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# Keynes's probability: An introduction to the theory of logical groups

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## Abstract

The present work is intended to be an informal introduction to the theory of *abstract logical groups*. This particular formalization stems from some concepts of abstract algebra and the Johnson-Keynes's theory of groups. Therefore the aim of this paper is that of provide the readers with the logical reasoning behind this brand new theory. I shall depict the philosophical notions as bases of the Keynes's probability and then I shall explain it in terms of group. Furthermore we shall see, albeit roughly, a first definition of abstract groups.

1. In order to comprehend the probability's meaning of Keynes, we have to deal with a precise definition of knowledge. In treating this concept, it is chiefly plain the influence of the Russell's philosophy during the early 20<sup>th</sup> century. I shall not concern about a thorough examination on the strict philosophical relations betwixt Mr. Keynes and Mr. Russell, but I do have to clarify some notions about them inasmuch as we are dealing with the logical framework of the Keynes's probability. By studying Russell's philosophy it is unavoidable to come across both the notion of knowledge and acquaintance. The knowledge by acquaintance may be considered as a foundation of all our knowledge [CtR]. Thus, the 'knowledge' is divided into *of things* and *of truths*. Furthermore, 'of things' is divided into *by acquaintance* and *by description*, whereas 'of truth' is divided into *intuitive* and *derivative*. The relation between the knowledge *of things* and the knowledge *of truths* gives us the logical conclusion that *by acquaintance* is the *primitive* knowledge. Thus, by [CtR]<sup>1</sup>:

$$K^d \rightarrow K' \rightarrow K_i \rightarrow K^a.$$

When one deals with Mr. Keynes's probability,  $K^a$  can be thought of as a *direct* knowledge, rather even in Keynes there is a concept of purest knowledge. The knowledge of things is thus linked to an experience of them, hence it is connected to the acquaintance. It is worthwhile noting that the acquisition of some kind of information by whatsoever experience it is not enough in so far as one is dealing with probability, one needs the intuition. Thus, intuition and acquaintance give rise a knowledge, in few words, they give rise a level of rational belief: certainty. In this case we have the highest degree of rational belief in so far we have a knowledge directly from an experience that has been perceived. This is linked to 'by truths', rather we know a proposition when it is in fact true, otherwise we cannot know it. In [KeP], Keynes identifies three different ways of acquisition by *direct* acquaintance: sensation, meaning and perception[FSP]. We have not a knowledge yet, albeit they give rise our knowledge. We have a sensation when we have an experience of a direct acquaintance, we understand ideas or meaning about which we have thoughts, we perceive facts, characteristics, relation of sense-data or meaning [KeP]. Whilst these are the objects of direct acquaintance, the objects of knowledge and belief are termed *propositions*. Thus the objects of them are intensely different from each other. We call the space of the objects of acquaintance *direct space* and we denote it by  $\mathbb{K}^a$  and the space of proposition  $\mathbb{P}^k$ . One ought to note that  $\mathbb{K}^a \cup \mathbb{P}^k = \emptyset$  but:

$$\vdash \mathbb{K}^a \supset \vdash \mathbb{P}^k$$

thus as a matter of definition of acquaintance:

$$\therefore \phi(\kappa). \supset_{\kappa} \psi(\kappa). \kappa \in \mathbb{V} \quad \text{Df.}$$

It is obvious that  $\kappa$  is an object of acquaintance thereby it is always true (namely it belongs to  $\mathbb{V}$ ). Therefore if the propositional function  $\phi(\kappa) \in \mathbb{V}$  implies  $\psi(\kappa) \in \mathbb{V}$ . By defining this implication there

<sup>1</sup>For the sake of elegance we denote: 'by description' by  $K^d$ , 'derivative' by  $K'$ , 'intuitive' by  $K_i$  and 'by acquaintance' by  $K^a$  and the 'depends on' relation by  $\rightarrow$ .

is a novel concept that may pass unnoticed:  $\phi(\cdot)$ . This *semi*-propositional function does not concern sentences like ' $\kappa$  is tall' but perceptions, understanding and so forth. It is the ground upon which  $\psi(\cdot)$  acquires a meaning. We do not have to call  $\phi(\cdot)$  a propositional function, firstly because of its nature, it is not belonged to  $\mathbb{P}^k$ . Consequentially one cannot give a value to an object of  $\mathbb{K}^a$ , and thus  $\kappa \in \mathbb{K}^a$  is not even a variable. I reckon that  $\phi(\kappa)$  could be seen as a collection of some objects of direct-knowledge:

$$\phi(\kappa) = \bigcup_i^n \kappa_i$$

and it is straightforward that  $\phi(\cdot)$  is a collection of (what I like to call) sparks. The notion of the *spark*-object says that one has a[the] pure[st] perception (lato sensu) which is not well-defined in meaning but it is fundamental for his understanding, he has not a knowledge, he is staring something which is sparkling through the darkness before his mind, before his capacity of comprehension. It is possible to obtain a knowledge *of* a spark by using  $\psi(\cdot)$ . The latter is undoubtedly a propositional function, rather it settles who is  $\kappa$ . The figurative existence of spark-objects is justified by *a priori* truths which are true in any possible world [CtR].

The symbol of probability is  $a/h$ ; our premisses consist of any set of propositions  $h$  and our conclusion consist of any set of propositions  $a$ . This symbol is of utmost importance because of its clearness, rather it conveys that there is a *probability relation* between  $a$  and  $h$ . Thus, as Russell has said, ' $h$  in this symbol represents and entails much of Mr. Keynes's philosophy'. Rather it is in  $h$  that knowledge plays its significant role. Each relation, each kind of knowledge, each spark and each value is encompassed in  $h$ . Thus, whilst the subjective part is played by  $h$ , the objective one is expressed by  $a/h$ . This statment is important and says that the probability relation is objective once the subjective ground has been built by our knowledge (from our acquaintance and thus by the sparks). We have to look at  $a/h$  as a particular kind of  $a\mathcal{R}b$ . As a matter of fact when one infers as *of*  $h$ , he uses a particular path which it is not as general as  $\mathcal{R}$ . We could state that  $a/h$  is a sub-relation of  $\mathcal{R}$ .

In Keynes's probability we have to distinguish between:

|                  |   |
|------------------|---|
| <b>Belief</b>    | rational and irrational; certain in degree and only probable.   |
| <b>Knowledge</b> | direct or indirect; of primary or of secondary proposition; according as it is <i>of</i> or merely <i>about</i> its object. |

Thus, it is preferable to regard knowledge as fundamental and to define rational belief by reference to it. Thereby, we have to focus on the important meaning of 'rational'. In [KeP], Keynes depicted rational beliefs as follows:

If a man believes something for a reason which is preposterous or for no reason at all, and what he believes turns out to be true for some reason not known to him, he cannot be said to believe it *rationally*, although he believes it and it is in fact true. On the other

hand, a man may rationally believe a proposition to be *probable* when it is in fact false. The distinction between rational belief and mere belief, is not the same as the distinction between true beliefs and false beliefs. The highest degree of rational belief, which is termed *certain* rational beliefs, corresponds to *knowledge* [KeP].

In order to comprehend his reasoning it would be better off considering an axiomatization of the actual meaning of 'rational':

- $x\mathcal{R}y.y\mathcal{R}z. \supset_{x,y} .x\mathcal{R}z$  or  $\vdash .x = y.y = z. \supset .x = z$  (transitive);
- $\vdash .x = x$  (reflexive);
- $\vdash .x = y. \equiv .y = x$  (symmetrical).

Thus a reasoning based upon a direct contemplation of a spark which is regarding the three axioms above is called *rational*. It is important to notice that one has to know by *what* his reasoning is made of, *how* he has built it up and *why* he infers something. Therefore he can prove that his reasoning entails the three axioms. A rational belief in its highest degree is called *certain*, thereby we could obtain a knowledge *of* a proposition. This reasoning evokes the notion of primary and secondary proposition and thus the notion of *argument*. Therefore a proposition *b* that we know is different from the proposition *p* in which we have probable degree *q* of rational belief. Now given an *h*, in the sense specified above, then *b* is the proposition *p* which bears the probability-relation of degree *q* to the set of proposition *h*; and this knowledge justifies us in a rational belief of degree *q* in the proposition *p* [KeP]. It is plain that a knowledge of a proposition implies the certainty of rational belief and the truth of its proposition. In Keynes's words:

'The peculiarity of certainty is that knowledge of secondary proposition involving certainty, together with knowledge of what stands in this secondary proposition in the position of evidence, leads to *knowledge of*, and not merely *about*, the corresponding primary proposition. Knowledge, on the other hand, of a secondary proposition involving a degree of probability lower than certainty, together with knowledge of the premiss of the secondary proposition, leads only to *rational belief of the appropriate degree* in the primary proposition' [KeP].

It is worthwhile noting that the direct knowledge is the result of contemplating those objects belonged to  $\mathbb{K}^a$  and the indirect fashion is by means of the so called *argument*, thereby through perceiving the probability-relation of the proposition, about which we seek knowledge, to other proposition [KeP]. Thus, by argument, we know a secondary proposition involving the proposition we are looking for, hence it is a knowledge *about*. It is of utmost importance to understand that one passes from a knowledge of a proposition *x* to a knowledge about a proposition *y* by perceiving a logical relation between them which one can contemplate by direct acquaintance.

2. Keynes says that 'the theory of probability deals with the relation between two sets of propositions, such that, if the first set is known to be true, the second can be known with the appropriate degree of probability by argument from the first. The relation, however, also exists when the first set is not known to be true and is hypothetical' [KeP]. Thus Keynes talks about the collection of propositions logically involved in the premisses and he calls it *group*. Actually he has acquired the notion of group from Sir. Johnson, rather he wrote in a footnote [JWE]: the term group is here used in its precise mathematical meaning. In [JWE] we come across the notion of compound propositions as those propositions constructed out of two or more *simple* propositions. The different forms of the main (compound) proposition hinge on the different *conjunctions* it may assume. It is thus important the difference between *conjunctive* and *conjuncts*. The former are relations' meaning and symbols *stricto sensu*, i.e. *and* is par excellence *conjunctive*, whereas the latter are the components. Therefore the proposition 'p and q' assumes a certain form by the *conjunctive* 'and' and the *conjuncts* 'p,q'.

The use of the *conjunctive* 'and' is not the same as the use of the *enumerative* 'and'. Following [JWE], whilst the former treats components combinatorially, the latter treats them severally. Therefore 'p and q' are two proposition by using the enumerative 'and', whereas it is one proposition by using the conjunctive 'and'. In conjoining two or more propositions we are realising their joint force. Thus, following [JWE], when one infers from the conjunctive proposition 'p and q' a set of propositions, these cannot be inferred from neither *p* alone or *q* alone. Given the *joint*-value, the term group appears. With 'p,q' as components we can generate several groups: the two groups of propositions implied by *p*, by *q* respectively, the two groups of propositions implied by 'p and q', 'q and p'; if there were three components 'q,p,r' we would have another group consisting of propositions implied by 'p and q and r'. The importance of the notion of group in [KeP] can be understood in Keynes's words: 'we define a group as *containing* all the propositions logically involved in any of the premisses or in any conjunction of them; and as *excluding* all the propositions the contradictories of which are logically involved in any of the premisses or in any of the conjunction of them. To say, therefore, that a proposition follows from a premiss, is the same thing as to say that it belongs to the group which the premiss specifies' [KeP]. Yet by the idea of group one can define either logical *consistency* or logical *inconsistency* of premisses: 'if any part of the premisses specifies a group containing a proposition, the contradictory of which is contained in a group specified by some other part, the premisses are logically inconsistent; otherwise they are logically consistent. In short, premisses are inconsistent if a proposition *follows from* one part of them, and its contradictory from another part' [KeP].

The meaning of *follows from* or *logically involved in* stems from the Mr. Russell's *Logic of implication*. This kind of implication is called *formal implication*, it states that for all possible values of *x*, if the hypothesis  $\phi x$  is true, the conclusion  $\psi x$  is true [RWP]. Thus a formal implication

$$(x) : \phi x. \supset .\psi x$$

may be interpreted as: all values of *x* which satisfy  $\phi x$  satisfy  $\psi x$  [RWP], where a value *x* is said to *satisfy*  $\phi x$  or  $\phi \hat{x}$  when  $\phi x$  is true for *that* value of *x* [RWP]. Thus, since  $\phi x. \supset .\psi x$  will always be true when  $\phi x$  is false, it is only the values of *x* that make  $\phi x$  true that are *important* in a formal

implication; what is effectively stated is that, for all these values,  $\psi x$  is true [RWP]. We may rewrite the formal implication

$$\phi x. \supset_x . \psi x :=: (x) : \phi x. \supset . \psi x$$

Now it is important to define what Keynes precisely intends when one says that one proposition follows from another. Thus following [KeP], a group of propositions is a set of propositions such that:

- if the proposition 'p is formally true' belongs to a group, all propositions which are instances of the same formal propositional function also belong to it;
- if the proposition  $p$  and the proposition 'p implies q' both belong to it, then the proposition  $q$  also belongs to it;
- if any proposition  $p$  belongs to it, then the contradictory of  $p$  is excluded from it.

It is thus important to define the *fundamental* propositions which characterize a group. Therefore the propositions  $p_1, p_2, \dots, p_n$  are called *fundamental* to the group  $h$  when they themselves belong to the group, if between them they completely specify a group; and if none of them belong to the group specified by the rest (for if  $p_r$  belongs to the group specified by the rest, this term is redundant) [KeP]. It is now surely plain that a proposition *follows from* given premiss if it is included in the group which the premiss specifies, that is to say, a proposition is *certain* ( $a/h = 1$ ) in relation to a given premiss. It is *impossible* ( $a/h = 0$ ) if its contradictory follows from the premiss. Given an evidence, two propositions are contradictory to one another when they are in juxtaposition with that evidence and they are inconsistent given that they do not belong to the same group. Furthermore, we face a degree of probability when a proposition is not itself included in a group specified by the premiss and whose contradictory is not included either [KeP].

In [JWE] there is a difference between *certified* and *uncertified* proposition, albeit it is a subjective one, given that an uncertified proposition may become certified. This may happen, for example, with increased observations and the distinction is temporal and relative to individuals and their means of acquiring knowledge [JWE]. This statement is very important so as to make a probabilistic reasoning. When we face certified proposition we have to clarify which kind of certification it is: *formally* or *experientially* certified. The former occurs when the truth of a proposition is certified by pure thought or reason, whereas the latter is certified on the ground of actual experience [JWE]. Now the reasoning is: 'while we define a formally certifiable proposition as one which can be certified by thought or reason alone, we do not define experiential propositions as those which can be certified by experience alone, but rather as those which can only be certified with the aid of experience. In this way we imply that experience alone would be inadequate' [JWE]. Thus because of Sir. Johnson that Keynes take into account the intuition besides the experience.

Foregoing statements are treated in [KeP], but Keynes divides the moment of certification into *hypothetical groups* and *real groups*. Furthermore there is the important notion of *requirement*. It is a first cousin of the concept of necessity in [JWE]. The requirement resembles that of the kantian necessity

and it is not the Johnson's *nomic*. Yet it deals with the laws of nature taken alone do not necessitate any event whatever; we should have rather to say that a law of nature necessitates that the happening of some one thing should necessitate the happening of a certain other thing [JWE]. Rather the Johnson's *nomic* and *contingent* characterise a fact whereas certified and uncertified characterise a proposition. Thus the *requirement* deals with formal certification. We say that 'p requires q' when we cannot know p to be true unless we already know q. It is obvious that if  $p/h = 1$  and  $q/h \neq 1$  then p does not require q given that p belongs to the real group and q does not belong. Furthermore we say that p requires q when there is no real group to which p belongs and q does not belong. Now, if the fundamental group is *uniquely* determined, a group  $h'$  is a sub-group to the ground  $h$ , if the set fundamental to  $h'$  is included in the set fundamental to  $h$  [KeP]. Given that we can say that p does not require q within the group  $h$  if  $p \in h$  and  $h' \subset h$  (in this case it means subset and not a logical implication) then  $p \in h'$  and  $q \notin h'$ . In other words if  $h'/h = 1$ ,  $p/h' = 1$  and  $q/h' \neq 1$ . Furthermore, we say that the probability relation of p does not require q within the group  $h$ , if given  $h'' \subset h' \subset h$  (intended as subsets) and that  $h'/h'' = 1$ ,  $h''/h = 1$ , we have  $p/h'' = p/h'$  and  $q/h' \neq q/h$ .

The logical priority follows the requirement concept, rather p is prior to q when p does not require q but q does require p. We say that p is prior to q within the group  $h$  when p does not require q within the group whereas q requires p within the group. From these statements follow that there is no proposition prior to a fundamental one.

**3.** We have seen that the Keynes's probability can be understood by means of those particular *groups*. For example a premiss of an argument must be formally or experientially certified, hence our premiss and our argument must be of the same group. In this reasoning it is plain the importance of groups, rather they define what premiss is worthwhile and thus what proposition is suitable for our aim. Now it would be very interesting to define what I called *abstract logical groups*. Of course I shall not describe this formalization in its most formal fashion here, but who wants to read this topic thoroughly, he can have a glance at [LGr].

As we have already seen (2) the certainty 'premiss-proposition' means that both of them have to be included into the same group, the group specified by the premisses. We may formalize this statement by giving an *algebraic structure* to this kind of matter. Given our *simple* propositions 'p,q' we can form a set and we want to define a *law of composition*. If  $p \in A$  and  $q \in A$ , this law can be seen as a function defined on  $A \times A$  and taking its value in A. Thus the value of a propositional function  $f(\hat{p}, \hat{q})$  for an ordered pair  $(p, q) \in A \times A$ , is the composition of this pair under our law of composition. We call *magma* a set endowed with a law of composition. Thus when we have a *propositional composition* we have not a group yet. Now, given T, a law of composition  $(p, q) \rightarrow pTq$  on A is associative if for all the propositions 'p,q,r' in A

$$(pTq)Tr = pT(qTr).$$

We call A stable under the law T if  $ATA \subset A$  (in this case it means subset).

In order to give the definition of group, we need the *unit class*. We denote by  $t'p$  the class whose only

member is  $p$  [RWP]:

$$l'p = \hat{q}(q = p) \quad \text{Df}$$

thus,  $l'p$  means the class of objects which are identical with  $p$ . An *identity element* can be seen as the highest degree of rational belief  $e$ , thus  $pTe = eTp = p$ . In this case we call our magma a *unital magma* and given the associative law we call it *monoid*. A very important property is that of *invertibility*. We have to bear in mind that in an abstract meaning every proposition ( $p$ ) is invertible ( $p'$ ), that it to say  $p'Tp = e$  and  $pTp' = e$ .

Besides the notion of *internal* composition there is the notion of *action* or *external* composition. Thus, given two sets  $\Omega$  and  $A$ , a mapping of  $\Omega$  into the set  $A^A$  of mappings of  $A$  into itself is called an action of  $\Omega$  on  $A$ . The propositional function  $f_\alpha(p)$  of  $A$ , where  $\alpha \in \Omega$  and  $p \in A$  is called the *transformation* of  $p$  under  $\alpha$ . Thus the element of  $\Omega$  are called *operators*.

Now a monoid in which every element is invertible is called a *group* and a group with an action  $\Omega$  which is distributive with respect to the group law, is called a group with operators in  $\Omega$ .

The Keynes's probability may be built up from abstract logical groups. A very important kind of consequence is the notion of *logical homotheties* which is the endomorphism of the underlying group structure. In particular, given a fundamental proposition we can build a more complex proposition's framework up, thus an operator in  $\Omega$  gives the *homothety logic-ratio*. This is an operator that is able to enlarge a prior proposition given a certain ratio. For example a ratio ( $p$  and  $\cdot$ ) or ( $p \vee qT\cdot$ ) et cetera.

## Appendix

### a. Knowledge

- derivative** everything that we can deduce from self-evident truths by the use of self-evident principles of deduction [PoP]. It is couched in terms of what 'we *can* deduce' and not in terms of what 'we *have* deduced' [CtR]. It does not give rise by inference, but by a *psychological* inference which characterizes the connection betwixt beliefs. It is exemplified by the way in which a newspaper reader forms beliefs [CtR].
- intuitive** is self-evident 'in the first and most absolute sense' [PoP]. Alongside this kind of intuitive knowledge, there is a special kind of it which is capable of degree, hence it is not *perfect* but there may be a doubt in underlying beliefs (such as memories, ethical judgements and so forth). A knowledge thought of as this kind of intuition implies that there is no sharp distinction between knowledge and probable opinion.
- by description** 'This is knowledge which involves knowing that there is some one thing with certain properties even though we do not know who or what it is, in the sense that we are not acquainted with it' [CtR].

### b. Some logical symbols

- $\vdash$  it is the assertion-sign (assertion, axiom, theorem);
- $., : , \dots$  et cetera are the same of  $(, [, \{$ ;
- $\vee, \supset, \sim, \equiv, \dots$  are *or, implication, negation, if and only if, and* respectively;
- $\mathcal{R}$  denoted a relation;
- Df. means *definition*;
- the hat, i.e.  $\hat{x}$ , means a constant, a certain value of  $x$ , a defined one;
- $\psi$  and  $\phi$  are variables which range over propositional functions;
- $\exists$  and  $()$  means *there exists* and *for all*.

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