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SOME EQUIVALENTS OF BROUWER'S FIXED POINT THEOREM AND THE EXISTENCE OF ECONOMIC EQUILIBRIUM

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Abstract. The primary goal of the paper is to deliver a simple proof of equivalence between Brouwer's fixed point theorem and the existence of equilibrium in a simple exchange model with monotonic consumers. To achieve this end, we discuss some equivalent formulations of Brouwer's theorem and prove additional ones, that are 'approximating' in character or seem to be better suited for economic applications than the standard results.

Keywords: existence of equilibrium, Brouwer's fixed point, simple exchange model.

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1. Introduction

The main issue addressed in this paper is the question of equivalence between the existence of equilibria in a simple exchange model with monotonic consumers and Brouwer's fixed point theorem. Probably the first equivalence result is included in [7] where it is proven that if every excess demand function (we use the same convention for discriminating between/defining the notions of *excess demand function* and *excess demand function generated by an (exchange) economy* as in [8]) defined on closed simplex possesses properly defined equilibrium then every continuous function from closed simplex to itself has a fixed point. The main problem with this approach is that the usually adopted assumption of monotonicity of consumers' preferences excludes the closed simplex as the domain of corresponding excess demand function. In other words: it is not known for what economy with monotonic consumers the excess demand function defined on closed simplex describes the economy's aggregate behaviour. It is shown in [8] that every excess demand function defined on a compact subset of closed the standard simplex is generated by an economy with non-satiated consumers, whose preferences are not necessarily monotonic (theorem 1). In the same paper, K.-Ch. Wong proves that the above mentioned result implies the equivalence between the existence of equilibria in simple exchange models with non-satiated consumers and Brouwer's theorem (Wong's theorem 2). Despite of generality of Wong's approach (namely, the fact that he allows for non-satiation) the following problem arises: if the excess demand function is defined in the interior of the standard simplex and satisfies usual boundary condition, then Wong's theorem 1 does not apply - and it is possible to generate the function by an economy only on ε-simplices. Thus, after Wong's paper the question of equivalence mentioned at the very beginning still remained open. The last remark may seem paradoxical a bit since monotonic consumers are non-satiated. But one should be cautious: to prove theorem 2 Wong used his theorem 1, one of the underlying assumptions for which is the compactness of the domain. Recently, a successful attempt to prove the initial equivalence has been undertaken in [6], though the result is not the main one in Toda's paper. To achieve the goal, Toda shows that for every excess demand function there is a sequence of economies with monotonic consumers such that the limsup of the equilibria is contained in equilibrium set of the excess demand - meanwhile Mas-Colell's argument is used [4]. We proceed in a different way: we prove some 'approximating' equivalents of Brouwer's theorem - including the one in which the boundary condition holds (our theorem 6) and then apply Mas-Colell's results so as to get the final result in almost a trivial way. The novelty of the paper lies in the way we approach the initial problem and in that we offer two equivalents of Brouwer's theorem (theorems 3 and 6).

In the following part of the paper, we present the notation. Then, we introduce equivalents of Brouwer's theorem and proceed to address the initial problem.

2. Notation

For vectors $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, $y = (y_1, \ldots, y_n) \in \mathbf{R}^n$, we write $x \geq y$, when $x_i \geq y_i$, $i = 1, \ldots, n$; $x > y$ is for strict component-wise inequalities $x_i > y_i$, $i = 1, \ldots, n$. $x \in \mathbf{R}^n_+$ means $x \geq 0$; $x \in \mathbf{R}^n_{++}$ means $x > 0$. In what follows $S = \{x \in \mathbf{R}^n_{++} : \sum_{i=1}^n x_i = 1\}$ is the (relative) interior of the standard simplex, the closure of which is denoted as $\overline{S} = \{x \in \mathbf{R}^n_+ : \sum_{i=1}^n x_i = 1\}$. For vectors $x, y \in \mathbf{R}^n$, their scalar product is $xy = \sum_{i=1}^n x_i y_i$. $|a|$ is absolute value of $a \in \mathbf{R}$.

3. The equivalents

The classical version of Brouwer's theorem [2] is

Theorem 1. *If $F : \overline{S} \to \overline{S}$ is a continuous function, then there exists $x \in \overline{S} :\ F(x) = x$.*

Uzawa proved in [7] the following equivalent formulation of theorem 1

Theorem 2. *Let $F : \overline{S} \to \mathbf{R}^n$ be a continuous function satisfying Walras' Law*

$$\forall x \in \overline{S} \qquad xF(x) = 0. \tag{1}$$

There exists $x \in \overline{S}$ satisfying $F(x) \leq 0$.

It can be shown that the above theorems are equivalent [7]. We shall prove a bit different equivalent:

Theorem 3. *Let $F : S \to \mathbf{R}^n$ be a continuous function satisfying Walras' Law*

$$\forall x \in S \qquad xF(x) = 0 \tag{2}$$

and bounded from below:

$$\exists K > 0\, \forall x \in S \quad F_i(x) > -K,\ i = 1, \ldots, n.$$

There exists a sequence $\{x^q\}_{q=1}^\infty \subset S$ satisfying $\lim_{q \to \infty} F_i(x^q) \leq 0$, $i = 1, \ldots, n$.

of a bit different version of Brouwer's theorem [5]:

Theorem 4. *Let $F : S \to \overline{S}$ be a continuous function. There exists a sequence $\{x^q\}_{q=1}^{\infty} \subset S$ satisfying $\lim_{q\to\infty}(F_i(x^q) - x_i^q) = 0$, $i = 1, \dots, n$.*

Theorem 5. *Theorems 1-4 are equivalent.*

Proof. **(1⇔2)** was proven by Uzawa [7]. Obviously, implications **(3⇒2)** and **(4⇒1)** are true.
(1⇒4)[1] Let us define $F^\varepsilon : S_\varepsilon \to S_\varepsilon$ as $F_i^\varepsilon(x) = \frac{F_i(x)+\varepsilon}{n\varepsilon+1}$, where

$$S_\varepsilon = \left\{ x \in S : \, x_i \geq \frac{\varepsilon}{n\varepsilon + 1}, i = 1, \dots, n \right\}$$

and $\varepsilon > 0$ is sufficiently small, then by theorem 1 for each ε $x^\varepsilon \in S_\varepsilon$: $x^\varepsilon = F^\varepsilon(x^\varepsilon)$ exists. The thesis follows since for $i = 1, \dots, n$

$$|F_i(x^\varepsilon) - x_i^\varepsilon| = |F_i(x^\varepsilon) - F_i^\varepsilon(x^\varepsilon)| = \left| F_i(x^\varepsilon) - \frac{F_i(x^\varepsilon) + \varepsilon}{n\varepsilon + 1} \right| = \varepsilon \left| \frac{nF_i(x^\varepsilon) - 1}{n\varepsilon + 1} \right|$$

the right-hand side term of the above equality converges to 0, when ε converges to 0.
(4⇒3) [2] Suppose that theorem 4 is true. This implies that theorem 1 holds. Let F satisfy assumptions of theorem 3. For every $1 > \varepsilon > 0$ define set

$$S_\varepsilon = \left\{ x \in S : \, x_i \geq \frac{\varepsilon}{2n + 1}, \, i = 1, \dots, n \right\}$$

and function $F^\varepsilon : S_\varepsilon \to S_\varepsilon$ given by formula

$$\forall x \in S_\varepsilon \quad F_i^\varepsilon(x) = \frac{\varepsilon + x_i + \max\{0, \, \overline{F}_i(x)\}}{n\varepsilon + 1 + \sum_{i=1}^n \max\{0, \, \overline{F}_i(x)\}}, \; i = 1, \dots, n,$$

where $\overline{F}_i(x) = \min\{1, \, F_i(x)\}$. By theorem 1 for every such ε there exists $x^\varepsilon \in S_\varepsilon$ such that

$$x_i^\varepsilon = \frac{\varepsilon + x_i^\varepsilon + \max\{0, \, \overline{F}_i(x^\varepsilon)\}}{n\varepsilon + 1 + \sum_{i=1}^n \max\{0, \, \overline{F}_i(x^\varepsilon)\}}, \; i = 1, \dots, n$$

[1] See also [5].
[2] The proof stems - in almost unchanged form - from a 'standard' existence proof in [3, p. 193-194] - just a thorough investigation and minor changes are necessary.

which can be equivalently written as

$$(3)\quad n\varepsilon x_i^\varepsilon + x_i^\varepsilon \sum_{i=1}^{n} \max\{0,\, \overline{F}_i(x^\varepsilon)\} = \varepsilon + \max\{0,\, \overline{F}_i(x^\varepsilon)\},\ i = 1, \ldots, n.$$

Taking $\varepsilon \to 0^+$ we may assume that $\lim_{\varepsilon\to 0^+} x^\varepsilon = \overline{x} \in \overline{S}$. Let $A := \{i : \overline{x}_i = 0\}$. Consider two cases

$i \in A$: The left-hand side of (3) converges to 0. This implies

$$\lim_{\varepsilon\to 0^+} \max\{0,\, \overline{F}_i(x^\varepsilon)\} = 0,$$

so that $\limsup_{\varepsilon\to 0^+} \overline{F}_i(x^\varepsilon) \leq 0$, which allows us to write

$$\limsup_{\varepsilon\to 0^+} F_i(x^\varepsilon) \leq 0.$$

$i \notin A$: If $\liminf$ of the right-hand-side of (3) is 0, then the left-hand side term converges to 0, and

$$\liminf_{\varepsilon\to 0^+} \left(\sum_{i=1}^{n} \max\{0,\, \overline{F}_i(x^\varepsilon)\} \right) \leq 0,$$

which is possible only if $\liminf \overline{F}_i(x^\varepsilon) = 0$, $i = 1, \ldots, n$, which implies $\liminf F_i(x^\varepsilon) = 0$, $i = 1, \ldots, n$ - this would end the proof. The only left possibility is that for all $i \notin A$ $\liminf$ of the right-hand side of (3) is strictly greater than 0. This implies that $\liminf F_i(x^\varepsilon) > \delta > 0$. Application of Walras' Law (2) gives us $\forall \varepsilon$

$$\sum_{i\in A} x_i^\varepsilon F_i(x^\varepsilon) + \sum_{i\notin A} x_i^\varepsilon F_i(x^\varepsilon) = 0.$$

However, in the limit the left-hand side of the last equality is positive, while the right-hand side term equals 0, which cannot hold simultaneously.

We conclude that for all i $\liminf F_i(x^\varepsilon) \leq 0$, which proves the thesis. $\square$

4. Brouwer's theorem vs. economic equilibria

Another important and particularly suitable for economic applications equivalent of Brouwer's theorem is

Theorem 6. *Let $F : S \to \mathbf{R}^n$ be a continuous function, bounded from below and satisfying Walras Law and the boundary condition: if $\lim_{q\to\infty} x^q = x \in \overline{S}\backslash S$, then $\lim_{q\to\infty} \max_{i=1,\dots,n}\{F_i(x^q)\} = +\infty$ for each sequence $\{x^q\}_{q=1}^{\infty} \subset S$. There exists a sequence $\{x^q\}_{q=1}^{\infty} \subset S$ satisfying*

$$\lim_{q\to\infty} F_i(x^q) \le 0,\ i = 1, \dots, n.$$

Proof. Obviously all assumptions in theorem 3 are met, therefore the thesis holds. □

Theorem 7. *Theorems 3 and 6 are equivalent.*

Proof. We just need to prove that theorem 6 implies 3. Suppose that a function F satisfies the hypothesis of theorem 3. For every $\varepsilon > 0$ define $\overline{F} : S \to \mathbf{R}^n$ as $\overline{F}(x) = F(x) + \varepsilon G(x)$, where

$$G(x) = \left(\frac{1}{nx_1} - 1, \dots, \frac{1}{nx_n} - 1\right).$$

It can be easily checked that $\overline{F}$ satisfies assumptions of theorem 3. Whence, $\forall \varepsilon > 0\, \exists x^\varepsilon \in S$, $\overline{F}(x^\varepsilon) = 0$. The equality comes from the fact that if the sequence satisfying the assertion of theorem 3 converges to the boundary of S, then at least one of the values of $\overline{F}_i$ diverges to $+\infty$. Thus, the limit point is in S - therefore positive - and Walras' Law implies equalities. We have for $i = 1, \dots, n$

$$\forall \varepsilon > 0 \quad F_i(x^\varepsilon) = -\varepsilon\left(\frac{1}{nx_i^\varepsilon} - 1\right). \tag{4}$$

We can assume $\lim_{\varepsilon\to 0^+} x^\varepsilon = x \in \overline{S}$ (choose a subsequence if needed). If $x_i = 0$ then for small values of ε the right-hand-side term of (4) is negative so that in the limit the left-hand-side term must be non positive. If $x_i > 0$, then the limit of the left-hand-side of (4) is 0. The thesis follows. □

4.1. **The equivalence of the existence of economic equilibria and Brouwer's fixed point theorem.** So far we have presented purely mathematical results with no concern for economic interpretation. Here we dive into economics. By (pure) exchange economy we mean set $E = \{(\succcurlyeq^i, \omega^i, \mathbf{R}^n_+)\}_{i=1}^n$, where $\mathbf{R}^n_+$ is interpreted as a consumption set, $\succcurlyeq^i$ is a continuous, monotone, strictly convex preference relation (i.e. complete and transitive relation) on $\mathbf{R}^n_+$ and $\omega^i \in \mathbf{R}^n_{++}$ is initial endowment. Given an economy E, the (aggregate) excess demand function $F^E : S \to \mathbf{R}^n$ corresponding to E (E generates F_E) is defined as a function of prices $p \in S$:

$$F^E(p) := \sum_{i=1}^n \{x \in \mathbf{R}^n_+ : p\,x \leq p\,\omega^i, \text{ and } \big(y \in \mathbf{R}^n_+, p\,y \leq p\,\omega^i\big) \Rightarrow x \succcurlyeq^i y\} - \\ - \sum_{i=1}^n \omega^i$$

It is known from [1, p. 102] that F^E satisfies assumptions of theorem 6 (imposed on F therein). We call a vector $p \in S$ equilibrium of economy E if $F^E(p) = 0$. A theorem from [4, p. 118] implies the following

Theorem 8. *If $F : S \to \mathbf{R}^n$ satisfies assumptions of theorem 6, then there exists an economy E such that*

$$F^E(p) = 0 \Leftrightarrow F(p) = 0.$$

This allows us to state (see also [6]).

Theorem 9. *Brouwer's fixed point theorem is equivalent to existence of equilibrium for every exchange economy E.*

Proof. It is a consequence of equivalence of theorems 6 and 1 and theorem 8. □

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