The value of reliability

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Abstract

We derive the value of reliability in the scheduling of an activity of random duration, such as travel under congested conditions. Using a simple formulation of scheduling utility, we show that the maximal expected utility is linear in the mean and standard deviation of trip duration, regardless of the form of the standardised distribution of trip durations. This insight provides a unification of the scheduling model and models that include the standard deviation of trip duration directly as an argument in the cost or utility function. The results generalise approximately to the case where the mean and standard deviation of trip duration depend on the starting time. An empirical illustration is provided.

KEYWORDS: Welfare; Random duration; Time; Scheduling; Reliability; Variability
JEL codes: D01; D81

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1 Introduction

In this paper we consider the value of reliability for an agent who wishes to undertake an activity such as a trip of random duration and must decide when to initiate the activity, knowing only the distribution of its duration. We are concerned with the value of changes to the distribution of the trip duration. The value of a change in the mean duration is just the value of time, which is a concept with a long history in economics (Becker, 1965; Beesley, 1965; Johnson, 1966; DeSerpa, 1971) and there is a large literature on its measurement.[1] The concept of the value of reliability is less well established but about as important. In this paper we take the value of reliability to be the value of a change in the standard deviation of trip duration. Some contributions (Brownstone and Small, 2005; Lam and Small, 2001; Small et al., 2005) have instead defined reliability as the range between, e.g., the 0.5 and the 0.9 quantiles of the distribution of durations.

We incorporate reliability by building on the model of Small (1982)[2], who considered the scheduling of commuter work trips when the commuters have scheduling costs as well as time and monetary costs.[3] We formulate the scheduling costs as an opportunity cost per minute of starting early and a greater cost per minute of finishing late relative to some fixed deadline. In contrast to earlier contributions (e.g., Noland and Small, 1995), we are able to derive the optimal expected cost for a general distribution of trip durations. We obtain the simple result that the optimal departure time as well as the optimal expected cost depend linearly on the mean and standard deviation of the distribution of trip durations, provided the standardised distribution is fixed. Both the optimal departure time and the value of reliability depend in a simple way on the standardised distribution of trip durations and the optimal probability of being late, which in turn is given by the scheduling costs.

In this paper we are able to show that the expected utility is linear in standard deviation for any travel time distribution. We also show that the influence of the travel time distribution on valuation of reliability can be summarised in two factors: the standard deviation and a mean lateness factor, which can easily be calculated for any travel time distribution. This result is useful when one has preference parameters estimated for one travel time distribution (for instance a stated

[1] Some recent references on the measurement of the value of time are Small et al. (2005) and Fosgerau (2007).

[2] The model derives from Vickrey (1969).

[3] Some early contributions discussing the cost of stochastic delay are Douglas and Miller (1974) and Anderson and Kraus (1981).

preference study), and want to use them in an applied context with a different, empirical travel time distribution.

Although originally motivated in transport, the structure of the scheduling problem occurs in many situations; for example, deciding when to enter a queue or when to begin a search. For firms, some inventory stock problems are of similar structure, for instance when holding a deteriorating good in stock is costly, with a random cut-off quality with high cost and costs for delayed delivery (Liaoa et al., 2000; Hochman et al., 1990). In health economics, waiting times for patients are random and associated with a cost (Mataria et al., 2007). The structure of the scheduling problem also occurs in the decision of how durable to design a product, given that it should last for at least some fixed period and lacking knowledge of the intensity of use. Finally, the timing of activities is also the focus in the literature on the real option model (Dixit and Pindyck, 1994). However, while the option value model focuses on the timing of decision under increasing information, we address a situation where waiting does not allow more information to be gathered. Hence, our problem is not an optimal stopping problem in the sense of the real option model.

In transportation, similar problems occur in several different contexts. Airlines have to decide how much slack to allow in the scheduling of flights. In Brueckner (2004), the passenger must buy the air ticket before knowing his preferred arrival time, and expected scheduled delay differs between airlines. The scheduling problem with delays is also relevant in the tourism literature (Batabyal, 2007).

Increasingly congested road networks have caused travel times to become highly irregular in many places, which makes road transport an important case for the scheduling model. In this paper we find that travel time uncertainty accounts for about 15 per cent of time costs on a typical urban road. Considering the large share of individuals' time budgets that is spent on transport, it is clear that uncertainty of trip durations represents a significant cost to society in general. The concepts of the value of time and the value of reliability are both of crucial importance for decisions regarding capacity provision, operations, pricing and other regulation of transport networks. Both concepts are similarly important in urban economics, travel costs being a main determinant of urban spatial structure (e.g., Brueckner, 1987).

Transport is also a clear case of time dependent demand with sharp peaks in the morning and the afternoon. In our empirical illustration in Section 4 we test and accept that for practical purposes the distribution of travel times at different times of day is described completely through the mean and standard deviation of travel time and a fixed standardised distribution of travel time. In the theoretical analysis we also deal with the case where the mean and standard deviation of the distribution of trip durations depend on the departure time. In transport, this pertains to the case where a traveller decides to leave earlier or later along the slopes of the peak in order to avoid the worst. This kind of analysis is important for understanding the effects of pricing policies aimed at regulating traffic during the peak (peak spreading).

As a final application of the model, we consider the case of a scheduled service, where the departure time cannot be chosen freely, but must adhere to a fixed schedule. We are able to make some progress with this case, but find that the simple properties of the unconstrained case no longer hold. In particular, the expected utility is no longer linear in the standard deviation of trip duration.

It should be noted that we take the individual perspective, where the distribution of trip durations is seen as exogenous. This is in contrast with the literature that investigates the properties of equilibrium where the travel time distribution is dependent on the individual departure time choices, and where individuals apply scheduling considerations. Notably, there is the bottleneck model of Vickrey (1969), which has been developed, e.g., in Arnott et al. (1993). A few contributions include stochastic capacity and demand such that travel times become random, e.g., Daniel (1995) and Arnott et al. (1999). Furthermore, there is a literature on learning the equilibrium in congestion games (Sandholm, 2002, 2005) that goes some way in handling stochastic delays. However, none of the papers mentioned in this paragraph address the value of reliability.

The layout of the paper is as follows. Section 2 starts with a brief background from the existing transportation literature. Section 3 does the first bit of theoretical work presenting a simple model where the trip duration distribution is independent of the departure time. Then Section 4 measures some characteristics of observed travel times by car on a typical congested urban road. As this example illustrates, the standardised travel time distribution may be independent of the time of day but the mean and standard deviation are not. Therefore the model is extended in Section 5 to the cases where the mean and the standard deviation but not the standardised trip duration distribution depend on departure time as could be the case through a peak period. Section 6 considers briefly the case of a scheduled service, where the departure time cannot be chosen freely. Section 7 concludes the paper. The more complicated derivations are placed in the appendix.

2 Modelling background

We will first give a short review of the existing transportation literature on the value of reliability. There are basically two existing approaches.

In the first approach (*the mean-variance approach*), it is assumed that individuals have preferences against travel time uncertainty per se. The expected utility of an individual can be written (ignoring monetary cost terms)

$$E(U) = \eta\mu + \rho\sigma \tag{1}$$

where μ is the expected travel time and σ is the standard deviation of travel time, while η and ρ are preference parameters. For applications of this approach, see Hollander (2006) and Noland and Polak (2002). Some contributions have replaced the standard deviation with another measure of scale, an interquantile range (see e.g., Small et al., 2005, and the references in the introduction).

In the second approach (*the scheduling approach*), the individual holds preferences for timing of activities, which is more in line with transport demand as a derived demand. Utility is defined directly from outcomes, i.e. being early or late. Travel time variability affects individuals' utility to the extent that the arrival time at an activity (destination) is affected. The individual holds preferences for being early or late, compared to a preferred arrival time (PAT), see Small (1982) and Noland and Small (1995). Normalising an individual's PAT to be at time zero, let $-D$ be the departure time. Hence, earlier departure corresponds to a larger D.

Let T be the stochastic travel time. Then, in the scheduling approach, the utility is given by[4]

$$U(D,T) = \alpha T + \beta(T-D)^{-} + \gamma(T-D)^{+} + \theta 1_{T>D},$$

where $(T-D)^{-}$ is schedule delay early, $(T-D)^{+}$ is schedule delay late. The term $\theta \neq 0$ allows for a discontinuous penalty for lateness. Then the expected utility is given by

$$E(U(D)) = \alpha\mu + \beta E(T-D)^{-} + \gamma E(T-D)^{+} + \theta p_L, \tag{2}$$

where p_L is the probability of being late.

In the case of an exponential travel time distribution, Noland and Small (1995) derive the expected utility given optimal departure time. Written in the form of

[4]The + notation denotes the function, such that $x^{+} = x$ if $x > 0$ and zero otherwise, x^{-} is defined by $x = x^{+} - x^{-}$. The indicator $1_{T>D}$ is equal to one if $T > D$ and zero otherwise.

Bates et al. (2001), we have

$$E(U(D^*)) = \alpha\mu + \tilde{H}(\alpha, \beta, \gamma, \theta, \Delta, \sigma)\sigma + \theta p_L^* \tag{3}$$

where σ is the standard deviation (and mean) of the exponential travel time distribution, and p_L^* is the probability of lateness, given optimal departure time D^*. The term Δ, defined Bates et al. (2001), allows the travel time distribution to depend on departure time to a certain extent.

In many cases, the analysis is simplified by the assumption that $\theta = 0$ and that the travel time distribution is independent of the departure time ($\Delta = 0$). Then, Bates et al. (2001) argue that for a wide range of distributions, $\beta E(T - D^*)^- + \gamma E(T - D^*)^+$ is well approximated by $\tilde{H}(\beta, \gamma)\sigma$, where σ is the standard deviation of the travel time, and $\tilde{H}$ can be considered constant for any given combination of β and γ. They do not provide evidence to support this claim.

The main purpose of this paper is to show that this statement is true in general and how the factor H can be calculated for any travel time distribution. In particular, we will show that it is dependent on the shape of the travel time distribution.

3 A simple model

Consider an agent about to undertake a trip of uncertain duration. Express the travel time (duration) in the following convenient form.

$$T = \mu + \sigma X, \tag{4}$$

where X is a standardised random variable with mean 0, variance 1, density ϕ, and cumulative distribution Φ.[5] The problem of the agent is to choose when to depart given a preferred arrival time. As above, say that his preferred arrival time is 0, and that he departs at time $-D$.

We assume a utility function consisting of three terms:

$$U(D, T) = \lambda D + \omega T + \nu(T - D)^+, \tag{5}$$

where (λ, ω, ν) are preference parameters. The first term is the disutility of starting early. We may think of this as the opportunity cost of interrupting a prior

[5]We have assumed that the mean of X exists. We also assume that X has convex support such that the inverse distribution exists. The assumption that X has a variance is not strictly necessary. It is possible to replace σ by another measure of scale such as an interquantile range.

activity. The second term captures the disutility of time spent travelling. The third term is the disutility of being late. To keep things simple, we do not consider a cost term. Note that we use a different formulation than the customary (α, β, γ) formulation in, e.g., Bates et al. (2001) and Arnott et al. (1993). Below we shall show that they are in fact equivalent. The main reason is that our formulation makes it explicit how both the value of time and the value of reliability depend on preferences related to activities both earlier and later than the trip under consideration. A secondary reason is that our notation reduces clutter in mathematical derivations to follow.

To clarify the translation between our formulation and the customary (α, β, γ) formulation, note that the latter defines the utility function by

$$U = \alpha T + \beta (T - D)^{-} + \gamma (T - D)^{+}.$$

Note that

$$(T - D)^{-} = (T - D)^{+} - T + D$$

and this to write the utility function as

$$U = (\alpha - \beta) T + \beta D + (\beta + \gamma)(T - D)^{+}.$$

Then in our formulation we have

$$\begin{aligned} \omega &= \alpha - \beta, \\ \lambda &= \beta, \\ \nu &= \beta + \gamma. \end{aligned}$$

We do not include the discontinuous penalty θ for being late, nor do we include any disutility for uncertainty per se. Note that the formulation of utility may be viewed as a minimal way of introducing risk aversion. In general, risk aversion results from concavity of utility. Linear utility results in risk neutrality. In (5), utility is linear except for the lateness term that has a kink at the point $T = D$ where lateness kicks in. The results to be derived below will depend crucially on these assumptions, see Bates et al. (2001) for a discussion. Finally, note that the travel time distribution is independent of departure time. This assumption is strong and will be relaxed later in the paper.

We assume that the agent chooses D so as to maximise expected utility, i.e.

$$EU^{*} = \max_{D} EU(D, T) = \max_{D} \left[\lambda D + \omega \mu + \nu \int_{\frac{D-\mu}{\sigma}}^{\infty} (\mu + \sigma x - D) \phi(x) dx \right]. \tag{6}$$

Appendix A.1 shows that the first order condition for the agent's utility maximisation problem is

$$\Phi\left(\frac{D-\mu}{\sigma}\right) = 1 - \frac{\lambda}{\nu}. \tag{7}$$

We note from (7) that λ/ν is the optimal probability of being late (Bates et al., 2001). Rewriting the first order condition we find that the optimal departure time is

$$D^* = \mu + \sigma\Phi^{-1}\left(1 - \frac{\lambda}{\nu}\right). \tag{8}$$

This shows that the distribution of X only enters the optimal departure time through its $1 - \frac{\lambda}{\nu}$ quantile. So even though the scheduling utility function has a kink, the optimally chosen departure time is linear in μ and σ. By inserting the optimal departure time (8) into the expected utility it is possible to obtain the optimal maximum expected utility (see appendix A.1).

$$EU^* = (\lambda + \omega)\mu + \nu\sigma \int_{1-\frac{\lambda}{\nu}}^{1} \Phi^{-1}(s)ds$$

Now, define the functional

$$H(\phi, \frac{\lambda}{\nu}) = \int_{1-\frac{\lambda}{\nu}}^{1} \Phi^{-1}(s)ds \tag{9}$$

To interpret H, we note that H is the mean lateness in standardised travel time. Therefore, we will henceforth term H the *mean lateness factor*.

Given the definition of the mean lateness factor in (9), the expected utility of an agent who faces a given distribution of trip durations and who optimally chooses his departure time can be written as

$$EU^* = (\lambda + \omega)\mu + \nu H(\Phi, \frac{\lambda}{\nu})\sigma. \tag{10}$$

First, we identify in the first term $(\lambda + \omega)$ as the the value of travel time. Our model formulation highlights that the value of travel time depends on both the disutility of travel time itself, as well as the marginal disutility λ from interrupting a previous activity. The marginal disutility from interrupting a previous activity also affects the valuation of travel time reliability through its influence on the mean lateness factor H.

Second, we observe that the optimal expected utility is linear in the mean and standard deviation of the trip duration, provided the standardised distribution of durations Φ is constant. It should be emphasised that this holds for *any* fixed standardised (absolutely continuous) travel time distribution. This result has been hinted at in the literature, but until now it has only been established in some special cases (Noland and Small, 1995; Bates et al., 2001).

Third, the value of reliability, $\nu H(\phi, \frac{\lambda}{\nu})$, is not a constant preference parameter but depends on the scheduling parameters λ and ν, as well as on the standardised duration distribution ϕ. In particular, the mean lateness factor depends on the shape of the travel time distribution above its $1 - \frac{\lambda}{\nu}$ quantile.

Before proceeding, we shall state the main result in (10) in terms of the customary (α, β, γ) parameters. Substituting parameters, the optimal expected utility can be written as

$$EU^* = \alpha\mu + (\beta + \gamma)H(\Phi, \frac{\beta}{\beta + \gamma})\sigma$$

with

$$H(\Phi, \frac{\beta}{\beta + \gamma}) = \int_{\frac{\gamma}{\beta+\gamma}}^{1} \Phi^{-1}(s)ds.$$

In order to apply the model, one needs to know the preference parameters (ω, λ, ν) or alternatively (α, β, γ) and also the standardised travel time distribution ϕ. In some studies the parameters η and ρ in (1) are estimated directly. However, under the current model we have $\rho = \nu H$ which depends not only on preference parameters but also on the standardised travel time distribution. Therefore, the ρ parameter cannot be directly applied in a different setting, unless we assume that the standardised travel time distribution is the same in the first study and the application.[6]

Our results do indicate how one can translate models estimated in terms of η and ρ into (ω, λ, ν)-terms. We require knowledge of the standardised travel time distribution ϕ prevailing when estimating η and ρ and also knowledge of the optimal probability of being late λ/ν.[7] Using this information, we may then compute the value of H that prevailed in the estimation study. From $\rho = \nu H$ we may then find ν and from the optimal probability of being late λ/ν we may find λ. Finally, use $\eta = \lambda + \omega$ to find ω.

[6] As an example, the estimates of η and ρ in Eliasson (2004) have been applied directly in a cost benefit analysis of the Stockholm congestion scheme (Eliasson, 2009).

[7] One could estimate the average probability of being late from typical stated preference studies of the value of reliability.

4 Empirical illustration

In this section we use a travel time dataset to first estimate the mean and standard deviation of travel time as a function of the time of day. Results show that these functions are not constant as was the maintained assumption in the previous section. The standardised travel time distribution ϕ does however appear to be roughly constant over the day. We will use this information to take a look at the mean lateness factor H.

We use data recorded over the period January 16 to May 8, 2007 at a congested radial road in Greater Copenhagen. Based on timed license-plate matches, the data provide minute by minute observations of the average travel time in minutes for an 11.260 km section. We consider weekdays between 6 AM and 10 PM and discard observations where no traffic was recorded. We use data for the direction towards the city centre, where there is a distinct peak in the morning. This dataset has 24 271 observations. Label these by (T_i, t_i), where T_i is travel time in minutes for the i'th observation and t_i is the time of day in minutes since midnight.

Figure 1 shows first a nonparametric kernel regression of travel time against time of day.[8] The resulting curve is an estimate of $\mu(t)$. The figure also shows the 95 per cent pointwise confidence band. It is fairly tight indicating that μ is quite precisely estimated. There is a sharp morning peak at 8 AM and a smaller peak in the afternoon between 4 and 5 PM.

The lower curve estimates the standard deviation of travel time as a function of the time of day, $\sigma(t) = \sqrt{E[(T - \mu(t))^2]}$. This is achieved by performing a nonparametric regression of squared residuals $(T_i - \mu(t_i))^2$ against time of day and then taking the square root of the result.[9]

Using these estimated functions we have computed standardised travel times by $X_i = (T_i - \mu(t_i))/\sigma(t_i)$ such that these have zero mean and unit variance conditional on the time of day. Figure 2 shows a nonparametric estimate of the cumulative distribution of standardised travel time conditional on the time of day.[10] That is, at each time of day, the figure shows an estimate of the cumulative distri-

[8]This regression has been performed using a normal second-order kernel and a bandwidth of 15 minutes. This is larger than the bandwidth indicated by least squares cross-validation (Li and Racine, 2007) but the regression is visibly undersmoothed by that bandwidth. We note that the number of observations is large and the independent variable (time of day) is binned by minute such that the optimal bandwidth might seem to be too small. All programming is in Ox (Doornik, 2001). The code is available on request.

[9]The bandwidth is again taken to be 15 minutes, chosen by eye-balling.

[10]There is only smoothing in the time of day dimension, where the bandwidth of 15 minutes is still applied.

bution of standardised travel time. If the distribution of standardised travel time is independent of the time of day, then all these curves will be parallel.

A formal test rejects independence of the standardised travel time and time of day. However, that finding should probably not be emphasised too much since the dataset is large and small differences will lead to rejection of independence. Figure 2 does also visually reveal some dependency, especially around the morning peak, but the figure suggests the dependency is not strong.

We have then estimated the unconditional density of the standardised travel times.[11] The resulting estimate is shown in Figure 3, where the dashed curve indicates the lower bound of the 95 per cent pointwise confidence interval. The estimated density has a heavy right tail and bears some resemblance to a lognormal or a gamma density, but it is clear from the figure that it is neither.

We have computed $H(\phi, \frac{\lambda}{\nu})$ for various values of $\frac{\lambda}{\nu}$. We have done this both for the empirical distribution of standardised travel times and for the standard normal distribution. The results are shown in Table 1. In this example there are large differences in H between the normal and the empirical travel time distribution, showing the importance of accounting for the actual distribution of durations. The differences are largest for small values of $\frac{\lambda}{\nu}$ when the optimal probability of being late is low. This is due to the fat right tail of the empirical travel time distribution relative to the normal distribution. However, the difference is also quite large when the optimal probability of lateness is one half.

5 Time varying mean and standard deviation of duration

So far we have considered the distribution of travel time to be independent of the departure time. This is not true in general, as the previous section showed. Indeed, in that example, the mean and standard deviation of durations did depend on the departure time, while the distribution of standardised durations could still be assumed to be roughly independent of the departure time.

We will now relax the assumption of fixed mean and standard deviation of trip durations. We will consider a situation where the mean and standard deviation of durations depend linearly on the departure time. In the example considered in the previous section, we noted that μ and σ in the example seemed to vary (more or

[11] Using a normal second-order kernel and a bandwidth of 0.18162 chosen by least squares cross-validation.

less) linearly with the time of day on both sides of the peak. The main motivation for looking at this case is that we are considering small changes such that we may assume that the departure time changes in a continuous fashion. With large changes, travellers may decide to switch from one side of the peak to another in which case the linearity assumption is no longer useful.

We use the following parametrisation where linear functions are pivoted around the optimal departure time corresponding to constant values of μ and σ.

$$\begin{aligned} D_0 &= \mu_0 + \sigma_0 \Phi^{-1}\left(1 - \frac{\lambda}{\nu}\right) \\ \mu &= \mu_0 + \mu'(D - D_0) \\ \sigma &= \sigma_0 + \sigma'(D - D_0) \end{aligned}$$

The mathematical derivations for this case are somewhat involved and are given in Appendix A.2. It turns out that the optimal expected utility is still linear in mean duration. That is,

$$\frac{dEU^*}{d\mu_0} = (\lambda + \omega),$$

which is the same result as in the case where the distribution of durations is independent of the departure time. Thus in computing the marginal expected utility of mean duration we may ignore that the mean and standard deviation functions depend on the endogenous departure time D.

The corresponding result for the standard deviation is more complicated. Write the value of reliability, i.e. the derivative of the optimal expected utility with respect to σ_0, as a function of the slopes μ' and σ':

$$VoR(\mu', \sigma') = \frac{dEU^*}{d\sigma_0}.$$

Like the value of time, the value of reliability does not depend on the levels of the mean and standard deviation of duration in μ_0 and σ_0. Define for convenience the standardised departure times $Y = \frac{D-\mu}{\sigma}$ and $Y_0 = \Phi^{-1}(1 - \frac{\lambda}{\nu})$, where the latter is the optimal departure time in the case of constant $\mu = \mu_0$ and $\sigma = \sigma_0$. Then we find that the value of reliability is given by

$$VoR(\mu', \sigma') = \lambda Y_0 - \nu Y_0(1 - \Phi(Y)) + \nu \int_Y^\infty x\phi(x)dx.$$

As would be expected, this expression reduces to $VoR(0, 0) = \nu \int_{Y_0}^\infty x\phi(x)dx$, which is the same result as in Section 3. The appendix shows that $VoR(\mu', \sigma') \leq$

$VoR(0,0)$, such that the value of reliability is overestimated if dependency of the distribution of durations on the departure time is ignored. This is true regardless of the signs of μ' and σ', so it does not matter whether the upward or the downward slope of a peak is considered.

Using the independence assumption as an approximation may, however, not lead to a large error as can be seen from the following approximation, derived in Appendix A.2.

$$\frac{VoR(\mu',\sigma') - VoR(0,0)}{VoR(0,0)} \approx -\frac{1}{2\phi(Y_0)H}\left(\frac{\lambda+\omega}{\nu}\mu' + H\sigma'\right)^2$$

This formula may be used to correct an estimate of VoR based on constant μ and σ. If the discrepancy is small we may alternatively just use $VoR(0,0)$ and ignore the error. For the example in Section 4 we find the following figures. Observe from Figure 1 that $\mu' \approx 10/120 \approx 0.08$ and that $\sigma' \approx 4/120 \approx 0.03$. From Small (1982) we take the parameters as roughly $(\alpha,\beta,\gamma) = (2,1,4)$. Then we have $\omega = \alpha - \beta = 1$, $\lambda = \beta = 1$ and $\nu = \beta + \gamma = 5$.

From Table 1 we find $H \approx 0.29$. Furthermore, $\phi(Y_0) \approx 0.23$. With these numbers it turns out that the relative approximation error is about -0.012, which must be considered small given the precision with which the preference parameters can be estimated.

Given that the approximation error from applying the independence assumption is small, we may use (10) to compute the share of the time costs due to reliability for a traveller in the empirical example in the previous section. Figure 4 shows this share over the day. It varies around 15 per cent which must be considered significant. Even so, it is quite conceivable that this share is higher in places with more serious congestion.

6 The case of a scheduled service

It is conceivable that the technique used to generate the above results could also be used in the case of a scheduled service where the agent is not able to choose his departure time freely but has to choose from a fixed set of departure times. This section demonstrates by means of a counter-example, that the scheduling utility in (5) does not in general lead to an optimal expected utility that is linear in the mean and standard deviation of trip duration. We will assume a fixed interval (headway) of $2h$. This case arises for example when the issue is reliability of rail or bus services. We assume that the timing of departure times is unrelated to his preferred completion time and retain the assumption that duration is random given by $\mu + \sigma X$, where μ and σ are now again fixed. Unfortunately, in this case, as in Bates et al. (2001), it seems not possible to solve the utility maximisation problem explicitly for general travel time distributions.

Still, it is possible to say something. Consider the expected utility function as a function of departure time D:

$$EU(D) = \lambda D + \omega\mu + \nu \int_{\frac{D-\mu}{\sigma}}^{\infty} (\mu + \sigma x - D)\phi(x)dx.$$

This is globally convex since $\frac{d^2 EU(D)}{dD^2} = \frac{\nu}{\sigma}\phi(\frac{D-\mu}{\sigma}) > 0$. The expected utility maximising departure time will therefore always be the one in the interval defined by the equation

$$EU(D_h - h) = EU(D_h + h),$$

since this equation identifies the interval of length $2h$ of maximal expected utility. We have fixed the preferred arrival time at time 0 and taken the scheduling of departure times to be independent of everything else. We may therefore view the scheduling of departure time as a uniformly distributed random variable over the interval $[D_h - h, D_h + h]$. The expected utility under such a schedule is therefore given by the following expression.[12]

$$E(EU(D)) = \frac{1}{2h} \int_{D_h - h}^{D_h + h} EU(D)dD$$

It seems not possible to find a general explicit solution for this for a general duration distribution. However, it is possible in some cases to derive $E(EU(D))$ under specific assumptions about the duration distribution. It turns out that the resulting expression for $E(EU(D))$ is rather complex, and in particular it is not in general linear in the mean and standard deviation of duration. Appendix A.3 presents an example of this for the case of an exponentially distributed travel time.

7 Conclusion

We have established a simple relationship between the fundamental quantities from which cost or disutility is derived in Small's scheduling model and the mean

[12] Expectation is formed both with respect to the location of the schedule of departure time and with respect to the duration distribution.

and standard deviation of a distribution of durations under the optimally chosen departure time. Given the marginal utilities in the scheduling model, it is then possible to compute the value of reliability for any given travel time distribution. Moreover, it is possible to translate the value of reliability from one travel time distribution to another. The result remains a good approximation when the mean and standard deviation of travel time depend on the departure time while the standardised distribution must be constant.

Our analysis is subject to a number of caveats. Most importantly, we assume that the travel time distribution is fixed and known by the decision maker, and we also assume linear scheduling utility with no discontinuous disutility of lateness, nor any disutility of uncertainty per se. The validity of these assumptions needs to be assessed empirically in future research. Whether these assumptions may be relaxed theoretically should also be addressed in future research. We do, however, believe them to be essential for the simple results obtained in this paper. Another important issue, that we have not begun to touch, is the consequences of deviations from expected utility theory as described by, e.g., Kahneman and Tversky (1979); Starmer (2000).

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A Mathematical appendix

A.1 A simple model

This appendix refers to section 3. The first point is to find the first order condition for the maximisation of the expected utility in (6). Recall the following general formula:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = b'(x)f(x,b) - a'(x)f(x,a) + \int_{a(x)}^{b(x)} \frac{df(x,t)}{dx}dt.$$

Use this to differentiate (6) with respect to the departure time D and set to zero to find the first order condition. Note here that the derivative with respect to the lower integral limit is zero, since the integrand is zero at the lower bound. The derivative with respect to the upper integral limit is also zero, since the upper integral limit is constant. The first order condition then becomes

$$\lambda = \nu\left(1 - \Phi\left(\frac{D-\mu}{\sigma}\right)\right).$$

Insert the optimal departure time (8) into the expected utility to obtain the optimal expected utility.

$$
\begin{aligned}
EU^* = \quad & \lambda\left[\mu+\sigma\Phi^{-1}\left(1-\frac{\lambda}{\nu}\right)\right]+\omega\mu+\nu\int_{\Phi^{-1}\left(1-\frac{\lambda}{\nu}\right)}^{\infty}\left(\sigma x-\sigma\Phi^{-1}\left(1-\frac{\lambda}{\nu}\right)\right)\phi(x)dx \\
= \quad & (\lambda+\omega)\mu+\lambda\sigma\Phi^{-1}\left(1-\frac{\lambda}{\nu}\right) \\
& -\nu\sigma\Phi^{-1}\left(1-\frac{\lambda}{\nu}\right)\int_{\Phi^{-1}\left(1-\frac{\lambda}{\nu}\right)}^{\infty}\phi(x)dx+\nu\sigma\int_{\Phi^{-1}\left(1-\frac{\lambda}{\nu}\right)}^{\infty}x\phi(x)dx \\
= \quad & (\lambda+\omega)\mu+\lambda\sigma\Phi^{-1}\left(1-\frac{\lambda}{\nu}\right) \\
& -\nu\sigma\Phi^{-1}\left(1-\frac{\lambda}{\nu}\right)\left(1-\Phi\left(\Phi^{-1}\left(1-\frac{\lambda}{\nu}\right)\right)\right)+\nu\sigma\int_{\Phi^{-1}\left(1-\frac{\lambda}{\nu}\right)}^{\infty}x\phi(x)dx \\
= \quad & (\lambda+\omega)\mu+\nu\sigma\int_{1-\frac{\lambda}{\nu}}^{1}\Phi^{-1}(s)ds
\end{aligned}
$$

A.2 Approximation to the value of reliability

This appendix refers to Section 5 where the mean and standard deviation of duration are linear in the departure time. We define standardised departure time $Y=\frac{D-\mu}{\sigma}$ and $Y_0=\Phi^{-1}(1-\frac{\lambda}{\nu})$. The first-order condition for the choice of departure time can be expressed in compact form.

$$
0=\lambda+\omega\mu'+\nu\int_{Y}^{\infty}(\mu'+\sigma'x-1)\phi(x)dx
$$

It is only Y in this expression that depends on μ_0 and σ_0. So we can conclude that the derivatives of Y with respect to μ_0 and σ_0 are zero. This insight allows us to derive the marginal expected utilities of μ_0 and σ_0. Multiply the first-order condition by $D-D_0$ and subtract from the expected utility (6) to obtain

$$
\begin{aligned}
EU^* &= \lambda D_0+\omega\mu_0+\nu\int_{Y}^{\infty}(\mu_0+\sigma_0 x-D_0)\phi(x)dx \\
&= (\lambda+\omega)\mu_0+\lambda\sigma_0 Y_0-\nu\sigma_0 Y_0(1-\Phi(Y))+\nu\sigma_0\int_{Y}^{\infty}x\phi(x)dx
\end{aligned}
$$

We find that $\frac{dEU^*}{d\mu_0}=(\lambda+\omega)$ for any value of μ',σ'. This is the same result as in the case when μ and σ are constant.

The next point is to find the value of reliability. Differentiate the expected utility above with respect to σ_0 to obtain

$$\frac{dEU^*}{d\sigma_0} = \lambda Y_0 - \nu Y_0(1-\Phi(Y)) + \nu \int_Y^\infty x\phi(x)dx$$

Recall that the value of reliability is $\nu \int_{Y_0}^\infty x\phi(x)dx$ in the case when $\mu' = \sigma' = 0$ and note that the expression above reduces to this when $\mu' = \sigma' = 0$. As an approximation to $\frac{dEU^*}{d\sigma_0}$ it is natural to consider using $\nu \int_{Y_0}^\infty x\phi(x)dx$ since this does not require computation of Y. It is therefore of interest to consider the size of the error in using such an approximation

Denote the value of reliability by $VoR(\mu', \sigma') = \frac{dEU^*}{d\sigma_0}$. We are then concerned with the relative difference $\frac{VoR(\mu',\sigma') - VoR(0,0)}{VoR(0,0)}$ and we would like to show that this is small when μ', σ' are small.

We may obtain from the FOC that

$$\frac{dY}{d\mu'}(\mu' = \sigma' = 0) = -\frac{\lambda + \omega}{\nu\phi(Y_0)} < 0$$

and

$$\frac{dY}{d\sigma'}(\mu' = \sigma' = 0) = -\frac{\int_{1-\frac{\lambda}{\nu}}^1 \Phi^{-1}(s)ds}{\phi(Y_0)} < 0.$$

Let z_1 be one of μ', σ'. Then

$$\frac{dVoR}{dz_1} = -\nu(Y - Y_0)\phi(Y)\frac{dY}{dz_1}.$$

This is zero at $\mu' = \sigma' = 0$ since then $Y = Y_0$. Hence the change in the marginal utility of standard deviation is small when μ', σ' are small. Differentiate again to find

$$\frac{d^2VoR}{dz_1 dz_2} = -\nu\phi(Y)\frac{dY}{dz_1}\frac{dY}{dz_2} - \nu(Y - Y_0)\phi'(Y)\frac{dY}{dz_1}\frac{dY}{dz_2} - \nu(Y - Y_0)\phi(Y)\frac{d^2Y}{dz_1 dz_2}.$$

At $Y = Y_0$ this equals $-\nu\phi(Y_0)\frac{dY}{dz_1}\frac{dY}{dz_2} < 0$, so the value of reliability is locally concave in z with a local maximum at $z = 0$. This means that the value of reliability at $z \neq 0$ is overestimated by using the value at $z = 0$, regardless of the signs of z.

Given that we will be making a systematic error by using the formula derived under constant μ and σ, it is useful to assess the size of the error if we use the value of reliability at Y_0 at small values of z. Using a quadratic approximation we find that

$$\begin{aligned}
\frac{VoR(\mu',\sigma') - VoR(0,0)}{VoR(0,0)} &\approx \frac{1}{2VoR(0,0)}\left(\frac{d^2VoR}{d\mu'^2}\mu'^2 + 2\frac{d^2VoR}{d\mu' d\sigma'}\mu'\sigma' + \frac{d^2VoR}{d\sigma'^2}\sigma'^2\right) \\
&= -\frac{\nu\phi(Y_0)}{2\nu H}\left(\frac{dY}{d\mu'}\mu' + \frac{dY}{d\sigma'}\sigma'\right)^2 \\
&= -\frac{1}{2\phi(Y_0)H}\left(\frac{\lambda+\omega}{\nu}\mu' + H\sigma'\right)^2.
\end{aligned}$$

A.3 Example with a scheduled service

This appendix presents an example of a scheduled service with an exponentially distributed duration, where $T = \mu + X$ and $X \sim \phi(x) = \xi e^{-\xi x}$. We note that $\Phi(x) = 1 - e^{-\xi x}$ and that $\Psi(x) := \int_0^x x\phi(x)dx = \frac{1-e^{-\xi x}}{\xi} - xe^{-\xi x}$. Then it may be verified that ($\omega = 0$ and $\lambda = 1$ are omitted)

$$EU(D) = D + \frac{\nu}{\xi}e^{-\xi(D-\mu)}.$$

Moreover the midpoint of the interval from which departure time is chosen is

$$\mu + \frac{1}{\xi}log(\frac{\nu}{2\xi h}(e^{\xi h} - e^{-\xi h})),$$

such that the expected expected utility becomes

$$E(EU(D)) = (\mu + \frac{1}{\xi}) + \frac{1}{\xi}log(\frac{\nu}{2\xi h}(e^{\xi h} - e^{-\xi h})).$$

We may interpret the first term as relating to the mean duration while the second term relates to the standard deviation of duration $\frac{1}{\xi}$. But we note that the parameter ξ that characterizes the exponential distribution also appears inside a complicated expression that multiplies the standard deviation. So in contrast to the case when departure time can be chosen freely, we do not obtain that expected utility is linear in the standard deviation of duration in the case of a scheduled service.

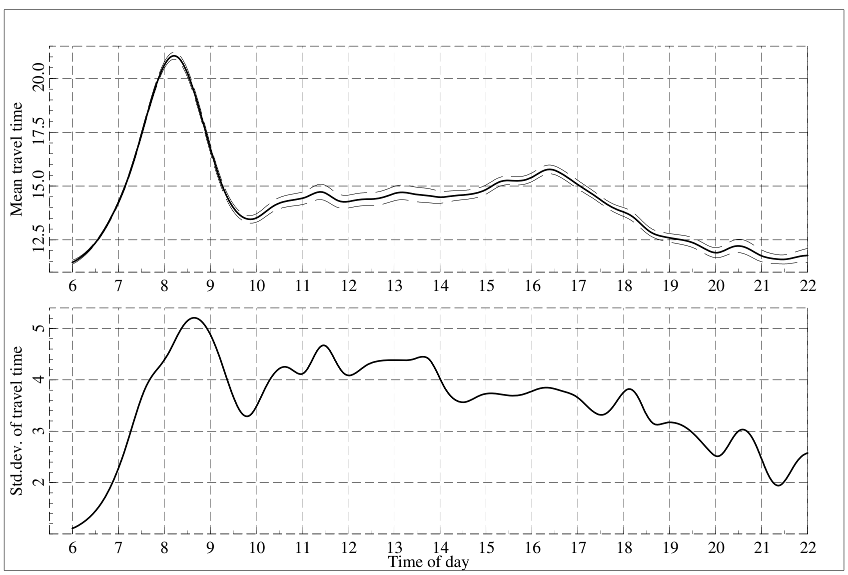

Figure 1: Mean and standard deviation of travel time (in minutes) over a weekday

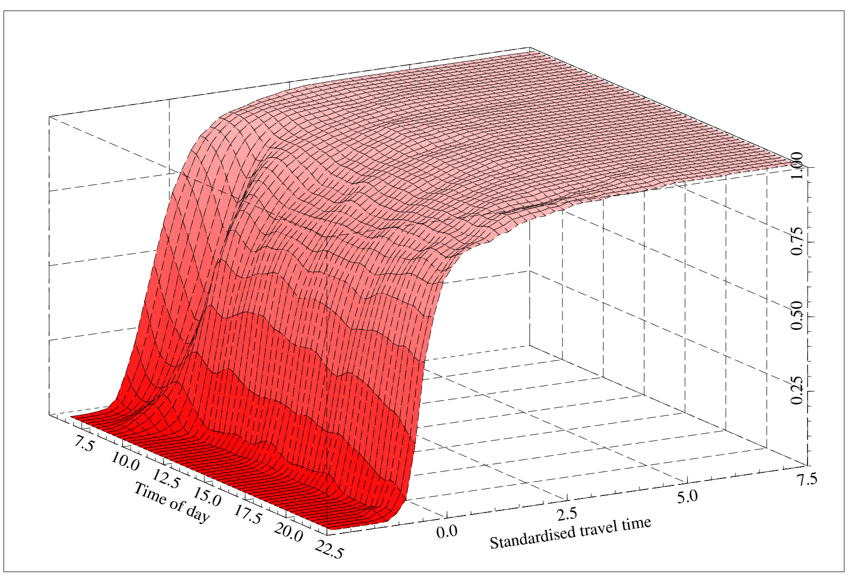

Figure 2: Cumulative distribution of standardised travel time conditional on time of day

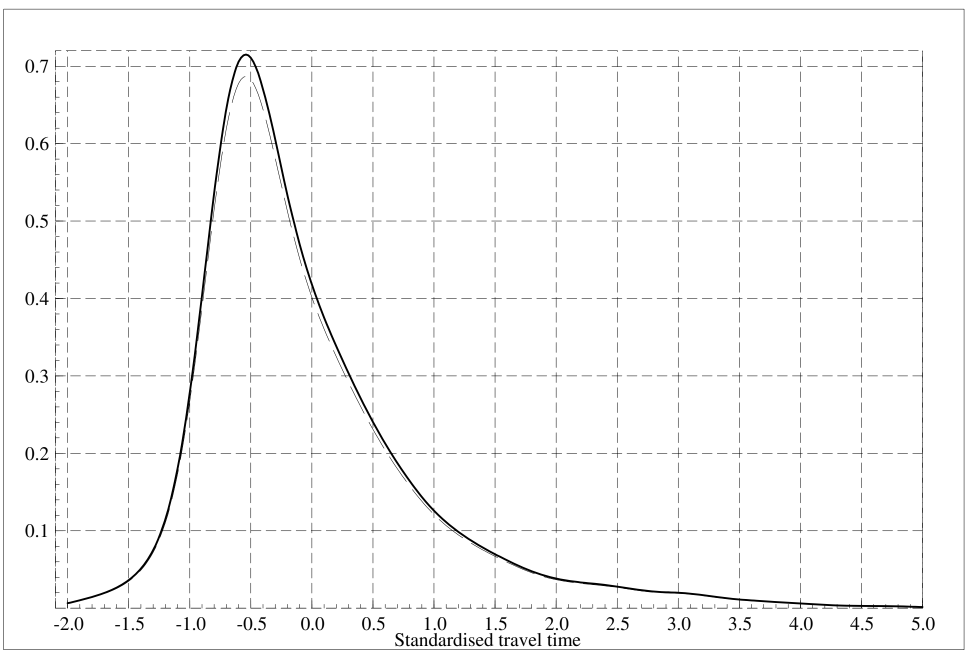

Figure 3: Density of standardised travel time

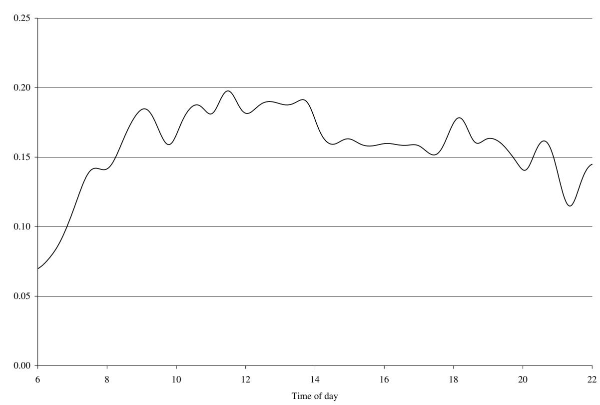

Figure 4: The share of the value of reliability in the total time cost

Table 1: H at various values of $\frac{\lambda}{\nu}$

$\frac{\lambda}{\nu}$	Normal	Empirical
0.50	0.39545	0.30857
0.25	0.31561	0.31297
0.20	0.27809	0.29250
0.15	0.23154	0.26105
0.10	0.17419	0.21445
0.05	0.10264	0.14221