbances). To reconcile such slow adjustment with the standard neoclassical growth model (the Ramsey model with exogenous technological change), an output elasticity with respect to capital as high as 0.75–0.8 is needed. Mankiw et al. (1992) showed that including human capital in the accumulation process along with physical capital brings the theoretical speed of convergence in line with the empirical estimate of around 2% a year. Newer studies show that strictly convex capital installation costs also tend to reduce the implied speed of convergence (Ortigueira and Santos, 1997). In Eicher and Turnovsky (1999) it is demonstrated that the speed of convergence is substantially reduced by adding an R&D sector to the model. However, Turnovsky (2002) finds that the elasticity of substitution in production between capital and labor significantly affects the speed of convergence in the Ramsey model. A reduction in the elasticity of factor substitution from the benchmark level of one to a lower (empirically realistic) level, however, increases the model's implied speed of convergence. Dalgaard (2003), followed by Chatterjee (2005), finds that the convergence speed critically depends on capital utilization rates and that models with full capital utilization may overstate the speed of convergence.

The overall conclusion from this theoretical literature is that "natural" extensions of the standard neoclassical growth model (except with regard to the elasticity of substitution in production between capital and labor) tend to bring down the asymptotic speed of convergence closer to the empirical estimate of 2% a year found by Barro and Sala-i-Martin (1992) and Mankiw et al. (1992). In turn, some empirical studies questioned these low estimates of the convergence speed, arguing that a number of econometric issues, like endogeneity of explanatory variables and country-specific fixed effects, had been ignored. Evidence has been put forward that the speed of convergence significantly varies across periods and groups of countries. Some studies provide estimates for a convergence speed of approximately 6% (Evans, 1997) and of 4.7% for a sample of 75 countries and 9.3% for OECD countries (Islam, 1995). Recently, the cross-country study by McQuinn and Whelan (2007), based on data for changes in the capital-output ratio, suggests convergence speeds of about 7% per year.

So we may say that the theoretical and empirical convergence literature has shown "convergence" with each other. Yet several factors of importance for the speed of convergence have not received much attention in the literature. The contribution of the present paper is to examine how the *composition* of technical change affects the speed of convergence - both in an asymptotic sense and in finite distances from the steady state. We consider the composition of technical change along three dimensions. The *first* relates to the *form* of technical change, i.e., the degree in which technical change is embodied rather than disembodied.[1] The *second* dimension relates to the *source* of technical change, where we contrast exogeneity with endogeneity in the form of learning by doing in the Arrow (1962) sense (that is, learning from investment experience). The *third* dimension involves the *vehicle* of investment experience. What role does it play whether the vehicle through which

[1]Following Solow (1960), technical change is said to be *embodied* if taking advantage of new technical knowledge requires construction of new investment goods. The newest technology is incorporated in the design of newly produced equipment; and this equipment will not participate in subsequent technical progress.

learning occurs is *gross* investment rather than *net* investment?

Studying the role of the composition of technical change in this context is motivated by two facts. First, based on data for the U.S. 1950-1990, Greenwood et al. (1997) estimate that embodied technical progress explains about 60% of the growth in output per man hour, the remaining 40% being accounted for by disembodied technical progress. So, empirically, *embodied* technical progress seems to play the dominant role. Furthermore, there are signs of an *increased* importance of embodiment of technical change in the wake of the computer revolution, as signified by a sharper fall in the quality-adjusted relative price of capital equipment (Greenwood and Jovanovic 2001; Jovanovic and Rousseau, 2002; Sakellaris and Wilson, 2004).[2] This raises the question how a shift in the relative importance of disembodied and embodied technical progress is likely to affect the speed of convergence. Second, most of the learning-by-investing literature has assumed that it is experience from *net* investment that drives learning.[3] The distinction between learning from *gross* rather than *net* investment has not received much attention so far.

To disentangle these issues, we set up a dynamic general equilibrium model in continuous time. The model builds on the framework on embodied technical change laid out by Greenwood et al. (1997).[4] By introducing endogenous learning from investment, our model essentially follows one of the "future directions" suggested by these authors. We depart, however, by allowing learning to imply scale effects on productivity *levels*. Such effects seem plausible in view of spillovers and the non-rival character of knowledge. On the other hand, we simplify by ignoring structures. We focus on the robust case of semi-endogenous growth rather than the knife-edge case of fully endogenous growth.

Within this framework, the paper presents four main results. First, endogenizing a fraction of the productivity increases as coming from learning by investing substantially lowers the speed of convergence. Intuitively, the presence of learn-

[2]For a survey, see Hornstein et al. (2005).

[3]Leading textbooks such as Acemoglu (2009), Aghion and Howitt (1998, 2009), Barro and Sala-i-Martin (2004), de la Croix and Michel (2002), and Valdez (1999), concentrate on this case — likely because of its mathematical simplicity. Probably for the same reason this literature predominantly disregard learning in the embodied form.

[4]The authors use the alternative term *investment-specific* technical change.

ing by investing adds a slowly moving complementary kind of capital ("investment experience") to the dynamic system — thereby slowing down the speed of convergence.

Second, the distinction between gross- and net investment as the vehicle through which learning by investing occurs is significant with regard to transitional dynamics and the speed of convergence. If net investment is the vehicle, then cumulative investment experience coincides with the capital stock. If, however, gross investment is the vehicle, cumulative investment experience becomes an additional stock variable, and the dimensionality of the dynamic system rises by one. This has two implications. As learning by investing becomes operative, the speed of convergence exhibits a discrete fall. This feature is absent if net investment is the vehicle of learning. Moreover, the speed of convergence is lower when the vehicle is gross rather than net investment. Intuitively, when the vehicle is gross investment, there is more overhang from the past, which slows down the speed of convergence. The notion that the vehicle of learning is gross investment is in our view more intuitive. It also accords better with the original ideas of Arrow who emphasized both *embodiment* of technical progress and learning from *gross* investment.[5]

Third, whether embodiment speeds up convergence as "old growth theory" concluded, turns out to depend critically on whether technical progress is exogenous or driven by learning. There is an early literature (Phelps, 1962; Sato, 1966; Williams and Crouch, 1972) which, within Solow-style neoclassical growth models, showed that for a given exogenous rate of technical progress, a higher degree of embodiment results in *faster* convergence. Since, to our knowledge this issue has not so far been taken up within Ramsey-style neoclassical growth models with an endogenous saving rate and not in models with endogenous technical change, we address the issue here. Somewhat surprisingly, the "classical" result that embodiment speeds up convergence turns out not to hold when growth is driven by learning. Hence, we conclude that a rising relative importance of embodied technical change in the wake of the computer revolution need not speed up the pace of adjustment. If

[5]In Arrow's words: "each new machine produced and put into use is capable of changing the environment in which production takes place, so that learning is taking place with continually new stimuli" (Arrow, 1962).

accompanied by a rising relative importance of learning by investing, the computer revolution may even slow down the speed of convergence.

Fourth, a series of numerical simulations gives a quantification of the theoretical results mentioned above. Specifically, the numerical simulations point to a speed of convergence on the small side of 2% per year.

Our paper is related to Jovanovic and Rousseau (2002) who also set up a model with embodied technical change and show that a greater ability to learn from investment experience slows down the speed of convergence. The present model departs by allowing multi-facetted technical change with learning based on gross rather than net investment and by assuming strictly concave utility (so that the interest rate is not fixed). Our focus on transitional dynamics and speed of convergence issues in a semi-endogenous growth setup is the main difference vis-à-vis the model named "Solow (1960) meets Arrow (1962)" in the survey by Greenwood and Jovanovic (2001). One of the models in Groth et al. (2010) also deals with embodied learning from investment. That paper aims at exploring conditions leading to less-than-exponential growth. In contrast, the present paper studies speed of convergence to a balanced growth path. Section 4 contains further comparisons with the existing literature.

The paper is organized as follows. Section 2 develops the gross-investment based version of the model, which we refer to as the "benchmark model". This version leads to a three-dimensional dynamic system the steady-state and stability properties of which are studied in Sections 3.1 and 3.2, respectively. Different measures of the speed of convergence are introduced in Section 3.3. Section 3.4 shows the novel result, linked to the distinction between decomposable and indecomposable dynamics, that as soon as learning becomes part of the growth engine, the asymptotic speed of convergence displays a discrete fall. Section 4 describes the case of learning based on net investment. This "alternative model" leads to two-dimensional dynamics and the appealing discontinuity disappears. By numerical simulations, Section 5 quantifies the mentioned discontinuity implied by the benchmark model. In addition, Section 5 explores the otherwise smooth dependency of different measures of the speed of convergence on the composition of technical change along the three dimensions described above. Finally, Section 6 concludes.

2 A benchmark model

2.1 Disembodied and embodied learning by investing

The learning-by-investing hypothesis is that variant of the learning-by-doing hypothesis that sees the *source* of learning as being primarily experience in the investment goods sector. This experience embraces know-how concerning how to produce the capital goods in a cost-efficient way and how to design them so that in combination with labor they are more productive in their applications. The simplest model exploring this hypothesis is in textbooks sometimes called the Arrow-Romer model and is a unified framework building on Arrow (1962) and Romer (1986). The key parameter is a learning parameter which in the "Arrow case" is less than one and in the "Romer case" equals one.[6] Whatever the size of the learning parameter, the model assumes that learning generates non-appropriable new knowledge that via knowledge spillovers across firms provides an engine of productivity growth for the major sectors of the economy. Summaries of the empirical evidence for learning and spillovers is contained in Jovanovic (1997) and Greenwood and Jovanovic (2001).

In the Arrow-Romer model firms benefit from recent advances in technical knowledge irrespective of whether they acquire new equipment or not. That is, technical change is assumed to be *disembodied:* new technical knowledge improves the combined productivity of capital and labor independently of whether the workers operate old or new machines. No new investment is needed to take advantage of the recent technological or organizational developments.

In contrast we say that technical change is *embodied*, if taking advantage of new technical knowledge requires construction of new investment goods. The newest technology is incorporated in the design of newly produced equipment; and this equipment will not participate in subsequent technical progress. An example: only the most recent vintage of a computer series incorporates the most recent advance in information technology. In this way investment becomes an important bearer of

[6] See, e.g., Valdés (1999) and Barro and Sala-i-Martin (2004).

the productivity increases which this new knowledge makes possible. This view is consistent with the findings in the cross-country studies by DeLong and Summers (1991), Levine and Renelt (1992), and Sala-i-Martin (1997). In the Levine and Renelt (1992) study, among over 50 different regressors only the share of investment in GDP, other than initial income, is found to be strongly correlated with growth.

Let the aggregate production function be

$$Y_t = K_t{}^{\alpha}(A_t L_t)^{1-\alpha}, \qquad 0 < \alpha < 1, \tag{1}$$

where Y_t is output, K_t capital input (measured in efficiency units), L_t labor input, and A_t labor-augmenting productivity originating in *disembodied* technical change, all at time t. Time is continuous. We consider two sources of growth in A_t, an endogenous source, accumulated *investment experience,* represented by the variable J_t, and an unspecified exogenous source, $e^{\gamma t}$:

$$A_t = J_t^{\beta} e^{\gamma t},\ 0 \leq \beta < 1,\ \gamma \geq 0. \tag{2}$$

The parameter β indicates the elasticity of labor-augmenting productivity w.r.t. investment experience and is thus a measure of the strength of disembodied learning. For short we name β the *disembodied learning parameter*. The upper bound on β is brought in to avoid explosive growth. In our benchmark model we assume that investment experience, J_t, is proportional to cumulative aggregate gross investment,

$$J_t = \int_{-\infty}^{t} I_\tau d\tau, \tag{3}$$

where I_τ is aggregate gross investment at time τ and we have normalized the factor of proportionality to one. The parameter γ in (2) is the rate of exogenous disembodied technical progress.

We consider a closed economy so that national income accounting implies

$$Y_t = I_t + C_t, \tag{4}$$

where C_t is aggregate consumption. We shall assume that, once produced, capital goods can never be used for consumption. So gross investment, I_t, is always non-negative.

The *embodied* component of technical progress, explaining about 60% of productivity growth according to Greenwood et al. (1997), is modeled in the following way:

$$\dot{K}_t = Q_t I_t - \delta K_t, \qquad \delta > 0, \tag{5}$$

where a dot over a variable indicates the time derivative, and Q_t measures investment-augmenting productivity, for short just the "quality", of newly produced investment goods. The growing level of technology implies rising Q_t. A given level of investment thus gives rise to a greater and greater addition to the effective capital stock. For realism and to allow a difference between gross and net investment we have the rate, δ, of physical capital depreciation strictly positive.

As with growth in A_t, there are also two potential sources of growth in Q_t. One is an endogenous source in the form of the investment experience J_t. The other is an exogenous source represented by the factor $e^{\psi t}$. Specifically, we assume that

$$Q_t = J_t{}^{\lambda} e^{\psi t}, \qquad 0 \leq \lambda < \frac{1-\alpha}{\alpha}(1-\beta), \ \psi \geq 0. \tag{6}$$

That is, the quality Q_t of investment goods of the current vintage is determined by cumulative experience which in turn reflects cumulative aggregate gross investment. The parameter λ indicates the elasticity of the quality of newly produced investment goods w.r.t. investment experience and is thus a measure of the strength of embodied learning. For short we name λ the *embodied learning parameter*. The upper bound on λ is brought in to avoid explosive growth.[7]

TABLE 1
TECHNICAL CHANGE PARAMETERS

Form of technical change	Source of technical change: Exogenous	Learning
Disembodied	γ	β
Embodied	ψ	λ

Table 1 summarizes how the technical change parameters relate to the form and the source, respectively, of technical progress. The third dimension of technical

[7] If, as in Greenwood and Jovanovic (2001), Q_t is assumed to be an isoelastic function of cumulative investment in *efficiency units*, the upper bound on λ will be $(1-\alpha)(1-\beta)$ instead.

change on which we focus relates to whether the *vehicle* of investment experience is cumulative *gross* investment or *net* investment. As the model structure is rather different in these two cases, we treat them separately, namely as the present "benchmark model" and the "alternative model" of Section 4, respectively.

We now embed the described technology in a market economy with perfect competition where learning effects appear as externalities. That is, each firm is too small to have any recognizable effect on the technology level variables A_t and Q_t.[8]

Let the output good be the numeraire. The representative firm chooses inputs so as to maximize the profit $\Pi_t = K_t^\alpha (A_t L_t)^{1-\alpha} - R_t K_t - w_t L_t$, where R_t is real cost per unit of capital services (the rental rate) and w_t is the real wage. Given equilibrium in the factor markets, the rental rate must satisfy

$$R_t = \alpha \tilde{k}_t^{\alpha-1} = \alpha \frac{Y_t}{K_t}, \tag{7}$$

where $\tilde{k}_t$ is the effective capital-labor ratio, $k_t/A_t \equiv K_t/(A_t L_t)$, as given from the supply side. We assume labor supply is inelastic and grows at the constant rate $n \geq 0$.

Since Q_t units of the capital good can be produced at the same minimum cost as one unit of the consumption good, the equilibrium price of the capital good in terms of the consumption good is

$$p_t = \frac{1}{Q_t}. \tag{8}$$

Denoting the real interest rate in the market for loans, r_t, we have the no-arbitrage condition

$$\frac{R_t - (\delta p_t - \dot{p}_t)}{p_t} = r_t, \tag{9}$$

where $\delta p_t - \dot{p}_t$ is the true economic depreciation of the capital good per time unit. So, given the interest cost, $p_t r_t$, the rental rate (or user cost) of capital is higher, the faster p_t falls, that is, the faster the quality of investment goods rises.

[8]This view of learning as a pure externality is of course a simplification. In practice firms' investment decisions bear in mind that adoption of new technology takes time and requires learning. The productivity slowdown in the 1970s has by some been seen as reflecting not a slowdown in the pace of technical progress but rather a speed-up in embodied technical change resulting in a temporary productivity delay (see, e.g., Hornstein and Krusell, 1996).

2.2 Dynamics of the production sector

From now the dating of the variables is suppressed when not needed for clarity. Let the growth rate of an arbitrary variable $x > 0$ be denoted $g_x \equiv \dot{x}/x$. Let z and x denote the output-capital ratio and the consumption-capital ratio, respectively, both in value terms, that is, $z \equiv Y/(pK)$ and $x \equiv C/(pK)$. Then, substituting (4) into (5), the growth rate of capital can be written

$$g_K = z - x - \delta. \tag{10}$$

In view of (8), $g_p = -g_Q$, and so, using (1), the growth rate of the output-capital ratio in value terms can be written

$$g_z = g_Y - g_p - g_K = (\alpha - 1)g_K + (1-\alpha)(g_A + n) + g_Q,$$

where

$$g_A = \beta g_J + \gamma, \tag{11}$$
$$g_Q = \lambda g_J + \psi, \tag{12}$$

and $n \geq 0$ is the constant growth rate of the labor force (full employment is assumed). By taking the time derivative on both sides of (3) we get $\dot{J} = I$ so that

$$g_J = \frac{I}{J} \equiv su, \tag{13}$$

where s is the saving-output ratio, i.e., $s \equiv I/Y \in [0,1]$, and u is the output-experience ratio, i.e., $u \equiv Y/J$.

It follows that

$$g_z = -(1-\alpha)(z - x - \delta) + [(1-\alpha)\beta + \lambda]\, su + (1-\alpha)(\gamma + n) + \psi, \tag{14}$$

and

$$g_u = g_Y - g_J = \alpha(z - x - \delta) - [1 - (1-\alpha)\beta]\, su + (1-\alpha)(\gamma + n), \tag{15}$$

where we have applied (1), (10), (11), (12), and (13). In these two equations we can substitute $s \equiv I/Y = 1 - x/z$, by (4) and the definitions of x and z. As a result the dynamics of the production sector is described in terms of the three endogenous variables z, x, and u. The role of the household sector is represented by x, which depends on households' consumption.

2.3 A representative household

There is a representative household with L_t members, each supplying one unit of labor inelastically per time unit. As indicated above, the growth rate of L_t is n. The household has a constant rate of time preference $\rho > 0$ and an instantaneous CRRA utility function with absolute elasticity of marginal utility of consumption equal to $\theta > 0$. Facing given market prices and equipped with perfect foresight the household chooses a plan $(c_t)_{t=0}^{\infty}$ so as to

$$\begin{aligned} \max U_0 &= \int_0^\infty \frac{{c_t}^{1-\theta}}{1-\theta} L_t e^{-\rho t} dt \qquad \text{s.t.} && (16) \\ \dot{V}_t &= r_t V_t + w_t L_t - c_t L_t, \qquad V_0 \text{ given, and} && (17) \\ \lim_{t\to\infty} V_t e^{-\int_0^t r_s ds} &\geq 0, && (18) \end{aligned}$$

where $c \equiv C/L$ is per capita consumption, $V = pK$ is financial wealth, and (18) is the No-Ponzi-Game condition.[9] Again, letting the dating of the variables be implicit, an interior solution satisfies the Keynes-Ramsey rule,

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(r-\rho) = \frac{1}{\theta}(\alpha z - \delta - g_Q - \rho), \qquad (19)$$

and the transversality condition that the No-Ponzi-Game condition holds with strict equality:

$$\lim_{t\to\infty} V_t e^{-\int_0^t r_s ds} = 0. \qquad (20)$$

The last equality in (19) follows from (9), (8), and (7).

3 The implied dynamic system

Log-differentiating the consumption-capital ratio $x = cL/(pK)$ w.r.t. t and applying (19) and (8) gives

$$\begin{aligned} g_x &= \frac{1}{\theta}(\alpha z - \delta - g_Q - \rho) + n + g_Q - g_K \\ &= \frac{1}{\theta}(\alpha z - \delta - \rho) - (z - x - \delta) + n + (1 - \frac{1}{\theta})(\lambda s u + \psi), \qquad (21) \end{aligned}$$

where $s \equiv 1 - x/z$.

[9] In case $\theta = 1$, the instantaneous utility function in (16) should be interpreted as $\ln c_t$.

The dynamics of the economy are described by the three differential equations, (21), (14), and (15), in the endogenous variables, x, z, and u. There are two predetermined variables, z and u, and one jump variable, x. A (non-trivial) *steady state* of the system is a point (x^*, z^*, u^*), with all coordinates strictly positive, such that $(x, z, u) = (x^*, z^*, u^*)$ implies $\dot{x} = \dot{z} = \dot{u} = 0$.[10] We now study existence and properties of such a steady state.

3.1 Steady state

The economy will in steady state follow a balanced growth path (BGP for short), defined as a path along which K, Q, Y, and c grow at constant rates, not necessarily positive. To ensure positive growth we need the assumption

$$\gamma + \psi + n > 0. \tag{A1}$$

This requires that at least one of these nonnegative exogenous parameters is strictly positive. Moreover, it turns out that this is needed to ensure that a viable economy (one with $Y > 0$) can be situated in a steady state.

In steady state we have $g_u = 0$. So by definition of u we get $g_Y^* = g_J^* = s^* u^*$ from (13). By setting the right-hand sides of (14) and (15) equal to nil and solving for g_Y^* $(= s^* u^*)$ and g_K^* $(= z^* - x^* - \delta)$ we thus find

$$g_Y^* = s^* u^* = \frac{\alpha\psi + (1-\alpha)(\gamma + n)}{(1-\alpha)(1-\beta) - \alpha\lambda} > 0, \tag{22}$$

and

$$g_K^* = \frac{[1 - (1-\alpha)\beta]\,\psi + (1+\lambda)(1-\alpha)(\gamma+n)}{(1-\alpha)(1-\beta) - \alpha\lambda} > 0. \tag{23}$$

That the two growth rates are strictly positive is due to (A1) combined with the restriction imposed in (6) on the embodied learning parameter λ. We see that $g_K^* \geq g_Y^*$ always. Strict inequality holds if and only if ψ (embodied exogenous technical change) or λ (embodied learning) is positive.[11] Thus, when technical progress has an embodied component, K grows faster than Y. This outcome is in line with the empirical evidence presented in, e.g., Greenwood et al. (1997).

[10] Generally, steady state values of variables will be marked by an asterisk.

[11] We have $1 - (1-\alpha)\beta > \alpha$ in view of $\alpha, \beta \in (0, 1)$.

According to (12), (13), and (22),

$$g_Q^* = \frac{(1-\alpha)\left[(1-\beta)\psi + \lambda(\gamma+n)\right]}{(1-\alpha)(1-\beta)-\alpha\lambda}. \tag{24}$$

Given (A1), we have $g_Q^* > 0$ if and only if ψ (embodied exogenous technical change) or λ (embodied learning) is positive. A mirror image of this is that the price p ($\equiv 1/Q$) of the capital good in terms of the consumption good is falling whenever there is embodied technical progress. Indeed,

$$g_p^* = -g_Q^* = -\frac{(1-\alpha)\left[(1-\beta)\psi + \lambda(\gamma+n)\right]}{(1-\alpha)(1-\beta)-\alpha\lambda}. \tag{25}$$

Whether or not Y/K is falling, the output-capital ratio in *value* terms, $Y/(pK) = z^*$, stays constant along a BGP.

By constancy of $x^*/z^* = (cL/Y)^*$ we conclude that cL is proportionate to Y in steady state. Hence $g_c^* = g_Y^* - n$ so that, combining (19) and (22), we find

$$g_c^* = \frac{1}{\theta}(\alpha z^* - \delta - g_Q^* - \rho) = \frac{(1-\alpha)\gamma + \alpha\psi + \left[(1-\alpha)\beta + \alpha\lambda\right]n}{(1-\alpha)(1-\beta)-\alpha\lambda} > 0, \tag{26}$$

where the inequality is due to (A1). The learning processes, whether in disembodied or embodied form, represented by β and λ, respectively, create and diffuse a nonrival good, technical knowledge. So learning by investing brings about a tendency to *increasing returns* to scale in the system. The way n appears in (26) indicates that the positive effect of β and λ on the growth rate of per capita consumption gets a boost via interaction with an expanding labor force, which signifies a rising scale of the economy.[12] In contrast, the disembodied and embodied *exogenous* sources of productivity growth, represented by γ and ψ, respectively, affect per capita growth independently of growth in the labor force.

To ensure boundedness of the discounted utility integral we shall throughout impose the parameter restriction

$$\rho - n > (1-\theta)\frac{(1-\alpha)\gamma + \alpha\psi + \left[(1-\alpha)\beta + \alpha\lambda\right]n}{(1-\alpha)(1-\beta)-\alpha\lambda}. \tag{A2}$$

[12]In view of cross-border technology diffusion, the growth-enhancing role of labor force growth inherent in knowledge-based growth models should not be seen as a prediction about individual countries in an internationalized world, but rather as pertaining to larger regions, perhaps the world economy.

This condition is equivalent to $\rho - n > (1-\theta)g_c^*$.

From (26) and (24) we find

$$\begin{aligned} z^* &= \frac{[(1-\alpha)\gamma + \alpha\psi]\,\theta + (1-\alpha)\,[\lambda\gamma + (1-\beta)\psi] + \{[(1-\alpha)\beta + \alpha\lambda]\,\theta + (1-\alpha)\lambda\}\,n}{\alpha\,[(1-\alpha)(1-\beta) - \alpha\lambda]} \\ &\quad + \frac{\rho+\delta}{\alpha} > 0. \end{aligned} \tag{27}$$

By (10), the steady state value of the consumption-capital ratio is $x^* = z^* - g_K^* - \delta$; into this expression (27) and (23) can be substituted (the resulting formula is huge, cf.Appendix A). The saving rate in steady state is $s^* = 1 - x^*/z^* > 0$ (see Proposition 1 below). By substituting this into (22) we get the output-experience ratio as $u^* = g_Y^*/s^*$.

Finally, by (19) the real interest rate in steady state is

$$r^* = \alpha z^* - \delta - g_Q^* = \theta g_c^* + \rho = \theta \frac{(1-\alpha)\gamma + \alpha\psi + [(1-\alpha)\beta + \alpha\lambda]\,n}{(1-\alpha)(1-\beta) - \alpha\lambda} + \rho. \tag{28}$$

The parameter restriction (A2) ensures that the transversality condition of the household is satisfied in the steady state. Indeed, from (A2) we have $r^* = \theta g_c^* + \rho > g_c^* + n = g_Y^* = g_p^* + g_K^* = g_V^*$ since $z \equiv Y/(pK) \equiv Y/V = z^*$ in steady state. It follows that the transversality condition of the household also holds along any path converging to the steady state

The following proposition summarizes the steady state properties.

Proposition 1. Assume (A1) and (A2). Then a (non-trivial) steady state, (x^*, z^*, u^*), exists, is unique, and satisfies the transversality condition (20). The steady state is associated with a BGP with the properties:
(i) $g_Y^* > 0$, $g_K^* > 0$, and $g_c^* > 0$; all three growth rates are increasing functions of the technical change parameters, γ, β, ψ, and λ, and, when learning occurs (β or λ positive), also of n;
(ii) $g_K^* \geq g_Y^*$ with strict inequality if and only if $\psi > 0$ or $\lambda > 0$;
(iii) $g_p^* < 0$ when $\psi > 0$ or $\lambda > 0$; $|g_p^*|$ is an increasing function of ψ and λ; and of γ if $\lambda > 0$; and of β if $\psi > 0$ or $\lambda > 0$;
(iv) the saving rate is $s^* = (g_K^* + \delta)/z^*$ and satisfies $0 < s^* < \alpha$;
(v) $(1-\alpha)z^* < x^* < z^*$;

(vi) $0 < u^* < z^*/(1+\lambda)$.

Proof. Existence and uniqueness was shown above, provided $s^* > 0$, which we show in connection with (iv). (i) follows immediately from (22), (23), and (26). (ii) was shown above. (iii) follows immediately from (25). (iv) is an application of $s \equiv I/Y = (\dot{K} + \delta K)/(QY) = (g_K + \delta)/z$, which follows from (5) and the definition of z. In steady state

$$s = s^* = \frac{g_K^* + \delta}{z^*} = \alpha \frac{g_K^* + \delta}{\theta g_c^* + \rho + g_Q^* + \delta} < \alpha \frac{g_K^* + \delta}{g_Y^* + g_Q^* + \delta} = \alpha, \qquad \text{(by (28))}$$

where $g_Y^* + g_Q^* = g_K^*$ follows from constancy of z and the inequality is implied by (A2), which in view of (26) is equivalent to $\theta g_c^* + \rho > g_c^* + n = g_Y^*$. The inequality $s^* > 0$ in (iv) follows from (i), (iii), and $\delta > 0$. (v) is implied by (iv) since $s^* = 1 - x^*/z^*$ and $0 < \alpha < 1$. The first inequality in (vi) follows from $u^* = g_Y^*/s^*$ together with (i) and (iv); in view of (22) and (10) we have $u^*/z^* = s^*u^*/(s^*z^*) = g_Y^*/(g_K^* + \delta) = (g_K^* - \psi)/\left[(1+\lambda)(g_K^* + \delta)\right]$, see Appendix A. As $\psi \geq 0$ and $\delta > 0$, the second inequality in (vi) follows. We have already shown that $\theta g_c^* + \rho > g_Y^*$. This inequality implies, by (28) and constancy of $z \equiv Y/(pK) \equiv Y/V$ in steady state, that $r^* > g_V^*$. The latter inequality ensures that the transversality condition (20) holds in the steady state. $\square$

Remark. As long as (A2) holds, all the formulas derived above for growth rates and for x^*, z^*, u^*, s^*, and r^* are valid for any combination of parameter values within the allowed ranges, including the limiting case $\gamma = \beta = \lambda = \psi = n = 0$. But in the absence of (A1), that is, when $\gamma = \psi = n = 0$, the steady state (x^*, z^*, u^*) is only an *asymptotic* steady state. Indeed, it has $0 < x^* < z^*$, but $u^* = 0$ because, while Y is growing at a diminishing rate, the denominator in $u \equiv Y/J$ goes to infinity at a faster speed. So, a viable economy (one with $Y > 0$ and $J < \infty$) cannot be situated in a steady state with $u^* = 0$, but it can approach it for $t \to \infty$ (and will in fact do so when (A2) holds). Thus, when (A1) is not satisfied, the formulas should be interpreted as pertaining to the asymptotic values of the corresponding ratios. And in contrast to (i) of Proposition 1, we get $g_Y^* = g_K^* = g_c^* = 0$. This should not be interpreted as if stagnation is the ultimate outcome, however. It is an example of less-than-exponential, but sustained quasi-arithmetic growth (see Groth et al., 2010). Since we are in this paper interested in the speed of convergence to a balanced growth path, we shall concentrate on the case where both (A1) and (A2) hold.

Note that violation of the upper bound on λ in (6) implies a growth potential so enormous that a steady state of the system is infeasible and the growth rate of the economy tends to be forever rising. To allow existence of a non-negative λ satisfying the parameter inequality in (6) we need $\beta < 1$, as was assumed in (2).

3.2 Stability

The steady-state properties would of course be less interesting if stability could not be established. We have, however:

Proposition 2. Assume (A1) and (A2). Let $z_0 = \bar{z}_0$ and $u_0 = \bar{u}_0$, where $\bar{z}_0$ and $\bar{u}_0$ are given positive numbers. Then there is a neighborhood of (z^*, u^*) such that for $(\bar{z}_0, \bar{u}_0)$ belonging to this neighborhood, there exists a unique equilibrium path $(x_t, z_t, u_t)_{t=0}^{\infty}$. The equilibrium path has the property $(x_t, z_t, u_t) \to (x^*, z^*, u^*)$ for $t \to \infty$.

Proof. In Appendix B it is shown that the Jacobian matrix associated with the dynamic system, evaluated in the steady state, has two eigenvalues with negative real part and one positive eigenvalue. There are two predetermined variables, z and u, and one jump variable, x. It is shown in Appendix C that the structure of the Jacobian matrix implies that for $(\bar{z}_0, \bar{u}_0)$ belonging to a small neighborhood of (z^*, u^*) there always is a unique $x_0 > 0$ such that there exists a solution, $(x_t, z_t, u_t)_{t=0}^{\infty}$, of the differential equations, (21), (14), and (15), starting from $(x_0, \bar{z}_0, \bar{u}_0)$ at $t = 0$ and converging to the steady state for $t \to \infty$. By (A2) and Proposition 1, the transversality condition (20) holds in the steady state. Hence it also holds along the converging path, which is thus an equilibrium path. All other solution paths consistent with the given initial values, $\bar{z}_0$ and $\bar{u}_0$, of the state variables diverge from the steady-state point and violate the transversality condition of the household and/or the non-negativity constraint on K for $t \to \infty$. Hence they can be ruled out as equilibrium paths of the economy. $\square$

In brief, the unique steady state is a saddle point and is saddle-point stable.

3.3 Speed of convergence

As implied by Proposition 2, two and just two eigenvalues have negative real part. In general these eigenvalues can be either real or complex conjugate numbers. In our simulations for a broad range of parameter values we never encountered complex eigenvalues. Similarly, the simulations suggested that repeated real negative eigenvalues will never arise for parameter values within a reasonable range. Hence we concentrate on the case of three real distinct eigenvalues two of which are negative. We name the three eigenvalues such that $\eta_1 < \eta_2 < 0 < \eta_3$.

Let the vector (x_t, z_t, u_t) be denoted (x_{1t}, x_{2t}, x_{3t}). The general formula for the solution to the approximating linear system is $x_{it} = C_{1i}e^{\eta_1 t} + C_{2i}e^{\eta_2 t} + C_{3i}e^{\eta_3 t} + x_i^*$, where C_{1i}, C_{2i}, and C_{3i} are constants that depend on (x_{10}, x_{20}, x_{30}). For the equilibrium path of the economy we have $C_{3i} = 0$, $i = 1, 2, 3$, so that

$$x_{it} = C_{1i}e^{\eta_1 t} + C_{2i}e^{\eta_2 t} + x_i^*, \qquad i = 1, 2, 3, \tag{29}$$

where C_{1i} and C_{2i} are constants that depend on the given initial condition $(x_{20}, x_{30}) = (\bar{z}_0, \bar{u}_0)$.

Let $\Delta_{it} \equiv x_{it} - x_i^*$, $i = 1, 2, 3$. Then the distance between the variable x_i, $i = 1, 2, 3$, at time t and its steady state value can be written $|\Delta_{it}|$. At a given t for which $|\Delta_{it}| \neq 0$ the instantaneous (proportionate) rate of decline of $|\Delta_{it}|$ is

$$-\frac{\frac{d|\Delta_{it}|}{dt}}{|\Delta_{it}|} = -\frac{\frac{d\Delta_{it}}{dt}}{\Delta_{it}} = \begin{cases} -\frac{C_{1i}e^{\eta_1 t}\eta_1 + C_{2i}e^{\eta_2 t}\eta_2}{C_{1i}e^{\eta_1 t} + C_{2i}e^{\eta_2 t}} = -\frac{\frac{C_{1i}}{C_{2i}}e^{(\eta_1-\eta_2)t}\eta_1 + \eta_2}{\frac{C_{1i}}{C_{2i}}e^{(\eta_1-\eta_2)t} + 1}, & \text{if } C_{2i} \neq 0, \\ -\eta_1, \quad \text{if } C_{2i} = 0 \text{ and } C_{1i} \neq 0. \end{cases}$$

In view of $\eta_1 < \eta_2 < 0$, for $C_{2i} \neq 0$ there exists a t_1 large enough such that for all $t > t_1$, the absolute value of $\frac{C_{1i}}{C_{2i}}e^{(\eta_1-\eta_2)t}$ is less than 1 and thereby $\Delta_{it} \neq 0$. Defining the *asymptotic speed of convergence* of x_i, denoted σ_i, as the limit of the proportionate rate of decline of $|\Delta_{it}|$ for $t \to \infty$, we thus have

$$\sigma_i = \begin{cases} -\eta_2 \text{ if } C_{2i} \neq 0, \\ -\eta_1 \text{ if } C_{2i} = 0 \text{ and } C_{1i} \neq 0. \end{cases} \tag{30}$$

When both C_{1i} and C_{2i} differ from zero, both negative eigenvalues enter the formula, (29), for the asymptotic solution, but the negative eigenvalue which is smallest in absolute value, here η_2, is the *dominant* eigenvalue.

The speed of convergence on which the empirical literature, reviewed in the introduction, first and foremost has focused is the speed of convergence of per capita output relative to trend, that is, the ratio y_t/y_t^*, where $y_t \equiv Y_t/L_t$. The asymptotic speed of convergence of this ratio is the same as that for the output-capital ratio (in value terms) in our model, namely σ_z ($\equiv \sigma_2$ as defined above).[13] Indeed, defining the trend level, y_t^*, as the level y_t would have if, given the capital-labor ratio (in value terms) $p_t k_t$, the output-capital ratio were equal to its long-run value, z^*, we have

$$\frac{y_t}{y_t^*} = \frac{y_t}{p_t k_t z^*} = \frac{z_t}{z^*}. \tag{31}$$

It follows that the ratio y_t/y_t^* has the same asymptotic speed of convergence as z_t itself.

The asymptotic speed of convergence need not generally be a good approximation to the instantaneous rate of decline of the distance of a variable to its steady-state value at a given point in time. Hence in the numerical simulations in Section 5 we shall pay some attention also to the average speed of convergence, μ_i, $i = x, z, u$, during certain time intervals. For a fixed $\varepsilon \in (0, 1)$, the *average speed of convergence* of, for instance, z during the time interval needed for the fraction $1 - \varepsilon$ of the initial distance from the steady-state value to be made good forever, is defined as the number μ_z satisfying

$$|z_{t_\varepsilon} - z^*| = |z_0 - z^*|\, e^{-\mu_z t_\varepsilon}. \tag{32}$$

where t_ε is the minimum real number such that $|z_t - z^*| < \varepsilon \cdot |z_0 - z^*|$ for all $t > t_\varepsilon$.[14] Two circumstances tend to make the average speed of convergence different from the asymptotic speed of convergence. First, in a finite distance from the steady state, the nonlinearities of the dynamic system play a role. Second, even the approximating linear dynamic system will have its average speed of convergence affected by (i) the initial conditions, (ii) both negative eigenvalues, cf. (29), and (iii) the allowed maximum proportionate distance ε. This ambiguity of μ_z explains

[13]As $(x_1, x_2, x_3) = (x, z, u)$, when convenient, we use the more concrete notation, σ_x, σ_z, and σ_u, rather than σ_1, σ_2, and σ_3, respectively.

[14]As the sign of $z_t - z^*$ may change during the adjustment process, the definition refers to absolute values.

the popularity of the asymptotic speed of convergence as a benchmark indicator in the literature.

A further complication arises because two alternative situations are possible: the situation where the dynamic system, (21), (14), and (15), is indecomposable and the situation where it is not. We say the dynamic system is *indecomposable* if all three variables, x, z, and u, are mutually dependent. On the other hand the system is *decomposable* if one or two of the three differential equations are uncoupled from the remaining part of the system. By inspection of the right-hand sides of (21), (14), and (15), we see that, apart from $s \equiv 1-x/z$, only four parameters enter the coefficients of x, z, and u, namely λ, β, α, and θ. The values of these parameters govern whether the dynamic system is indecomposable or decomposable. Two parameter value combinations lead to decomposable situations, namely *Case* $\mathfrak{D}$*1:* $\lambda = 0 = \beta$, $\theta \neq \alpha$; and *Case* $\mathfrak{D}$*2:* $\lambda = 0$, $\beta \geq 0$, $\theta = \alpha$ ($\mathfrak{D}$ for decomposability).[15]

When learning is operative ($\lambda > 0$ or $\beta > 0$), the dynamic system is indecomposable (at least when $\theta \neq \alpha$). Consequently the key variables, x, z, and u, have the same asymptotic speed of convergence. Indeed:

Proposition 3. Assume (A1) and (A2). Let $x_{i0} \neq x_i^*$, $i = 1, 2, 3$. If $\lambda > 0$ or ($\beta > 0$ and $\theta \neq \alpha$), then generically $C_{2i} \neq 0$, $i = 1, 2, 3$, and so the same asymptotic speed of convergence, $-\eta_2$, applies to all three variables in the dynamic system. This will also be the asymptotic speed of convergence of y_t/y_t^*.

Proof. See Appendix D.

The explanation of this result is that as long as at least part of technical progress is due to learning by investing, the laws of movement for the output-capital ratio, z, and (at least when $\theta \neq \alpha$) the consumption-capital ratio, x, are coupled to the law of movement of the output-experience ratio, u. So the dominant eigenvalue for the z and x dynamics is the same as that for the u dynamics, namely η_2.

[15]In Appendix D the concepts of decomposability and indecomposability are formally defined in terms of properties of the Jacobian matrix associated with the dynamic system.

3.4 Discontinuity of the asymptotic speed of convergence for x and z when learning disappears

When the dynamic system is decomposable, however, the movement of x and z is no longer linked to the slowly adjusting output-experience ratio and therefore, as we shall see, x and z adjust considerably faster. To be specific, consider first the Case $\mathfrak{D}1$. Here learning by investing is not operative, neither in embodied nor in disembodied form. Then the differential equations for the consumption-capital ratio, x, and the output-capital ratio, z, are uncoupled from the dynamics of the output-experience ratio, u. The evolution of x and z is entirely independent of that of u which in turn, however, *depends* on the evolution of x and z. In any event, x and z are the two variables of primary economic interest, whereas u is of economic interest only to the extent that its movement affects that of x and z; in Case $\mathfrak{D}1$ it does not. As $\theta \neq \alpha$, the (x, z) subsystem cannot be decomposed further.

Case $\mathfrak{D}2$ is the case where, due to the knife-edge condition $\theta = \alpha$, the dynamics of the jump variable x become independent of the dynamics of *both* state variables, z and u, when $\lambda = 0$, i.e., when embodied learning is absent. Indeed, with $\theta = \alpha$ and $\lambda = 0$, the differential equation for x reduces to $\dot{x} = (x - (\delta+\rho)/\alpha + \delta + n + (1 - 1/\alpha)\psi)x$. Then the transversality condition of the household can only be satisfied if $x = x^*$ for all t. A shift in a parameter affecting x^* implies an instantaneous jump of x to the new x^*. In this case we define the speed of convergence of x as infinite. The state variables z and u will still adjust only sluggishly.

An interesting question is how the asymptotic speed of convergence of an endogenous variable changes when a parameter value changes so that the system shifts from being indecomposable to being decomposable. To spell this out we need more notation. Consider again Case $\mathfrak{D}1$ where learning of any form is absent and $\theta \neq \alpha$. We let the eigenvalues associated with the subsystem for x and z in this case be $\eta_1 = \bar{\eta}_1$ and $\eta_3 = \bar{\eta}_3$, where $\bar{\eta}_1 < 0 < \bar{\eta}_3$. The third eigenvalue, η_2, belongs to the total system but does not in this case influence the x and z dynamics; it is denoted $\bar{\eta}_2$ and turns out to equal $-g_Y^* < 0$ (see Appendix E). In the sub-case of $\mathfrak{D}2$ where $\beta = 0$ in addition to $\theta = \alpha$ and $\lambda = 0$, we let the values taken by the eigenvalues be denoted $\tilde{\eta}_1$, $\tilde{\eta}_2$, and $\tilde{\eta}_3$.

As documented in Table 3 below and Appendix G, for realistic parameter values, $\bar{\eta}_2$ and $\tilde{\eta}_2$ are smaller in absolute value than $\bar{\eta}_1$ and $\tilde{\eta}_1$, respectively. That is, from an empirical point of view we can assume $\bar{\eta}_1 < \bar{\eta}_2 < 0 < \bar{\eta}_3$ as well as $\tilde{\eta}_1 < \tilde{\eta}_2 < 0 < \tilde{\eta}_3$. Given these inequalities, the asymptotic speed of convergence of one or more of the variables changes discontinuously as learning, embodied as well as disembodied, tends to vanish:

Proposition 4. Assume (A1) and (A2). Let $\bar{\eta}_1 < \bar{\eta}_2 < 0 < \bar{\eta}_3$ and $\tilde{\eta}_1 < \tilde{\eta}_2 < 0 < \tilde{\eta}_3$. We have:
(i) If $\theta \neq \alpha$, then, for $(\beta, \lambda) \to (0,0)^+$, in the limit where learning disappears, an upward switch occurs in the asymptotic speed of convergence for x and z from the value $-\bar{\eta}_2$ to $-\bar{\eta}_1$.
(ii) If $\theta = \alpha$, $\beta = 0$, and $\lambda > 0$, then, for $\lambda \to 0^+$, in the limit where learning disappears, two upward switches occur. The asymptotic speed of convergence for x shifts from the value $-\tilde{\eta}_2$ to infinity. And the asymptotic speed of convergence for z shifts from the value $-\tilde{\eta}_2$ to $-\tilde{\eta}_1 > -\tilde{\eta}_2$.
(iii) If $\theta = \alpha$, $\lambda = 0$, and $\beta \geq 0$, the asymptotic speed of convergence for x is always infinite. But for $\beta \to 0^+$, in the limit where learning disappears, the asymptotic speed of convergence for z switches from the value $-\tilde{\eta}_2$ to $-\tilde{\eta}_1 > -\tilde{\eta}_2$.

Proof. See Appendix E. $\square$

Result (i) is the generic result on which our numerical calculations concentrate. The intuition behind result (i) is that as long as at least a part of technical progress is due to learning by investing (either λ or β positive), the laws of movement for x and z are generically coupled to the law of movement of the sluggish output-experience ratio, u. Indeed, convergence is slow when physical capital accumulation is coupled to a slow-moving second kind of "capital", knowledge from investment experience. When learning by investing disappears, however, the movement of x and z is no longer hampered by this slow-adjusting factor and therefore x and z adjust much faster. In for instance Figure 1 below, for $\beta = \psi = 0$ and with the baseline parameter combination indicated in Table 2 below, this discontinuity in the limit shows up as a jump in the convergence speed for x and z from 0.03 to above 0.08 when $\lambda \to 0^+$.

The intuition behind result (ii) is similar, except that here the dynamics become fully recursive in the limit. This has two implications. First, the jump variable, x, ceases to be influenced by the movement of the state variables, z and u, and can therefore adjust with infinite speed. Second, z ceases to be influenced by the slow-adjusting u. Result (iii) refers to a situation where the asymptotic speed of convergence of the jump variable x is infinite even for $\beta > 0$ (that is, when disembodied learning is present) and remains so in the limit for $\beta \to 0^+$. Moreover, in the limit z ceases to be influenced by the slow-adjusting u and so the asymptotic speed of convergence of z jumps.

Most empirical evidence suggests $\theta \geq 1 > \alpha$. So the results (ii) and (iii), relying on the knife-edge case $\theta = \alpha$, are of limited interest. On the other hand, this case allows an explicit solution for the time path of one or more of the variables. Therefore at several occasions this case has received attention in the literature, for example in connection with the Lucas (1988) human capital accumulation model (see Xie (1994) and Boucekkine and Ruiz-Tamarit (2004)).

For mathematical convenience this section has talked about limiting values of the asymptotic speed of convergence for the two forms of learning approaching zero. We may turn the viewpoint round and end this section with the conclusion that as soon as learning from gross investment becomes *positive,* and thereby part of the growth engine, the asymptotic speed of convergence displays a discrete fall.

4 Alternative model: Learning from *net* investment

The benchmark model above assumes that learning stems from gross investment. What difference does it make if instead the vehicle of learning, whether embodied or disembodied, is *net* investment? To provide an answer, we now describe the case where it is the experience originating in cumulative *net* investment that drives productivity. This case seems less plausible, since presumably the *total* amount of newly produced equipment provides new stimuli and experience from which to learn, whatever the depreciation on existing equipment.[16] Yet the net investment case is certainly the more popular case in the literature, probably because of its mathematical simplicity.[17]

We replace (3) by $J_t = \int_{-\infty}^{t} I_\tau^n d\tau$, where I_τ^n denotes net investment (measured in efficiency units), $Q_\tau I_\tau - \delta K_\tau$, at time τ.[18] So $\dot{K}_\tau = I_\tau^n$, and by integration follows that J_t, the indicator of cumulative investment experience, now equals K_t. From this we see a reason why the net investment approach appears less plausible than the gross investment approach. If for some time interval capital depreciation should exceed gross investment, so that net investment is negative, then the experience index J goes down straight away in spite of the arrival of newly produced equipment embodying up-to-date technology.

Now (11) and (12) become $g_A = \beta g_K + \gamma$ and $g_Q = \lambda g_K + \psi$, respectively. To avoid growth explosion, we need that λ satisfies $0 \leq \lambda < (1-\alpha)(1-\beta)$, which is sharper than the restriction in (6). Since J is no longer distinct from K, the dynamic system reduces to two dimensions:

$$g_x = \frac{1}{\theta}(\alpha z - \delta - \rho) - \left[1 - (1 - \frac{1}{\theta})\lambda\right](z - x - \delta) + n + (1 - \frac{1}{\theta})\psi, \quad (33)$$

$$g_z = -\left[(1-\alpha)(1-\beta) - \lambda\right](z - x - \delta) + (1-\alpha)(\gamma + n) + \psi, \quad (34)$$

where, as before, $x \equiv C/(pK)$ and $z \equiv Y/(pK)$.

Also this simpler model has a unique saddle-point stable steady state (see Appendix F). The long-run growth rate of per capita consumption is

$$g_c^* = \frac{(1-\lambda)(1-\alpha)\gamma + \left[\alpha + (1-\alpha)\beta\right]\psi + \left[(1-\alpha)\beta + \alpha\lambda\right]n}{(1-\alpha)(1-\beta) - \lambda}.$$

To ensure that the discounted utility integral is bounded and the transversality condition satisfied, we need that $\rho - n > (1-\theta)g_c^*$. We assume the parameter values are such that this inequality is fulfilled.

[16] In Solow's words "even the 'Titanic' is still contributing to maritime productivity" (Solow, 1967, p. 39).

[17] As mentioned in the introduction, leading textbooks concentrate on this case and predominantly on learning in the disembodied form.

[18] We define net investment this way to get a framework nesting a series of available models in the literature.

Again, the relative price of capital equipment is falling if there is embodied technical progress. Indeed,

$$g_p^* = -g_Q^* = -\frac{(1-\alpha)\left[(1-\beta)\psi + \lambda(\gamma+n)\right]}{(1-\alpha)(1-\beta)-\lambda} < 0,$$

if $\psi > 0$ or $\lambda > 0$. Embodied technical progress also leads to a falling Y/K so that ultimately the output-capital ratio in value terms, $Y/(pK) \equiv z$, stays constant.

This model subsumes several models in the literature as special cases:

1. The simple neoclassical growth model: $\gamma > 0, \beta = \lambda = \psi = 0$.

2. Arrow-Romer model, the "Arrow version": $0 < \beta < 1, \gamma = \lambda = \psi = 0$.

3. Arrow-Romer model, the "Romer version": $\beta = 1, n = \gamma = \lambda = \psi = 0$.

4. Jovanovic and Rousseau (2002): $0 < \lambda < 1 - \alpha$, $\delta = \gamma = \beta = \psi = \theta = 0$.[19]

5. Boucekkine et al. (2003): knife-edge link between λ and β: $\lambda = (1-\alpha)(1-\beta), \gamma = \psi = 0$.[20]

Number 2 and 3 in the list are the standard textbook models of learning by investing referred to in the first paragraph of Section 2. The original contributions in Arrow (1962) and Romer (1986) are more sophisticated than these popular models from textbooks; moreover, Arrow (1962) in fact studied the case of learning from gross investment. Going into detail with this would take us too far, however.

We now return to the general version of the net-investment based learning model, summarized in (33) and (34). The case $\theta > \alpha$ is the empirically plausible case to be considered in the numerical simulations below. In this case (in fact whenever $\theta \neq \alpha$) the dynamic system is indecomposable even for $\lambda = 0$. The absolute value of the unique negative eigenvalue is the common asymptotic speed of convergence for x and z.

[19] The authors assume linear utility ($\theta = 0$), so that $r = \rho$ in equilibrium. On the other hand, the authors extend the model by incorporating a second capital good (like structures), not taking part in the embodied learning. And it is only in the theoretical analysis that the simplifying assumption that learning comes from net investment is relied upon.

[20] Strictly speaking, this description of Boucekkine et al. (2003) only covers the case $n = 0$. By letting the learning effects come from net investment *per capita,* the authors can allow $n > 0$ without growth explosion, unlike the "Romer version" above.

Contrary to the benchmark model of the preceding sections, this model version exhibits no discontinuity in the asymptotic speed of convergence in the limit as $(\beta, \lambda) \to (0, 0)^+$, i.e., as learning disappears. Indeed, when learning originates in net investment, the variable that drives productivity is cumulative net investment and thereby simply the capital stock. The dynamics of the capital stock is part of the dynamics of x and z whether or not any learning parameter is positive. It is otherwise in the benchmark model where as soon as a learning parameter becomes positive, the dynamics of x and z is coupled to the dynamics of an entirely new variable, cumulative *gross* investment. In the limiting case of $\beta = \lambda = 0$, i.e., no learning, the two models are of course identical.

We are now ready to consider numerical results for both the benchmark model of the preceding sections and the present simpler, alternative model.

5 Results from simulations

Proposition 4 implies the qualitative result that as soon as *learning* from gross investment becomes part of the growth engine, the asymptotic speed of convergence of x and z drops. Considering reasonable calibrations, four main *quantitative* questions suggest themselves. First, by how much does the *introduction* of learning lower speed of convergence? Second, if *more weight* is put on learning and less weight on unspecified exogenous sources of technical progress, by how much is the speed of convergence affected? Third, how much does it matter whether learning is based on gross or net investment? Fourth, when technical change is endogenous through learning, does *embodiment* of this technical change then *raise* the speed of convergence, as growth theory from the 1960s would predict? Numerical simulations, addressing these questions, are presented in the following.

What we call baseline values of the background parameters are listed in Table 2. Tables and graphs below are based on these parameter values which may be considered standard and noncontroversial. Appendix G contains sensitivity analysis, in particular with respect to the value of θ, since this parameter affects the asymptotic speed of convergence considerably.

TABLE 2

BASELINE VALUES OF BACKGROUND PARAMETERS

Preference parameters	$\rho = 0.02$, $\theta = 1.75$
Production parameters	$\alpha = 0.324$, $\delta = 0.05$
Population growth	$n = 0.01$

NOTE. The time unit is one year.

The parameters of primary interest are the technical change parameters: β, γ, λ, and ψ. The empirical literature does not provide firm conclusions as to the relative importance of learning by investing (including learning spillovers) versus other sources of long-run growth and the relative importance of embodied learning vs. disembodied learning. To clarify the potential quantitative role of these parameters for the speed of convergence, we vary them in pairs in the simulations so as to hold constant the growth rate of per capita consumption. Specifically, if one technical change parameter is increased, another technical change parameter is decreased so as to ensure $g_c^* = 0.02$. In this way we can study the role of the composition of technical progress without interference from the size of the growth rate.

5.1 The role of embodied learning

Panel A of Table 3 presents major results for the case where the strength, λ, of embodied learning vis-a-vis the strength, γ, of disembodied exogenous progress is in focus (at the same time as $\beta = \psi = 0$). The baseline combination of λ and γ appears in the second row. With this combination together with the baseline values of the background parameters, cf. Table 2, important stylized facts for a modern industrialized economy are reproduced by the model. Per capita consumption grows at a rate of 2% per year, 26% of output is devoted to investment,[21] and the output-capital ratio is 0.40. Moreover, embodied technical change accounts for 60% of the growth in per capita output, leaving the remaining 40% as due to disembodied technical change ($\gamma/g_c^* = 0.4$). This corresponds to the estimates by Greenwood et al. (1997). With $g_p^* = -0.03$, the baseline case roughly captures the observation

[21] When taking investment in consumer durables into account in addition to fixed capital investment, an investment share of GDP of around one fourth is empirically realistic.

that the relative price of capital equipment vis-a-vis consumption goods has in the US declined at a yearly rate of 3% in the period 1950-1990 (Greenwood et al. 1997).[22] The *asymptotic* speed of convergence, σ_i, $i = x, z, u$, amounts to about 1.6% per year, which in fact corresponds to estimates in the seminal studies by Mankiw et al. (1992) and Barro and Sala-i-Martin (2004).

As argued in Section 3.3. the asymptotic speed of convergence need not generally be a good approximation to the speed of convergence at a *finite* distance from the steady state. Hence, Panels A, B, and C of Table 3 also report μ_i, $i = x, z, u$, which are *average* speeds of convergence, in percentage points, during the first *half-life* of the initial distance to the steady state when z and u initially are 10% below their steady-state values.[23] For this case the baseline row indicates average speeds of convergence of x and z close to 4% per year and thus somewhat above the asymptotic speed of convergence. For u, however, the average speed of convergence is in this case slightly below the asymptotic speed of convergence.

Comparing the rows in Panel A of Table 3, we see the impact of raising embodied learning as a source of technical change while lowering disembodied exogenous technical change so as to hold constant the per capita consumption growth rate at 2% per year. For small λ the main source of technical progress is thus disembodied exogenous technical change, while for large λ it is embodied learning from gross investment.

[22] We only say "roughly captures" because in our model, p is the relative price of an *aggregate* capital good, whereas the 3% from Greenwood et al. (1997) excludes structures from the price index. On the other hand, studies by Jovanovic and Rousseau (2002) and Sakellaris and Wilson (2004) suggest a speed up of the fall in the relative price of capital equipment due to the expanding role of computers and IT-related technology.

[23] Because convergence may be non-monotonic, we define the *half-life* of the distance to the steady state as the time needed for half of the initial distance to the steady state to be made good *forever*.

TABLE 3

Speed of convergence as the embodied learning parameter, λ, rises

	λ	γ	g_c^*	r^*	s^*	z^*	g_p^*	γ/g_c^*	σ_x, σ_z	σ_u	μ_x	μ_z	μ_u
A. Adjustment of γ such that $g_c^* = 0.02$													
	0.00	0.02	0.02	0.055	0.25	0.32	-0.00	1.00	8.77	3.00	8.11	8.07	1.94
BL	0.83	0.01	0.02	0.055	0.26	0.40	-0.03	0.40	1.57	1.57	3.87	4.28	1.12
	1.39	0.00	0.02	0.055	0.27	0.45	-0.04	0.00	0.80	0.80	1.55	1.90	0.61
B. No adjustment of γ													
	0.00	0.02	0.02	0.055	0.25	0.32	-0.00	1.00	8.77	3.00	8.11	8.07	1.94
	0.83	0.02	0.04	0.090	0.25	0.56	-0.04	0.50	2.60	2.60	4.57	4.95	1.97
	1.39	0.02	0.08	0.160	0.26	1.03	-0.13	0.25	2.33	2.33	2.74	2.88	1.95
C. Learning from I^n; adjustment of γ such that $g_c^* = 0.02$													
	0.00	0.02	0.02	0.055	0.25	0.32	-0.00	1.00	8.77	–	8.13	8.09	–
	0.46	0.01	0.02	0.055	0.26	0.40	-0.03	0.40	2.75	–	2.54	2.54	–
	0.58	0.00	0.02	0.055	0.27	0.45	-0.04	0.00	1.20	–	1.12	1.12	–

Notes. Baseline values of background parameters as given in Table 2; β=0, ψ=0; σ_i is the asymptotic speed of convergence for $i = x, z, u$, and μ_i is the corresponding average speed of convergence, during the first halflife of the distance to steady state when z and u are initially 10% lower than their steady-state values; σ_i and μ_i shown in percentage points. BL = baseline case; Panels A and B: embodied learning from gross investment; Panel C: embodied learning from net investment.

Several features are worth mentioning. First, if $\lambda = 0$ (the standard neoclassical growth model), the asymptotic speed of convergence for x and z equals 8.78% and the average speed of convergence is at the same level. With the indicated baseline value of λ, however, all the measures of convergence speed take on significantly lower values. To obtain an asymptotic speed of convergence at the indicated level of 2%, the standard neoclassical growth model requires an output elasticity with respect to capital as high as $\alpha = 0.75$ (interpreted as reflecting the productive role of an expanded measure of capital including human capital, cf. Barro and Sala-i-Martin, 2004, p. 110). Table 3 shows that with embodied learning from investment accounting for 60% of the growth in per capita output (or consumption), an asymptotic speed of convergence of around 2% is obtained without requiring the output elasticity with respect to capital to exceed its "standard value" of one third.

Second, the impact of raising embodied learning further while lowering disembodied exogenous technical change results in still lower speeds of convergence. The explanation is that a higher relative weight of learning in the "growth engine" means a higher relative weight of the slow-adjusting cumulative investment experience that feeds learning.

The reason that we adjust γ downwards when raising λ is that otherwise the values of several key variables would not remain within ranges that seem empirically relevant (from a historical perspective). To document this, Panel B of Table 3 leaves γ fixed. The result is that the growth rate of per capita consumption rises to 8%; the rate of interest rises to 16%; and the output-capital ratio rises to a value above 1. Since such values are far away from what we have observed in the data, the associated speeds of convergence (higher than in Panel A) are of limited interest. Of course, here we take a backward-looking perspective. It can not be ruled out that the shift to a higher λ which seems associated with the computer revolution will result in higher *future* per capita growth, as conjectured by, e.g., Jovanovic and Rousseau (2002), thus speeding up the adjustment process.[24]

In Panel C of Table 3 learning stems from net investment rather than gross investment as in the model of Section 4. The second row of Panel C shows that for $\lambda = 0.455$ this model reproduces the same magnitudes of key endogenous variables as the baseline row in Panel A. Again, along with a rise in the fraction of the given g_c^* that is due to embodied learning there is a decline in the different measures of the speed of convergence.

With regard to the average speed of convergence, we have experimented with other initial distances from the steady state and with larger values of the fraction, $1-\varepsilon$, of the initial distance from the steady state to be made good forever (in the last three columns of Table 3, we had $\varepsilon = \frac{1}{2}$).[25]

As Table 4 shows: a) the average speeds of convergence tend to be somewhat larger than the asymptotic speeds of convergence, reflecting that in addition to the

[24]The last row in Panel B, including the sizeable $-g_p^*$, is not far from the (informal) forecast of growth "in the coming decades" suggested by Jovanovic and Rousseau (2002). For the case of linear utility (i.e., $\theta = 0$) and $\gamma = \beta = \psi = 0$, Jovanovic and Rousseau derive an explicit formula showing the speed of convergence to be decreasing in β. But since the authors do not adjust any other parameter, also growth is rising in the exercise.

[25]With regard to for example the variable z, let t_ε be the minimum real number such that $|z_t - z^*| < \varepsilon \cdot |z_0 - z^*|$ for all $t > t_\varepsilon$. Then, simulating the dynamic system by the Relaxation Algorithm, described in Trimborn et al. (2008), we estimate t_ε. Finally, we apply the formula $\mu_z = -(\ln \varepsilon)/t_\varepsilon$, cf. (32).

negative eigenvalue smallest in absolute value also the other negative eigenvalue is operative; b) the average speeds of convergence tend to be closer to the asymptotic speed of convergence when both predetermined variables, z and u, start out at the same side of their respective steady-state values rather than at the opposite side and when the required fraction, $1-\varepsilon$, is large; and c) the pattern of dependency on the relative strength of embodied learning is qualitatively the same (at least "roughly") for the average speed of convergence as for the asymptotic speed of convergence (the more so the larger the required distance reduction).

TABLE 4.
ASYMPTOTIC AND AVERAGE SPEED OF CONVERGENCE AS THE EMBODIED LEARNING PARAMETER, λ, RISES AND γ IS ADJUSTED. ALTERNATIVE INITIAL CONDITIONS AND REQUIRED DISTANCE REDUCTIONS

					$1-\varepsilon=0.90$			$1-\varepsilon=0.99$		
λ	γ	g_c^*	σ_x, σ_z	σ_u	μ_x	μ_z	μ_u	μ_x	μ_z	μ_u
A. $z_0/z^*=0.9$, $u_0/u^*=0.9$										
0.00	0.02	0.02	8.77	3.00	8.42	8.39	2.51	8.56	8.56	2.73
0.83	0.01	0.02	1.57	1.57	2.01	2.10	1.38	1.76	1.78	1.49
1.39	0.00	0.02	0.80	0.80	0.91	0.94	0.72	0.84	0.86	0.76
B. $z_0/z^*=1.1$, $u_0/u^*=1.1$										
0.00	0.02	0.02	8.77	3.00	9.12	9.15	2.72	8.97	8.97	2.87
0.83	0.01	0.02	1.57	1.57	2.22	2.32	1.50	1.85	1.89	1.54
1.39	0.00	0.02	0.80	0.80	0.99	1.03	0.78	0.89	0.91	0.79
C. $z_0/z^*=0.9$ $u_0/u^*=1.1$										
0.00	0.02	0.02	8.77	3.00	8.42	8.39	4.06	8.56	8.56	3.48
0.83	0.01	0.02	1.57	1.57	15.30	14.36	2.67	2.45	2.60	1.98
1.39	0.00	0.02	0.80	0.80	2.01	2.68	1.45	1.15	1.24	1.03
D. $z_0/z^*=1.1$, $u_0/u^*=0.9$										
0.00	0.02	0.02	8.77	3.00	9.12	9.15	3.61	8.97	8.97	3.29
0.83	0.01	0.02	1.57	1.57	3.79	17.30	2.20	2.19	2.35	1.83
1.39	0.00	0.02	0.80	0.80	1.48	1.91	1.17	1.03	1.12	0.95

NOTES. Baseline values of background parameters as given in Table 2; β=0, ψ=0; σ_i is the asymptotic speed of convergence for $i=x,z,u$, and μ_i is the corresponding average speed of convergence in different situations; all speeds of convergence in percentage points. Learning is based on gross investment.

Keeping this in mind, we shall from now on concentrate on the asymptotic speed of convergence of x and z, henceforth abbreviated SOC. Figure 1 gives a detailed portrait of the dependency of SOC on both the relative weight of embodied learning

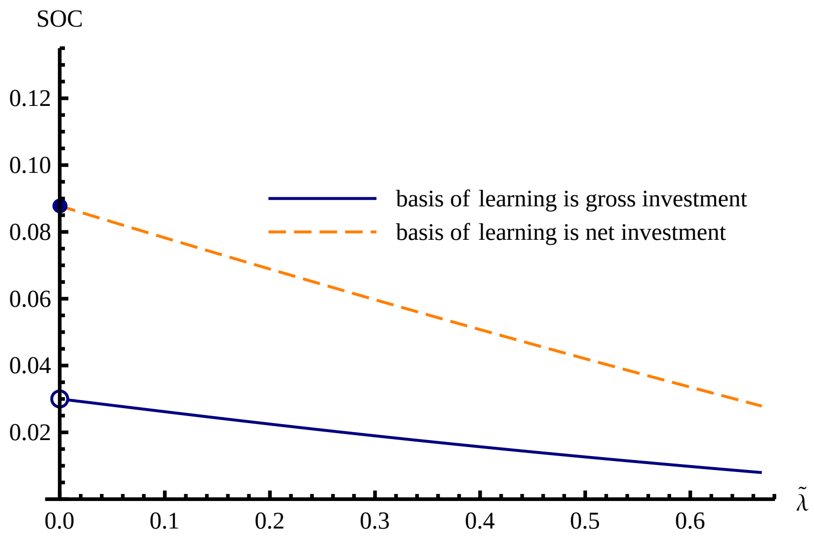

Figure 1: Asymptotic speed of convergence as the normalized embodied learning parameter, $\tilde{\lambda}$, rises and γ is adjusted so as to maintain $g_c^* = 0.02$. Note: $\beta = 0, \psi = 0; \alpha = 0.324$.

in the growth engine and the vehicle of learning. The solid curve shows SOC when the vehicle of learning is gross investment. At a significantly higher position is the dashed curve which shows SOC when the vehicle of learning is net investment. The variable along the horizontal axis, named $\tilde{\lambda}$, is the learning parameter normalized so as to ensure a common support, i.e., $\tilde{\lambda} \in [0, 1]$, for the two cases. Specifically, $\tilde{\lambda} \equiv \lambda\alpha/[(1-\alpha)(1-\beta)]$ when learning is based on gross investment; and $\tilde{\lambda} \equiv \lambda/[(1-\alpha)(1-\beta)]$ when learning is based on net investment. The range for $\tilde{\lambda}$ shown in the figure does not go beyond 0.67 because higher values would require a negative value of γ to maintain $g_c^* = 0.02$.

The intuition behind that SOC is lower when the basis of learning is gross investment than when it is net investment, is that the former basis involves more overhang from the past. Thereby the transitional dynamics becomes more sluggish.

Figure 1 also displays the pronounced discontinuity in SOC for x and z as learning from gross investment becomes positive. This discontinuity, drawn attention to in Proposition 4, appears as a conspicuous drop from the solid bullet on the vertical axis in Figure 1 to the hollow bullet. The solid bullet is situated where the dashed curve hits the vertical axis. This is because the two models are identical in the special case of no learning. As we already know from Section 4, when the learning parameter in the "net-investment framework" shifts from nil to positive, no discontinuity in SOC arises. In contrast, in the "gross-investment framework" such a shift couples the dynamics of x and z to that of a variable not involved before, namely the slow-adjusting cumulative gross investment.

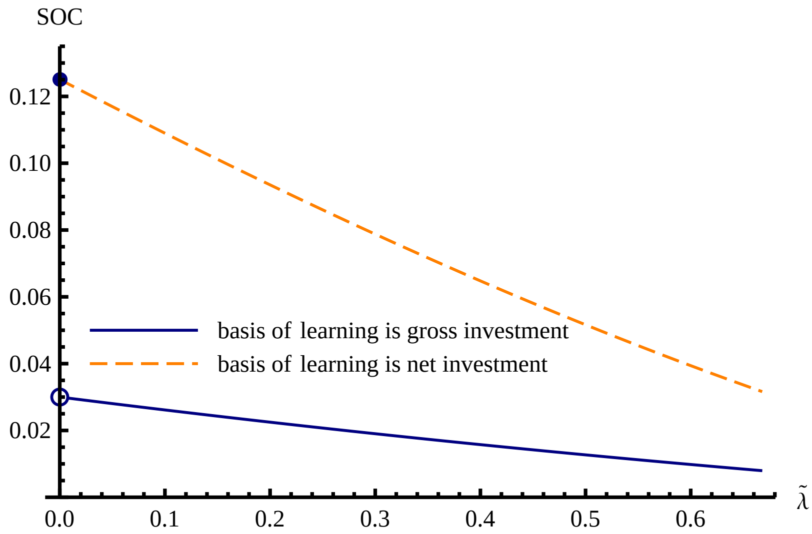

Figure 2: Asymptotic speed of convergence as the normalized embodied learning parameter, $\tilde{\lambda}$, rises and ψ is adjusted so as to maintain $g_c^* = 0.02$. Note: $\beta = 0, \gamma = 0; \alpha = 0.324$.

Figure 2 is analogue to Figure 1 except that it is not γ but the *embodied* exogenous technical change parameter, ψ, that is adjusted when the normalized embodied learning parameter rises (while $\gamma = \beta = 0$). The resulting pattern is rather similar to that in Figure 1. SOC is quite sensitive to the fraction of embodied productivity increases coming from learning rather than from unspecified exogenous factors. And the vertical distance between the two curves is again substantial, in fact even larger than before. That is, when a combination of embodied learning and embodied exogenous technical change drives productivity increases, SOC is very sensitive to whether learning is based on net or gross investment.

5.2 The role of disembodied learning

Although, for example, Greenwood et al. (1997) found that disembodied technical change accounts for only about 40% of the growth in output per hours worked, still the impact of whether its source is learning or exogenous, i.e., originating in factors outside the model, is of interest. Figure 3 shows how SOC changes as the strength, β, of disembodied learning is raised at the same time as disembodied exogenous technical change is lowered so as to hold constant g_c^* (while $\lambda = \psi = 0$).[26] The pattern is quite similar to that in Figure 1 for the embodied learning case: a) a rise in the fraction of disembodied technical change coming from learning rather than being exogenous lowers SOC; b) there is a substantial drop in SOC for x and z as learning from gross investment becomes positive; and c) going from the stippled "net-investment curve" to the solid "gross-investment curve" entails more than a halving of SOC.

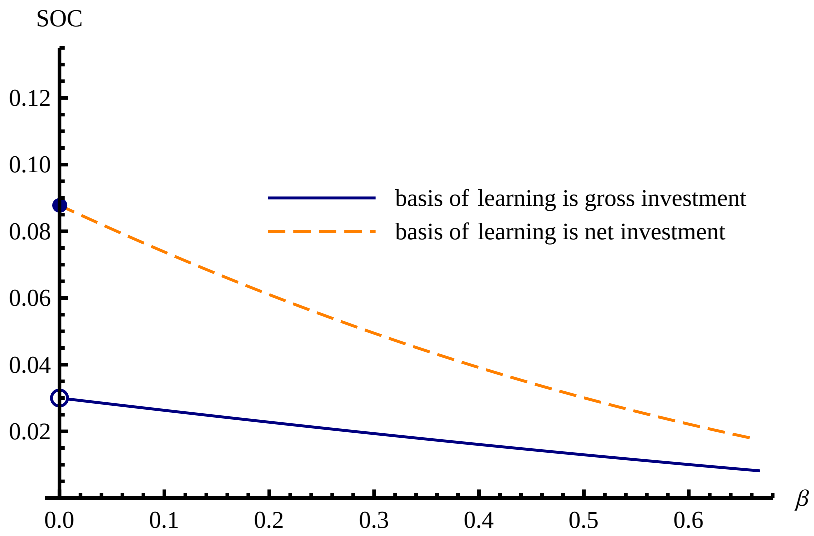

Figure 3: Asymptotic speed of convergence as the disembodied learning parameter, β, rises and γ is adjusted so as to maintain $g_c^* = 0.02$. Note: $\lambda = 0, \psi = 0; \alpha = 0.324$.

In Figure 4 it is instead the strength, ψ, of *embodied* exogenous technical change that is adjusted as β rises (while $\gamma = \lambda = 0$). Again we see: a) a falling SOC; b) a significant discontinuity as learning becomes operative; and c) a persistent difference in the level of the two curves.

The overall conclusion from this and the previous subsection is that the *source* of technical change and the *vehicle* (basis) of learning matter a lot for SOC. That learning slows down SOC reflects that the tendency of, say, a high output-capital

[26] Again the range of the abscissa is limited to values not requiring the adjusting variable to take on a negative value to maintain $g_c^* = 0.02$. This principle is also followed in the ensuing figures.

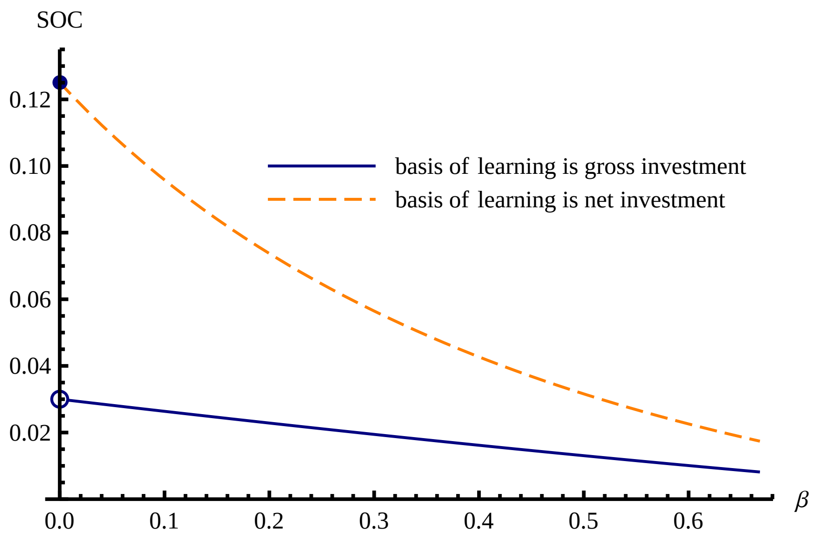

Figure 4: Asymptotic speed of convergence as the disembodied learning parameter, β, rises and ψ is adjusted so as to maintain $g_c^* = 0.02$. Note: $\lambda = 0, \gamma = 0; \alpha = 0.324$.

ratio to speed up capital deepening and thereby diminish itself is partly offset through the concomitant speed up of the productivity advances generated by the investment. That this offsetting influence is stronger when gross investment is the vehicle of learning rather than net investment reflects that in the former case an additional, slow-moving state variable, cumulative gross investment, interferes with the dynamics. Moreover, these features go through whether technical change is of embodied or disembodied form.

5.3 The role of embodiment as such

Empirical studies by, e.g., Jovanovic and Rousseau (2002) and Sakellaris and Wilson (2004) find that ICT technologies result in faster decline in the relative price of capital equipment vis-a-vis consumption goods than earlier technology revolutions. This can be seen as reflecting a rising tendency for technical change to take the embodied form.[27]

Is such a tendency likely to result in a higher speed of convergence for the

[27] Tables A, D, E, and F in the appendix show that g_p^* is quite sensitive to a rise in the fraction of technical change that is embodied. On the other hand, if embodied exogenous technical change, ψ, is the adjusting parameter when embodied learning rises (Table B), g_p^* is unaffected (but high since all technical change is in this case embodied). Indeed, the constancy of g_p^* in *this* case follows analytically from the formula (25) with $\gamma = \beta = 0$ and ψ as a function of λ so that $g_c^* = 0.02$.

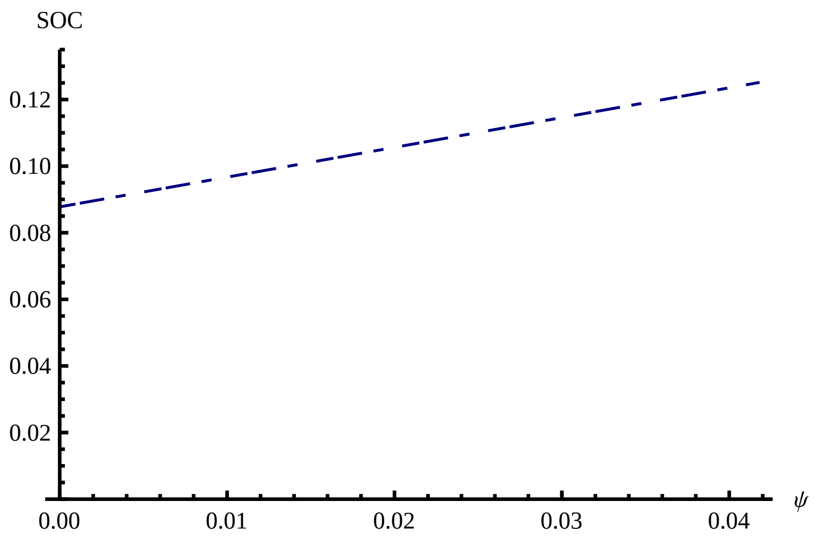

Figure 5: Asymptotic speed of convergence as the embodied exogenous change parameter, ψ, rises and γ is adjusted so as to maintain $g_c^* = 0.02$. Note: $\lambda = 0, \beta = 0; \alpha = 0.324$.

economy? As mentioned in the introduction, the literature from the 1960s leads to the presumption that the answer is yes. For Solow-style models with a constant saving rate, Phelps (1962), Sato (1966), and Williams and Crouch (1972) showed that when a higher fraction of exogenous productivity increases are embodied, a *higher* SOC appears.

By disentangling the impact of the form of technical progress from that of its source, we now examine whether embodiment *generally* has such an effect. Figure 5, where all technical progress is exogenous, is in accordance with the result from the early literature. SOC is seen to be an increasing function of the fraction of the exogenous productivity increases which are embodied. The intuition is that a higher degree of embodiment of a given amount of exogenous technical progress implies faster economic depreciation of the value of the capital stock and thereby less overhang from the past. (As learning is absent in Figure 5, the distinction as to the basis of learning from the earlier figures is irrelevant and only one curve appears; the benchmark model and the alternative model coincide.)

When the source of technical progress is instead endogenous in the form of learning, however, embodiment does *not* increase SOC. In Figure 6 all productivity growth is due to learning by investing. Not only does this generate a low SOC for

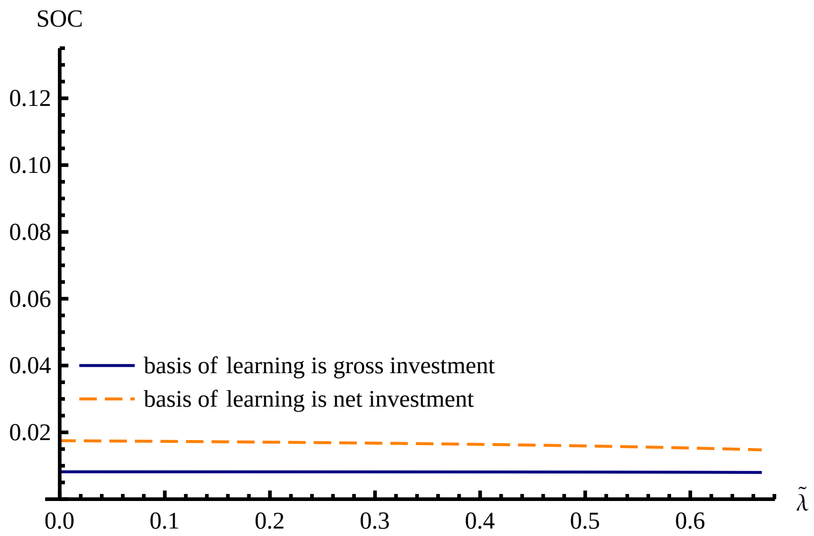

Figure 6: Asymptotic speed of convergence as the normalized embodied learning parameter, $\tilde{\lambda}$, rises and β is adjusted so as to maintain $g_c^* = 0.02$. Note: $\gamma = 0, \psi = 0; \alpha = 0.324$.

the reason explained at the end of the previous subsection. It also neutralizes the tendency of fast economic depreciation to raise SOC. Indeed, in Figure 6 SOC is essentially independent of the fraction of the learning taking the embodied form rather than the disembodied form.[28]This is so whether it is gross or net investment that drives learning (but the usual level difference between these two cases is again visible). The intuition is that when the economic depreciation due to embodiment of technical progress is linked to learning by investing, it is linked to a slow-moving endogenous force which offsets the speeding up of SOC through the boosted economic depreciation.

We conclude that a rising degree of embodiment of technical change in the wake of the computer revolution does *not* seem likely to bring about a rising SOC, at least not as long as the overall productivity growth rate is unaffected. If a rising degree of embodiment is accompanied by higher growth, however, a rising SOC can be expected, as witnessed by Table 3 above.

[28]See also Table E in Appendix G.

5.4 Other aspects

It is well-known that a rise in the output elasticity with respect to capital, everything else equal, tends to decrease the speed of convergence. A high output elasticity with respect to capital makes the output-capital ratio and interest rate less sensitive to changes in the capital intensity. Hence, if a disturbance for instance raises the output-capital ratio and the interest rate temporarily above their steady state levels and therefore induces a high saving and investment level, the adjustment will be relatively slow if the output elasticity with respect to capital is high.

When the vehicle of learning is net investment, the effective output elasticity with respect to capital is $\alpha + (1-\alpha)\,\beta$ rather than just α. This raises the question whether the negative slope of the stippled curve in for example Figure 3 is due to the capital-elasticity effect of a rising β on the effective output elasticity with respect to capital rather than to the learning effect. The stippled curve in Figure 7 shows that the answer is affirmative: along with the rising β, we here adjust not only γ so as to maintain $g_c^* = 0.02$, but also α so as to maintain $\alpha + (1-\alpha)\,\beta = 0.5$; as a result SOC is more or less constant, in fact slightly increasing. When the vehicle of learning is gross investment, however, a similar adjustment of α does not change the pattern qualitatively, but makes the slope less steep (compare the solid curve in Figure 7 with that in Figure 3).[29]

It is also well-known that the speed of convergence in a growth model generally tends to slow down as the desire for consumption smoothing, θ, rises and the population growth rate falls, respectively.[30] As expected, this holds in the present framework as well. At the same time, as documented in Appendix G, the qualitative patterns displayed by the graphs above go through for alternative values of θ and n, respectively. These patterns are also generally robust with respect to variation in values of the other background parameters, as long as restrictions (A1) and (A2) are observed.[31] Moreover, both qualitatively and quantitatively similar results are obtained when the household sector is instead described within a Blanchard-Yaari type of overlapping generations framework.

[29] See also Table C in Appendix G.

[30] See, e.g., Barro and Sala-i-Martin (2004, p. 112) and Turnovsky (2002).

[31] Sensitivity analysis w.r.t. α, δ, ρ, and n is available from the authors upon request.

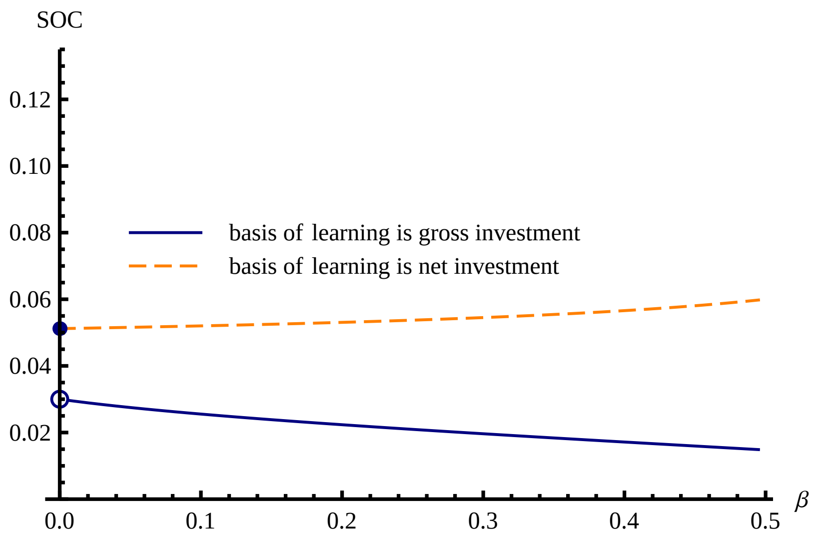

Figure 7: Asymptotic speed of convergence as the disembodied learning parameter, β, rises and γ is adjusted so as to maintain $g_c^* = 0.02$, while α is adjusted so as to maintain $\alpha + (1 - \alpha)\,\beta = 0.5$. Note: $\lambda = 0, \psi = 0$.

6 Conclusion

Based on a dynamic general equilibrium model we have studied how the composition of technical progress, along three dimensions, affects transitional dynamics, with an emphasis on the speed of convergence. The three dimensions are, first, the degree to which technical change is embodied, second, the extent to which an endogenous source, learning, drives productivity advances, and, third, the extent to which the vehicle of learning is gross investment rather than net investment.

A theoretical accomplishment is the result, linked to the distinction between decomposable and indecomposable dynamics, that as soon as learning from *gross* investment becomes part of the growth engine, the asymptotic speed of convergence displays a discrete fall. Such a succinct role for learning does not seem noticed within New Growth theory which predominantly has treated learning as originating in *net* investment so that it is cumulative net investment and thereby simply the capital stock which drives productivity. Since the dynamics of the capital stock is part of the overall economic dynamics whether or not any learning parameter is positive, learning becomes less pithy in that setting.

Our numerical simulations point to an asymptotic speed of convergence in a closed economy on the small side of 2% per year and possibly tending to a lower level in the future due to the rising importance of investment-specific learning in the wake of the computer revolution as the empirical evidence suggests.

The analysis shows that the speed of convergence, both ultimately and in a finite distance from the steady state, depends strongly and negatively on the importance of learning in the growth engine and on gross investment being the vehicle of learning rather than net investment. Finally, in contrast to a presumption implied by "old growth theory", a rising degree of embodiment of learning in the wake of the computer revolution is *not* likely to raise the speed of convergence.

7 Appendix

A. Steady state

By (10), the steady state value of the consumption-capital ratio is $x^* = z^* - g_K^* - \delta$. By substituting (27) and (23) into this expression, we get

$$\begin{aligned} x^* &= \frac{[(1-\alpha)\gamma + \alpha\psi]\,\theta - \{\alpha\,[1-(1-\alpha)\beta] - (1-\alpha)(1-\beta)\}\,\psi + (1-\alpha)\gamma\,[(1-\alpha)\lambda - \alpha]}{\alpha\,[(1-\alpha)(1-\beta) - \alpha\lambda]} \\ &\quad + \frac{\{[(1-\alpha)\beta + \alpha\lambda]\,\theta + (1-\alpha)\,[(1-\alpha)\lambda - \alpha]\}\,n}{\alpha\,[(1-\alpha)(1-\beta) - \alpha\lambda]} + \frac{\rho + (1-\alpha)\delta}{\alpha}. \end{aligned}$$

For the proof of (vi) of Proposition 1 we need:

Lemma A1. Assume (A1) and (A2). Then $g_K^* = (1+\lambda)g_Y^* + \psi$.

Proof. From (23) follows

$$\begin{aligned} g_K^* - \psi &= \frac{[1-(1-\alpha)\beta]\,\psi + (1+\lambda)(1-\alpha)(\gamma+n) - [(1-\alpha)(1-\beta) - \alpha\lambda]\,\psi}{(1-\alpha)(1-\beta) - \alpha\lambda} \\ &= \frac{(1+\lambda)\alpha\psi + (1+\lambda)(1-\alpha)(\gamma+n)}{(1-\alpha)(1-\beta) - \alpha\lambda} = (1+\lambda)g_Y^*, \end{aligned}$$

by (22). $\square$

B. Eigenvalues

Assume (A1) and (A2). Then, by Proposition 1, $s^*x^*z^*u^* > 0$. The Jacobian matrix associated with the system (21), (14), and (15) evaluated in the steady state, is $\mathbf{A} =$

$$\begin{bmatrix} x^*(1 - \frac{\theta-1}{\theta}\lambda\frac{u^*}{z^*}) & x^*\left(\frac{\alpha}{\theta} - 1 + \frac{\theta-1}{\theta}\lambda\frac{x^*u^*}{z^{*2}}\right) & x^*\frac{\theta-1}{\theta}\lambda s^* \\ z^*\left[1-\alpha-((1-\alpha)\beta+\lambda)\frac{u^*}{z^*}\right] & z^*\left[\alpha - 1 + ((1-\alpha)\beta+\lambda)\frac{x^*u^*}{z^{*2}}\right] & z^*\left((1-\alpha)\beta+\lambda\right)s^* \\ u^*\left[-\alpha + (1-(1-\alpha)\beta)\frac{u^*}{z^*}\right] & u^*\left[\alpha - (1-(1-\alpha)\beta)\frac{x^*u^*}{z^{*2}}\right] & -u^*\left(1-(1-\alpha)\beta\right)s^* \end{bmatrix},$$

where $s^* \equiv 1 - x^*/z^*$. The expression for the determinant can be reduced to

$$\det \mathbf{A} = \frac{\alpha}{\theta}\left[(1-\alpha)(1-\beta) - \alpha\lambda\right] s^* x^* z^* u^* > 0,$$

where the inequality follows from the parameter restriction in (6) and the positivity of $s^* x^* z^* u^*$. Thus either there are two eigenvalues with negative real part and one positive eigenvalue or all three eigenvalues, η_1, η_2, and η_3, have positive real part. Since the dynamic system has two pre-determined variables, z and u, and one jump variable, x, saddle-point stability requires that the latter possibility can be ruled out. And indeed it can. Consider

$$b \equiv \sum_{j>i} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix},$$

where a_{ij} is the element in the i'th row and j'th column of $\mathbf{A}$. From matrix algebra we know that $b = \eta_1\eta_2 + \eta_1\eta_3 + \eta_2\eta_3$. By Lemma B1 below, $b < 0$, and so the possibility that all three eigenvalues have positive real part can be ruled out.[32]

Lemma B1. Assume (A1) and (A2). Then $b < 0$.

Proof. From the definition of $\mathbf{A}$ follows

$$\begin{aligned}
\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} &= \left\{-\frac{\alpha}{\theta}(1-\alpha) - \left[(1-\alpha)\beta + \frac{\lambda}{\theta}\right] s^* \frac{u^*}{z^*} + \left[(1-\alpha)\beta + \lambda\right] \frac{\alpha}{\theta}\frac{u^*}{z^*} \right. \\
&\quad \left. +(\frac{1}{\theta} - 1)\alpha\lambda s^* \frac{u^*}{z^*}\right\} x^* z^*, \\
\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} &= \left[(1-\alpha)\beta - 1 + (\frac{1}{\theta} - 1)\alpha\right] s^* x^* u^*, \\
\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} &= \left[(1-\alpha)(1-\beta) - \alpha\lambda\right] s^* z^* u^*.
\end{aligned}$$

[32] Lemma B1 is a slight generalization of a similar result in Groth (2010).

By summation and ordering,

$$\begin{aligned}
b &= \left\{ -\frac{\alpha}{\theta}(1-\alpha)x^* + \frac{1}{\theta}\left[\lambda(\alpha - s^*) + \alpha(1-\alpha)\beta\right]\frac{u^*}{z^*}x^* \right. \\
&\quad \left. - s^*\frac{u^*}{z^*}x^* + \left[(1-\alpha)(1-\beta) - \alpha\lambda\right] s^* u^* \right\} z^* \\
&= \left\{ \frac{1}{\theta}\left[-\alpha(1-\alpha) + \left(\lambda(\alpha - s^*) + \alpha(1-\alpha)\beta\right)\frac{u^*}{z^*}\right] x^* \right. \\
&\quad \left. - \left[\frac{x^*}{z^*} - (1-\alpha)(1-\beta) + \alpha\lambda\right] s^* u^* \right\} z^* \\
&< \left\{ \frac{1}{\theta(1+\lambda)}\left[-\alpha(1-\alpha)(1+\lambda) + \lambda(\alpha - s^*) + \alpha(1-\alpha)\beta\right] x^* \right. \\
&\quad \left. - \left[\frac{x^*}{z^*} - (1-\alpha)(1-\beta) + \alpha\lambda\right] s^* u^* \right\} z^* \\
&< \left\{ -\frac{1}{\theta(1+\lambda)}\left[\alpha((1-\alpha)(1-\beta) - \alpha\lambda) + \lambda s^*\right] x^* \right. \\
&\quad \left. - \left[1 - \alpha - (1-\alpha)(1-\beta) + \alpha\lambda\right] s^* u^* \right\} z^* \\
&= \left\{ -\frac{1}{\theta(1+\lambda)}\left[\alpha((1-\alpha)(1-\beta) - \alpha\lambda) + \lambda s^*\right] x^* - \left[(1-\alpha)\beta + \alpha\lambda\right] s^* u^* \right\} z^* < 0,
\end{aligned}$$

where the first inequality is due to $s^* < \alpha$ and $(1+\lambda)u^*/z^* < 1$ by (iv) and (vi) of Proposition 1, respectively, the second inequality to $x^*/z^* = 1 - s^* > 1 - \alpha$, by (iv) of Proposition 1, and the last inequality to the restriction on λ in (6). $\square$

C. Local existence and uniqueness of a convergent solution

From Appendix B follows that the steady state has a two-dimensional stable manifold. Our numerical simulations suggest that the cases of repeated real eigenvalues or complex conjugate eigenvalues never arise for parameter values within a reasonable range. Hence we concentrate on the case of two distinct real negative eigenvalues, η_1 and η_2, where $\eta_1 < \eta_2 < 0$. Then any convergent solution is, in a neighborhood of (x^*, z^*, u^*), approximately of the form given in (29) which we repeat here for convenience:

$$x_{it} = C_{1i}e^{\eta_1 t} + C_{2i}e^{\eta_2 t} + x_i^*, \qquad i = 1, 2, 3, \tag{35}$$

where the constants C_{1i} and C_{2i} depend on initial conditions. Let $\mathbf{v^1} = (v_1^1, v_2^1, v_3^1)$

be an eigenvector associated with η_1. That is, $\mathbf{v^1} \neq (0,0,0)$ satisfies

$$\begin{aligned}(a_{11}-\eta_1)v_1^1 + a_{12}v_2^1 + a_{13}v_3^1 &= 0,\\ a_{21}v_1^1 + (a_{22}-\eta_1)v_2^1 + a_{23}v_3^1 &= 0,\\ a_{31}v_1^1 + a_{32}v_2^1 + (a_{33}-\eta_1)v_3^1 &= 0,\end{aligned} \tag{36}$$

where one of the equations is redundant. Similarly, let $\mathbf{v^2} = (v_1^2, v_2^2, v_3^2)$ be an eigenvector associated with η_2. Then, with η_1 replaced by η_2 in (36), these equations hold for (v_1^1, v_2^1, v_3^1) replaced by (v_1^2, v_2^2, v_3^2). Moreover, as $\eta_1 \neq \eta_2$, $\mathbf{v^1}$ and $\mathbf{v^2}$ are linearly independent. The C_i's in (35) are related to this in the following way:

$$C_{ji} = c_j v_i^j, \qquad j = 1,2, \quad i = 1,2,3, \tag{37}$$

where c_j, $j = 1,2$, are constants to be determined by the given initial condition $(x_{20}, x_{30}) = (\bar{z}_0, \bar{u}_0)$.

Returning to our original variable notation ($x_{1t} = x_t$, $x_{2t} = z_t$, and $x_{3t} = u_t$), (35) together with (37) implies, for $t = 0$ and $(z_0, u_0) = (\bar{z}_0, \bar{u}_0)$,

$$\begin{aligned}v_1^1 c_1 + v_1^2 c_2 - x_0 &= -x^*,\\ v_2^1 c_1 + v_2^2 c_2 + 0 &= \bar{z}_0 - z^*,\\ v_3^1 c_1 + v_3^2 c_2 + 0 &= \bar{u}_0 - u^*,\end{aligned} \tag{38}$$

where $\bar{z}_0$ and $\bar{u}_0$ are given whereas c_1, c_2, and x_0 are the unknowns. For the steady state to be saddle-point stable the structure of $\mathbf{A}$ must be such that this system has a unique solution (c_1, c_2, x_0). This is the case if and only if the vector $\mathbf{h} = (-1,0,0)$ does not belong to the linear subspace, $Sp(\mathbf{v^1}, \mathbf{v^2})$, spanned by the linearly independent eigenvectors $\mathbf{v^1}$ and $\mathbf{v^2}$. Our claim is that this condition is satisfied. We prove this by showing that the opposite leads to a contradiction.

Suppose that, contrary to our claim, there exist constants α_1 and α_2 such that

$$\alpha_1 \mathbf{v^1} + \alpha_2 \mathbf{v^2} = \mathbf{h} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}. \tag{39}$$

Multiplying from the left by $\mathbf{A}$ gives

$$\alpha_1 \mathbf{A}\mathbf{v^1} + \alpha_2 \mathbf{A}\mathbf{v^2} = \alpha_1\eta_1 \mathbf{v^1} + \alpha_2\eta_2 \mathbf{v^2} = \mathbf{A}\mathbf{h} = \begin{pmatrix} -a_{11} \\ -a_{21} \\ -a_{31} \end{pmatrix}, \tag{40}$$

where we have used the definition of eigenvalues. By (39) follow $\alpha_2 v_2^2 = -\alpha_1 v_2^1$ and $\alpha_2 v_3^2 = -\alpha_1 v_3^1$. Substituting into (40) yields

$$\begin{aligned} \alpha_1 v_2^1 \eta_1 - \alpha_1 v_2^1 \eta_2 &= -a_{21}, \\ \alpha_1 v_3^1 \eta_1 - \alpha_1 v_3^1 \eta_2 &= -a_{31}, \end{aligned}$$

so that

$$\alpha_1 v_2^1 = -\alpha_2 v_2^2 = \frac{a_{21}}{\eta_2 - \eta_1}, \tag{41}$$

$$\alpha_1 v_3^1 = -\alpha_2 v_3^2 = \frac{a_{31}}{\eta_2 - \eta_1}, \tag{42}$$

where $\eta_2 - \eta_1 > 0$.

Lemma C1. Assume (A1) and (A2). Then $a_{11} > 0, a_{21} > 0$, $a_{22} < 0$, $a_{33} < 0$, and $a_{31} + a_{32} > 0$.

Proof. Assume (A1) and (A2). Then, by Proposition 1, $s^* x^* z^* u^* > 0$. From the definition of $\mathbf{A}$ in Appendix B we have, first, $a_{11} = x^* [1 - (1 - \theta^{-1}) \lambda u^*/z^*] > x^*(1 - \lambda u^*/z^*) > 0$, where the last inequality follows from $u^*/z^* < 1/(1+\lambda)$, cf. (v) of Proposition 1; second, $a_{21} = z^* [1 - \alpha - ((1-\alpha)\beta + \lambda) u^*/z^*] > 0$ by (v) of Proposition 1 and the restriction on λ in (6); third, $a_{22} = z^* [\alpha - 1 + ((1-\alpha)\beta + \lambda)(1 - s^*) u^*/z^*]$ $= -a_{21} - z^*((1-\alpha)\beta + \lambda) s^* u^*/z^* < 0$, since $a_{21} > 0$; fourth, we immediately have $a_{33} < 0$; finally, $a_{31} + a_{32} = u^* [1 - (1-\alpha)\beta] s^* u^*/z^* > 0$. □

By Lemma C1, $a_{21} \neq 0$ and so (42) together with (41) implies that

$$v_3^1 = a_{31} v_2^1 / a_{21}, \tag{43}$$

and that $v_2^1 \neq 0$ (and $v_2^2 \neq 0$). Multiplying the second equation in (36) by a_{31} and the third by a_{21} and subtracting yields

$$[a_{31}(a_{22} - \eta_1) - a_{21} a_{32}] v_2^1 + [a_{31} a_{23} - a_{21}(a_{33} - \eta_1)] v_3^1 = 0.$$

Substituting (43) into this, v_2^1 cancels out. Ordering gives

$$a_{32} a_{21}^2 - a_{23} a_{31}^2 - a_{21} a_{31} (a_{22} - a_{33}) = 0. \tag{44}$$

It remains to show that (44) implies a contradiction.

Let $k_1 \equiv 1-(1-\alpha)\beta > 0$ and $k_2 \equiv (1-\alpha)\beta + \lambda \geq 0$. Insert the elements of $\mathbf{A}$ into the left-hand side of (44) to get $a_{32}a_{21}^2 \ -a_{23}a_{31}^2 \ -a_{21}a_{31}(a_{22}-a_{33})$

$$= z^* u^* \left\{ (\alpha - k_1 \frac{x^* u^*}{z^{*2}}) z^* (1-\alpha-k_2\frac{u^*}{z^*})^2 \right.$$

$$\left. -k_2 s^* u^* (k_1 \frac{u^*}{z^*} - \alpha)^2 - (1-\alpha-k_2\frac{u^*}{z^*})(k_1\frac{u^*}{z^*}-\alpha)\left[(\alpha-1)z^* + k_1 s^* u^* + k_2 \frac{x^* u^*}{z^*}\right]\right\}$$

$$= s^* z^* u^{*2} k_1 \left\{ (1-\alpha)\left[1-(1+\lambda)\frac{u^*}{z^*}\right] + \alpha k_2 \frac{u^*}{z^*}\right\} > s^* z^* u^{*2} k_1 \alpha k_2 \frac{u^*}{z^*} \geq 0,$$

where the first inequality is implied by $\alpha < 1$ and (v) of Proposition 1. Having hereby falsified (44), we conclude that $\mathbf{h} \notin Sp(\mathbf{v^1}, \mathbf{v^2})$, implying existence of a unique convergent solution.

D. When A is indecomposable, generically the same asymptotic speed of convergence applies to all three variables in the dynamic system

Consider an $n \times n$ matrix $\mathbf{M}$, $n \geq 2$. Let the element in the i'th row and j'th column of $\mathbf{M}$ be denoted a_{ij}. Let S be a subset of the row (and column) indices $N = \{1, 2, \ldots, n\}$ and let S^c be the complement of S. Then $\mathbf{M}$ is defined as *decomposable* if there exists a subset S of N such that $a_{ij} = 0$ for $i \in S$, $j \in S^c$. Thus, when the matrix $\mathbf{M}$ is decomposable, then by interchanging some rows as well as the corresponding columns it is possible to obtain a *lower block-triangular matrix*, that is, a matrix with a null submatrix in the upper right corner. A special case of a decomposable matrix $\mathbf{M}$ is the case where by interchanging some rows as well as the corresponding columns it is possible to obtain a *lower triangular matrix*, that is, a matrix with zeros everywhere above the main diagonal.

If $\mathbf{M}$ is decomposable, any subset S of the row indices such that $a_{ij} = 0$ for $i \in S$, $j \in S^c$, is called an *independent subset.* If a quadratic matrix is not decomposable, it is called *indecomposable.*

By inspection of the Jacobian matrix $\mathbf{A}$ defined in Appendix B we check under what circumstances $\mathbf{A}$ is decomposable. We have $N = \{1, 2, 3\}$. Using Lemma C1 we first see that the only row number that can by itself be an independent subset is $\{1\}$, which requires $a_{12} = a_{13} = 0$. This will hold if and only if $\lambda = 0$ and $\theta = \alpha$. Next we check when a *pair* of rows constitutes an independent subset. If $\{1, 2\}$ is an independent subset, we must have $a_{13} = a_{23} = 0$. This will hold if and only if $\lambda = \beta = 0$. The pair $\{2, 3\}$ can not be an independent subset since $a_{21} \neq 0$, by Lemma C1. Finally, if $\{1, 3\}$ should be an independent subset, we should have $a_{12} = a_{32} = 0$. It is easily shown that necessary (but not sufficient) for $a_{12} = 0$ is that $\theta \leq \alpha$. And $a_{32} = 0$ is only possible for very special combinations of parameter values involving *all* parameters of the system. So from a generic point of view we can rule out this case, which is not of much interest anyway because $\theta \leq \alpha$ is not empirically plausible.

We are left with two decomposable cases: *Case* $\mathfrak{D}$*1:* $\lambda = 0 = \beta$, $\theta \neq \alpha$; and *Case* $\mathfrak{D}$*2:* $\lambda = 0$, $\beta \geq 0$, $\theta = \alpha$. These cases are treated in Appendix E.

Here we consider the complement of the union of these cases, that is, the case where $\lambda > 0$ or ($\beta > 0$ and $\theta \neq \alpha$), implying that the Jacobian matrix $\mathbf{A}$ is generically indecomposable.

Regarding the eigenvalues of $\mathbf{A}$, as above we concentrate on the case of two distinct real negative eigenvalues, η_1 and η_2, where $\eta_1 < \eta_2 < 0$, and one positive eigenvalue, η_3.

Lemma D1. Assume (A1) and (A2). Let $\mathbf{v}^2 = (v_1^2, v_2^2, v_3^2)$ be an eigenvector associated with η_2, where $\eta_1 < \eta_2 < 0$. If $\lambda > 0$ or ($\beta > 0$ and $\theta \neq \alpha$), then $v_2^2 \neq 0$, and, generically, $v_i^2 \neq 0$, for $i = 1, 3$.

Proof. Assume (A1) and (A2) and that $\lambda > 0$ or ($\beta > 0$ and $\theta \neq \alpha$). It immediately follows that $a_{23} > 0$. By definition of η_2 and $\mathbf{v}^2$,

$$(a_{11} - \eta_2)v_1^2 + a_{12}v_2^2 + a_{13}v_3^2 = 0\,, \tag{45}$$

$$a_{21}v_1^2 + (a_{22} - \eta_2)v_2^2 + a_{23}v_3^2 = 0\,, \tag{46}$$

$$a_{31}v_1^2 + a_{32}v_2^2 + (a_{33} - \eta_2)v_3^2 = 0\,. \tag{47}$$

That $v_2^2 \neq 0$ is shown by contradiction. Suppose $v_2^2 = 0$. Then, by (45) and (46),

$$\begin{bmatrix} a_{11} - \eta_2 & a_{13} \\ a_{21} & a_{23} \end{bmatrix} \begin{pmatrix} v_1^2 \\ v_3^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $v_1^2 \neq 0$ or $v_3^2 \neq 0$, since $\mathbf{v}^2$ is an eigenvector. Consequently, the determinant of the 2×2 matrix must be vanishing, i.e., $(a_{11} - \eta_2)a_{23} - a_{21}a_{13} = 0$. But, considering matrix $\mathbf{A}$ we have, after ordering,

$$(a_{11}-\eta_2)a_{23}-a_{21}a_{13}=\frac{s^*z^*}{\theta}\{(1-\alpha)\beta\theta(x^*-\eta_2)+\lambda[(1-\alpha+\alpha\theta)x^*-\theta\eta_2]\}>0,$$

where the inequality follows from $\eta_2<0$ and the assumption that $\lambda>0$ or $\beta>0$. From this contradiction we conclude that $v_2^2\neq 0$.

Now suppose $v_1^2=0$. Then, by (45) and (46),

$$\begin{bmatrix} a_{12} & a_{13} \\ a_{22}-\eta_2 & a_{23} \end{bmatrix}\begin{pmatrix} v_2^2 \\ v_3^2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $v_2^2\neq 0$, the determinant of the 2×2 matrix must be vanishing:

$$a_{12}a_{23}-a_{13}(a_{22}-\eta_2)=0. \tag{48}$$

But, as noted above, $a_{23}>0$; and since by assumption, if $\lambda=0$, we have $\theta\neq\alpha$, a_{12} and a_{13} cannot be nil at the same time. Consequently, in no dense open subset in the relevant parameter space does (48) hold. This proves the genericity of $v_1^2\neq 0$.

Finally, suppose $v_3^2=0$. Then, by (45) and (47),

$$\begin{bmatrix} a_{11}-\eta_2 & a_{12} \\ a_{31} & a_{32} \end{bmatrix}\begin{pmatrix} v_1^2 \\ v_2^2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $v_2^2\neq 0$, the determinant of the 2×2 matrix must be vanishing:

$$(a_{11}-\eta_2)a_{32}-a_{31}a_{12}=0. \tag{49}$$

But $a_{11}-\eta_2>0$ and, by Lemma C1, a_{31} and a_{32} cannot be nil at the same time. Consequently, in no dense open subset in the relevant parameter space does (49) hold. This proves the genericity of $v_3^2\neq 0$. □

Lemma D2. Assume (A1) and (A2). Let $x_{i0}\neq x_i^*$, $i=1,2,3$. If $\lambda>0$ or ($\beta>0$ and $\theta\neq\alpha$), then c_2 in (37) differs generically from 0.

Proof. In Appendix C we showed that (38) has a unique solution (c_1,c_2,x_0). By Cramer's rule

$$c_2=-\frac{(z_0-z^*)v_3^1-(u_0-u^*)v_2^1}{v_2^1v_3^2-v_2^2v_3^1},$$

where $v_2^1v_3^2-v_2^2v_3^1\neq 0$, that is, $(v_2^1,v_3^1)\neq(0,0)$ and $(v_2^2,v_3^2)\neq(0,0)$. Let $z_0\neq z^*$ and $u_0\neq u^*$. Suppose $c_2=0$. Then $(z_0-z^*)v_3^1=(u_0-u^*)v_2^1$, which is possible only if $v_2^1 \neq 0$, $v_3^1 \neq 0$, and the pair (z_0, u_0) satisfies $(z_0 - z^*)/(u_0 - u^*) = v_2^1/v_3^1$. Such pairs, however, do not constitute a dense open subset in the (z, u)-plane, as was to be shown. □

Combining Lemma D1 and D2 we have that when (A1) and (A2) hold together with $\lambda > 0$ or ($\beta > 0$ and $\theta \neq \alpha$), then generically $C_{2i} = c_2 v_i^2 \neq 0$, $i = 1, 2, 3$. In the light of (30) it follows that in this case the same asymptotic speed of convergence, $-\eta_2$, applies to all three variables in the dynamic system. That this will also be the asymptotic speed of convergence of y_t/y_t^* follows by (31). This proves Proposition 3.

E. Discontinuity of the dominant eigenvalue for the x and z dynamics when learning disappears

We assume throughout that (A1) and (A2) hold so that, by Proposition 1, x^*, z^*, u^*, and s^* are all strictly positive.

Decomposable case $\mathfrak{D}$*1:* $\lambda = 0 = \beta$, $\theta \neq \alpha$. In this case $a_{13} = 0 = a_{23}$. So the Jacobian matrix $\mathbf{A}$ is lower block-triangular, implying that its eigenvalues coincide with the eigenvalues of the upper left 2 x 2 submatrix on the main diagonal of $\mathbf{A}$ and the lower right diagonal element, $a_{33} < 0$. Let $\mathbf{A_{11}}$ denote the upper left 2 x 2 submatrix.

Decomposable case $\mathfrak{D}$*2:* $\lambda = 0$, $\beta \geq 0$, $\theta = \alpha$. In this case (and only in this case) $a_{12} = 0 = a_{13}$. So $\mathbf{A}$ is again lower block-triangular, but this time with the positive eigenvalue equal to $a_{11} = x^* > 0$, whereas the two negative eigenvalues are associated with the lower right 2 x 2 submatrix of $\mathbf{A}$. Let this submatrix be denoted $\mathbf{A_{22}}$. As long as $\beta > 0$, $a_{23} \neq 0$ and $\mathbf{A}$ is not further decomposable. In case $\beta = 0$, also $a_{23} = 0$. Then $\mathbf{A_{22}}$, hence also $\mathbf{A}$, is lower triangular with the eigenvalues appearing on the main diagonal.

As a preparation for the proof of Proposition 4, which involves both case $\mathfrak{D}1$ and $\mathfrak{D}2$, we need three lemmas concerning case $\mathfrak{D}1$. For case $\mathfrak{D}1$ we have

$$\mathbf{A} = \left[\begin{array}{ccc} \multicolumn{2}{c}{\mathbf{A_{11}}} & 0 \\ & & 0 \\ a_{31} & a_{32} & a_{33} \end{array} \right] = \left[\begin{array}{ccc} x^* & \left(\frac{\alpha}{\theta} - 1\right) x^* & 0 \\ (1-\alpha) z^* & (\alpha - 1) z^* & 0 \\ \left(\frac{u^*}{z^*} - \alpha\right) u^* & \left(\alpha - \frac{x^* u^*}{z^{*2}}\right) u^* & -s^* u^* \end{array} \right] \quad (50)$$

The submatrix $\mathbf{A_{11}}$ has determinant $\det \mathbf{A_{11}} = -(1-\alpha)\frac{\alpha}{\theta}x^* z^* < 0$. The eigenvalues are $\bar{\eta}_1$ and $\bar{\eta}_3$, where $\bar{\eta}_1 < 0 < \bar{\eta}_3$. The third eigenvalue of $\mathbf{A}$ is $\bar{\eta}_2 = -s^* u^* = -g_Y^* < 0$. For realistic parameter values we have $\bar{\eta}_1 < \bar{\eta}_2 < 0$.

Lemma E1. Let $\lambda = 0 = \beta$ and $\theta \neq \alpha$. Let $z_0 = \bar{z}_0 > 0$ be given. Then the unique convergent approximating solution for the (x, z) subsystem is

$$x_t = cv_1^1 e^{\bar{\eta}_1 t} + x^*, \tag{51}$$

$$z_t = cv_2^1 e^{\bar{\eta}_1 t} + z^*, \tag{52}$$

where $\bar{\eta}_1$ is the negative eigenvalue of $\mathbf{A_{11}}$, $v_1^1 = 1$, $v_2^1 = -(x^* - \bar{\eta}_1)/a_{12} \neq 0$, and $c = (\bar{z}_0 - z^*)/v_2^1$.

Proof. From Lemma C1 we know that $a_{21} \neq 0$ and since $\lambda = 0$ is combined with $\theta \neq \alpha$, $a_{12} \neq 0$. So $\mathbf{A}_{11}$ is not decomposable. As $x^* > 0$ and $\bar{\eta}_1 < 0$, we have $a_{12}v_2^1 = -(x^* - \bar{\eta}_1) < 0$, which implies $v_2^1 \neq 0$. So $c = (\bar{z}_0 - z^*)/v_2^1$ is well-defined and ensures, when combined with (52), that $z_0 = \bar{z}_0$. Finally, since $x^* = a_{11}$, by construction (v_1^1, v_2^1) satisfies the equation $(a_{11} - \bar{\eta}_1)v_1^1 + a_{12}v_2^1 = 0$. Thus, (v_1^1, v_2^1) $\neq (0, 0)$ is an eigenvector of A_{11} associated with $\bar{\eta}_1$; and (51)-(52) thereby constitutes the unique convergent approximating solution for the (x, z) subsystem. $\square$

Lemma E2. Let $\lambda = 0 = \beta$ and $\theta \neq \alpha$. Let the two negative eigenvalues of $\mathbf{A}$, $\bar{\eta}_1$ and $\bar{\eta}_2$, satisfy $\bar{\eta}_1 < \bar{\eta}_2 < 0$. Define $\mathbf{v^1} = (v_1^1, v_2^1, v_3^1)$, where (v_1^1, v_2^1) is as given in Lemma E1, and $v_3^1 = (a_{31}v_1^1 + a_{32}v_2^1)/(\bar{\eta}_1 - a_{33})$. Then $\mathbf{v^1}$ is an eigenvector of $\mathbf{A}$ associated with the eigenvalue $\bar{\eta}_1$. Further, $\mathbf{v^2} = (v_1^2, v_2^2, v_3^2) = (0, 0, 1)$ is an eigenvector of $\mathbf{A}$ associated with the eigenvalue $\bar{\eta}_2$.

Proof. Since $a_{33} = \bar{\eta}_2 > \bar{\eta}_1$, $\bar{\eta}_1 - a_{33} < 0$. Then v_3^1 is well-defined and by construction $\mathbf{v^1}$ satisfies (36) with $\eta_1 = \bar{\eta}_1$ in view of $a_{13} = a_{23} = 0$. Let $\mathbf{w} = (w_1, w_2, w_3)$ be an arbitrary eigenvector of $\mathbf{A}$ associated with the eigenvalue $\bar{\eta}_2$:

$$\begin{aligned}
(a_{11} - \bar{\eta}_2)w_1 + a_{12}w_2 + 0 &= 0, \\
a_{21}w_1 + (a_{22} - \bar{\eta}_2)w_2 + 0 &= 0, \\
a_{31}w_1 + a_{32}w_2 + (a_{33} - \bar{\eta}_2)w_3 &= 0.
\end{aligned}$$

The eigenvalues of $\mathbf{A_{11}}$ are $\bar{\eta}_1 < 0$ and $\bar{\eta}_3 > 0$, and since $\bar{\eta}_1 < \bar{\eta}_2 < 0$, $\bar{\eta}_2$ cannot be an eigenvalue of $\mathbf{A_{11}}$. Hence, $w_1 = 0 = w_2$. As $\bar{\eta}_2 = a_{33}$, this implies that $w_3 \neq 0$ is arbitrary and can be set equal to 1. Thereby $\mathbf{v^2} = \mathbf{w}$. $\square$

Lemma E3. Let $\lambda = 0 = \beta$ and $\theta \neq \alpha$. Let $z_0 = \bar{z}_0 > 0$ and $u_0 = \bar{u}_0 > 0$ be given. Let c be defined as in Lemma E1 and $\mathbf{v^1}$ and $\mathbf{v^2}$ as in Lemma E2. Then the unique convergent approximating solution for the total system is given by (51), (52), and

$$u_t = c_1 v_3^1 e^{\bar{\eta}_1 t} + c_2 v_3^2 e^{\bar{\eta}_2 t} + u^*, \tag{53}$$

with $c_1 = c = (\bar{z}_0 - z^*)/v_2^1$ and $c_2 = \bar{u}_0 - u^* - c_1 v_3^1$. The speed of convergence of x and z is $-\bar{\eta}_1$, whereas that of u is $-\bar{\eta}_2$.

Proof. In Lemma E2 it was shown that $\mathbf{v^1}$ and $\mathbf{v^2}$ are eigenvectors of $\mathbf{A}$ associated with the eigenvalues $\bar{\eta}_1$ and $\bar{\eta}_2$, respectively. We show that the solution formula (35) with $\eta_1 = \bar{\eta}_1$, $\eta_2 = \bar{\eta}_2$, and $C_{ji} = c_j v_i^j$, $j = 1, 2$, $i = 1, 2, 3$, for all $t \geq 0$ implies the proposed solution. In view of $c_1 = c = (\bar{z}_0 - z^*)/v_2^1$ and $v_1^2 = 0$, (35) for $i = 1$ is the same as (51). In view of $c_1 = c$ and $v_2^2 = 0$, (35) for $i = 2$ is the same as (52). It follows that x and z share the same speed of convergence, $-\bar{\eta}_1$. Finally, in view of $c_2 = \bar{u}_0 - u^* - c_1 v_3^1$ and $v_3^2 = 1$, (35) for $i = 3$ is the same as (53). It remains to show that $\bar{\eta}_2$ is the dominant eigenvalue for the dynamics of u. Since $\bar{\eta}_1 < \bar{\eta}_2 < 0$, this is so if $C_{23} \equiv c_2 v_3^2 \neq 0$ generically. As $v_3^2 = 1$,

$$c_2 v_3^2 = c_2 = \bar{u}_0 - u^* - c_1 v_3^1 = \bar{u}_0 - u^* - (\bar{z}_0 - z^*) v_3^1 / v_2^1,$$

by the definition of c_1. Let $\bar{u}_0 \neq u^*$ and $\bar{z}_0 \neq z^*$. Suppose $c_2 = 0$. Then $(\bar{z}_0 - z^*) v_3^1 / v_2^1 = \bar{u}_0 - u^*$. Pairs $(\bar{z}_0, \bar{u}_0)$ satisfying this do not, however, constitute a dense open subset in the (z, u)-plane. Hence $c_2 v_3^2$ $(= c_2) \neq 0$ generically, as was to be shown. $\square$

Proof of Proposition 4 of Section 3.4. It is given that when $\lambda = 0 = \beta$ and $\theta \neq \alpha$, the eigenvalues of $\mathbf{A}$ are real numbers, $\bar{\eta}_1$, $\bar{\eta}_2$, and $\bar{\eta}_3$, that satisfy $\bar{\eta}_1 < \bar{\eta}_2 < 0 < \bar{\eta}_3$. Similarly, when $\lambda = 0 = \beta$ together with $\theta = \alpha$, the eigenvalues of $\mathbf{A}$ are real numbers, $\tilde{\eta}_1$, $\tilde{\eta}_2$, and $\tilde{\eta}_3$, that satisfy $\tilde{\eta}_1 < \tilde{\eta}_2 < 0 < \tilde{\eta}_3$.

(i): Suppose $\theta \neq \alpha$ and that λ or β (or both) are strictly positive but close to zero. By hyperbolicity of the steady state, the eigenvalues of $\mathbf{A}$, η_1, η_2, and η_3, are still real and, by continuity, close to $\bar{\eta}_1$, $\bar{\eta}_2$, and $\bar{\eta}_3$. Thus, maintaining numbering in accordance with size, we have $\eta_1 \approx \bar{\eta}_1 < \eta_2 \approx \bar{\eta}_2 < 0 < \eta_3 \approx \bar{\eta}_3$. In view of $\theta \neq \alpha$, as long as $\lambda > 0$ or $\beta > 0$, Proposition 3 applies. So the same asymptotic speed of convergence, $-\eta_2$, applies to all three variables. Let $(\beta, \lambda) \to (0,0)^+$. Then $-\eta_2 \to -\bar{\eta}_2$. In the limit Lemma E3 applies, that is, the equilibrium path for x and z is given by (51) and (52), respectively. Consequently, in the limit the speed of convergence of x and z shifts from the value $-\bar{\eta}_2$ to the value $-\bar{\eta}_1$.

(ii): Let $\theta = \alpha$ and $\beta = 0$. As long as $\lambda > 0$, $\mathbf{A}$ is indecomposable. Let $\lambda \to 0^+$. In the limit $\mathbf{A}$ takes the form given in (50) with $a_{12} = 0$, that is, $\mathbf{A}$ becomes lower triangular with eigenvalues $\tilde{\eta}_3 = x^* > 0$, $\tilde{\eta}_1 = (\alpha - 1)z^* < 0$, and $\tilde{\eta}_2 = -g_Y^* < 0$ where, by assumption, $\tilde{\eta}_1 < \tilde{\eta}_2$. As long as $\lambda > 0$, but close to zero, an argument analogue to that under (i) applies, except that in the limit it is only z that shifts to a higher *finite* speed of convergence. The jump variable x becomes in the limit independent of both z and u. Thus x becomes free to adjust instantaneously to its steady state value; that is, in the limit the speed of convergence of x is infinite.

(iii): Let $\theta = \alpha$ and $\lambda = 0$. Then, $a_{12} = a_{13} = 0$. Even for $\beta > 0$ the dynamic system belongs to the decomposable case $\mathfrak{D}2$ described above, and the jump variable x is independent of the dynamics of z and u. So the speed of convergence of x is infinite even for $\beta > 0$ and remains so in the limit for $\beta \to 0^+$. But the (z, u) dynamics is governed jointly by $\eta_1 \approx \tilde{\eta}_1$ and $\eta_2 \approx \tilde{\eta}_2$ as long as β is strictly positive but close to zero, where $\tilde{\eta}_1 < \tilde{\eta}_2 < 0$. In the limit for $\beta \to 0^+$, however, $\mathbf{A}$ becomes lower triangular and so the movement of z ceases to be influenced by the slow adjustment of u and is governed only by the eigenvalue $\tilde{\eta}_1 = (\alpha - 1)z^*$. The speed of convergence of z thus jumps from $-\tilde{\eta}_2$ to the higher value $-\tilde{\eta}_1$. $\square$

F. Saddle-point stability when learning is based on net investment

When learning is based on net investment, the dynamic system becomes two-dimensional, cf. the formulas for g_x and g_z in Section 4. To avoid explosive growth the parameter values are restricted as follows:

$$0 \leq \lambda < (1 - \alpha)(1 - \beta)\,. \tag{*}$$

The Jacobian matrix evaluated in steady state is

$$\mathbf{B} = \begin{bmatrix} x^*(1 - \frac{\theta-1}{\theta}\lambda) & x^*(\frac{\alpha}{\theta} + \frac{\theta-1}{\theta}\lambda - 1) \\ z^*[(1-\alpha)(1-\beta) - \lambda] & -z^*[(1-\alpha)(1-\beta) - \lambda] \end{bmatrix}.$$

We find $\det \mathbf{B} = -\frac{\alpha}{\theta}[(1-\alpha)(1-\beta) - \lambda]x^* z^* < 0$, where the inequality is implied by the parameter restriction (*). Thus the eigenvalues, η_1 and η_2, differ in sign, and the steady state is saddle-point stable.

The non-trivial steady state, (x^*, z^*), has consumption-capital ratio

$$x^* = z^* - \delta - \frac{(1-\alpha)(\gamma+n) + \psi}{(1-\alpha)(1-\beta) - \lambda}$$

and output-capital ratio

$$\begin{aligned} z^* \quad = \quad & \frac{\theta\left[(1-\alpha)\gamma + \alpha\psi\right] + (1-\alpha)\left[\lambda\gamma + (1-\beta)\psi + \theta(\beta\psi - \lambda\gamma)\right]}{\alpha\left[(1-\alpha)(1-\beta) - \lambda\right]} \\ & + \frac{\left\{\theta\left[(1-\alpha)\beta + \alpha\lambda\right] + (1-\alpha)\lambda\right\} n}{\alpha\left[(1-\alpha)(1-\beta) - \lambda\right]} + \frac{\delta + \rho}{\alpha}. \end{aligned}$$

G. Simulations

The numerical results in this appendix refer to the benchmark model with learning based on gross investment. "Speed of convergence" refers to the common asymptotic speed of convergence of x and z, i.e., $\sigma_x = \sigma_z$. By Proposition 4, in the *absence* of learning, $\sigma_u \neq \sigma_i$, $i \in \{x, z\}$. In the tables, numbers in parentheses indicate the speed of convergence, in percentage points, of u in the *absence* of learning. Unless otherwise specified, values of the background parameters are the baseline values specified in Table 2 of the text. The range of the parameter appearing in the first column of the tables is limited to values not requiring the adjusting variable to take on a negative value to maintain $g_c^* = 0.02$.

TABLE A
Asymptotic speed of convergence as the embodied learning parameter, λ, rises and γ is adjusted so as to maintain $g_c^* = 0.02$.

			Speed of Convergence in %				r^*	s^*	$(Y/(pK))^*$	g_p^*
			$\theta = 1$	$\theta = 1.75$	$\theta = 3$	$\theta = 4$	.... $\theta = 1.75$			
Panel A. $n = 0.01$										
λ	γ	α								
0.00	0.020	0.324	10.48	8.78	7.52	6.96	0.055	0.25	0.32	0.000
			(3.00)	(3.00)	(3.00)	(3.00)				
0.28	0.016	0.324	2.55	2.49	2.42	2.37	0.055	0.25	0.35	-0.008
0.56	0.012	0.324	2.10	2.01	1.91	1.85	0.055	0.26	0.38	-0.017
0.84	0.008	0.324	1.66	1.57	1.47	1.41	0.055	0.26	0.40	-0.025
1.11	0.004	0.324	1.25	1.17	1.08	1.03	0.055	0.27	0.43	-0.033
1.39	0.000	0.324	0.86	0.80	0.73	0.70	0.055	0.27	0.45	-0.042
Panel B. $n = 0.005$										
λ	γ	α								
0.00	0.020	0.324	10.38	8.67	7.40	6.85	0.055	0.23	0.32	0.000
			(2.50)	(2.50)	(2.50)	(2.50)				
0.32	0.016	0.324	2.10	2.05	2.00	1.97	0.055	0.24	0.32	-0.008
0.63	0.012	0.324	1.68	1.62	1.54	1.50	0.055	0.24	0.35	-0.016
0.95	0.009	0.324	1.27	1.21	1.14	1.13	0.055	0.25	0.37	-0.024
1.27	0.005	0.324	0.89	0.84	0.78	0.75	0.055	0.25	0.40	-0.032
1.58	0.000	0.324	0.53	0.49	0.45	0.43	0.055	0.26	0.45	-0.040
Panel C. $n = 0.001$										
λ	γ	α								
0.00	0.020	0.324	10.31	8.57	7.32	6.77	0.055	0.22	0.32	0.000
			(2.10)	(2.10)	(2.10)	(2.10)				
0.40	0.016	0.324	1.69	1.65	1.61	1.59	0.055	0.23	0.35	-0.008
0.79	0.012	0.324	1.26	1.21	1.16	1.13	0.055	0.23	0.38	-0.017
1.19	0.008	0.324	0.84	0.80	0.76	0.73	0.055	0.24	0.40	-0.025
1.59	0.004	0.324	0.46	0.43	0.40	0.38	0.055	0.24	0.43	-0.033
1.98	0.000	0.324	0.09	0.08	0.08	0.07	0.055	0.25	0.45	-0.042

Note: $\beta = 0$, $\psi = 0$. When $\lambda = 0$, u converges with a lower speed than (x, z). This lower speed is shown in brackets.

TABLE B

ASYMPTOTIC SPEED OF CONVERGENCE AS THE EMBODIED LEARNING PARAMETER, λ, RISES AND ψ IS ADJUSTED SO AS TO MAINTAIN $g_c^* = 0.02$

			Speed of Convergence in %				r^*	s^*	$(Y/(pK))^*$	g_p^*
			$\theta = 1$	$\theta = 1.75$	$\theta = 3$	$\theta = 4$	.... $\theta = 1.75$			
λ	ψ	α								
0.00	0.042	0.324	15.43 (3.00)	12.51 (3.00)	10.32 (3.00)	9.33 (3.00)	0.055	0.27	0.45	-0.042
0.28	0.033	0.324	2.54	2.49	2.42	2.38	0.055	0.27	0.45	-0.042
0.56	0.025	0.324	2.09	2.01	1.92	1.86	0.055	0.27	0.45	-0.042
0.84	0.017	0.324	1.66	1.58	1.48	1.42	0.055	0.27	0.45	-0.042
1.11	0.008	0.324	1.25	1.17	1.08	1.03	0.055	0.27	0.45	-0.042
1.39	0.000	0.324	0.86	0.80	0.73	0.70	0.055	0.27	0.45	-0.042

NOTE: $\beta = 0$, $\gamma = 0$.

TABLE C

ASYMPTOTIC SPEED OF CONVERGENCE AS THE DISEMBODIED LEARNING PARAMETER, β, RISES AND γ IS ADJUSTED SO AS TO MAINTAIN $g_c^* = 0.02$

			Speed of Convergence in %				r^*	s^*	$(Y/(pK))^*$	g_p^*
			$\theta = 1$	$\theta = 1.75$	$\theta = 3$	$\theta = 4$	.... $\theta = 1.75$			
Panel A.										
β	γ	α								
0.00	0.020	0.324	10.48 (3.00)	8.78 (3.00)	7.52 (3.00)	6.96 (3.00)	0.055	0.25	0.32	0.0
0.13	0.016	0.324	2.59	2.51	2.42	2.36	0.055	0.25	0.32	0.0
0.27	0.012	0.324	2.17	2.05	1.90	1.82	0.055	0.25	0.32	0.0
0.40	0.008	0.324	1.76	1.61	1.45	1.37	0.055	0.25	0.32	0.0
0.53	0.004	0.324	1.35	1.20	1.06	0.99	0.055	0.25	0.32	0.0
0.67	0.000	0.324	0.95	0.82	0.71	0.66	0.055	0.25	0.32	0.0
Panel B. $[\alpha + (1-\alpha)\beta] = 0.5$										
β	γ	α								
0.00	0.020	0.500	6.23 (3.00)	5.12 (3.00)	4.23 (3.00)	3.81 (3.00)	0.055	0.38	0.21	0.0
0.14	0.016	0.420	2.53	2.42	2.29	2.21	0.055	0.32	0.25	0.0
0.24	0.013	0.340	2.24	2.12	1.97	1.89	0.055	0.26	0.31	0.0
0.32	0.010	0.260	2.03	1.90	1.77	1.69	0.055	0.20	0.40	0.0
0.39	0.008	0.180	1.86	1.74	1.61	1.54	0.055	0.14	0.58	0.0
0.44	0.007	0.100	1.73	1.61	1.49	1.43	0.055	0.08	1.05	0.0

NOTE: $\lambda = 0$, $\psi = 0$.

TABLE D

Asymptotic speed of convergence as the disembodied learning parameter, β, rises and ψ is adjusted so as to maintain $g_c^* = 0.02$

			Speed of Convergence in %				r^*	s^*	$(Y/(pK))^*$	g_p^*
			$\theta = 1$	$\theta = 1.75$	$\theta = 3$	$\theta = 4$	.... $\theta = 1.75$			
Panel A.										
β	ψ	α								
0.00	0.042	0.324	15.4	12.5	10.3	9.33	0.055	0.27	0.45	-0.042
			(3.00)	(3.00)	(3.00)	(3.00)				
0.13	0.033	0.324	2.58	2.52	2.43	2.38	0.055	0.27	0.43	-0.033
0.27	0.025	0.324	2.16	2.05	1.92	1.85	0.055	0.26	0.40	-0.025
0.40	0.017	0.324	1.75	1.62	1.47	1.39	0.055	0.26	0.38	-0.017
0.53	0.008	0.324	1.34	1.20	1.06	1.00	0.055	0.25	0.35	-0.008
0.67	0.000	0.324	0.95	0.82	0.71	0.66	0.055	0.25	0.32	0.000
Panel B. $[\alpha + (1-\alpha)\beta] = 0.5$										
β	ψ	α								
0.00	0.020	0.500	7.64	6.18	5.00	4.45	0.055	0.40	0.25	-0.020
			(3.00)	(3.00)	(3.00)	(3.00)				
0.14	0.022	0.420	2.54	2.45	2.33	2.25	0.055	0.34	0.30	-0.022
0.24	0.025	0.340	2.23	2.13	2.00	1.92	0.055	0.28	0.38	-0.025
0.32	0.029	0.260	2.01	1.90	1.78	1.71	0.055	0.21	0.52	-0.029
0.39	0.038	0.180	1.83	1.73	1.62	1.55	0.055	0.15	0.79	-0.038
0.44	0.060	0.100	1.68	1.59	1.49	1.43	0.055	0.09	0.65	-0.060

Note: $\lambda = 0$, $\gamma = 0$.

TABLE E

Asymptotic speed of convergence as the disembodied learning parameter, β, rises and λ is adjusted so as to maintain $g_c^* = 0.02$

			Speed of Convergence in %				r^*	s^*	$(Y/(pK))^*$	g_p^*
			$\theta = 1$	$\theta = 1.75$	$\theta = 3$	$\theta = 4$	.... $\theta = 1.75$			
β	λ	α								
0.00	1.39	0.324	0.86	0.80	0.73	0.70	0.055	0.27	0.45	-0.042
0.13	1.11	0.324	0.87	0.80	0.73	0.69	0.055	0.27	0.43	-0.033
0.27	0.84	0.324	0.89	0.81	0.73	0.69	0.055	0.26	0.40	-0.025
0.40	0.56	0.324	0.91	0.81	0.72	0.68	0.055	0.26	0.38	-0.017
0.53	0.28	0.324	0.92	0.82	0.72	0.67	0.055	0.25	0.35	-0.008
0.67	0.00	0.324	0.95	0.82	0.71	0.66	0.055	0.25	0.32	0.000

Note: $\gamma = 0$, $\psi = 0$.

TABLE F

Asymptotic speed of convergence as the exogenous embodied change parameter, ψ, rises and γ is adjusted so as to maintain $g_c^* = 0.02$

			Speed of Convergence of (x, z) in %				r^*	s^*	$(Y/(pK))^*$	g_p^*
			$\theta = 1$	$\theta = 1.75$	$\theta = 3$	$\theta = 4$	.... $\theta = 1.75$			
ψ	γ	α								
0.000	0.020	0.324	10.48	8.78	7.52	6.96	0.055	0.25	0.32	0.000
0.008	0.016	0.324	11.47	9.52	8.08	7.43	0.055	0.25	0.35	-0.008
0.017	0.012	0.324	12.46	10.27	8.63	7.90	0.055	0.26	0.38	-0.017
0.025	0.008	0.324	13.45	11.01	9.19	8.37	0.055	0.26	0.40	-0.025
0.033	0.004	0.324	14.44	11.76	9.75	8.85	0.055	0.27	0.43	-0.033
0.042	0.000	0.324	15.43	12.51	10.32	9.33	0.055	0.27	0.45	-0.042

Note: $\beta = 0$, $\lambda = 0$. In the decomposable case, the SOC of u equals the constant $g_Y^* = 3.00\%$.

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