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Abstract

Random Utitly Models (RUMs) are a particularly convenient way of modelling product differentiation. In this paper we demonstrate that they can be used to examine the possibilities of creating quality measures from data on prices and sales volumes. We formulate conditions sufficient for the existence of quality measures in two broad families of RUMs: additive random utility models and pure vertical differentiation models.

JEL codes: C60, D43, L10, L13, L15

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1.Introduction

Random utility models have a wide range of applications in psychology, social science, economics and natural science. They have also been subject to a considerable amount of theoretical work. This work has been partly inspired by application problems, but is also, to a large extent, 'autonomous' in nature (see the review by Marley (2002)). The problem we pose and partly solve in this paper, belongs in the first category. It is motivated by our work on the theory of consumer choice in a market with product differentiation.

One of the aspects of product differentiation is 'quality'. Assuming the quality of individual product varieties is unobservable, it might be tempting to infer it from data on sales volumes and prices. Examples of such works are Khandelwal (2010) and Hallak and Schott (2011) who create (and estimate) indicators of quality in foreign trade. Both papers use specific demand functions (nested logit and a two-tier CES function, respectively), which can be inverted so that the vector of qualities is isolated. The question we would like to ask is, *when is such an inversion possible in general*? Or in other words: when can the quality measure be theoretically identified, given the vector prices and sales volumes?

It is particularly convenient to formulate this problem in terms of random utility theory. In this line of RUM applications, the deterministic components of the conditional utility function are interpreted as price and quality of the product variety, whereas the stochastic component stands for consumers' subjective tastes. We demonstrate that the conditions sufficient for the existence of quality measures can be expressed in terms of the specification of the random utility function and the distribution of the stochastic component.

The rest of the paper is structured as follows. In Section 2 we formulate the problem in strict terms. In Section 3 we demonstrate the results for some broad classes of random utility functions, while in Section 4 we offer conclusions.

2. Problem Formulation

Assume a market with a differentiated product wherein the demand for different varieties of the product is a result of the aggregation of individual choices made according to a random utility model (RUM). We use the notation by Anderson et al. (1992)[1] to describe these choices by the individual conditional indirect utility function:

$$V_i = V\left(p_i, a_i, \varepsilon_i\right) \qquad (2.1)$$

[1] This is slightly different from an econometrician's notation. For the latter see e.g. Walker and Ben-Akiva (2002)

– where V_i is the utility of a the consumer from consuming the variety (model) i of a differentiated good ($i = 1, \ldots, n$). V_i is a function of three variables: the price of the i-th model $p_i \in \mathbf{R}^+$, the quality of the model $a_i \in \mathbf{R}$, and ε_i which is a random variable interpreted as the individual satisfaction of the consumer from buying the variety. By definition, p_i is nonnegative. Function V is increasing in a_i and ε_i, and decreasing in p_i. Variable a_i represents the vertical dimension of product differentiation while the random variable ε_i represents the horizontal dimension. A special case of (2.1) is the additive random utility model (ARUM):

$$V_i = w(p_i, a_i) + \varepsilon_i \qquad (2.2a)$$

where $w: \mathbf{R}^2 \mapsto \mathbf{R}$ is decreasing in the first argument and increasing in the second argument; or, even more specifically a linear random utility model (LRUM):

$$V_i = -p_i + a_i + \varepsilon_i \qquad (2.2b)$$

Regardless of the form of the V function, it is assumed that the consumer makes a discrete choice, i.e. she chooses only one model of the n varieties available: the one that yields her the biggest utility V_i.

Usually it is further assumed that she buys only one unit of the preferred variety, which has the implication that choice probabilities are the same as market shares[2]. We keep this assumption for expositional convenience, but it is not critical for our results, as long as choice probabilities can be easily translated into market shares.

Without loss of generality, we consider only one consumer (alternatively, we could assume a finite number of identical consumers). The choice probability for variety i equals:

$$S_i(\text{p,a}) = P_\varepsilon\left(T_i\right) \qquad (2.3)$$

where $T_i = \left\{ e = (e_1, \ldots, e_n) : V\left(p_i, a_i, e_i\right) = \max_j V\left(p_j, a_j, e_j\right) \right\}$ and P_ε is the probability of the joint distribution: $\varepsilon = \left(\varepsilon_1, \ldots, \varepsilon_n\right)$.

It will be useful to adopt the following working definition:

[2] Provided all the consumers are identical

Definition 1. *We call the set* $T_i = \left\{ e = (e_1,\ldots,e_n) : V\left(p_i, a_i, e_i\right) = \max_j V\left(p_j, a_j, e_j\right)\right\}$ *the* ***bearing set*** *of variety i.*

For $S_1,\ldots,S_n$ to be choice probabilities we have to make one more assumption:

$$P_\varepsilon\left(T_{i1} \cap T_{i2}\right) = 0 \qquad \text{if } i_1 \neq i_2 \qquad (2.4)$$

i.e. the probability of the consumer being indifferent when choosing between any two varieties is zero.

We are interested in inferring qualities from the data on prices and choice probabilities. Note however that in additive models, increasing all the qualities by a constant does not change the choice probabilities (with prices given). Consequently, the following definition of a quality measure takes into account its 'relative' character.

Definition 2. *Let* $V : \left(\mathbf{R}^+\right)^n \times \mathbf{R}^n \mapsto \mathbf{R}^n$ *be a random utility function and let* $H = \mathrm{Im}(V)$. *Function* $m_V^k : \left(\mathbf{R}^+\right)^n \times H \times \mathbf{R} \mapsto \mathbf{R}^n$ *where* $k \in \{1,...,n\}$ i*s a* ***quality measure*** *for the random utility model generated by* V*, if for any vectors* a=$(a_1,...,a_n)$, p=$(p_1,..., p_n)$ and S=$(S_1,...,S_n)$ *complying with equality (2.3) the following equality holds:*

$$m_V^k(\mathrm{p},\mathrm{S},a_k) = \mathrm{a} \qquad (2.5)$$

The quality of any variety i is measured relatively to the quality of variety k, which can be thought of as a kind of quality *numeraire*. The principal problem we are addressing in this paper is the following: *which conditions imposed on the RUM allow for the respective quality measure to exist?*

Example (*multinomial logit model*).

For the multionomial logit model (MNL), a quality measure exists. By the Holman-Marley theorem, MNL is a LRUM model (type (2.2b)) in which the variables $\varepsilon_1,\ldots,\varepsilon_n$ are i.i.d. extreme value (type 1) distributed (cf. Anderson et al. 1992, p.38). The multinomial logit model has a convenient closed form for choice probabilities:

$$S_i(\text{p,a}) = \frac{\exp\left(\dfrac{-p_i + a_i}{\mu}\right)}{\sum_j \exp\left(\dfrac{-p_j + a_j}{\mu}\right)}$$

where $\mu > 0$ is a parameter of the extreme value distribution. By implication, for any two varieties i_1 and i_2,

$$a_{i1} - a_{i2} = (p_{i1} - p_{i2}) + \mu(\ln S_{i1} - \ln S_{i2})$$

Let the function $m_V^k = (m_1, ..., m_n)$ be defined as follows.

$$m_i = \begin{cases} a_k & \text{if} \quad i = k \\ a_k + p_i - p_k + \mu(\ln S_i - \ln S_k) & \text{if} \quad i \neq k \end{cases}$$

It can be verified that m_V^k meets condition (2.5) and hence it is a quality measure.

In the above example the quality measure could be explicitly constructed, which will not always be the case. However, the multinomial logit model has another property that is worth generalizing about. It was said above that shifting the quality vector in a LRUM leaves the choice probabilities intact. If this is the only manipulation of the qualities that has this effect in an additive RUM, then a quality measure exists:

Lemma 1.*Consider an additive random utility model (ARUM):*

$$V_i = w(p_i, a_i) + \varepsilon_i$$

if it has the following property:

$$S(\text{p,a}) = S(\text{p,a'}) \quad \Rightarrow \quad \text{w'} = \text{w} + c\mathbf{I} \quad (2.6)$$

where $\text{w'}_\text{i} = w(p_i, a'_i)$, $\mathbf{I}$ *is the identity matrix, and* $c \in \mathbf{R}$, *then a quality measure exists for this model.*

Proof. Let $k \in \{1, ..., n\}$ be any number and let the function $h_V^k : \left(\mathbf{R}^+\right)^n \times H \times \mathbf{R} \mapsto 2^{\mathbf{R}^n}$, be defined as follows:

$$h_V^k(\text{p,S,}a_k) = A$$

where A is the set of all the vectors $\mathrm{a}=(a_1,...,a_k,...,a_n)$, such that vectors p, S and a comply with equality (2.3). We shall demonstrate that A consists of only one element and hence it can be identified with m_V^k. Indeed, assume that for a given triple $(\mathrm{p},\mathrm{S},a_k)$, A consists of at least two elements: $\mathrm{a}, \mathrm{a}' \in A$. But since there must be $a_k = a'_k$, and hence also $w_k = w'_k$ assumption (2.6) implies that $\mathrm{c}=0$ and so $\mathrm{a}=\mathrm{a}'$

■

Generally, quality measures need not exist, as the following example shows.

Example (*LRUM with a 'gap' in the support of ε*).

Consider a linear RUM ($w_i = -p_i + a_i$) with just two varieties. The probability of choosing option 1 in such a model equals $S_1(\mathrm{p,a}) = P(T_1)$ where the set $T_1 = \{e = (e_1, e_2) : e_1 - e_2 \geq -w_1 + w_2\}$ is the shaded area in Figure 1.

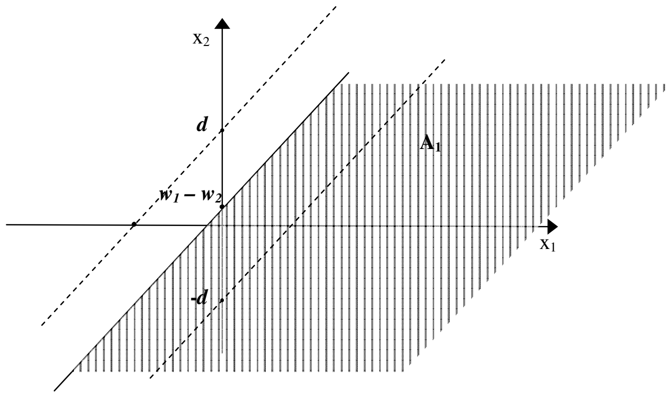

Figure 1. The probability of choosing variety 1

Assume further that the consumer might like variety 1 more or she might like variety 2 more, but either way, her preference for the favoured variety is very strong. Hence, for some $d > 0$:

$$P(|\varepsilon_1 - \varepsilon_2| < d) = 0$$

The area between the dotted lines in Figure 1 has zero probability. This implies that as long as the difference $w_1 - w_2$ remains in the interval $(-d, d)$, the choice probability S_1 is unchanged. Consequently, there can be no quality measure, because with prices and choice probabilities given, qualities can still be manipulated and hence function m_V^k cannot be well defined. Observe that this counterexample works for several distributions of ε, as long as the measure of the area between the dotted lines is zero.

3. Conditions sufficient for the existence of quality measures

We will discuss the conditions sufficient for the quality measure to exist in two families of RUMs that are quite popular in the economic literature: additive random utility models and what we call the 'pure vertical differentiation model'. It can be demonstrated that in both cases, the assumption that makes the existence of a quality measure possible eliminates the non-convexity of the bearing sets of the kind that generate the counterexample in the previous section.

3.1. Additive Random Utility Model

Our first major result is the following.

Theorem 1. *If the market demand function is generated by an ARUM model $V_i = w(p_i, a_i) + \varepsilon_i$ and the support of the random variable $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ is identical with the $\mathbf{R}^n$ space, then a quality measure exists.*

Proof. We use the notation $w_i = w(p_i, a_i)$ and let $f: \mathbf{R}^n \mapsto \mathbf{R}^n$ be the choice probability: $f(\mathrm{w}) \equiv S(\mathrm{p,a})$. We will demonstrate that if $f(\mathrm{w}) = f(\mathrm{w'})$, then there exists a number $c \in \mathbf{R}$, such that:

$$\mathrm{w'} = \mathrm{w} + c\mathbf{I} \qquad (3.1)$$

- which by Lemma 1 guarantees the existence of a quality measure.

The proof is by contradiction. Assume that for some vectors w and w' holds $f(w) = f(w')$ but not (3.1). Consequently, the difference $w'_j - w_j$ is not constant for all j. Let us define

$$w'_r - w_r = \max_j \{w'_j - w_j\} \qquad (3.2)$$

$$w'_s - w_s = \min_j \{w'_j - w_j\} \qquad (3.3)$$

We shall prove that $f_r(w) < f_r(w')$, which contradicts the assumption. By analogy to T_i let us define $T'_i = \{(x_1, \ldots, x_n) : x_i + w'_i \geq x_j + w'_j \text{ for } j = 1, \ldots, n\}$. Observe that $f_r(w) = P(T_r)$ and $f_r(w') = P(T'_r)$. We will demonstrate that $T_r \subseteq T'_r$ and the difference $T'_r \backslash T_r$ has a positive probability implying $P(T_r) < P(T'_r)$.

To see that $T_r \subseteq T'_r$ take any $x = (x_1, \ldots, x_n) \in T_r$. Then for any j the following inequality holds:

$$w_r - w_j \geq x_j - x_r \qquad (3.4)$$

On the other hand (3.2) implies

$$w'_r - w'_j \geq w_r - w_j \qquad (3.5)$$

(3.4) and (3.5) together yield:

$$w'_r - w'_j \geq x_j - x_r, \text{ hence}$$

$$x_r + w'_r \geq x_j + w'_j$$

which is equivalent to $x \in T'_r$.

Now, let $d = w'_r - w_r - w'_s + w_s$ (by (3.2) and (3.3) this number is positive) and let T_0 be an open ball of radius $d/4$ centered at $y = (y_1, \ldots, y_n)$ where:

$$y_r = \frac{w_s + w'_s}{2},$$

$$y_s = \frac{w_r + w'_r}{2},$$

$$y_i = -M \quad \text{for } i \neq r, s$$

where M is any positive number satisfying the following condition:

$$M > \max_j \left\{ w'_j - \tfrac{1}{2}(w'_r + w_r) - w'_s \right\}$$

We will demonstrate that $T_0 \subseteq T'_r$ but $T_0 \cap T_r = \varnothing$.

The proof of inclusion $T_0 \subseteq T'_r$

Let $x = (x_1, \ldots, x_n) \in T_0$. To show that $x \in T'_k$ we have to demonstrate that for any i:

$$x_i + w'_i \leq x_r + w'_r \qquad (3.6)$$

For $i = r$ (3.6) is obviously true. We will consider two other cases: $r \neq i \neq s$ and $i = s$.

Assume $i \neq s$. If $x \in T_0$, then $|x - y| < d/4$, and in particular for any i $|x_i - y_i| < d/4$, hence

$$-{}^{d}\!/_{4} \overset{(A)}{<} x_r - y_r < {}^{d}\!/_{4}$$

$$-{}^{d}\!/_{4} < x_i - y_i \overset{(B)}{<} {}^{d}\!/_{4}$$

Adding both sides of (A) and (B) and transforming the result we arrive at:

$$x_i - y_i < {}^{d}\!/_{2} + x_r - y_r \qquad (3.7)$$

Using the definitions of y_i, y_r and d we can substitute:

$$x_i + M < \tfrac{1}{2}\left(w'_r - w_r - w'_s + w_s\right) + x_r - \tfrac{1}{2}\left(w'_s + w_s\right)$$

$$x_i + M < \tfrac{1}{2}\left(w'_r - w_r\right) + x_r - w'_s$$

Adding w'_i to both sides yields

$$x_i + w'_i < -M + \tfrac{1}{2}\left(w'_r - w_r\right) + x_r - w'_s + w'_i$$

Which can be transformed as follows

$$x_i + w'_i < x_r + w'_r + \left(-\tfrac{1}{2}\left(w'_r + w_r\right) - w'_s + w'_i - M\right)$$

By the definition of M , the expression in brackets must be negative, so

$$x_i + w'_i < x_r + w'_r$$

which proves the inclusion $T_0 \subseteq T'_r$ for $r \neq i \neq s$.

Now suppose that $i = s$. Inequality (3.7) is still true implying that:

$$x_s - y_s < {}^{d}\!/_{2} + x_r - y_r$$

Substituting for y_s, y_r and d yields:

$$x_r - \tfrac{1}{2}(w_r + w'_r) < \tfrac{1}{2}\left(w'_r - w_r - w'_s + w_s\right) + x_r - \tfrac{1}{2}(w'_s + w_s)$$

$$x_s + w'_s < x_r + w'_r$$

which completes the proof of the inclusion $T_0 \subseteq T'_r$.

To demonstrate that $T_0 \subseteq T'_r \backslash T_r$ we still have to show that $T_0 \cap T_r = \varnothing$**.**

The proof of $T_0 \cap T_r = \varnothing$

Let $x \in T_0$. Then:

$$-\tfrac{d}{4} < x_r - y_r \overset{(C)}{<} \tfrac{d}{4}$$

$$-\tfrac{d}{4} \overset{(D)}{<} x_s - y_s < \tfrac{d}{4}$$

Adding both sides of (C) and (D) and rearranging the result we arrive at:

$$x_r - y_r < x_s - y_s + \tfrac{d}{2}$$

Substituting for y_r, y_s and d:

$$x_r - \tfrac{1}{2}\left(w'_s + w_s\right) < x_s - \tfrac{1}{2}\left(w'_r + w_r\right) + \tfrac{1}{2}\left(w'_r - w_r - w'_s + w_s\right)$$

$$x_r + w_r < x_s + w_s$$

But this contradicts the very definition of T_r, because $x_r + w_r \geq x_i + w_i$ must hold for any $x \in T_r$, in particular $i = s$. Consequently if $x \in T_0$ then $x \notin T_r$, so $T_0 \cap T_r = \varnothing$. And since we have demonstrated already that $T_0 \subseteq T'_r$, then $T_0 \subseteq T'_r \setminus T_r$.

However, given that T_0 is an open ball, its measure is positive ($P\left(T_0\right) > 0$) by the assumption that the support of ε is the entire space R^n. This reasoning is illustrated by Figure 3.

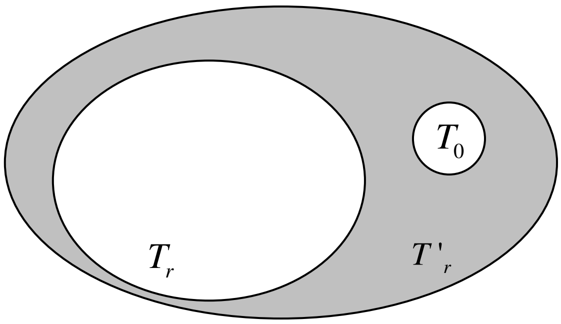

Figure 2. Proof of Theorem 1

We conclude that:

$$f_r(w) = P\left(T_r\right) < P\left(T_r\right) + P\left(T_0\right) = P\left(T_r \cup T_0\right) \leq P\left(T'_r\right) < f_r(w')$$

contradicting $f(\mathrm{w}) = f(\mathrm{w}')$.

■

What does it mean that the support of ε is identical to the space R^n? It is equivalent to the assumption that each nontrivial n-cube $(a_1,b_1)\times(a_2,b_2)\times\ldots\times(a_n,b_n)$ has a positive probability. The interpretation is that no configuration of tastes for different varieties is impossible. If the rationale behind employing a RUM is consumer heterogeneity (as it often is in empirical studies of demand), Theorem 1 requires that there is a nonnegligible group of consumers exhibiting any combination of preferences one can think of. One family of demand functions that meet this assumption are those generated by McFadden's Theorem of General Extreme Value (cf. Anderson et al 1992, p. 48). They include, in particular, the logit model and nested logit models that fulfill certain additional conditions[3].

From Theorem 1 almost immediately follows:

Corollary 1. *If the market demand function is generated by an ARUM model* $V_i = w_i(p_i, a_i) + \varepsilon_i$, *random variables* $\varepsilon_1,\ldots,\varepsilon_n$ *are independent and for each* ε_i *its support is identical with the* $\mathbf{R}$ *space, then a quality measure exists*[4]*.*

Relevant examples include the CES function[5] and (again) the logit model. Note, however, that neither Theorem 1 nor Corollary 1 require that variables $\varepsilon_1,\ldots,\varepsilon_n$ are identically distributed, so quality measures can exist even for the demand functions for which varieties are unequally popular with customers.

3.2. Pure Vertical Differentiation Models

The second class of models for which we can prove that quality measures exist is one that has been quite intensively worked on in industrial organization: pure vertical differentiation models (PVDM).

Definition 3*. A Pure Vertical Differentiation Model (PVDM) is a random utility model in which* $\varepsilon_1 = \ldots = \varepsilon_n = \bar{\varepsilon}$.

[3] The condition is that the heterogeneity of preferences at the higher level (nest) is at least as great as at the lower level. Otherwise a nested multinomial logit model might not be generated by a RUM (Anderson et al 1992, p. 48).

[4] Note that this result adds to our knowledge on independent RUMs, advanced among others by Suck (2002).

[5] It can be demonstrated that the CES (constant elasticity function) is generated by the following ARUM: $V_i = \ln a_i - \ln p_i + \varepsilon_i$, where $\varepsilon_1,\ldots,\varepsilon_n$ are i.i.d. extreme value (type 1) distributed, provided that consumers buy a certain variable amount of the differentiated good (cf. Anderson et. al 1992, p. 86-88). In this case, choice probabilities are different than market shares.

Classic examples of PVDMs are those by Shaked and Sutton (1983) who consider $V_i = (\overline{\varepsilon} - p_i)a_i$ and Mussa and Rosen (1978) who consider[6] $V_i = \overline{\varepsilon}a_i - p_i$. Admittedly, neither of the models is specified in terms of random utility. Instead they are formulated more in the spirit of location models: there is no random variable but a parameter representing the heterogeneity of consumers. In Shaked and Sutton's paper, consumers differ in income while in Mussa and Rosen's they differ in the 'intensity of a consumer's taste for quality'. However these models can be interpreted as RUMs as well[7]. PVDM is a special case of a random utility model, though, with some particularly 'good' properties, as the following lemma shows

Lemma 2. *If all choice probabilities in a PVDM are positive, then:* $a_i > a_j \Leftrightarrow p_i > p_j$.

Proof. Consider the random utility function, which in the case of a PVDM is:

$$V_i = V(p_i, a_i, \overline{\varepsilon})$$

V is decreasing in the first argument but increasing in the other two. Consequently if $a_i > a_j$ but $p_i < p_j$, then $V_i > V_j$ for any value of $\overline{\varepsilon}$ implying $S_j = 0$.

■

Corollary. *If all choice probabilities are positive then* $a_i = a_j \Leftrightarrow p_i = p_j$.

■

In the rest of the paper we assume that the qualities $a_1 \dots a_n$ are pairwisely different, which is consistent with assumption (2.4).

Interestingly, also for pure vertical differentiation models, the characteristics that are key to the existence of a quality measure are related to the support of the stochastic component. On one hand, the sufficient condition is weaker than it is in the case of an ARUM: it is enough that the support of $\overline{\varepsilon}$ is a convex subset of $\mathbf{R}$. On the other hand there is also an additional condition related to the utility function. Before our second major result is introduced we need the following lemma.

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[6] Strictly speaking, their model is more general: is not even a discrete choice model. Nevertheless the implied conditional utility function is this. Indeed, it is invoked as by other authors studying discrete choice models who refer to Mussa's and Rosen's article (cf. Cremer and Thisse 1991).

[7] On the other hand, we know of no prior work that would define the vertical differentiation model as a RUM**.**

Lemma 3. *Assume a PVDM and let* $a_1 < a_2 < ... < a_n$. *If the support of* $\overline{\varepsilon}$ *is a convex subset of* $\mathbf{R}$ *and the model has the following property*

$$\textit{For any } i > j, e' \geq e: \; V\left(p_i, a_i, e\right) \geq V\left(p_j, a_j, e\right) \Rightarrow V\left(p_i, a_i, e'\right) \geq V\left(p_j, a_j, e'\right) \qquad (3.8)$$

– then each bearing set T_i *is convex.*

Proof. Suppose that $x, y \in T_i$ with $x < y$ and $z = \alpha x + (1-\alpha) y$ for some $\alpha \in [0,1]$. We will demonstrate that $z \notin T_j$ for any $j \neq i$, which implies $z \in T_j$ by the convexity of support T. Consider the case when $j < i$. Since $x < z$, by property (3.8) there must be $z \notin T_j$. Now assume $j > i$ and suppose $z \in T_j$. Given that $z < y$, property (3.8) implies that $y \in T_j$ which contradicts the definition of y. Hence $z \notin T_j$

■

Property (3.8) is important but not too restrictive from a practical point of view. Indeed it is quite intuitive and consistent with the interpretation of PVDMs in industrial organization, where higher values of $\overline{\varepsilon}$ are associated with higher income or a stronger inclination to buy higher quality products. Hence if a lower realization of $\overline{\varepsilon}$ implies that among two given varieties, the higher-quality option is preferred, then it is plausible to assume that higher quality will also be preferred for a higher realization of $\overline{\varepsilon}$. Also the assumption about the convexity of the support of $\overline{\varepsilon}$ is consistent with the literature in industrial organization, where the support is routinely assumed to be an interval.

Theorem 2. *If a PVDM has property (3.8), the support of* $\overline{\varepsilon}$ is convex, *and all choice probabilities are positive, then a quality measure exists.*

Proof. Let the varieties be renumbered so that $a_1 < a_2 < ... < a_n$. By Lemma 2 we also have $p_1 < p_2 < ... < p_n$. We know from Lemma 3 that all the bearing sets are convex. Since they are all subsets of $\mathbf{R}$, they must be either intervals or half-lines (we assumed away empty sets). This can be easily verified by invoking a reasoning similar to the Lemma 3 proof that they must be located on the $\mathbf{R}$ line in the same order as the varieties (cf. Figure 3).

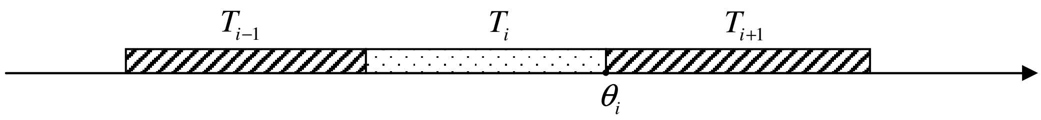

Figure 3. Bearing sets in a PVDM

Note that any two 'consecutive' bearing sets T_i, T_{i+1} have at most one common point. Indeed, since they are convex, their intersection has to be convex too. By assumption (2.4) this intersection has to have zero measure. But since the support of $\overline{\varepsilon}$ is convex, it means that $T_i \cap T_{i+1}$ is either empty or it is a single point. In practice, either T_i is right-closed and T_{i+1} is left-closed and so $T_i \cap T_{i+1} = \{\theta_i\}$ for some number $\theta_i \in \mathbf{R}$ (cf. Figure 3) or θ_i is in only one of these sets while the other is right-open (or left-open). In any case we can define numbers $\theta_1, \ldots, \theta_{n-1}$ as points that separate consecutive bearing sets. Let us also define $\theta_0 = \min T_1$ and $\theta_n = \max T_n$. Note that:

$$S_i = P_{\overline{\varepsilon}}\left(T_i\right) = G(\theta_{i+1}) - G(\theta_i)$$

where G is the cdf of $\overline{\varepsilon}$. Hence:

$$G(\theta_i) = G(\theta_0) + \sum_{j=1}^{i-1} S_j = \sum_{j=1}^{i-1} S_j \qquad (3.9)$$

The convexity of support T implies that G is strictly increasing, so there exists an inverted function G^{-1} and (3.9) yields:

$$\theta_i = G^{-1}\left(\sum_{j=1}^{i-1} S_j\right)$$

Consequently, with the vector S known, vector $\theta = \left(\theta_1, \ldots, \theta_n\right)$ can be unequivocally determined. This suggests a way of defining the quality measure m_V^k. Suppose that $k < n$.The definition of θ_k implies:

$$V\left(p_k, a_k, \theta_k\right) = V\left(p_{k+1}, a_{k+1}, \theta_k\right)$$

a_{k+1} is the solution of this equation, which is unique, because V is increasing in its second argument. By applying the same procedure to a_{k+1} and a_{k+2}, one can unequivocally define a_{k+2} and so on. Obviously one can also define a_{k-1} by considering

$$V\left(p_{k-1}, a_{k-1}, \theta_{k-1}\right) = V\left(p_k, a_k, \theta_{k-1}\right)$$

and, by analogy all the other elements of the quality vector left to a_k. The method of constructing a quality measure has been demonstrated.

■

4. Conclusions

Random utility models are a particularly convenient way of modelling product differentiation. In this paper we demonstrated that they can be used to examine the possibilities of creating quality indicators. We formulated conditions sufficient for the existence of quality measures in two broad families of RUMs: additive random utility models and pure vertical differentiation models. In both cases, the conditions proved to be strongly related to the convexity of the support of the model's random component.

Our study contributes to the characteristics of RUMs: in terms of the research classification outlined by Marley (2002), we add to the 'characterization' problems. Obviously there is considerable room for improvement in our results, from the sufficient conditions for other types of RUMs (or generalized RUMs as defined by Walker and Ben-Akiva (2002)) to a full characterization of necessary conditions. Nevertheless, we demonstrated that the family of demand functions for which quality measures exist is much wider than the specific functions used so far in applied studies.

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