Some Notes on Inconsistence and Indecisiveness in the Analytic Hierarchy Process

Mazurek Jiri

Silesian University - School of Business Administration

9 November 2012

Online at https://mpra.ub.uni-muenchen.de/42543/
MPRA Paper No. 42543, posted 12 November 2012 10:55 UTC
Some Notes on Inconsistence and Indecisiveness in the Analytic Hierarchy Process

Jiří Mazurek

Abstract: The aim of the article is to introduce a mathematical concept of indecisiveness into the analytic hierarchy process (AHP) framework. Indecisiveness can be useful in two ways: first, decision makers with high indecisiveness (higher than a given threshold value) can be excluded from a decision making process in its early stages as low-competent and replaced by other, more competent DMs; second, indecisiveness along with consistency index C.I. can be used for the calculation of (aposteriori) DMs’ weights without any additional information about DMs’ age, formal knowledge, social status, etc. The proposed approach is demonstrated on examples.

Keywords: AHP, inconsistence, decision making, indecisiveness.

JEL classification: D79.

1. Introduction

Decision making processes are omnipresent in human life as well as in many areas of economics such as marketing, management, human resources, logistics, etc. Decision making methods range from the simple majority rule to more sophisticated approaches such as the analytic hierarchy/network process (AHP/ANP) by T. L. Saaty, see e.g. [3], [4], [5] or [6], computing with words ([7] or [8]) or decision making in the fuzzy environment ([1] or [2]).

In general, a multi-criteria decision making process consists of the following steps:

1. Identification of the problem.
2. Analysis of the problem.
3. Establishing sets of alternatives, criteria and experts who will evaluate alternatives.
4. Evaluation of alternatives by experts with respect to given criteria.
5. Selection of the best alternative (which includes aggregation of experts’ preferences and ordering of alternatives from the best to the worst).

In the literature the step 5 is of the most interest, as there is vast number of methods proposed for various kinds of decision making processes. This article focuses on the step 4, and more specifically on the problem of ‘quality’ of decision makers’ (DMs’) preferences in the group decision making. So far, DMs’ preferences were thoroughly examined in terms of their (in)consistence (that is existence of a contradiction) because human judgment is imperfect due to many reasons such as time pressure, imprecise information, lack of knowledge, prejudices, etc.

DMs’ preferences are crucial part of each decision making, as they represent problem’s input, hence they determine problem’s solution. Apart from (in)consistence, which is of great importance without no doubt, DMs’ preferences are not studied in a more detail, though there are at least two other often neglected features important for problem’s solution associated with DMs’ preferences:

- differences among DMs’ preferences (degree of a conflict among DMs),
- decisiveness of each DM.

As for the former, conflicting attitudes of DMs might result in no consensus under some circumstances (consider e.g. situation, where two equally strong groups of DMs are in
opposition). Group decision making is always associated with less or more conflict among DMs, which must be resolved finally if a consensus is going to be achieved.

As for the latter, decision makers can express their preferences on a set of alternatives with a different degree of intensity (or confidence). In AHP, the intensity of preference is expressed on Saaty’s scale from 1 to 9 (see Table 1). In real decision situations, some decision makers can express stronger preferences than others. It seems natural to assign higher weights to a DM who has strong opinion on a topic (a DM is strongly decisive possibly due to better knowledge or experience), and lower weight to a DM with weak opinion. As shown in Section 2, in extreme cases a DM can express preferences that are absolutely consistent (see Table 2), but completely indecisive, thus useless. Clearly, inconsistence is not sufficient measure of quality DMs’ preferences.

Therefore, the aim of the article is to propose the measure of indecisiveness in the group AHP framework. This new measure can be useful in two ways:

- DMs with high indecisiveness can be excluded from a decision making process as low-competent immediately in its early stages and replaced by other, more competent DMs.

- Indecisiveness along with Saaty’s consistency index C.I can be used for the calculation of (aposteriori) DMs’ weights. Decision makers with larger consistency and decisiveness are assigned larger weights and vice versa. The advantage of this approach rests in the fact that only information given by DMs is used, and no additional knowledge ‘from outside’ about DMs’ age, formal knowledge, social status, etc., is required.

The article is organized as follows: in Section 2 AHP and consistency index C.I are briefly described, in Section 3 the measure of indecisiveness is introduced along with derivation of weights of DMs, and in Section 4 numerical examples are provided. Conclusions close the article.

2. Analytic hierarchy process

Analytic hierarchy process (AHP) was proposed by T. L. Saaty in 1980 [4]. Its fundamental part consists of pair-wise comparisons of objects on the $k^{th}$ level of hierarchy with regard to objects on the immediately higher $(k-1)^{th}$ level. Typically, the highest level is a goal, the second level form criteria and the lowest level consists of alternatives. The aim of AHP lies in the selection of the best alternative. As the article focuses on indecisiveness of decision makers in evaluating alternatives, only comparisons of alternatives with regard to a given set of criteria by one or more DMs will be considered.

Alternatives are compared on the scale from 1 to 9, where 1 denotes equal importance and 9 extreme importance of one alternative over another (see Table 1); $s_{ij}$ denotes preference of an alternative $i$ over an alternative $j$. Preferences $s_{ij}$ are reciprocal: if an alternative $A$ is moderately preferred over an alternative $B$, then $s_{AB} = 3$, and by definition $s_{BA} = 1/3$.

All pair-wise preferences $s_{ij}$ form a reciprocal matrix $P$ with elements $s_{ij} = \frac{1}{s_{ji}}; \forall i,j$.

Pair-wise preferences are consistent, if $s_{ij} \cdot s_{jk} = s_{ik}, \forall i,j,k$; that is the transitive property of DM’s preferences is preserved.

Consistency of DMs’ preferences is expressed by consistency index C.I. given as [4]:
\[ C.I. = \frac{\lambda_{\text{max}} - n}{n-1}, \]  

where \( \lambda_{\text{max}} \) is the largest (positive) eigenvalue of the matrix \( P \), \( n \) is the order of \( P \), and \( C.I. \geq 0 \). The value \( C.I. = 0 \) indicates absolute consistency of preferences; the larger is \( C.I. \), the more inconsistent preferences are. According to Saaty [6], human judgment is inconsistent by nature, so \( C.I. < 0.1 \) is tolerated.

Consistency index is the only measure of ‘quality’ of decision makers’ preferences in AHP. However, consider a preference matrix \( A \) shown in Table 2. It is easy to verify that \( \lambda_{\text{max}}(A) = 5 \), so \( C.I. = 0 \). Preferences expressed by \( A \) are absolutely consistent, but they are also absolutely useless, as no alternative is preferred over any other alternative (because a decision maker is absolutely indecisive). This example shows that consistency index alone is not a sufficient tool for preferences assessment.

<table>
<thead>
<tr>
<th>Intensity of Importance</th>
<th>Definition</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Equal Importance</td>
<td>Two activities contribute equally to the objective</td>
</tr>
<tr>
<td>2</td>
<td>Weak or slight</td>
<td>Experience and judgment slightly favor one activity over another</td>
</tr>
<tr>
<td>3</td>
<td>Moderate importance</td>
<td>Experience and judgment strongly favor one activity over another</td>
</tr>
<tr>
<td>4</td>
<td>Moderate plus</td>
<td>An activity is favored very strongly over another; its dominance demonstrated in practice</td>
</tr>
<tr>
<td>5</td>
<td>Strong importance</td>
<td>The evidence favoring one activity over another is of the highest possible order of affirmation</td>
</tr>
<tr>
<td>6</td>
<td>Strong plus</td>
<td>A better alternative way to assigning the small decimals is to compare two close activities with other widely contrasting ones, favoring the larger one a little over the smaller one when using the 1–9 values.</td>
</tr>
<tr>
<td>7</td>
<td>Very strong or demonstrated importance</td>
<td>Measurements from ratio scales</td>
</tr>
<tr>
<td>8</td>
<td>Very, very strong</td>
<td>Reciprocals of above</td>
</tr>
<tr>
<td>9</td>
<td>Extreme importance</td>
<td>If activity ( i ) has one of the above nonzero numbers assigned to it when compared with activity ( j ), then ( j ) has the reciprocal value when compared with ( i )</td>
</tr>
</tbody>
</table>

Table 1. Saaty’s scale in AHP. Source: [6].
Table 2. An example of preferences of an absolutely indecisive DM.

3. The measure of indecisiveness

As shown in Table 2, a DM is absolutely indecisive, if all \( s_{ij} = 1 \). Let 1 denote the matrix with such elements. Then decisiveness of a DM with preferences given by the matrix \( A(a_{ij}) \) can be evaluated from an absolute difference matrix \( D(d_{ij}) \): \( d_{ij} = |a_{ij} - 1| \).

The larger are elements \( d_{ij} \), that is differences \( a_{ij} - 1 \), the higher is DM’s decisiveness. For a decisiveness evaluation, the entrywise \( p \)-norm applied on \( D \) can be used:

\[
\|D\|_p = \left( \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij}^p \right)^{1/p}
\]

where \( p \in \mathbb{N} \). The special case \( p = 2 \) is called Frobenius norm (it is an equivalent of the Euclidean norm for vectors), and will be used henceforward.

The larger is \( \|D\|_p \) the greater is the decisiveness of a decision maker. It is possible to state some minimal value of \( \|D\|_p \) (the threshold of decisiveness – \( \tau \)) dependent on \( n \), so that DMs with decisiveness \( \|D\|_p \) smaller than \( \tau \) would be excluded from a decision making process.

Weights \( w_i \) of decision makers can be derived with the use of formulas (1) and (2) combined into one overall index. Weights of DMs are functions of indecisiveness and consistency index: \( w = w(\|D\|_p, C.I.) \), and should satisfy the following properties:

- The function \( w \) is non-negative.
- The function \( w \) is strictly increasing in \( \|D\|_p \).
- The function \( w \) is strictly decreasing in \( C.I. \).

In addition, because \( C.I. \) acts as a penalization factor in \( w \), it is reasonable to require no penalization for absolute consistency (\( C.I. = 0 \)). This condition is satisfied e.g. for the exponential function \( e^0 = 1 \). One simple function \( w \) satisfying all three conditions, and additional condition mentioned above, can be given as:

\[
w = \|D\|_2 \cdot e^{-10C.I.}
\]
most cases. Also, it follows from (3) that for the limit value of acceptable consistency $C.I. = 0.1$, the weight $w$ takes especially simple form: $w = \frac{\lVert D \rVert_p}{e}$.

Usually, DMs compare alternatives with regard to more than one criterion. In such a case, indecisiveness and consistency index $C.I.$ is estimated for each criterion separately, and then the average indecisiveness along with average $C.I.$ is used to calculate (average) DMs’ weights by relation (3).

Numerical examples illustrating weight evaluation with the use of (3) are provided in the following section. Nowadays, there are many software tools which facilitate computation of $C.I.$ or $\lambda_{\text{max}}$:

- ExpertChoice (www.expertchoice.com),
- Mathematica (www.wolfram.com/mathematica/),
- Meta-numerics (www.meta-numerics.net),
- WolframAlpha (www.wolframalpha.com), etc.

ExpertChoice is a commercial product for AHP solutions; it computes $C.I.$ directly. Other mathematical tools are capable of computing an eigenvalue system of the matrix $P$. The last two products are free of charge.

4. Numerical examples

In this Section two numerical examples are provided. In Example 1 alternatives are compared by one criterion and in Example 2 by two criteria. In both cases consistency index $C.I.$, indecisiveness $\lVert D \rVert_p$ and weights $w$ of all DMs are evaluated.

**Example 1.** Let $A$ and $B$ be preference matrices of decision makers $\text{DM}_1$ and $\text{DM}_2$ respectively on three alternatives (see Table 3) by a given criterion. We will evaluate weights of both decision makers using the formula (3) and the Frobenius norm ($p = 2$).

$$
A = \begin{pmatrix}
1 & 9 & 7 \\
1/9 & 1 & 3 \\
1/7 & 1/3 & 1
\end{pmatrix} \\
B = \begin{pmatrix}
1 & 2 & 3 \\
1/2 & 1 & 2 \\
1/3 & 1/2 & 1
\end{pmatrix}
$$

Table 3. Preferences of $\text{DM}_1$ (matrix $A$) and $\text{DM}_2$ (matrix $B$).

Consistency index $C.I.$ (1) for both DMs:

$$
\lambda_{\text{max}}(A) = 3.206, \ n = 3 \ \Rightarrow \ C.I.(\text{DM}_1) = \frac{3.206 - 3}{2} = 0.103.
$$

$$
\lambda_{\text{max}}(B) = 3.008, \ n = 3 \ \Rightarrow \ C.I.(\text{DM}_1) = \frac{3.008 - 3}{2} = 0.004.
$$

Hence, $\text{DM}_2$ is more consistent in his evaluation of alternatives. On the other hand, $\text{DM}_1$ is much more decisive than $\text{DM}_2$: 
Finally, we compute weights \( w_1 \) and \( w_2 \), and normalized weights \( w_1' \) and \( w_2' \) of DM\(_1\) and DM\(_2\) respectively using (3):

\[
w_1 = 10.29 \cdot e^{-10.103} = 3.67.
\]

\[
w_2 = 2.64 \cdot e^{-10.004} = 2.63.
\]

\[
w_1' = 0.583, \quad w_2' = 0.417.
\]

The first decision maker is assigned slightly larger weight due to his greater decisiveness.

**Example 2.** Let \( A \) and \( B \) be the same preference matrices of decision makers DM\(_1\) and DM\(_2\) respectively on three alternatives (see Table 3) by a criterion \( C_1 \) as in Example 1. Let \( C \) and \( D \) be preference matrices of decision makers DM\(_1\) and DM\(_2\) respectively on three alternatives by a criterion \( C_2 \) (see Table 4).

\[
C = \begin{pmatrix}
1 & 2 & 3 \\
1/3 & 1 & 4 \\
1/2 & 1/4 & 1
\end{pmatrix} \quad D = \begin{pmatrix}
1 & 4 & 1/2 \\
1/4 & 1 & 1/5 \\
2 & 5 & 1
\end{pmatrix}
\]

**Table 4.** Preferences of DM\(_1\) (matrix \( C \)) and DM\(_2\) (matrix \( D \)) with regard to the criterion \( C_2 \).

Consistency index \( C.I. (1) \) of preferences with regard to the criterion \( C_2 \):

\[
\lambda_{\text{max}} (A) = 3.107, \quad n = 3 \quad \Rightarrow \quad C.I. (DM_1) = \frac{3.107 - 3}{2} = 0.054.
\]

\[
\lambda_{\text{max}} (B) = 3.025, \quad n = 3 \quad \Rightarrow \quad C.I. (DM_1) = \frac{3.025 - 3}{2} = 0.013.
\]

Again, DM\(_2\) is more consistent in his evaluation. As for decisiveness, DM\(_2\) is more decisive:

\[
\|P\|_2 (DM_1) = \left[ 1^2 + 2^2 + 3^2 + \left( \frac{2}{3} \right)^2 + \left( \frac{1}{2} \right)^2 + \left( \frac{3}{4} \right)^2 \right]^{1/2} = 3.906,
\]

\[
\|P\|_2 (DM_2) = 5.239.
\]

Weights \( w_1^{(2)} \) and \( w_2^{(2)} \), and normalized weights \( w_1^{(2)'} \) and \( w_2^{(2)'} \) of DM\(_1\) and DM\(_2\) respectively, are:

\[
w_1^{(2)} = 3.906 \cdot e^{-10.054} = 2.276.
\]

\[
w_2 = 5.239 \cdot e^{-10.013} = 4.600.
\]

\[
w_1^{(2)'} = 0.331 \quad w_2^{(2)'} = 0.669.
\]
From Example 1 we know, that weights of DMs with regard to the criterion $C_1$ are:

\[ w_1^{(1)} = 0.583, \quad w_2^{(1)} = 0.417 \]

By averaging we can assign final weights to both DMs: \( w_1 = 0.457, \quad w_2 = 0.543 \).

With computed weights the decision making can proceed into the aggregation phase and then to the selection of the best alternative.

5. Conclusions

The aim of the article was to introduce a concept of indecisiveness into the group analytic hierarchy process (AHP) framework and to show how weights of decision makers can be derived only from their preferences with the use of indecisiveness and Saaty’s consistency index $C.I$. The advantage of this approach is that no additional information about decision makers’ experience or knowledge ‘from outside’ is required. Further work may focus on differences (conflicts) among decision makers’ preferences in the group AHP and the influence of the differences on a possibility of an existence of a group consensus.

References:


Mgr. Jiří Mazurek, Ph.D.
Assistant Professor
Department of Mathematical Methods in Economics
School of Business Administration in Karviná,
Silesian University in Opava
e-mail: mazurek@opf.slu.cz