Additive measures of travel time variability

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Abstract

This paper derives a measure of travel time variability for travellers equipped with scheduling preferences defined in terms of time-varying utility rates, and who choose departure time optimally. The corresponding value of travel time variability is a constant that depends only on preference parameters. The measure is unique in being additive with respect to independent parts of a trip. It has the variance of travel time as a special case. Extension is provided to the case of travellers who use a scheduled service with fixed headway.

Keywords: travel time variability, scheduling preferences, reliability, additivity

1. Introduction

During recent years, travel unreliability due to random variability of travel time has become an important issue for planners and decision makers. To incorporate the consequences of random travel time variability in planning, it is necessary to be able to predict the response of travellers to changes in the distribution of random travel times in real networks. Therefore travel time variability must be incorporated in transportation network models. This requires generally that measures of travel time variability are calculated for each road link or public transit route and then aggregated along paths. This is facilitated if the cost corresponding to a path is simply the sum of link costs. Shortest path algorithms are generally and essentially based on additivity.

Non-additive shortest path search is a difficult problem (e.g. Hutson and Shier, 2007) and is not available in any commercial transportation planning software. The mean travel time is additive in this sense but does not account for travel time variability. The standard deviation of travel time is a popular measure of travel time variability but it is not additive.

This paper characterises measures of random travel time cost (including the cost of variability) that possess the following additivity property. Consider a measure $C(\cdot)$ that assigns a value on the extended real line to any distribution of random travel time. Write for convenience $C(T)$, where it is understood that T represents the travel time distribution and not a realisation. Then C is additive if $C(T_1+T_2)=C(T_1)+C(T_2)$ whenever the random travel times T_1, T_2 are independent and $C(T_1)$, $C(T_2)$ and $C(T_1+T_2)$ are finite. Additivity then implies the desired property that the cost associated with a path is the sum of the cost associated to links, when travel times are random and independent across links.

In general, any functional that depends in a nontrivial way on the travel time distribution may be called a measure of travel time cost. But it is clear that not all measures are equally relevant for describing behaviour. We consider measures of travel time cost based on Vickrey (1973) scheduling preferences. In brief this means that travellers are viewed as deriving utility at specific time-varying rates from being at the origin or at the destination of a trip. Faced with a distribution of random travel time, travellers choose departure time to maximise expected utility. An expression for the maximal expected utility (or minimal expected cost) is used as a measure of the travel time cost. We seek a form of scheduling preferences that provide additive measures of travel time cost.

Apply the decomposition $T=\mu+X$ where $ET=\mu$ is the mean travel time and $EX=0$. In a nutshell, the main result of this paper is that any (nontrivial) additive cost measure based on smooth scheduling preferences is of the form

$$C(T)=H_0\mu+\frac{\gamma}{\beta^2}\ln Ee^{\beta X},$$

where $\gamma>0$, H_0 and β are real numbers. This formula exhausts all possibilities. When $\beta\to 0$, then the measure becomes $C(T)=H_0\mu+\frac{\gamma}{2}\sigma^2$, which is linear in the mean and variance σ^2 of travel time. For $\beta\neq 0$ the measure is sensitive to the skewness of the distribution with positive skewness being costly as β increases from zero. In this expression, H_0 is the value of mean travel time, which is just the derivative of C with respect to μ. Defining similarly a measure of travel time variability as $(1/\beta^2)\cdot\ln Ee^{\beta X}$, the corresponding value of travel time variability is simply the constant γ.

A plethora of travel time variability measures have been proposed before. Travel time percentiles, buffer index, planning time index and misery index are often recommended for monitoring and communicating achieved or predicted advances in reliability due to various improvements of the transport system (Lomax et al, 2003). The most commonly used measure of travel time variability is the standard deviation of travel time (Batley et al, 2008). None of these measures are additive and hence they are not convenient for network modelling purposes.

Expected utility theory (EUT; Von Neumann and Morgenstern 1944) can be used to derive a travel time cost measure given random travel time for rational travellers. The rationality assumption is useful as a benchmark and safeguards against the dangers of ad hoc theory. Under EUT, it is possible to specify a disutility function of trip attributes and derive the travel time cost as expected disutility. Travellers may be risk averse or risk prone depending on the curvature of the utility function. It is possible to follow EUT and define disutility as a function of travel time alone. E.g., De Palma and Picard (2005) estimated the distribution of a risk aversion parameter in the context of route choice. Cheu et al (2008) have demonstrated that additivity requires exponential disutility functions for modelling route choice in stochastic networks. A drawback of these approaches is a weak theoretical base for choice of the utility functions.

Another, more fundamental approach takes scheduling considerations into account in a structural model of departure time choice and derives a cost expression as a reduced form. It treats the disutility of travel time as opportunity cost of time that could be spent at the origin or at the destination of the trip. Therefore, a general scheduling utility function is a function of two variables, the departure time and the arrival time. Vickrey (1973) considered a separable utility function which is a sum of utility derived from time spent at the origin and time spent at the destination of a trip[1]. This portrays travellers who like to leave late and arrive early. Using such a formulation of utility it is possible to consider travellers who choose departure time optimally to maximise expected utility when faced with uncertain travel time. This explicitly recognises that the consequences of travel time variability are the pertinent risk of arriving late at the destination and the cost of departing early from the origin. Noland and Small (1995), Bates et al (2001), Fosgerau and Karlström (2010), Fosgerau and Engelson (2011) and Engelson (2011) have shown how measures of travel time variability can be derived in this way from travellers' scheduling preferences. The optimal expected total travel time cost takes different forms depending on the temporal profiles of the marginal utility of time (MUT) at origin and at destination and on the distribution of travel time.

A popular formulation of scheduling preferences is so-called $\alpha-\beta-\gamma$ preferences, where the MUT at the origin is constant and the MUT at the destination is a step function. This formulation derives from Vickrey (1969) and Small (1982) and is often used in bottleneck

[1] This is a generalisation of Vickrey (1969) who adopted a constant home utility function and a piecewise-constant work utility function.

models of congestion (e.g. Arnott et al., 1993). Noland and Small (1995) showed how $\alpha-\beta-\gamma$ preferences lead to a travel time cost measure that is linear in (μ,σ), the mean and the standard deviation of travel time, when the travel time distribution is exponential or uniform. Fosgerau and Karlström (2010) generalised this result to (essentially) any travel time distribution. However, in addition to the nonadditivity of the standard deviation, the result carries the inconvenience that the marginal cost of standard deviation depends on the shape of the distribution of travel time.

Fosgerau and Engelson (2011) used linearly varying utility rates to find that the cost of random travel time becomes linear in (μ,μ^2,σ^2). This formulation is convenient since the marginal costs of μ, μ^2 and σ^2 do not depend on the shape of the travel time distribution. If the MUT at the origin is constant, then the term involving the square of the mean travel time disappears and the cost measure is additive.

Engelson (2011) derived the optimal expected travel time cost formulae for the two cases when both MUT at origin and destination are either quadratic or exponential and demonstrated special cases when the travel time cost is additive. In this paper we show that the additivity requires that the MUT at the origin is constant and that the MUT at the destination has an exponential form, which has linear MUT as a limiting case.

In the next section 2 we introduce a general form of scheduling preferences and establish necessary and sufficient conditions for the additivity of minimal travel time cost when the travel time is certain. Section 3 explains the method of deriving the optimal cost of random travel time from scheduling preferences and establishes the existence of such cost. Section 4 establishes necessary and sufficient conditions for the additivity of travel time cost with random travel times. Section 5 describes the properties of the obtained measure of travel time variability and compares it with other measures based on the scheduling approach. Section 6 extends the results of section 4 to the case of scheduled service. Section 7 illustrates the calculation of the new measure of travel time variability from real world data. The last section presents conclusions and suggests directions for further research.

2. Scheduling preferences

Following Vickrey (1973), we consider an individual about to travel between two locations. She has time varying marginal utilities of time (MUT) $H(t)$ and $W(t)$ associated with activities that can be performed respectively at the origin and at the destination of the trip.[2] The MUT are measured in monetary units per time unit, say Euro per minute. $H(t)$ and $W(t)$ may be interpreted respectively as the utility of one minute later departure and the utility of one minute

[2] The notation is chosen to remind the reader of the home to work commute, but the model applies to any trip.

earlier arrival compared to one minute spent during travel. For mathematical convenience, the two functions are assumed to be defined for all real t.

Throughout the paper we assume that there is a time t_0 such that the individual prefers being at the origin before this time and at the destination after this time. This is comprised in the following assumption.

Assumption 1. $H(t)$ is non-increasing, $W(t)$ is non-decreasing, and there is an instance t_0 such that $H(t) > W(t)$ for all $t < t_0$ and $H(t) < W(t)$ for all $t > t_0$.

The assumption is illustrated on Figure 1. Let the trip from the origin to the destination take time $\mu \geq 0$ with certainty and independently of the departure time. If the trip occurs within the time interval $[a,b]$ then, given departure time t, the total utility derived from the activities during $[a,b]$ is equal to

$$\int_a^t H(\tau)d\tau + \int_{t+\mu}^b W(\tau)d\tau, \tag{1}$$

where the first integral is total utility of time spent at the origin since time a, and the second integral is the total utility of time spent at the destination until time b. The maximal possible utility,

$$\int_a^{t_0} H(\tau)d\tau + \int_{t_0}^b W(\tau)d\tau, \tag{2}$$

would be obtained if the trip occurred instantly at time t_0. The travel time cost is therefore defined as the difference between (2) and (1):[3]

$$c(t,\mu) = \int_t^{t_0} H(\tau)d\tau + \int_{t_0}^{t+\mu} W(\tau)d\tau. \tag{3}$$

Figure 1 here.

For any $\mu \geq 0$, lower utility is derived from being at origin after time t_0 than from being at destination after time $t_0+\mu$, therefore it is worthwhile to travel. Given the travel duration μ, the

[3] Note that a and b cancel out. Their values imply no bound on μ.

traveller's departure time choice problem is minimisation of the total cost (3) with respect to t.[4] Denote

$$C(\mu) = \min_t c(t, \mu). \tag{4}$$

If H and W are continuous functions then the first order condition for the minimisation problem is

$$H(t) = W(t + \mu), \tag{5}$$

which means that the last minute at the origin is as valuable as the first minute at the destination. This condition is illustrated in Figure 1 where $t^*(\mu)$ is the solution to the optimal departure time choice problem given travel duration μ. The corresponding travel time cost $C(\mu) = c(t^*(\mu), \mu)$ is the area *CEDdc*. In order to include step functions in our consideration, we do not assume here continuity of H or W. The departure time choice problem (4) still has an optimal solution for any non-negative μ but condition (5) need not be satisfied. The following theorem establishes existence of a solution and puts the problem in a simpler form using the function $V(t) = \max(H(t), W(t))$.

Theorem 1. *For any $\mu \geq 0$, the minimal value of $c(t, \mu)$ is attained at some $t = t^*$ such that $t_0 - \mu \leq t^* \leq t_0$. The minimal value is equal to*

$$C(\mu) = \min_{t_0 - \mu \leq t \leq t_0} \int_t^{t+\mu} V(\tau) d\tau. \tag{6}$$

See Appendix A for proof.

Now we will establish necessary and sufficient conditions for additivity when travel times are certain. This is of interest as a step towards the case of random travel times, as any condition that is necessary for additivity with certain travel times is also necessary for the more general case of random travel times. For the case of certain travel times, additivity becomes simply $C(\mu_1 + \mu_2) = C(\mu_1) + C(\mu_2)$ for any non-negative μ_1 and μ_2.

With constant H or W it is easy to see that the minimal cost value is linear and hence additive in μ. Indeed, with $H(t) \equiv H_0$, the optimal solution is $t^*(\mu) = t_0 - \mu$ and the minimal

[4] A special case of this problem with constant $H(t) \equiv \alpha$ and the two-valued function $W(t) = \alpha - \beta$ for $t < t_0$ and $W(t) = \alpha + \gamma$ for $t \geq t_0$ leads to the popular α–β–γ formulation of the scheduling preferences discussed in the introduction.

value is $C(\mu) = H_0\mu$, while for $W(t) \equiv W_0$ we have $t^*(\mu) = t_0$ and $C(\mu) = W_0\mu$. The following theorem demonstrates that either *H* or *W* must necessarily be constant on the respective half axis when the minimal cost is additive in μ.

Theorem 2. *The following three conditions are equivalent:*

(i) $C(\mu)$ *is additive in* μ*;*

(ii) (A) $H(t) = H_0$ *is constant for all* $t < t_0$ *and* $H_0 \leq W(t)$ *for all* $t > t_0$*, or*

(B) $W(t) = W_0$ *is constant for all* $t > t_0$ *and* $W_0 \leq H(t)$ *for all* $t < t_0$*;*

(iii) $V(t) \leq V_0$ *for all* $t < t_0$*, or* $V(t) \leq V_0$ *for all* $t > t_0$*, where* $V_0 = \operatorname{ess\,inf} V$.[5]

The proof can be found in Appendix A.

Having established necessary and sufficient conditions for the additivity of travel time cost in the case of certain travel time, we go on in the next section to the case of random travel times.

3. The expected travel time cost with random travel time

Following Noland and Small (1995), Bates et al (2001), Fosgerau and Karlström (2010) and Fosgerau and Engelson (2011), we now define the travel time cost for a traveller with random travel time *T* departing at time *t* as expected disutility

$$c(t,T) = E\left[\int_t^{t_0} H(\tau)d\tau + \int_{t_0}^{t+T} W(\tau)d\tau\right], \tag{7}$$

which is a natural generalisation of (3). The arguments of $c(t,T)$ are the departure time and the distribution of travel time which is assumed to have a mean value $\mu = ET$. The distribution of travel time is assumed to be fixed i.e. exogenous and independent of departure time. These assumptions are plausible for public transport and, in inter-peak periods, for individual transport, and are introduced for mathematical tractability. Fosgerau and Karlström (2010) derived an approximate value of reliability under α–β–γ scheduling preferences when the mean and standard deviation of the travel time depend linearly on the departure time.

[5] Essential infimum, $\operatorname{ess\,inf} V$ is defined as the least upper bound of the set of all such numbers *b* that $V(t) \geq b$ a. e.

For each realisation of T, the expression in brackets is convex in t as sum of two convex functions (because H is non-increasing and W is non-decreasing). It is therefore bounded from below by some affine function of t. This implies that $c(t,T)$ may be a finite number or plus infinity. Moreover, $c(t,T)$ is convex, the domain of t for which $c(t,T)$ is finite is convex, and $c(t,T)$ is continuous in the interior of this domain. If T has a compact support then $c(t,T)$ is finite and continuous for any t.

Similarly to the case of certain travel time, the traveller chooses departure time t to minimise the expected cost. Given the distribution of travel time T independent of the departure time, we are concerned with the optimal travel time cost

$$C(T) = \min_t c(t,T). \tag{8}$$

This generalises (4). It is a functional defined on all distributions of travel time T. If W and T are non-negative and $c(t,T)$ is finite for some t, then $C(T)$ is finite because $c(t,T) \geq c(t,0)$. The following theorem establishes existence of solution to problem (8) without assumptions on the signs of W and T. [6]

Theorem 3. *Assume that the $c(t,T)$ is finite for any t. Then there exists a real number t^* such that $c(t^*,T) \leq c(t,T)$ for any t.*

The proof can be found in Appendix A.

Since W is increasing, $\int_{t_0}^{t+s} W(\tau)d\tau$ is convex as a function of s and, due to Jensen's inequality, $c(t,T) \geq c(t,\mu)$ for any t. Therefore $C(T) = \min_t c(t,T) \geq \min_t c(t,\mu) = C(\mu)$ i.e. the travellers are risk averse.

When both functions H and W are continuous, the optimal value of departure time $t^*(T)$ can be obtained from the first order optimality condition

$$H(t) = EW(t+T) \tag{9}$$

and then substituted into (7) to obtain $C(T) = c(t^*(T),T)$. Due to the convexity of $c(t,T)$ in t, condition (9) is sufficient for the minimum. With continuous H and monotone W, the method still can be applied provided that the distribution of T is continuous.

[6] Findings by Tseng and Verhoef (2008) indicate that negative values of W in the neighborhood of t_0 are quite possible. For practical purposes the distribution of travel time is sometimes assumed to be normal which implies negative travel times with positive probability.

4. Additivity with random travel time

This section is devoted to finding conditions on the functions *H* and *W* that guarantee additivity of travel time costs. The definition of additivity was given in the introduction. Necessary and sufficient conditions for additivity of travel time costs with non-negative certain travel time were obtained in the section 2. Since certain travel time can be considered as a special case of random travel time and since any two certain travel times are statistically independent, Theorem 2 provides a necessary condition for the additivity of $C(T)$. Namely, the functional $C(T)$ can only be additive in the two cases:

(A) $H(t) = H_0$ is constant for all $t < t_0$ and $H_0 \le W(t)$ for all $t > t_0$, or

(B) $W(t) = W_0$ is constant for all $t > t_0$ and $W_0 \le H(t)$ for all $t < t_0$.

Case (B) is uninteresting, since the optimal departure time in this case is t_0 whenever travel time is positive (a. s.) and the cost measure becomes a constant times the mean travel time. This measure is trivially additive but does not account for travel time variability. We therefore dismiss this case.

Case (A) potentially comprises cases where the MUT functions are not smooth. In the popular case of $\alpha-\beta-\gamma$ scheduling preferences discussed in footnote 4 above and in the introduction, $H(t) \equiv H_0$ while the function *W* is a step function. The expected travel time cost has the form

$$C(T) = H_0\mu + \eta\sigma \tag{10}$$

(Fosgerau and Karlström, 2010), where η depends on the standardised distribution of T. Consider T_1 and T_2 both standard normally distributed. Then $C(T_1) = C(T_2) = \eta$, while $C(T_1 + T_2) = \sqrt{2}\cdot\eta$, which shows that this measure is not additive. This does not rule out the possibility that additivity may hold for non-smooth MUT, although we believe that to be unlikely. The following theorem establishes a necessary condition for additivity for the case of smooth MUT. Below we show that this condition is also sufficient.

Theorem 4. *Let W be thrice differentiable and have positive first derivative on the open interval (a,b). Let the optimal expected travel time cost* $C(T)$ *be additive for any constant* $H(t) \equiv H_0 \in W((a,b))$. *Then either*

$$W(t) = u_0 + \gamma t \tag{11}$$

for any $t \in (a,b)$, *where* u_0 *and* γ *are constants or*

$$W(t) = W_0 + me^{\beta t} \tag{12}$$

for any $t \in (a,b)$, *where* W_0, *m and* β *are constants.*

The proof can be found in Appendix A.

The two forms of the MUT function at destination (11) and (12) can be conveniently presented by one equation

$$W(t) = H_0 + \frac{\gamma}{\beta}\left[e^{\beta(t-t_0)} - 1\right] \tag{13}$$

which defines a family of exponential-plus-constant functions. All these functions satisfy conditions $W(t_0) = H_0$ and $W'(t_0) = \gamma$. In order to satisfy Assumption 1, γ must be positive while t_0 and H_0 can take any value. Equation (13) can be obtained from (12) by setting $m = \frac{\gamma}{\beta}\exp(-\beta t_0)$ and $W_0 = H_0 - \frac{\gamma}{\beta}$. By letting β tend to 0 in (13), the affine function $W(t) = H_0 + \gamma(t - t_0)$ is obtained which is the same as (11) with $u_0 = H_0 - \gamma t_0$. Thus the affine MUT at the destination is a limiting case of the exponential-plus-constant MUT. Figure 2 shows a bundle of function plots with fixed γ, t_0 and H_0 and different β.

Figure 2 here.

In order to find the optimal travel time cost, substitute (13) into (7), obtaining

$$\begin{aligned} c(t,T) &= \int_t^{t_0} H_0 d\tau + E\int_{t_0}^{t+T}\left[H_0 + \frac{\gamma}{\beta}e^{\beta(\tau-t_0)} - \frac{\gamma}{\beta}\right]d\tau = \\ &= H_0\mu - \frac{\gamma}{\beta}(t - t_0 + \mu) + \frac{\gamma}{\beta^2}\left[e^{\beta(t-t_0)}Ee^{\beta T} - 1\right]. \end{aligned} \tag{14}$$

The first order condition for optimal departure time

$$-\frac{\gamma}{\beta} + \frac{\gamma}{\beta}e^{\beta(t-t_0)}Ee^{\beta T} = 0$$

has a unique solution

$$t^* = t_0 - \frac{1}{\beta}\ln Ee^{\beta T},$$

which after substitution into (14) provides the minimal expected travel time cost

$$C(T) = H_0\mu + \frac{\gamma}{\beta^2}\ln Ee^{\beta(T-\mu)}. \tag{15}$$

Note that this expression does not involve t_0. This is because changing t_0 does not affect the shape of the scheduling preferences but only shifts the function W horisontally. The travel time cost (15) is indeed additive by independent parts of the trip because for independent T_1 and T_2 with mean values μ_1 and μ_2

$$C(T_1+T_2) = H_0(\mu_1+\mu_2) + \frac{\gamma}{\beta^2}\ln E\left[\exp\left(\beta(T_1-\mu_1)\right)\exp\left(\beta(T_2-\mu_2)\right)\right] =$$

$$= H_0(\mu_1+\mu_2) + \frac{\gamma}{\beta^2}\left[\ln E\exp\left(\beta(T_1-\mu_1)\right) + \ln E\exp\left(\beta(T_2-\mu_2)\right)\right] = C(T_1) + C(T_2).$$

Note that $g_T(\beta) = \ln E\exp(\beta T)$ is called the cumulant generating function (CGF) because its derivatives at $\beta = 0$ are the cumulants of T (Lukacs, 1970). Thus the cost of travel time variability is measured in (15) by the CGF of the centralised travel time distribution $g_{T-\mu}(\beta)$.

When the variance σ^2 of the travel time exists, the limit of the travel time cost (15) as $\beta \to 0$ can be calculated by using l'Hôpital's rule twice as

$$\lim_{\beta\to 0} C(T) = H_0\mu + \frac{\gamma}{2}\sigma^2, \tag{16}$$

which of course is also additive. It is a special case of affine scheduling preferences considered in Fosgerau and Engelson (2011).

To summarise the result of this section, there is just one smooth form of scheduling preferences defined in terms of MUT that provides non-trivial optimal expected travel time cost with the desirable additivity property. That form is (13) with constant $H(t) = H_0$ and the corresponding travel time cost is given by (15). The limiting case as $\beta \to 0$ corresponds to affine W and the travel time cost equal to the linear combination of the mean and the variance of travel time.

5. Properties of the CGF measure of travel time variability

Equation (15) shows that the optimal expected travel time cost can be represented as a sum of the cost of mean travel time and the cost of travel time reliability. The measure of travel time reliability is the value of the CGF of the centralised travel time distribution, evaluated at β and divided by β^2.

Similarly to the variance, the CGF of the centralised random variable is never negative and is zero only when the variable is constant a.e.; this is a simple corollary of the Jensen's inequality and the strict convexity of the exponential function (Lehmann and Casella, 1998).

The travel time cost (15) may be infinite for some β and some distributions of travel time. The following three cases are of interest for applications.

1. If the travel time distribution has compact support then the cost (15) is finite for any β and can be presented as the convergent Taylor series

$$C(T) = H_0\mu + \gamma\frac{\sigma^2}{2} + \gamma\sum_{n=3}^{\infty}\frac{\beta^{n-2}k_n}{n!} \tag{17}$$

where k_n is the cumulant of order n of the travel time distribution (Lukacs, 1970). In particular, $\frac{d}{d\beta}C(T)|_{\beta=0} = \frac{\gamma k_3}{6}$, where $k_3/\sigma^3 = E\left[(T-\mu)^3\right]/\sigma^3$ is the skewness of the travel time distribution. Hence, the travel time cost is sensitive to the skewness of the distribution with positive skewness being costly as β increases from zero.

2. If the distribution of travel time does not have compact support but is concentrated on the right half-axis (e.g. if the travel time is never negative) then the travel time cost (15) is finite for any negative β. Existence of the cost (15) for positive β depends on probability of long travel times. For example, if the travel time is Gamma distributed with shape parameter θ then the travel time cost $C(T)$ is finite only for $\beta < 1/\theta$. If the distribution is lognormal then $C(T)$ is infinite for any $\beta > 0$.

3. For the normal distribution, $C(T) = H_0\mu + \gamma\frac{\sigma^2}{2}$ for any β. For the non-Gaussian stable distribution with the stability parameter $\alpha \le 1$, the mean value is infinite and so is the cost $C(T)$. When $\alpha > 1$, $C(T)$ is finite for positive β only if the skewness parameter is -1 and for negative β only if the skewness parameter is 1.[7]

[7] Fosgerau and Fukuda (2008) fitted the stable distribution with $1.1 \le \alpha \le 1.3$ and the skewness parameter between 0.8 and 1 for travel time data in Copenhagen.

The travel time cost in (15) based on exponential MUT and the limiting case in (16) have two advantages over the equation (10) based on the α–β–γ scheduling preferences. First, the coefficient of the variability measure does not depend on the shape of the travel time distribution. This implies that the value of travel time variability can be transferred from one situation to another without a need to consider the difference in travel time distributions. The second advantage is the additivity with respect to parts of trip with independent travel times.

Compared to the variance of travel time, the general CGF measure has the advantage of the added flexibility given by the parameter β, which allows adjusting the form of MUT at the destination. The travel time variance as a measure of travel time reliability has been criticised for not taking into account the skewness of the travel time distribution (van Lint et al., 2008). The CGF does depend on skewness of the travel time distribution for non-zero β.

6. Scheduled services

This section extends the previous analysis of exponential scheduling preferences to the case of scheduled services. At the same time it extends the results in Fosgerau and Engelson (2011) for the traveller using scheduled services and having affine MUT to the case when MUT at the origin of the trip is constant and MUT at the destination is exponential plus a constant.

Consider a traveller who chooses between departures of a scheduled service having fixed headway h. The service is assumed to always depart according to the timetable but the in-vehicle time T is randomly distributed.[8] The traveller values waiting time as travel time and has constant MUT H_0 at the origin and MUT defined by equation (13) at the destination. The traveller may choose to consult the timetable and thereby acquire the information about exact departure times of the services. This planning does however involve effort on behalf of the traveller, which carries a cost. We aim at expressing the optimal expected total travel time cost (including planning cost) via the headway and the distribution of the in-vehicle travel time.

If the traveller chooses not to consult the time table but instead shows up at the station to wait for the next departure then the total travel time consists of the in-vehicle time T and the waiting time w. From the perspective of the traveller, the waiting time is random, uniformly distributed on $[0,h]$. With $H(t) \equiv H_0$ and $W(t)$ defined by (13) the optimal expected travel cost can be obtained from (15) by replacing the mean travel time μ by the sum $\mu + h/2$ of mean in-vehicle time and mean waiting time and replacing the random travel time T by the sum $T + w$ of in-vehicle time and waiting time. Thus,

[8]The access time from the origin to the boarding station and egress time from the alighting station to the destination can be included in T.

$$C_n(T) = H_0\left(\mu + \frac{h}{2}\right) + \frac{\gamma}{\beta^2}\ln Ee^{\beta(T+w-\mu-h/2)}$$

where subscript n indicates that the traveller does not plan. Since waiting time and in-vehicle time are independent, the optimal expected travel cost is

$$C_n(T) = H_0\left(\mu + \frac{h}{2}\right) + \frac{\gamma}{\beta^2}\ln Ee^{\beta(T-\mu)} + \frac{\gamma}{\beta^2}\ln\left(\frac{1}{h}\int_0^h e^{\beta x}dx\right) - \frac{\gamma h}{2\beta} =$$
$$= C(T) + \frac{\gamma}{\beta^2}\ln\frac{e^{\beta h}-1}{\beta h} - \frac{\gamma h}{2\beta} + H_0\frac{h}{2}$$

A traveller who knows the timetable can time her arrival to the station exactly and thereby avoid waiting time. By the virtue of (14), her expected travel time cost associated with departure time t is equal to

$$c_p(t,T) = H_0\mu - \frac{\gamma}{\beta}(t - t_0 + \mu) + \frac{\gamma}{\beta^2}\left[e^{\beta(t-t_0)}Ee^{\beta T} - 1\right], \tag{18}$$

where the subscript p refers to the planning traveller. Since this function is convex in t, the traveller chooses the unique departure in interval $[s - h/2, s + h/2]$ defined by $c_p(s - h/2, T) = c_p(s + h/2, T)$. By solving this equation, the midpoint of the interval is obtained as

$$s = t_0 - \frac{1}{\beta}\ln Ee^{\beta T} - \frac{1}{\beta}\ln\frac{\exp(\beta h/2) - \exp(-\beta h/2)}{\beta h}\ .$$

Before the timetable is known, the departure time of the scheduled service is uniformly distributed over $[s - h/2, s + h/2]$ from the perspective of the traveller, Therefore the expectation of the optimal travel time cost (18) of the traveller who consults the timetable is

$$C_p(T) = \frac{1}{h}\int_{s-h/2}^{s+h/2} c_p(t,T)\,dt =$$
$$= H_0\mu + \frac{\gamma}{\beta^2}\ln Ee^{\beta(T-\mu)} + \frac{\gamma}{\beta^2}\ln\frac{\exp(\beta h/2) - \exp(-\beta h/2)}{\beta h} = \tag{19}$$
$$= C(T) + \frac{\gamma}{\beta^2}\ln\frac{e^{\beta h}-1}{\beta h} - \frac{\gamma h}{2\beta}.$$

In addition to the travel time cost (19), the planning traveller incurs the planning cost $\varsigma > 0$. Therefore a rational traveller will choose to plan if and only if $\varsigma < C_n(T) - C_p(T) = H_0\frac{h}{2}$

where the right hand side is the gain from planning equal to the relative utility of the mean waiting time $h/2$ at the origin. The gain is positive if the traveller values time at origin higher than the travel time. Thus the total expected travel cost for the rational traveller is

$$C_s(T) = \min\left[C_n(T), C_p(T) + \varsigma\right] = C(T) + \frac{\gamma}{\beta^2}\ln\frac{e^{\beta h}-1}{\beta h} - \frac{\gamma h}{2\beta} + \min\left(H_0\frac{h}{2}, \varsigma\right). \qquad (20)$$

The influence of travel time reliability on the total travel cost is exactly as in the unscheduled case. The sum of the last three terms in (20) indicates the cost for the rational traveller of being restricted to a schedule. It tends to 0 when $h \to 0$ and behaves as $\varsigma + \frac{\gamma h}{2|\beta|} - \frac{\gamma}{\beta^2}\ln|\beta h|$ for large h. The marginal cost of headway is $H_0/2$ at very short headways and $\frac{\gamma}{2|\beta|}$ at very long headways.

When $\beta \to 0$, equation (20) becomes $C_s(T) = C(T) + \gamma h^2/24 + \min(H_0 h/2, \varsigma)$, which is quite similar to the affine scheduling preferences in Fosgerau and Engelson (2011) when the MUT at the origin is constant.

7. Empirical illustration

In order to describe day-to-day variability, we employ data related to the same time intervals on a number of different days. So let the travel time be observed for the same origin and destination and the same time of day for N days. The proposed CGF measure of travel time variability with $\beta \neq 0$ can be estimated from this sample as $v_\beta = \frac{1}{\beta^2}\ln\frac{1}{N}\sum_{n=1}^{N}\exp\left[\beta\left(t_n - \bar{t}\right)\right]$ where t_n is the observed travel time on day n and $\bar{t}$ is the sample mean. When $\beta = 0$, the measure is estimated as half the sample variance. Figure 3 shows the result of such estimation based on data collected through the period January 16 – May 8, 2007 on 91.1 km of freeway network in South-East Denmark. We use travel times in minutes per kilometre for light vehicles, measured using automatic number plate recognition. Cameras are placed near each intersection, dividing the network into 15 pieces, with data recorded separately for the two directions giving observations for each of 30 one-way links. The links range from 1.7 to 11.9 kilometres in length and two to three lanes in width. Data are recorded in five-minute intervals. After dropping observations with missing information we have 606494 observations, pooling observations from different links. We estimate the CGF measure of travel time variability for different times of day and different values of β. To produce the plots in Figure 3, we have employed some smoothing replacing each travel time observation t^p in period p by the weighted average $0.25t^{p-1} + 0.5t^p + 0.25t^{p+1}$.[9]

Figure 3 here.

It can be observed from the figure that the CGF estimate increases with β, which indicates that the travel time distribution is positively skewed. The cost measure is correlated with the mean travel time. In spite of the smoothing, the cost measure still exhibits small fluctuations which seem likely to be caused by random fluctuations in data rather than by systematic variations in the distribution of travel time.

8. Conclusion

This paper has derived a measure of random travel time variability that is founded in microeconomic theory of traveller behaviour. We consider scheduling preferences given in terms of time varying utility rates at the origin and at the destination of a trip and we consider travellers who can freely choose departure time. We show that there is just one (smooth) form of such scheduling preferences that leads to a family of nontrivial additive measures of expected travel time cost that accounts for travel time variability. A measure from this family applies also to scheduled services and additivity continues to hold in this case.

The additive measure of random travel time variability is a value of the cumulant generating function (CGF) divided by the square of the argument. This is a function with a number of convenient features as discussed in the paper. The CGF measure depends on a parameter, β, which determines the curvature of the MUT at the destination. When $\beta = 0$, the measure is just the variance of travel time and hence the measure can be considered a generalisation of the variance.

As we have argued, additivity is a very useful property of a measure of expected travel time cost. This does not, however, imply that an additive measure is the one that best describes the actual preferences of travellers. It is an empirical question whether scheduling preferences are such that additivity can be assumed to hold. Parametric estimation of piecewise linear scheduling preferences have been performed using both revealed preference and stated preference methods; see Bates (2008) for review. The exponential scheduling preferences

[9] For the purpose of this paper it is sufficient to perform this simple analysis. It is, of course, possible to perform a much more detailed analysis of our data.

presented in this paper can be estimated in a similar way in order to support the consistent framework for convenient incorporation of travel time reliability in transportation models. This topic is currently under study by the authors.

Another issue is the empirical estimation of the CGF measure. Our empirical illustration showed that the estimated standard deviation of travel time (and hence the estimated cost measure) had peaks at some times of day that seemed unlikely to be the product of systematic variation in the distribution of travel time. Such peaks might diminish as the size of the sample increases. However, this indicates that it is desirable to know something about the asymptotics of the estimation of the CGF measure from finite samples. It might be possible to develop statistical procedures to assess the uncertainty of an estimate of the CGF measure such as is routinely done for other estimates such as the mean travel time. This is left for further research.

Finally, we would like to direct attention to the empirical question of whether travel times on different links in a road network can be considered to be independent. It seems likely that they will generally not be independent. Positive correlations take place due to queue spillbacks from downstream to upstream links and due to events that influence travel time in the whole study area, like weather conditions or public transport disturbances. Negative correlations may exist as well, for example between the travel times on a bottleneck link and the downstream link. Eliasson (2007) and Fosgerau and Fukuda (2008) demonstrated that standard deviations, respectively distributions of travel time obtained along routes assuming independent travel times on adjacent links are rather close to the standard deviations and distributions obtained by travel time measurements on the whole route. This indicates that sum of the reliability measures considered in this paper across the links may be an acceptable approximation of the reliability measures for the whole trip.

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Appendix A. Proofs of theorems.

Proof of Theorem 1. Since $c(t,\mu)$ is continuous in *t*, it attains a minimal value *A* on the interval $[t_0-\mu, t_0]$ by the Weierstrass theorem. It remains to show that *A* is also a global minimum. For any $t \geq t_0$,

$$c(t,\mu)-c(t_0,\mu)=\int_t^{t_0} H(\tau)d\tau+\int_{t_0}^{t+\mu} W(\tau)d\tau-\int_{t_0}^{t_0+\mu} W(\tau)d\tau=\int_{t_0+\mu}^{t+\mu} W(\tau)d\tau-\int_{t_0}^{t} H(\tau)d\tau .$$

Using Assumption 1, each of the two integrals in the right hand side can be estimated separately as

$$\int_{t_0}^{t} H(\tau)d\tau \leq (t-t_0)H(t_0+0) \text{ and } \int_{t_0+\mu}^{t+\mu} W(\tau)d\tau \geq (t-t_0)W(t_0+\mu)$$

whence

$$c(t,\mu)-c(t_0,\mu) \geq (t-t_0)[W(t_0+\mu)-H(t_0+0)] \geq 0 .$$

Similarly, $c(t,\mu)-c(t_0-\mu,\mu)\geq 0$ for $t\leq t_0-\mu$. Therefore *A* is the minimal value of $c(t,\mu)$ over all *t*.

It follows that $C(\mu)=\min\limits_{t_0-\mu\leq t\leq t_0}\left[\int_t^{t_0} H(\tau)d\tau+\int_{t_0}^{t+\mu} W(\tau)d\tau\right]$. However, $H(\tau)=V(\tau)$ for $\tau<t_0$, and $W(\tau)=V(\tau)$ for $\tau>t_0$, which implies the assertion of the theorem. ■

Proof of Theorem 2. First note that $C(\mu)$ is bounded on [0,1] due to boundedness of $V(\tau)$ for $t_0-\mu-1\leq\tau\leq t_0+\mu+1$.

Assume (*i*). By Theorem 1 of Section 2.1.1 in Aczél (2006), additivity and boundedness of $C(\mu)$ on an interval together imply linearity i. e.

$$C(\mu)=h\mu \tag{A.1}$$

for some *h* and for all $\mu\geq 0$.

Assume that (*iii*) is false, i. e. there exist $t_1<t_0$ and $t_2>t_0$ such that $V(t_1)>V_0$ and $V(t_2)>V_0$. Choose $\varepsilon>0$ such that $V_0+\varepsilon<\min[V(t_1),V(t_2)]$ and denote $A_\varepsilon=\{t:V(t)\leq V_0+\varepsilon\}$ and $\mu(\varepsilon)=mes(A_\varepsilon)$ where *mes* denotes Lebesgue measure. Then, due to the monotonicity of *V* for $t<t_0$ and for $t>t_0$, A_ε is an interval contained in $[t_1,t_2]$. Therefore $C(\mu(\varepsilon))\leq\int_{A_\varepsilon} V(\tau)d\tau\leq(V_0+\varepsilon)\mu(\varepsilon)$, which together with (A.1) implies $h\leq V_0+\varepsilon$. As this is valid for any $\varepsilon>0$ small enough, the inequality

$$h\leq V_0 \tag{A.2}$$

follows.

Now let $\mu'=\mu(0)+1$ and denote by *t** the value at which the minimal value of $c(t,\mu')$ is attained. Then

$$C(\mu')=\int_{t^*}^{t^*+\mu} V(\tau)d\tau=\int_{[t^*,t^*+\mu']\cap A_0} V(\tau)d\tau+\int_{[t^*,t^*+\mu']\setminus A_0} V(\tau)d\tau$$

where the first integral on the right hand side equals $V_0 mes([t^*, t^*+\mu'] \cap A_0)$ because $V(t) = V_0$ for almost all $\tau \in [t^*, t^*+\mu'] \cap A_0$ by the definitions of V_0 and A_0. As regards the second integral on the right hand side, note that, by definition of A_0, all values of the integrand are greater than V_0 on the set of integration and that the set has positive measure because $mes(A_0) = \mu(0) < \mu'$. Therefore $\int_{[t^*, t^*+\mu'] \setminus A_0} V(\tau)d\tau > V_0 mes([t^*, t^*+\mu'] \setminus A_0)$ and

$$C(\mu') > V_0 mes([t^*, t^*+\mu'] \cap A_0) + V_0 mes([t^*, t^*+\mu'] \setminus A_0) = V_0 \mu'$$

which implies $h\mu' > V_0 \mu'$ in violation of (A.2). Hence the last assumption is false, and (*i*) implies (*iii*).

Now assume (*iii*). Note that, due to the monotonicity, $V(t) \geq V_0$ for all $t \neq t_0$. Therefore $V(t) = V_0$ for all $t > t_0$ or $V(t) = V_0$ for all $t < t_0$. In the first case, the objective in (6) attains its minimal value at $t = t_0$ because for $t \in [t_0 - \mu, t_0]$

$$\int_t^{t+\mu} V(\tau)d\tau - \int_{t_0}^{t_0+\mu} V(\tau)d\tau = \int_t^{t_0} V(\tau)d\tau - \int_{t+\mu}^{t_0+\mu} V(\tau)d\tau \geq V_0(t_0 - t) - V_0(t_0 - t) = 0$$

whence

$$C(\mu) = \int_{t_0}^{t_0+\mu} V(\tau)d\tau = V_0 \mu .$$

In the second case, similar reasoning leads to the same expression which obviously satisfies (*i*).

Finally, the equivalence of (*ii*) and (*iii*) is due to the equality

$$V_0 = \min(\inf\{H(t) : t < t_0\}, \inf\{W(t) : t > t_0\})$$

which follows from the fact that $V(t)$ coincides with $H(t)$ for all $t < t_0$ and with $W(t)$ for all $t > t_0$. ■

Proof of Theorem 3. For any *t*,

$$c(t+1,T) - c(t,T) = \int_{-\infty}^{1-t} \left[\int_{t+s}^{t+s+1} W(\tau)d\tau \right] dF(s) + \int_{1-t}^{+\infty} \left[\int_{t+s}^{t+s+1} W(\tau)d\tau \right] dF(s) + \int_{t+1}^{t} H(\tau)d\tau \qquad \text{(A.3)}$$

where F is the cumulative distribution function of travel time. Since W is non-decreasing, the first integral in (A.3) can be estimated as

$$\int_{-\infty}^{1-t}\left[\int_{t+s}^{t+s+1} W(\tau)\,d\tau\right]dF(s) \geq \int_{-\infty}^{1-t}\left[\int_{s}^{s+1} W(\tau)\,d\tau\right]dF(s)$$

where the right hand side vanishes as $t \to +\infty$. In the second integral in (A.3), $s \geq 1-t$ implies $\tau \geq 1$, and, together with the monotonicity of W,

$$\int_{1-t}^{+\infty}\left[\int_{t+s}^{t+s+1} W(\tau)\,d\tau\right]dF(s) \geq \int_{1-t}^{+\infty}\left[\int_{t+s}^{t+s+1} W(1)\,d\tau\right]dF(s) = W(1)\left[1-F(1-t)\right]$$

where the right hand side tends to $W(1)$ as $t \to +\infty$. Finally, the third integral can be estimated for $t>1$ as

$$\int_{t+1}^{t} H(\tau)\,d\tau \geq -H(t) \geq -H(1)$$

because H is non-increasing. Thus $c(t+1,T)-c(t,T)$ is bounded from below by a function that tends to the positive number $W(1)-H(1)$ as $t \to +\infty$. This means that there exists t_+ such that $c(t+1,T)-c(t,T) > 0$ for all $t > t_+$. Due to the convexity of $c(t,T)$, $c(t,T) \geq c(t_+ + 1,T)$ for all $t > t_+ + 1$.

It can be shown in a similar manner that there exists t_- such that $c(t,T) \geq c(t_-,T)$ for all $t < t_-$. Application of the Weierstrass theorem on the interval $[t_-, t_+]$ completes the proof. ■

Proof of theorem 4. Choose any $t_0 \in (a,b)$ and set $H(t) \equiv H_0 = W(t_0)$. Let X be a random variable with compact support, $EX = 0$ and $E\left(X^2\right) > 0$. For any real σ, let $t^*(\sigma)$ be an optimal solution to problem (8) with $T = \sigma X$, i.e. $C(\sigma X) = c(t^*(\sigma), \sigma X)$. The existence of $t^*(\sigma)$ follows from Theorem 3. For any σ the first order condition (9),

$$EW(t^*(\sigma) + \sigma X) = H_0$$

is satisfied. By differentiating this we obtain

$$\frac{d}{d\sigma} t^*(\sigma) = -\frac{E\left[X \cdot W'(t^*(\sigma) + \sigma X)\right]}{EW'(t^*(\sigma) + \sigma X)}. \tag{A.4}$$

Due to the envelope theorem,

$$\frac{d}{d\sigma}C(\sigma X) = \frac{\partial}{\partial\sigma}c(t,\sigma X)\bigg|_{t=t^*(\sigma)} = \frac{\partial}{\partial\sigma}E\int_{t_0}^{t+\sigma X}W(\tau)d\tau\bigg|_{t=t^*(\sigma)} = E\left[XW\left(t^*(\sigma)+\sigma X\right)\right].$$

Using the chain rule and (A.4), the second derivative can be obtained as

$$\frac{d^2}{d\sigma^2}C(\sigma X) = E\left[X\cdot W'\left(t^*(\sigma)+\sigma X\right)\left(\frac{\partial}{\partial\sigma}t^*(\sigma)+X\right)\right]$$
$$= E\left[X^2W'\left(t^*(\sigma)+\sigma X\right)\right] - \frac{\left\{E\left[X\cdot W'\left(t^*(\sigma)+\sigma X\right)\right]\right\}^2}{E\left[W'\left(t^*(\sigma)+\sigma X\right)\right]}.$$

In the similar way, rather long expressions for the third and the fourth derivatives can be derived. Substituting $\sigma = 0$ and using the facts that $EX = 0$ and $t^*(0) = t_0$, one obtains

$$\begin{aligned}\frac{d^4}{d\sigma^4}C(\sigma X)\bigg|_{\sigma=0} &= W'''(t_0)E\left(X^4\right) - \frac{3\left\{E\left[X^2\cdot W''(t_0)\right]\right\}^2}{W'(t_0)} = \\ &= W'''(t_0)k_4(X) + 3\left\{W'''(t_0) - \frac{\left[W''(t_0)\right]^2}{W'(t_0)}\right\}\sigma^4(X)\end{aligned} \qquad \text{(A.5)}$$

where $\sigma^4(X)$ is the squared variance and $k_4(X) = E\left(X^4\right) - 3\sigma^4(X)$ is the fourth cumulant. Let X_1 and X_2 be independent random variables distributed as *X*. Since the functional $C(T)$ is additive the equation $C(\sigma X_1 + \sigma X_2) = C(\sigma X_1) + C(\sigma X_2)$ is fulfilled for any σ and therefore $\frac{d^4}{d\sigma^4}C\left(\sigma(X_1+X_2)\right)\Big|_{\sigma=0} - \frac{d^4}{d\sigma^4}C(\sigma X_1)\Big|_{\sigma=0} - \frac{d^4}{d\sigma^4}C(\sigma X_2)\Big|_{\sigma=0} = 0$. Since $k_4(X_1+X_2) - k_4(X_1) - k_4(X_2) = 0$ and $\sigma^4(X_1+X_2) - \sigma^4(X_1) - \sigma^4(X_2) = 2\sigma^2(X_1)\sigma^2(X_2) > 0$ it follows from (A.5) that $W'''(t_0) - \frac{\left[W''(t_0)\right]^2}{W'(t_0)} = 0$ for any $t_0 \in (a,b)$, i.e. $\frac{d}{dt}\left(\frac{W''}{W'}\right) = 0$. Solving this differential equation one obtains $W' = Me^{\beta t}$ with arbitrary constants β and *M*, which implies that *W* must be either an affine function (if $\beta = 0$) or a sum of an exponential function and a constant. ■

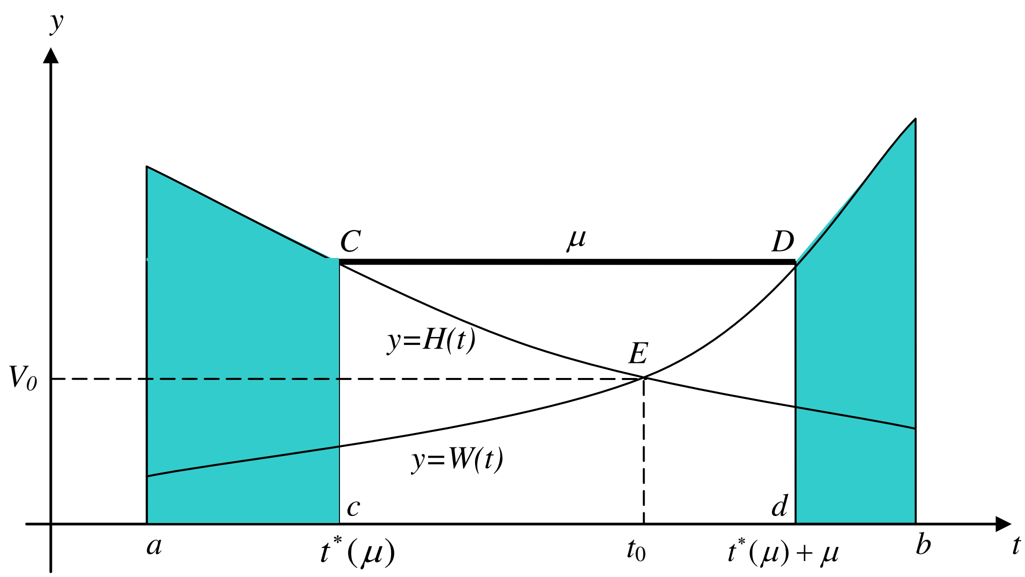

Figure 1. The total utility and the disutility of travel duration at optimal choice of departure time.

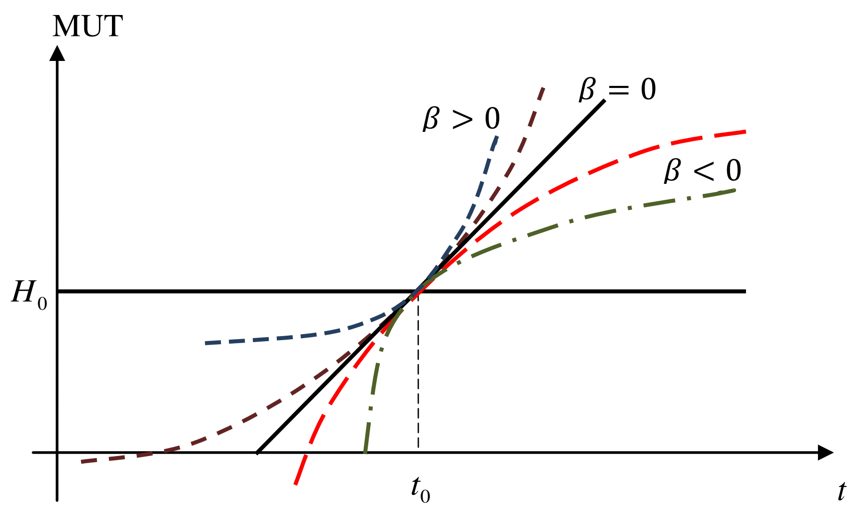

Figure 2. The family of MUT functions providing additive travel time cost. The function H is constant while the exponential-plus-constant functions W defined by equation (13) are increasing, convex for $\beta > 0$, and concave for $\beta < 0$.

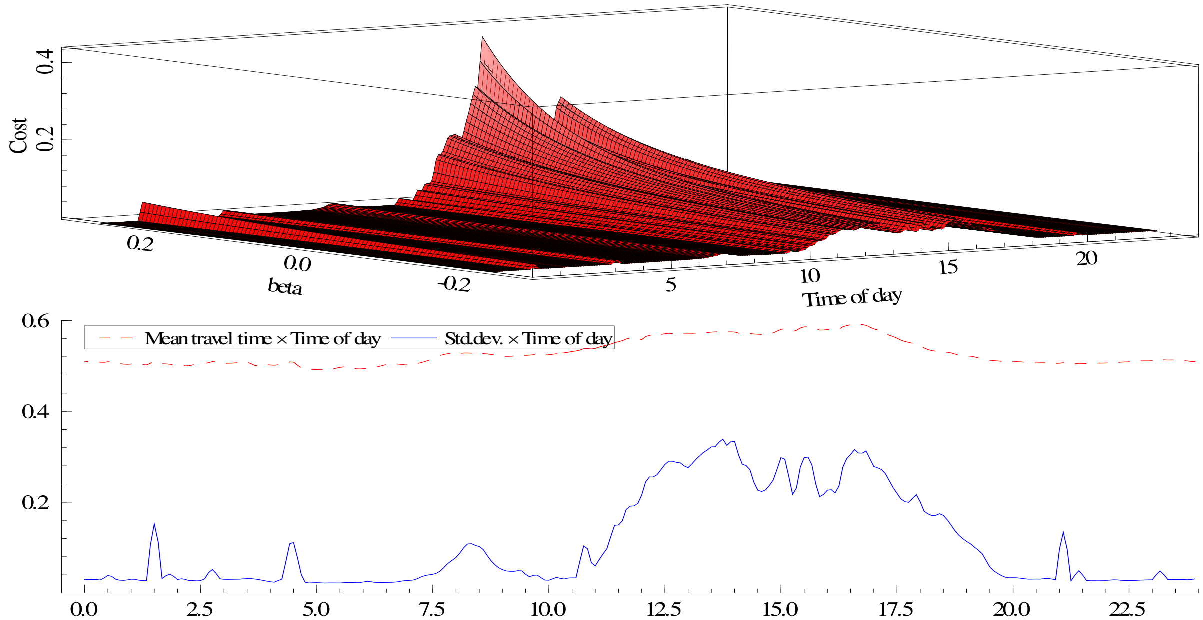

Figure 3. Illustrative computation of measures of travel time variability for Danish freeways. The lower plot shows the sample mean travel time and the sample standard deviation of travel time by time of day. The upper plot is a 3D diagram of the estimated CGF measure.