Some Notes on Inconsistence and Indecisiveness in the Analytic Hierarchy Process

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Abstract: The aim of the article is to introduce a mathematical concept of indecisiveness into the analytic hierarchy process (AHP) framework. Indecisiveness can be useful in two ways: first, decision makers with high indecisiveness (higher than a given threshold value) can be excluded from a decision making process in its early stages as low-competent and replaced by other, more competent DMs; second, indecisiveness along with consistency index *C.I.* can be used for the calculation of (aposteriori) DMs' weights without any additional information about DMs' age, formal knowledge, social status, etc. The proposed approach is demonstrated on examples.

Keywords: AHP, inconsistence, decision making, indecisiveness.

JEL classification: D79.

1. Introduction

Decision making processes are omnipresent in human life as well as in many areas of economics such as marketing, management, human resources, logistics, etc. Decision making methods range from the simple majority rule to more sophisticated approaches such as the analytic hierarchy/network process (AHP/ANP) by T. L. Saaty, see e.g. [3], [4], [5] or [6], computing with words ([7] or [8]) or decision making in the fuzzy environment ([1] or [2]).

In general, a multi-criteria decision making process consists of the following steps:

1. Identification of the problem.
2. Analysis of the problem.
3. Establishing sets of alternatives, criteria and experts who will evaluate alternatives.
4. Evaluation of alternatives by experts with respect to given criteria.
5. Selection of the best alternative (which includes aggregation of experts' preferences and ordering of alternatives from the best to the worst).

In the literature the step 5 is of the most interest, as there is vast number of methods proposed for various kinds of decision making processes. This article focuses on the step 4, and more specifically on the problem of 'quality' of decision makers' (DMs') preferences in the group decision making. So far, DMs' preferences were thoroughly examined in terms of their (in)consistence (that is existence of a contradiction) because human judgment is imperfect due to many reasons such as time pressure, imprecise information, lack of knowledge, prejudices, etc.

DMs' preferences are crucial part of each decision making, as they represent problem's input, hence they determine problem's solution. Apart from (in)consistence, which is of great importance without no doubt, DMs' preferences are not studied in a more detail, though there are at least two other often neglected features important for problem's solution associated with DMs' preferences:

- differences among DMs' preferences (degree of a conflict among DMs),
- decisiveness of each DM.

As for the former, conflicting attitudes of DMs might result in no consensus under some circumstances (consider e.g. situation, where two equally strong groups of DMs are in opposition). Group decision making is always associated with less or more conflict among DMs, which must be resolved finally if a consensus is going to be achieved.

As for the latter, decision makers can express their preferences on a set of alternatives with a different degree of intensity (or confidence). In AHP, the intensity of preference is expressed on Saaty's scale from 1 to 9 (see Table 1). In real decision situations, some decision makers can express stronger preferences than others. It seems natural to assign higher weights to a DM who has strong opinion on a topic (a DM is strongly decisive possibly due to better knowledge or experience), and lower weight to a DM with weak opinion. As shown in Section 2, in extreme cases a DM can express preferences that are absolutely consistent (see Table 2), but completely indecisive, thus useless. Clearly, inconsistence is not sufficient measure of quality DMs' preferences.

Therefore, the aim of the article is to propose the measure of indecisiveness in the group AHP framework. This new measure can be useful in two ways:

- DMs with high indecisiveness can be excluded from a decision making process as low-competent immediately in its early stages and replaced by other, more competent DMs.

- Indecisiveness along with Saaty's consistency index *C.I.* can be used for the calculation of (aposteriori) DMs' weights. Decision makers with larger consistency and decisiveness are assigned larger weights and vice versa. The advantage of this approach rests in the fact that only information given by DMs is used, and no additional knowledge 'from outside' about DMs' age, formal knowledge, social status, etc., is required.

The article is organized as follows: in Section 2 AHP and consistency index *C.I.* are briefly described, in Section 3 the measure of indecisiveness is introduced along with derivation of weights of DMs, and in Section 4 numerical examples are provided. Conclusions close the article.

2. Analytic hierarchy process

Analytic hierarchy process (AHP) was proposed by T. L. Saaty in 1980 [4]. Its fundamental part consists of pair-wise comparisons of objects on the k^{th} level of hierarchy with regard to objects on the immediately higher $(k - 1)^{th}$ level. Typically, the highest level is a *goal*, the second level form *criteria* and the lowest level consists of *alternatives*. The aim of AHP lies in the selection of the best alternative. As the article focuses on indecisiveness of decision makers in evaluating alternatives, only comparisons of alternatives with regard to a given set of criteria by one or more DMs will be considered.

Alternatives are compared on the scale from 1 to 9, where 1 denotes equal importance and 9 extreme importance of one alternative over another (see Table 1); s_{ij} denotes preference of an alternative i over an alternative j. Preferences s_{ij} are reciprocal: if an alternative A is moderately preferred over an alternative B, then $s_{AB} = 3$, and by definition $s_{BA} = 1/3$.

All pair-wise preferences s_{ij} form a reciprocal matrix P with elements $s_{ij} = \frac{1}{s_{ji}}; \forall i, j$.

Pair-wise preferences are consistent, if $s_{ij} \cdot s_{jk} = s_{ik}$, $\forall i, j, k$; that is the transitive property of DM's preferences is preserved.

Consistency of DMs' preferences is expressed by *consistency index C.I.* given as [4]:

$$C.I. = \frac{\lambda_{\max} - n}{n-1}, \qquad (1)$$

where $\lambda_{\max}$ is the largest (positive) eigenvalue of the matrix *P*, *n* is the order of *P*, and $C.I. \geq 0$. The value $C.I. = 0$ indicates absolute consistency of preferences; the larger is *C.I.*, the more inconsistent preferences are. According to Saaty [6], human judgment is inconsistent by nature, so $C.I. < 0.1$ is tolerated.

Consistency index is the only measure of 'quality' of decision makers' preferences in AHP. However, consider a preference matrix *A* shown in Table 2. It is easy to verify that $\lambda_{\max}(A) = 5$, so $C.I. = 0$. Preferences expressed by *A* are absolutely consistent, but they are also absolutely useless, as no alternative is preferred over any other alternative (because a decision maker is absolutely indecisive). This example shows that consistency index alone is not a sufficient tool for preferences assessment.

Intensity of Importance	*Definition*	*Explanation*
1	Equal Importance	Two activities contribute equally to the objective
2	Weak or slight	
3	Moderate importance	Experience and judgment slightly favor one activity over another
4	Moderate plus	
5	Strong importance	Experience and judgment strongly favor one activity over another
6	Strong plus	
7	Very strong or demonstrated importance	An activity is favored very strongly over another; its dominance demonstrated in practice
8	Very,very strong	
9	Extreme importance	The evidence favoring one activity over another is of the highest possible order of affirmation
1.1–1.9	When activities are very close a decimal is added to 1 to show their difference as appropriate	A better alternative way to assigning the small decimals is to compare two close activities with other widely contrasting ones, favoring the larger one a little over the smaller one when using the 1–9 values.
Reciprocals of above	If activity *i* has one of the above nonzero numbers assigned to it when compared with activity *j*, then *j* has the reciprocal value when compared with *i*	A logical assumption
Measurements from ratio scales		When it is desired to use such numbers in physical applications. Alternatively, often one estimates the ratios of such magnitudes by using judgment

Table 1. Saaty's scale in AHP. Source: [6].

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Table 2. An example of preferences of an absolutely indecisive DM.

3. The measure of indecisiveness

As shown in Table 2, a DM is absolutely indecisive, if all s_{ij} =1. Let 1 denote the matrix with such elements. Then decisiveness of a DM with preferences given by the matrix $A(a_{ij})$ can be evaluated from an absolute difference matrix $D(d_{ij})$: $d_{ij} = |a_{ij} - 1|$.

The larger are elements d_{ij}, that is differences $a_{ij} - 1$, the higher is DM's decisiveness. For a decisiveness evaluation, the entrywise p-norm applied on D can be used:

$$\|D\|_p = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij}^p \right)^{1/p} \qquad (2)$$

where $p \in N$. The special case p = 2 is called Frobenius norm (it is an equivalent of the Euclidean norm for vectors), and will be used henceforward.

The larger is $\|D\|_p$ the greater is the decisiveness of a decision maker. It is possible to state some minimal value of $\|D\|_p$ (the threshold of decisiveness – τ) dependent on n, so that DMs with decisiveness $\|D\|_p$ smaller than τ would be excluded from a decision making process.

Weights w_i of decision makers can be derived with the use of formulas (1) and (2) combined into one overall index. Weights of DMs are functions of indecisiveness and consistency index: $w = w(\|D\|_p, C.I.)$, and should satisfy the following properties:

- The function w is non-negative.
- The function w is strictly increasing in $\|D\|_p$.
- The function w is strictly decreasing in $C.I.$

In addition, because $C.I.$ acts as a penalization factor in w, it is reasonable to require no penalization for absolute consistency ($C.I.$ = 0). This condition is satisfied e.g. for the exponential function ($e^0 = 1$). One simple function w satisfying all three conditions, and additional condition mentioned above, can be given as:

$$w = \|D\|_2 \cdot e^{-10C.I.} \qquad (3)$$

The factor of 10 in the exponential function was chosen to magnify the value of the inconsistency index given by (1), which is by definition much smaller than matrix elements in most cases. Also, it follows from (3) that for the limit value of acceptable consistency *C.I.* = 0.1, the weight w takes especially simple form: $w = \frac{\|D\|_2}{e}$.

Usually, DMs compare alternatives with regard to more than one criterion. In such a case, indecisiveness and consistency index *C.I.* is estimated for each criterion separately, and then the average indecisiveness along with average *C.I.* is used to calculate (average) DMs' weights by relation (3).

Numerical examples illustrating weight evaluation with the use of (3) are provided in the following section. Nowdays, there are many software tools which facilitate computation of *C.I.* or $\lambda_{\max}$:

- *ExpertChoice* (www.expertchoice.com),
- *Mathematica* (www.wolfram.com/mathematica/),
- *Meta-numerics* (www.meta-numerics.net),
- *WolframAlpha* (www.wolframalpha.com), etc.

ExpertChoice is a commercial product for AHP solutions; it computes *C.I.* directly. Other mathematical tools are capable of computing an eigenvalue system of the matrix P. The last two products are free of charge.

4. Numerical examples

In this Section two numerical examples are provided. In Example 1 alternatives are compared by one criterion and in Example 2 by two criteria. In both cases consistency index *C.I.*, indecisiveness $\|D\|_p$ and weights w of all DMs are evaluated.

Example 1. Let A and B be preference matrices of decision makers DM_1 and DM_2 respectively on three alternatives (see Table 3) by a given criterion. We will evaluate weights of both decision makers using the formula (3) and the Frobenius norm (p = 2).

$$A = \begin{pmatrix} 1 & 9 & 7 \\ 1/9 & 1 & 3 \\ 1/7 & 1/3 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 1/2 & 1 & 2 \\ 1/3 & 1/2 & 1 \end{pmatrix}$$

Table 3. Preferences of DM_1 (matrix A) and DM_2 (matrix B).

Consistency index *C.I.* (1) for both DMs:

$$\lambda_{\max}(A) = 3.206,\ n = 3 \Rightarrow C.I.(DM_1) = \frac{3.206-3}{2} = 0.103.$$

$$\lambda_{\max}(B) = 3.008,\ n = 3 \Rightarrow C.I.(DM_1) = \frac{3.008-3}{2} = 0.004.$$

Hence, DM_2 is more consistent in his evaluation of alternatives. On the other hand, DM_1 is much more decisive than DM_2:

$$\|D\|_2(DM_1)=\left[8^2+6^2+2^2+\left(\frac{8}{9}\right)^2+\left(\frac{6}{7}\right)^2+\left(\frac{2}{3}\right)^2\right]^{1/2}=10.29$$

$$\|D\|_2(DM_2)=2.64$$

Finally, we compute weights w_1 and w_2, and normalized weights w_1' and w_2' of DM_1 and DM_2 respectively using (3):

$$w_1=10.29\cdot e^{-10\cdot 0.103}=3.67\,.$$

$$w_2=2.64\cdot e^{-10\cdot 0.004}=2.63\,.$$

$$w_1'=0.583,\ w_2'=0.417\,.$$

The first decision maker is assigned slightly larger weight due to his greater decisiveness.

Example 2. Let *A* and *B* be the same preference matrices of decision makers DM_1 and DM_2 respectively on three alternatives (see Table 3) by a criterion C_1 as in Example 1. Let *C* and *D* be preference matrices of decision makers DM_1 and DM_2 respectively on three alternatives by a criterion C_2 (see Table 4).

$$C=\begin{pmatrix}1 & 2 & 3\\ 1/3 & 1 & 4\\ 1/2 & 1/4 & 1\end{pmatrix}\quad D=\begin{pmatrix}1 & 4 & 1/2\\ 1/4 & 1 & 1/5\\ 2 & 5 & 1\end{pmatrix}$$

Table 4. Preferences of DM_1 (matrix *C*) and DM_2 (matrix *D*) with regard to the criterion C_2.

Consistency index *C.I.* (1) of preferences with regard to the criterion C_2:

$$\lambda_{\max}(A)=3.107\,,\ n=3\ \Rightarrow\ C.I.(DM_1)=\frac{3.107-3}{2}=0.054\,.$$

$$\lambda_{\max}(B)=3.025\,,\ n=3\ \Rightarrow\ C.I.(DM_1)=\frac{3.025-3}{2}=0.013\,.$$

Again, DM_2 is more consistent in his evaluation. As for decisiveness, DM_2 is more decisive:

$$\|D\|_2(DM_1)=\left[1^2+2^2+3^2+\left(\frac{2}{3}\right)^2+\left(\frac{1}{2}\right)^2+\left(\frac{3}{4}\right)^2\right]^{1/2}=3.906\,,$$

$$\|D\|_2(DM_2)=5.239.$$

Weights $w_1^{(2)}$ and $w_2^{(2)}$, and normalized weights $w_1^{(2)\prime}$ and $w_2^{(2)\prime}$ of DM_1 and DM_2 respectively, are:

$$w_1^{(2)}=3.906\cdot e^{-10\cdot 0.054}=2.276\,.$$

$$w_2=5.239\cdot e^{-10\cdot 0.013}=4.600\,.$$

$$w_1^{(2)\prime}=0.331\ w_2^{(2)\prime}=0.669.$$

From Example 1 we know, that weights of DMs with regard to the criterion C_1 are:

$w_1^{(1)}{}' = 0.583$, $w_2^{(1)}{}' = 0.417$

By averaging we can assign final weights to both DMs: $w_1{}' = 0.457$, $w_2{}' = 0.543$.

With computed weights the decision making can proceed into the aggregation phase and then to the selection of the best alternative.

5. Conclusions

The aim of the article was to introduce a concept of indecisiveness into the group analytic hierarchy process (AHP) framework and to show how weights of decision makers can be derived only from their preferences with the use of indecisiveness and Saaty's consistency index *C.I.* The advantage of this approach is that no additional information about decision makers' experience or knowledge 'from outside' is required. Further work may focus on differences (conflicts) among decision makers' preferences in the group AHP and the influence of the differences on a possibility of an existence of a group consensus.

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