# Notes on GEKS and RGEKS indices

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## Comments on a method to generate transitive indices

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### Abstract/Summary

The paper is dealing with a proposal first made by Ivancic, Fox and Diewert (Ottawa Group Meeting in Neuchâtel 2009) to create transitive indices by averaging direct (not chained) Fisher indices which themselves, however, are known to lack transitivity. To require transitivity means that *all* indirect comparisons between A and B (via C etc.) are consistent with *the* [only] direct one, viz. $P_{AB}$. This restrictive property appears justified in the international framework – for which the original GEKS methodology was devised –, but it may be called in question as "over-ambitious" in the intertemporal situation where only some indirect comparisons are relevant, viz. those between adjacent intervals as they are used in the chain index method.
It is misleading to describe the task as removal of "chain drift" because here (as in the problem GEKS dealt with) no chain indices are involved but only direct Fisher price indices.
There are three types of index functions in the order of increasing complexity and data requirements, chain indices $\overline{P}_{0t}$, direct indices $P_{0t}$, and GEKS indices (where the formula also encompasses besides $P_{0t}$ a number of additional direct indices of the $P_{0k}$ and $P_{kt}$ type). So when it suffices to provide the chain-index-type of indirect intertemporal comparisons (over a sequence of adjacent intervals, like 0-1-2-3-4…rather than also indirect comparisons of the sort 0-5-3-8-2 …) it seems to be reasonable to confine onselef with the least demanding chain indices. To advocate (R)GEKS in favour of chain indices on the other hand requires good arguments as to their (alleged) advantages. However, we cannot see them, but see many disadvantages of the (R)GEKS approach instead.
As various values can be chosen for m, the number of periods taken into account in a GEKS index (or in RGEKS, its "rolling" variant) provides *a number of* different (and equally legitimate) indices $P_{st}$ for comparing the same two periods, s and t (by contrast to the other two index types which both yield a definite result here) leaving it open what should be viewed as *the* (?) "drift free" index series. The RGEKS method (of which the chain index is the special case of m = 2) is designed to avoid the re-computing of previously computed indices $P_{0t}$ or $P_{st}$ (s > 0) when a new period t+1 becomes available (a problem not arising with the other two index types). It does so, however, at the price of losing the transitivity property of GEKS (for a given m). Moreover, just like chain indices RGEKS indices depend on the frequency of updating, they can display fluctuations around a positively or negatively sloped trend when price movement is cyclical and has no trend and they can remain unchanged between two periods, s and t, although prices in s and t are different, and rightly reflected by $P_{st} \neq 1$.



Keywords: index numbers, transitivity, chain drift, Gini-Eltetö-Köves-Szulc, chain index

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## 1. Introduction

The paper is dealing with a proposal first made by Ivancic, Fox and Diewert (2009)[1] - or IFD for short - to handle the transitivity problem with chain indices (chained *price* indices in particular). The problem arose in the context of high-frequency (weekly, monthly) chaining of unit values and price indices based on scanner data. In order to ensure transitivity IFD proposed to adopt methods, initially developed for international comparisons, where transitivity is of paramount interest. The method in question to gain transitivity is known as GEKS-method, an acronym alluding to the four inventors of the method, Gini, Eltetö, Köves, and Szulc.[2] Also in use are the names RGEKS, RWGEKS, or RYGEKS, meaning "rolling", "rolling window", or "rolling year" GEKS. "Rolling" refers to a kind of moving average methodology, adopted in order to cope with the problem that GEKS indices are incomparable when calculated from time series of different length.

[1] Lorraine Ivancic, Kevin J. Fox, W. Erwin Diewert, Scanner Data, Time Aggregation and the Construction of Price Indexes, May 2009 (Ottawa Group Meeting in Neuchâtel).

[2] In the literature the practice to mention Gini in addition to EKS only came in use a bit later. IFD prefer to use the acronym GEKS (Gini - Eltetö - Köves - Szulc), however, GEKS is also used for "Generalized GEKS" where m normalized *country weights* are assigned to the m countries (see von der Lippe (2007), p. 554). Also the Dutch van Ijzeren 1956 is sometimes quoted as an inventor of this method. While the GEKS method is using Fisher-indices $P^F$, another method of deriving transitive parities, the CCD method (Caves - Christensen - Diewert) is built on Törnqvist indices $P^T$.

The paper examines the properties of the GEKS-approach to make a series of transitive indices $P_{0t}^{GEKS}$ from non- transitive direct Fisher price indices $P_{0t}^{F}$.[3]

Some problems with that method are quite obvious. To begin with the derivation of the GEKS-index-formula requires a time reversible index, like $P_{0t}^{F}$ (because $P_{t0}^{F} = \left(P_{0t}^{F}\right)^{-1}$) as "raw material" for the GEKS-formula.[4] In such index functions like $P^F$ use is made of quantities of both periods, $q_0$ *and* $q_t$ in a symmetric fashion.[5] Ideally both q's should refer to the same collection of goods. This may be less of a problem with a *chained* Fisher index where only "links", referring to adjacent periods, such as $P_1^F = P_{01}^F$, $P_2^F = P_{12}^F$ etc. are needed to create a "chain" $\overline{P}_{0t}^F = P_1^F P_2^F ... P_t^F$ by multiplying the links. By contrast, the *direct* index, say $P_{06}^F$ is much more difficult to compile, because it requires in period t = 6 quantities $q_6$ and $q_0$ and not only quantities $q_6$ and $q_5$, so that to ensure identity of goods may well prove problematic. And the GEKS method is even more difficult to implement, as calculating $P_{06}^{EKS}$ makes it necessary to have not only the *direct* index $P_{06}^F$ but also a number of additional direct indices like $P_{01}^F$, $P_{16}^F$, $P_{02}^F$, $P_{26}^F$ … as factors in the formula for $P_{06}^{GEKS}$.

So we may conclude as a first result that we have three kinds of index numbers in the order of increasing complexity and data requirements: first a chain index $\overline{P}_{06}^F = P_1^F P_2^F ... P_6^F$, then the direct index $P_{06}^F$,[6] and finally the GEKS-index $P_{06}^{GEKS}$, as a geometric mean of *a number of* such direct Fisher price indices. Hence GEKS indices are considerably more complicated than anything else, and possibly unduly complex if you try to infer their properties and to give an easy to understand interpretation to empirical results gained with them.

This may trigger the quite obvious question: is $P_{06}^{GEKS}$ really so much better than $\overline{P}_{06}^F$ (or also $P_{06}^F$) to justify this extra expense. To mention it right at the outset: the "message" of this paper is, that they are not worth coping with all these difficulties.

Moreover as a second result we may state, that there is no unique GEKS-index for any two periods compared, say 0 and 3. The result for $P_{03}^{GEKS}$ from a series going from 0 to 3 (so that m = 4 periods are involved), that is $P_{03(m=4)}^{GEKS}$ for example, will in general differ from the GEKS index for the same two periods, 0 and 3 when it is calculated from a series going to t = 4 or t =

3 In what follows the first subscript denotes the base, the second the reference period. The superscript denotes the kind of index function. A chain index is distinguished from the corresponding direct index of the same type and for the same interval by a bar. Our notation is a bit different from IFD's notation: IFD use P(i/j) as "level in i relative to j" in the sense of our symbols $P_{ji}$. In IFD's notation we have e.g. the vector P(j) = [P(1/j) P(2/j) … P(15/j)] when 15 months are considered (a preferred application of GEKS in IFD). This would be in our notation P(A) = [$P_{AA}$ $P_{AB}$ $P_{AC}$ …].

4 Note that transitivity implies identity and time reversibility, but the converse is not true (for which $P^F$ may serve as a good example).

5 This is in no small measure quite demanding as far as the data requirements are concerned. Other time reversal indices would be the Törnqvist index $P^T$ - as in the above mentioned CCD method - or the Walsh index ($P^W$). To my knowledge nobody yet studied the formulas we would get with a the GEKS method based on a much simpler (and thus possibly more readily available) index, like Laspeyres $P^L$ which, however, fails the time reversal test.

6 Note that for the chain index it is only necessary to have quantities for the same goods referring only to two adjacent periods at a time. This precisely is what is seen as one of the major advantages of chain indices. With them it is easier to handle the withdrawal of old and entry of new goods. in an index. However, this advantage vanishes in an GEKS index. The GEKS formula, say for $P_{06}$ requires availability of fully comparable quantities $q_0$, $q_1$, …, $q_6$, rendering it much less convenient an index design than the chain index formula.

5 etc. with consequently m = 5 or m = 6 etc. periods involved, so that $P^{GEKS}_{03(m=5)}$ will differ from $P^{GEKS}_{03(m=4)}$, and $P^{EKS}_{03(m=6)}$ will again differ from $P^{EKS}_{03(m=5)}$ as well as $P^{EKS}_{03(m=4)}$ etc.

Furthermore it turns out that the GEKS formulas become ever more complicated the longer the time series from which they are calculated (the greater m is). Of course all these indices for comparing 3 to 0 are equally legitimate and one as well reasoned as the other. Thus the GEKS method fails to provide a unique "drift-free" series of index numbers, unless m is fixed.

In order to overcome such difficulties and work uniformly with a fixed m it became common to combine the GEKS method with a "rolling" device so that the calculation is based on periods 0 to period m-1, then from 1 to m, from 2 to m+1 etc. The properties of such a modification of the GEKS method deserve a careful scrutiny. It does not come as a surprise, for example, that (unlike "standard" GEKS-indices) a series of rolling GEKS indices no longer satisfies transitivity(the property which to guarantee was the very purpose of developing the GEKS method). By adopting a "rolling" procedure the GEKS method also comes closer to a chain index, which is known for its "chain drift" (intransitivity). It may be interesting to note that the usual chain indices are just the limiting case of m = 2 of a rolling GEKS index.

Finally it is known from simulations and empirical studies that GEKS-indices may well yield some awkward and counter-intuitive results (in no small measure resulting from the generally known shortcomings of high-frequent chaining). It is well known in particular, that a chain index may rise or decline beyond limits undulating around a trend when the price movement follows a regular cycle without a trend, however. We will see (and demonstrate in a numerical example) that this is true also for an RGEKS index, when the length m of the window is different from (a multiple of) the length of the cycle.

The structure of this note therefore is as follows: It appears useful to start in sec. 2 with some explanations concerning the notion of "transitivity". The relatively short sec. 3 presents the various quite complicated formulas used to derive the GEKS indices from bilateral Fisher price indices $P^{F}_{0t}$. It may be confusing but the formula of $P^{GEKS}_{0t}$ can be written in many equivalent ways. In sec. 4 we demonstrate in more detail the complexity of the GEKS formulas, and the relations between them. In sec. 5 we take a closer look at the "rolling" methodology, and sec. 6 makes reference to some empirical results, and draws some conclusions.

The sections 2, 4, and 5 are the most important ones. A particularly detailed discussion of the notion of transitivity in sec. 2 may (hopefully) shed some light on the question: do we really need the GEKS approach in the intertemporal framework and why shouldn't the simpler chain indices do the job just as well? And sec. 4 and 5 contain the bulk of our criticism as regards GEKS-indices, and RGEKS-indices respectively.

Hence some details concerning the three sections (sec. 2, 4 and 5) seem to be opportune to be mentioned already right now. Ironically transitivity may be considered more essential a quality in the case of international rather than intertemporal comparisons.[7] This seems to be "ironical" only at first glance, because in the case of m countries there is no natural order, whereas with periods in time 0 precedes 1, which in turn precedes 2 etc. (a fact exploited in the idea of chaining). On the other hand this means that many more reasonable indirect comparisons are possible in the case of countries, when no natural order exists. And this is just what transitivity is about. In essence transitivity requires that *all* sorts of indirect comparisons between any two situations yield the same result; for example between countries say A and B via country

[7] A transitive international comparison between any four countries, say A, B, C, and D, means that the direct parity $P_{AB}$ coincides with all sorts of indirect parities as for example $P_{AC}P_{CB}$ or $P_{AD}P_{DC}P_{CB}$ etc.

C, or via countries D and F. In intertemporal comparisons, however, not all indirect comparisons appear equally meaningful. We may compare t = 0 with t = 3 via periods 1 and 2 (as done in a chain index), but with good reasons we will refrain from comparing 0 to 3 via periods in the future, like 5 or 6. So it is not surprising that transitivity is more important when dealing with countries rather than points (or intervals) in time, and that in the intertemporal framework a "need" for transitivity may well be questioned.[8]

Against this backdrop it is not at all a matter of course to view transitivity desirable in an inter*temporal* context and to consider a method designed for international comparisons to be transferable without modifications into the intertemporal situation.

Hence it appears useful to provide a more detailed analysis of the notion of "transitivity" (as done in sec. 2), because transitivity is the central concern in the GEKS-method. In this context it is also appropriate to recall the sometimes obscured relationship between "chaining" and "chainability" (as a synonym for transitivity). It sounds strange but a chain index is gained from chaining (multiplying links) but it is not "chainable". It is rather path dependent instead, and path dependence is just the very opposite of transitivity.

After presenting the GEKS formulas in sec. 3 we try to make clear in sec. 4 that the formulas (and the relations between them) are indeed quite complicated. They become more and more complicated, the greater the number m of periods taken into account for compiling such an index.[9] GEKS indices are in essence weighted geometric means of a number of Fisher indices relating to periods 0, 1, …, t. Moreover, when a new period t + 1 appears two problems arise

- to update the index: the relationship between $P_{0t}^{GEKS}$ and $P_{0,t+1}^{GEKS}$ is more complicated than with chain indices where only a "link" $P_{t,t+1} = P_{t+1}$ is needed to form $\overline{P}_{0,t+1} = \overline{P}_{0t} \cdot P_{t+1}$: to proceed for example from the series of indices $P_{01}$, $P_{02}$, $P_{03}$, with m = 4 periods (0, 1, 2, 3) to m = 5 periods with a new period t = 4 you need no less than four new direct indices $P_{04}^F$, $P_{14}^F$, $P_{24}^F$, and $P_{34}^F$ to calculate $P_{04(m=5)}$ from $P_{03(m=4)}$ which requires data for no less (in order to update $\overline{P}_{03}^F$ to $\overline{P}_{04}^F$ the only "new" index needed is $P_{34}^F = P_4^F$),
- you should also, as a second task, re-calculate all former indices because $P_{01(m=5)} \neq P_{01(m=4)}$, $P_{02(m=5)} \neq P_{02(m=4)}$ etc.[10] (there is no such re-calculation with direct or chain indices; this is a typical GEKS task; once $\overline{P}_{0t}^F$ is given there is no need to review $\overline{P}_{0,t-1}^F$, $\overline{P}_{0,t-2}^F$ etc. (and the same is true for the direct index $P_{0t}^F$).

It is just this need to re-calculate all previously compiled indices once a new period t + 1 appears, which gave rise to the "rolling" method to be discussed in sec. 5. By "rolling" we get rid of this task but unfortunately also of transitivity, the reason for which GEKS was made. Moreover nothing can be offered in the ways of a theory for the choice of m, the length (width) of the "window". It is hardly more than a convention when IFD recommend a window

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[8] By the way, this explains also why in our view also *country* reversibility (ensuring a unique parity between any two countries) is much better motivated than the famous "*time* reversal test".

[9] We call this number m (by analogy to the number of countries in the GEKS method). With a time series and periods 0, 1, 2, …, t the number m usually is m = t+1.

[10] "…when a new period of data becomes available all of the previous period parities must be recomputed" (IFD, p. 22). Note also that other indices will also change (although they may not be so interesting as $P_{01}$, $P_{02}$, …), for example $P_{13(m=5)} \neq P_{13(m=4)}$. When a new period 6 (in a series going from 0 to 5) is added to the time series, not only all 5 indices $P_{01}$, $P_{02}$, … $P_{05}$ (but also ten other indices like $P_{14}$, or $P_{35}$ etc.) will change. Note that the re-calculation of indices of the past (what we called the "second task") does not apply to chain indices, as there all former indices remain unchanged when a new period becomes available.

of m = 13 months. Given that different choices of m may be made the rolling as well as the standard GEKS method is unable to provide an unequivocal empirical result (by contrast to direct and chain indices). The method rather provides many different results as there are many different m's which to choose would be equally acceptable.

This brings us to the problem of defining the notion of "drift". IFD extensively made use of the concept of "drift" and of a "drift free" index. The term "drift" requires a well defined target track from which the actual track is "drifting" away. "Drift" in the context of chain indices usually means the extent to which a chain index like $\overline{P}_{0t}^{F}$ differs from the corresponding direct index ( $P_{0t}^{F}$ ). So the function $D_{0t}^{PF} = \overline{P}_{0t}^{F}/P_{0t}^{F}$ is known as drift-function for a Fisher price index. However, given that the GEKS method aims at a *transitive* index the direct Fisher index $P_{0t}^{F}$ which is *not* transitive cannot possibly serve as *the* target from which another index (which index?) is said to "drift" away. So "drift" in this case can only be meant as divergence of $\overline{P}_{0t}^{F}$ (and interestingly, now also of $P_{0t}^{F}$ ) from the index $P_{0t}^{GEKS}$ . The problem, however, is – as already mentioned – that there is not *the* unique $P_{0t}^{GEKS}$ but a number of different $P_{0t(m)}^{GEKS}$ indices depending on m, the periods taken into account.

## 2. Transitivity and "chain drift"

Although IFD speak of "chain drift"[11] their problem is a general transitivity issue of indices, irrespective of whether chain indices or direct indices[12] are involved. To exhibit a "drift" in the case of a chain index is only one of several aspects (or better, consequences, of non-transitivity. It is useful to make a distinction between three different situations in comparing indices

| | | first index | |
|---|---|---|---|
| | | direct | chain |
| second index | direct | 3 | 1 |
| | chain | 1 | 2 |

The common feature of all three variants of the same underlying intransitivity problem is that there is only one way of comparing two things directly[13] but as a rule there are many ways to compare them indirectly (via some third thing and by multiplying) and intransitivity (or "drift" if you like) is given when any one of the many indirect comparisons (no matter which one) diverges from the unique direct one.

Transitivity in this general definition covers the following three phenomena

1. *Drift as divergence of a chain index from a direct index*: the term "chain drift" is in general used in the sense that a chain-index $\overline{P}_{0t} = P_{01}P_{12}...P_{t-1,t}$ (or simply $\overline{P}_{0t} = P_1P_2...P_t$ ) is drifting away from its corresponding direct index $P_{0t}$ such that

[11] The term "chain drift" (instead of in-transitivity) seems to be misleading and seems to be used in the GEKS-literature simply as a synonym for intransitivity. Initially I thought the topic of the IFD-Paper would be about chain indices. This came as a surprise, because the GEKS method was not devised for chain indices but to remedy the in-transitivity of *direct* (not chained) Fisher indices.

[12] A "direct" index $P_{0t}$ is comparing two periods, 0 and t directly, using solely prices and quantities of these two periods and not also of intermediate periods, 1, 2, … , t-1 as this is usually done in the case of chain indices.

[13] We are disregarding for the moment the problem that a choice has to be made among a great variety of formulas (for example of Fisher, Laspeyres, Paasche etc.) for direct indices.

(1) $\overline{P}_{0t} = P_1P_2...P_t \neq P_{0t}$ (the term $DP_{0t} = \overline{P}_{0t}/P_{0t}$ may be called "drift" of the price index in question, and this definition may apply to all sorts of chain indices, e.g. for $\overline{P}^F_{0t}$ as compared to $P^F_{0t}$). As a consequence in particular the chain index can well fail the multiperiod identity (or circular) test $\overline{P}_{00} \neq 1$ (while the direct index satisfies, as $P_{00} = 1$).[14]

2. *Comparison of chain indices that differ in terms of partitioning of the interval and frequency of chaining:* transitivity requires the result for an interval [0,t] to be the same no matter how it is subdivided (partitioned) into two [0,s], [s,t],[15] three [0,r], [r,s], [s,t], or more sub-intervals, that is

(1) $P_{0t} = P_{0s}P_{st} = P_{0r}P_{rts}P_{st}$ (for *all* values of r, s,…).

The word "all" here cannot be emphasized too much.

A distinction now should be made between

2a. *The same interval but different frequencies (of updating)*: It is easy to see that e.g. the Laspeyres index (t = 4) is not transitive in the following sense

(2) $$\overline{P}^L_{04} = \frac{\sum p_1q_0}{\sum p_0q_0}\frac{\sum p_2q_1}{\sum p_1q_1}\frac{\sum p_3q_2}{\sum p_2q_2}\frac{\sum p_4q_3}{\sum p_3q_3} \neq \overline{P}^{L*}_{04} = \frac{\sum p_2q_0}{\sum p_0q_0}\frac{\sum p_4q_2}{\sum p_2q_2}.$$

This will be demonstrated in a numerical example in the appendix. Note also that both chain indices $\overline{P}^L_{04}$, and $\overline{P}^{L*}_{04}$ will in general differ (or "drift away") from the direct index $P^L_{04} = \sum p_4q_0 / \sum p_0q_0$ (a fact that refers back to point 1).

Interestingly it will be shown that, what we observed regarding $\overline{P}^L_{04}$, and $\overline{P}^{L*}_{04}$ as chain indices applies also to GEKS indices

Another situation is

2b. *The same frequency (of updating) but different partitions of the interval*: if the kind of chaining is the same (i.e. they are *uniformly* annually or biannually chained and use is made of the same formula for the links) then *any*[16] type of chain index fulfills

(3) $\overline{P}_{0t} = \overline{P}_{0s}\overline{P}_{st} = \overline{P}_{0r}\overline{P}_{rt}$ when $\overline{P}_{0s} = P_{01}P_{12}...P_{s-1,s}$, and $\overline{P}_{st} = P_{s,s+1}P_{s+1,s+2}...P_{t-1,t}$

"by construction". This should be kept distinct from the "chain drift" (discussed above under 1), that is the fact, that a chain index $\overline{P}_{0t}$ is in general not "chainable", although gained by multiplying (also known as "chaining" or "chainlinking").

3. *Intransitivity* also arises *when only direct indices are involved*. This may better be demonstrated in the interspatial (e.g. international) case with countries A, B, C, … For example a (direct) Fisher index comparing two countries, A and B is given by

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[14] When for all prices $p_{i0} = p_{it}$ applies, i = 1, … , n, then the index should yield $P_{0t} = 1$. Some authors define "chain drift" as violation of identity. It should be kept in mind that a chain index may violate identity (or more general proportionality) although its "links" (link indices) $P_{t-1,t}$ are index functions that are able to meet these criteria.

[15] The original notion of the "intercalation" criterion in index numbers (as "transitivity" formerly was called) was the idea that the two figures for the two half-year results should be consistent with the result for the full year. We may say then that s "intercalates" the interval [0,t] to form the two sub-intervals [0,s], [s,t]. Likewise the twelve figures for months and the four figures for quarters should be consistent with the result for the year as a whole.

[16] Therefore no superscript L (for Laspeyres) is involved in eq. 3.

$$P^F_{AB} = \sqrt{\frac{\mathbf{p}_B'\mathbf{q}_B}{\mathbf{p}_A'\mathbf{q}_A}}\sqrt{\frac{\mathbf{p}_B'\mathbf{q}_A}{\mathbf{p}_A'\mathbf{q}_B}} = \sqrt{V_{AB}}\sqrt{\frac{\mathbf{p}_B'\mathbf{q}_A}{\mathbf{p}_A'\mathbf{q}_B}},$$

where vector notation is used, and $V_{AB}$ is the "value ratio". Multiplied (or "chained") indirect comparisons via C as a "third country" will give a different results

$$\hat{P}^F_{AB(C)} = P^F_{AC}P^F_{CB} = \sqrt{\frac{(\mathbf{p}_C'\mathbf{q}_A)\cdot(\mathbf{p}_B'\mathbf{q}_C)\cdot(\mathbf{p}_B'\mathbf{q}_B)}{(\mathbf{p}_A'\mathbf{q}_A)\cdot(\mathbf{p}_A'\mathbf{q}_C)\cdot(\mathbf{p}_C'\mathbf{q}_B)}} = \sqrt{V_{AB}}\sqrt{\frac{(\mathbf{p}_C'\mathbf{q}_A)\cdot(\mathbf{p}_B'\mathbf{q}_C)}{(\mathbf{p}_A'\mathbf{q}_C)\cdot(\mathbf{p}_C'\mathbf{q}_B)}} \neq P^F_{AB},$$

such that (1) is violated because here $P^F_{AB} \neq P^F_{AC}P^F_{CB}$. In a similar vein we have

$$\hat{P}^F_{AB(D)} = P^F_{AD}P^F_{DB} = \sqrt{V_{AB}}\sqrt{\frac{(\mathbf{p}_D'\mathbf{q}_A)\cdot(\mathbf{p}_B'\mathbf{q}_D)}{(\mathbf{p}_A'\mathbf{q}_D)\cdot(\mathbf{p}_D'\mathbf{q}_B)}} \neq P^F_{AB}, \text{ and } \hat{P}^F_{AB(C,D)} = P^F_{AC}P^F_{CD}P^F_{DB} \neq P^F_{AB}.$$

So (direct) Fisher parities are clearly not transitive.[17] And it is this problem for which the GEKS method is designed.[18]

Point 3 is, however, not totally unrelated to situation 1 where we rightly speak of "chain drift". The reason is that the third concept clearly is the most general one of the three. The problems (1) and (2) would not occur with a transitive link indices (factors) in the chain (the product of such factors). For example with a Lowe index $P^{Lo}$ as link, we get

$$\overline{P}^{Lo}_{04} = \frac{\sum p_1 q}{\sum p_0 q}\frac{\sum p_2 q}{\sum p_1 q}\cdots\frac{\sum p_4 q}{\sum p_3 q} = \overline{P}^{Lo*}_{04} = \frac{\sum p_2 q}{\sum p_0 q}\frac{\sum p_4 q}{\sum p_2 q} = P^{Lo}_{04} = \frac{\sum p_4 q}{\sum p_0 q}$$ instead of (2), and the same is true with the transitive unweighted index $P^J_{0t} = \sqrt[n]{\prod \frac{p_{it}}{p_{i0}}}$ of Jevons.

It should be noted, that transitivity is tantamount to imposing very restrictive conditions on the matrix[19] $\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & \cdots & P_{0t} \\ P_{10} & P_{11} & \cdots & P_{1t} \\ \vdots & \vdots & \ddots & \vdots \\ P_{t0} & P_{t1} & \cdots & P_{tt} \end{bmatrix}$, to be more precise: they require **P** to be singular (so that $|\mathbf{P}| = 0$),[20] which brings us quite naturally to the question: Are these conditions possibly unnecessarily restrictive? I think they are indeed.

In the interspatial case (countries A, B, … ) it appears legitimate to consider *all* possible indirect comparisons equally important. It may be difficult if not impossible to decide that comparing France with Spain via Italy (so that $P_{FS} = P_{FI}P_{IS}$ holds) might be more reasonable than comparing France with Spain via Greece (so that $P_{FS} \neq P_{FG}P_{GS}$ may well be tolerable if only $P_{FS} = P_{FI}P_{IS}$ holds).[21] By contrast in the intertemporal case we have good reasons to view some

[17] *Transitivity* (concerning multinational comparisons) requires that *all* indirect comparisons between any two countries, A and B obtained by using other countries, like C as link should be equal to the direct index.

[18] IFD unfortunately spoke of "chain drift". I call this "unfortunately" because in this case (that is the EKS or GEKS method to make international comparisons transitive) originally no chain index was involved.

[19] Cf. von der Lippe (2007), p. 76 (I owe this insight concerning the matrix **P** to Pfouts).

[20] It can easily be seen that this is true if $P_{ij} = P_{ik}P_{kj}$ and therefore also $P_{ii} = 1$ and $P_{ij} = 1/P_{ji}$.

[21] With real indices (as opposed to the indices in **P** where $|\mathbf{P}| = 0$) this may well be possible, because it makes a difference whether we take an Italian or a Greek "basket" of consumer goods. So transitivity is in fact very (or unduly) restrictive.

indirect comparisons more important than other ones. For example we may argue that it seems reasonable to compare 2013 to 2010 indirectly only in one manner, viz. via 2011 and 2012 as done in chain indices, and that all other possible indirect comparisons, e.g. via 1868 and 2018 may be pointless. Thus, as aforesaid the intertemporal situation is different from the international one in that *only some* specific indirect comparisons, *but not all* need to be consistent.

In view of the reduced number of meaningful indirect comparisons in the intertemporal case one may ask: Is there an index function P for which holds $P_{0t} = P_{0s}P_{st}$ but at the same time *not* $P_{0t} = P_{0r}P_{rt}$ (so that $P_{0t} \neq P_{0r}P_{rt}$), so that some, but not all indirect comparisons are consistent in the sense of yielding the same result. To my knowledge the answer can only be no. Either all possible indirect comparisons are consistent and coincide with the direct index, or they are not. We only have index functions which are "full fledged" transitive or intransitive, there is no such thing as partial transitivity in between. One single inequality, as for example $P^F_{AD}P^F_{DB} \neq P^F_{AB}$ (and thus the existence of a drift $\hat{P}^F_{AB(D)} \big/ P^F_{AB} \neq 1$ where $\hat{P}^F_{AB(D)} = P^F_{AD}P^F_{DB}$) is sufficient to consider the index function (Fisher in this case) intransitive altogether. In this situation we can make a choice among two strategies to avoid ambiguity:

1. We may require "full" transitivity, which clearly seems more desirable in the interspatial case than in the intertemporal, or
2. we decide to make things unequivocal despite intransitivity, by simply declaring one (and only one) indirect comparison as the only "legitimate" one, for example $P_{06} = P_{01}P_{12} \dots P_{56}$ with the consequence of discarding all other possible indirect comparisons (e.g. $P_{05}P_{52}P_{26}$) as well as the (only) direct comparison (i.e. the direct index), that is we regard $P_{01}P_{12} \dots P_{56}$ as *the* "correct" index rather than $P_{06}$.

The second strategy amounts to the chain index approach in the intertemporal framework, and to the minimum spanning tree (MST) method in international comparisons. The selection of a unique sequence of indirect comparisons is based on chronology (providing a uniquely determined sequence of equally spaced adjacent intervals) in the case of chain indices, or on the similarity of weight-structures (e.g. consumption patterns in a CPI) in the MST case. The MST can be viewed as the "international" counterpart of the "intertemporal" chain index. Though the transitivity problem is not really solved (which appears acceptable as no "full" transitivity is required) by such a method, we at least got rid of the (from intransitivity) ensuing ambiguity, and this may be good enough as a result.

In intertemporal comparisons a sequence of adjacent and equally long intervals is clearly a more "natural" choice than any other indirect comparison. So chain indices cover the only practically important type of indirect comparisons. They are much simpler to implement than GEKS indices and this may well be more important a criterion than the fact that with GEKS indices some additional (and less relevant) indirect comparisons can consistently be made. In my view a sensible position regarding GEKS indices therefore is: search for their (preferably) *significant* advantages over chain indices and if you can't find them be satisfied with the much simpler chain indices.[22]

[22] In the case, they are found unsatisfactory; it might perhaps be a better idea to strive at a "pure" comparison with an appropriate direct index than to follow the RGEKS methodology.

## 3. Formulas for the GEKS index

### 3.1. A host of formulas for the same thing

Transitive GEKS parities are gained from averaging over all binary comparisons as regards prices or quantities respectively of m countries to be compared (GEKS and related methods like CCD therefore may be viewed as "generalizations of binary comparisons" (Balk). The problem of GEKS indices is firstly, that many apparently different and complicated formulas exist for them which in actual fact amount to the same and secondly that the derivation of the GEKS-parity by minimising the distance $\min_{P_1,\ldots,P_m} \sum_i \sum_k \left[\ln\left(P_{ik}^F\right) - \ln\left(P_k / P_i\right)\right]^2$ is rarely spelled out in detail,[23] so that the interpretation and rationale of GEKS might not be easily and well understood.

With m countries the GEKS parity between k (base country) and j (comparison country) is given by

$$(4) \qquad P_{kj}^{GEKS} = \left( \frac{P_{1j}^F}{P_{1k}^F} \frac{P_{2j}^F}{P_{2k}^F} \cdots \frac{P_{mj}^F}{P_{mk}^F} \right)^{1/m} = \left( \prod_i \frac{P_{ij}^F}{P_{ik}^F} \right)^{1/m},$$

which is due to the country reversal test[24] $P_{BA}^F = 1/P_{AB}^F$ equivalent to

$$(4a) \qquad P_{kj}^{GEKS} = \left( \prod_i P_{ki}^F P_{ij}^F \right)^{1/m} \text{ and}$$

$$(4b) \qquad P_{kj}^{GEKS} = \left( \prod_i P_{ki}^F \cdot \prod_i P_{ij}^F \right)^{1/m}.$$

We demonstrate this with m = 3 countries A, B, and C (k = A, j = C); then (4a) is

$$(4a^*) \qquad P_{AC}^{GEKS} = \left[ \underbrace{P_{AA}^F P_{AC}^F}_{i=A} \underbrace{P_{AB}^F P_{BC}^F}_{i=B} \underbrace{P_{AC}^F P_{CC}^F}_{i=C} \right]^{1/3} = \sqrt[3]{\left(P_{AC}^F\right)^2 P_{AB}^F P_{BC}^F}, \text{ and (4b) is}$$

$$(4b^*) \qquad P_{AC}^{GEKS} = \left[ \left(P_{AA}^F P_{AB}^F P_{AC}^F\right)\left(P_{AC}^F P_{BC}^F P_{CC}^F\right) \right]^{1/3} = \sqrt[3]{\left(P_{AC}^F\right)^2 P_{AB}^F P_{BC}^F}.$$

Because of the time reversibility (4) can also be written as

$$(4c) \qquad P_{kj}^{GEKS} = \left( \frac{P_{k1}^F}{P_{j1}^F} \frac{P_{k2}^F}{P_{j2}^F} \cdots \frac{P_{km}^F}{P_{jm}^F} \right)^{1/m},$$

and in the three countries example a presentation of the $P^{GEKS}$ formula analogous to (4) is

$$P_{AC}^{GEKS} = \left( \frac{P_{AC}^F}{P_{AA}^F} \frac{P_{BC}^F}{P_{BA}^F} \frac{P_{CC}^F}{P_{CA}^F} \right)^{1/3}, \text{ and to (4c) respectively } P_{AC}^{GEKS} = \left( \frac{P_{AA}^F}{P_{CA}^F} \frac{P_{AB}^F}{P_{CB}^F} \frac{P_{AC}^F}{P_{CC}^F} \right)^{1/3}.$$

Translated into the intertemporal context the equivalent of eq. 4b is

$$(5) \qquad P_{0T}^{GEKS} = \left[ \left(P_{0T}^F\right)^2 \prod_{t \neq 0} P_{0t}^F \prod_{t \neq T} P_{tT}^F \right]^{1/m},$$

[23] See for example von der Lippe (2007), p. 555.

[24] This is the interspatial counterpart to the time reversal test.

which appears to be quite convenient (though, as shown above, there exist many other equivalent formulas). So for example with three periods 0 and 2) we have (m = 3)

(5a) $P_{02}^{GEKS} = \sqrt[3]{\left(P_{02}^{F}\right)^{2} P_{01}^{F} P_{12}^{F}}$ ,

derived from (4b) $\sqrt[m]{\prod_i P_{ki}^{F} \prod_i P_{ij}^{F}} = \sqrt[3]{P_{00}^{F} P_{01}^{F} P_{02}^{F} \cdot P_{02}^{F} P_{12}^{F} P_{22}^{F}}$ which simplifies to (5a) as $P_{kk} = 1$. Another quite interesting way to present the formulas for GEKS indices can be found in tab. 2 below.

### 3.2. What makes the indices transitive?

It now also can easily be verified that the GEKS index meets transitivity[25] Taking three periods, 0, 1 and 2 we get $P_{02}^{GEKS} = \sqrt[3]{\left(P_{02}^{F}\right)^{2} P_{01}^{F} P_{12}^{F}}$ and

(5b) $P_{01}^{GEKS} P_{12}^{GEKS} = \sqrt[3]{\left(P_{01}^{F}\right)^{2} P_{02}^{F} P_{21}^{F}} \sqrt[3]{\left(P_{12}^{F}\right)^{2} P_{10}^{F} P_{02}^{F}}$ ,

the RHS of this eq. of course is $\sqrt[3]{\left(P_{01}^{F}\right)^{2} P_{02}^{F} \frac{1}{P_{12}^{F}} \left(P_{12}^{F}\right)^{2} \frac{1}{P_{01}^{F}} P_{02}^{F}}$ which in fact is equal to $P_{02}^{GEKS}$.

Note that the GEKS indices differ from the corresponding direct and chain indices. For example $P_{02}^{GEKS} = \sqrt[3]{\left(P_{02}^{F}\right)^{2} P_{01}^{F} P_{12}^{F}} = \sqrt[3]{\left(P_{02}^{F}\right)^{2} \overline{P}_{02}^{F}}$ differs from both, $P_{02}^{F}$ and $\overline{P}_{02}^{F} = P_{01}^{F} P_{12}^{F}$.

Eqs. 4b and 5b also explain why GEKS indices are transitive. It is well known that an index $P_{0t}$ is transitive when it can be written as a ratio of some sort of absolute levels, $P_{0t} = \Pi_t/\Pi_0$. So we have from (5b) $P_{02(m=3)}^{GEKS} = \sqrt[3]{\left(P_{02}^{F}\right)^{2} P_{01}^{F} P_{12}^{F}} = \frac{\sqrt[3]{P_{02}^{F} P_{12}^{F} P_{22}^{F}}}{\sqrt[3]{P_{00}^{F} P_{10}^{F} P_{20}^{F}}} = \frac{\Pi_{2(m=3)}}{\Pi_{0(m=3)}}$, and in the same manner we can write $P_{01(m=3)}^{GEKS} = \frac{\sqrt[3]{P_{01}^{F} P_{11}^{F} P_{21}^{F}}}{\sqrt[3]{P_{00}^{F} P_{10}^{F} P_{20}^{F}}} = \frac{\Pi_{1(m=3)}}{\Pi_{0(m=3)}}$, and $P_{12(m=3)}^{GEKS} = \frac{\sqrt[3]{P_{02}^{F} P_{12}^{F} P_{22}^{F}}}{\sqrt[3]{P_{01}^{F} P_{11}^{F} P_{21}^{F}}} = \frac{\Pi_{2(m=3)}}{\Pi_{1(m=3)}}$, so that numerator and denominator can be viewed as geometric means (or price levels). Now it can easily be verified that $P_{02(m=3)}^{GEKS} = P_{01(m=3)}^{GEKS} P_{12(m=3)}^{GEKS}$. For m > 3 the GEKS indices may of course also be viewed as ratios of price levels as follows: $P_{01(m=4)}^{GEKS} = \frac{\sqrt[4]{P_{01}^{F} P_{11}^{F} P_{21}^{F} P_{31}^{F}}}{\sqrt[4]{P_{00}^{F} P_{10}^{F} P_{20}^{F} P_{30}^{F}}} = \frac{\Pi_{1(m=4)}}{\Pi_{0(m=4)}}$, and $P_{01(m=5)}^{GEKS}$ written as (in analogy to (4b)) as product

(6) $P_{01(m=5)}^{GEKS} = \sqrt[5]{P_{00}^{F} P_{01}^{F} P_{02}^{F} P_{03}^{F} P_{04}^{F}} \cdot \sqrt[5]{P_{01}^{F} P_{11}^{F} P_{21}^{F} P_{31}^{F} P_{41}^{F}} = \left(\Pi_{0(m=5)}\right)^{-1} \Pi_{1(m=5)}$.

### 3.3. GEKS and chain indices

With m = 2 the index $P_{01(m=2)}^{GEKS} = P_{01}^{F}$ can also be viewed as ratio of price levels $\Pi_{1(m=2)} / \Pi_{0(m=2)}$ which are defined as $\frac{\Pi_{1(m=2)}}{\Pi_{0(m=2)}} = \frac{\sqrt[2]{P_{01}^{F} P_{11}^{F}}}{\sqrt[2]{P_{00}^{F} P_{10}^{F}}} = P_{01(m=2)}^{GEKS} = P_{01}^{F}$ in line with a general definition of the "level" $\Pi_{k(m)} = \sqrt[m]{P_{0k}^{F} P_{1k}^{F} P_{2k}^{F} \ldots P_{m-1,k}^{F}}$. That chain indices may be seen as a limiting case of GEKS

[25] GEKS parities also have some other useful properties. They pass the factor reversal test. which can, however, no longer be assumed if $P^{F}$ is replaced by another index function such as Törnquist $P^{T}$ (in the CCD method).

indices will yet become clearer when we look at the rolling GEKS method. With m = 2 we cannot express $P_{02(m=2)}^{GEKS}$ covering more than two periods, viz. 0, 1, and 2. This is where the rolling approach has to take place by providing the estimate $\hat{P}_{02(m=2)}^{GEKS} = P_{01(m=2)}^{GEKS} L_{12}$, with a "link" (or link index, because it is again a GEKS index) $L_{12}$ as will be seen later. Hence the usual chain index is simply a special variant of the RGEKS index.

### 3.4. The ambiguous notion of "drift"

While the notion of transitivity is quite clear (because there is only one direct index), this does not apply, however, to the concept of "drift" or "chain drift", a term unfortunately sometimes used synonymously to transitivity. "Drift" requires the decision to select a series as the relevant "drift free" or "target" series, and there is more than just one option.

Note also that $\overline{P}_{02}^{F} \neq P_{02}^{F}$ means that the chain index has a drift (measured against the direct index $P_{02}^{F}$). Here $P_{02}^{F}$ serves as target. But this cannot be satisfactory, because $P_{02}^{GEKS}$ also differs from $P_{02}^{F}$, and thus has a drift too (*when measured against the direct index*[26]). Thus with GEKS-indices divergence from the corresponding direct index cannot be the criterion. On the other hand as there are many values of m possible we may also define many GEKS indices as "drift free". There does not seem to be *the* "target" index against which a "drift" is defined.

Can we circumvent the problem to define a definite "target" by saying that $P_{02}^{GEKS}$ is "drift free" because $P_{02}^{GEKS} = P_{01}^{GEKS} P_{12}^{GEKS}$? This cannot be a solution for the simple reason that the much simpler chain index formula also satisfies $\overline{P}_{02}^{F} = \overline{P}_{01}^{F} \overline{P}_{12}^{F} = P_{1}^{F} P_{2}^{F}$, the equivalent equation.[27] Transitivity the outstanding feature of GEKS indices implies more than just this kind of indirect comparison between 0 an 2. But is this an advantage? Are the potentially feasible additional indirect comparisons relevant and worth being considered in practice? Moreover transitivity of GEKS indices is only given for indices with a common m. For example

$$P_{02(m=4)}^{GEKS} = \sqrt[4]{\left(P_{02}^{F}\right)^{2} P_{01}^{F} P_{12}^{F} P_{03}^{F} P_{32}^{F}} \neq P_{01(m=3)}^{GEKS} P_{12(m=3)}^{GEKS} = \sqrt[3]{\left(P_{01}^{F} P_{12}^{F}\right)^{2} P_{02}^{F} P_{21}^{F} P_{13}^{F} P_{32}^{F}} .$$

To sum up: Much like chain indices also GEKS-indices in the last analysis basically combine a number of binary indices. By contrast to chain indices, however, the GEKS-method gives us many different formulas which yet amount to the same formula, and which are much more complicated than the corresponding chain index formulas and which need for their compilation also direct indices referring to periods possibly wide apart and not just adjacent only.

For those who advocate GEKS-indices this may be well acceptable because such indices are designed to provide transitive, or "drift free" index numbers. It remains, however, a problem to define "drift free" and it should be noted that "standard" GEKS indices, are transitive only within a system of GEKS-indices of a given number m of periods taken into account in forming the GEKS-indices, and that this does not even apply to the so called "rolling" method, that is to RGEKS indices.

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26 It should be borne in mind that most of what is conceived as index theory (for example in the ways of utility maximization on a given preference function etc.) is aimed at a *direct* index $P_{0t}$ comparing 0 and t.

27 It is not easy to see why GEKS indices should be preferred over chain indices, when only specific (chain-type) indirect temporal comparisons are taken into consideration. The "advantages" of the GEKS approach become apparent only when an index, say $P_{08}$ is considered as generated by "links" like $P_{74}$ or $P_{36}$ etc. rather than links referring to adjacent periods $P_{01}P_{02}\ldots P_{78}$ only.

## 4. The (standard) GEKS-formulas are complicated and they depend on m

The GEKS index consists of 2m – 3 direct Fisher indices when m periods are involved, and therefore of 2(2m-3) ratios of sums of price-quantity products, a number which is rapidly growing. In the case of m = 15 (as it is preferred by IFD) we have to provide 27 indices and this means no less than 54 ratios with different price and quantity vectors for each period (eg. month) for which the index is to be compiled. As m increases the GEKS index will become ever more complicated and as already mentioned, "…when a new period of data becomes available all of the previous period parities must be recomputed" (IFD, p. 22).

### 4.1. Complexity of the GEKS-formulas

We begin with demonstrating how difficult GEKS indices are to implement in that they require combining quite a few direct Fisher price indices. It may be useful to present an example in which all price and quantity vectors are listed that enter a GEKS formula. For example with m = 6 the GEKS index to measure a price level $P_{06}$ is given by

$$\text{(7)} \qquad P^{GEKS}_{02(m=6)} = \sqrt[6]{\left(P^F_{02}\right)^2 P^F_{01}P^F_{12}P^F_{03}P^F_{32}P^F_{04}P^F_{42}P^F_{05}P^F_{52}}$$

It is built with 2m – 3 = 12 - 3 = 9 direct Fisher price indices as building blocs which in turn are made of 18 indices so that eq. 6 can be written with all 18 ratios of sums of products as follows

$$\text{(7a)} \qquad P^{GEKS}_{02(m=6)} = \sqrt[6]{\frac{\mathbf{p}_2'\mathbf{q}_0}{\mathbf{p}_0'\mathbf{q}_0}\frac{\mathbf{p}_2'\mathbf{q}_2}{\mathbf{p}_0'\mathbf{q}_2}\sqrt{\frac{\mathbf{p}_1'\mathbf{q}_0}{\mathbf{p}_0'\mathbf{q}_0}\frac{\mathbf{p}_1'\mathbf{q}_1}{\mathbf{p}_0'\mathbf{q}_1}\frac{\mathbf{p}_2'\mathbf{q}_1}{\mathbf{p}_1'\mathbf{q}_1}\frac{\mathbf{p}_2'\mathbf{q}_2}{\mathbf{p}_1'\mathbf{q}_2}\cdots\frac{\mathbf{p}_5'\mathbf{q}_0}{\mathbf{p}_0'\mathbf{q}_0}\frac{\mathbf{p}_5'\mathbf{q}_5}{\mathbf{p}_0'\mathbf{q}_5}\frac{\mathbf{p}_2'\mathbf{q}_5}{\mathbf{p}_5'\mathbf{q}_5}\frac{\mathbf{p}_2'\mathbf{q}_2}{\mathbf{p}_5'\mathbf{q}_2}}}$$

$$= \sqrt[6]{\left(P^F_{02}\right)^2 P^F_{01}P^F_{12}\ldots P^F_{05}P^F_{52}}$$

by contrast to $P^F_{02} = \sqrt{\frac{\mathbf{p}_2'\mathbf{q}_0}{\mathbf{p}_0'\mathbf{q}_0}\frac{\mathbf{p}_2'\mathbf{q}_2}{\mathbf{p}_0'\mathbf{q}_2}}$ and $\overline{P}^F_{02} = \sqrt{\frac{\mathbf{p}_1'\mathbf{q}_0}{\mathbf{p}_0'\mathbf{q}_0}\frac{\mathbf{p}_1'\mathbf{q}_1}{\mathbf{p}_0'\mathbf{q}_1}\frac{\mathbf{p}_2'\mathbf{q}_1}{\mathbf{p}_1'\mathbf{q}_1}\frac{\mathbf{p}_2'\mathbf{q}_2}{\mathbf{p}_1'\mathbf{q}_2}}$.

In the case of m = 15 we have 2m – 3 = 27 direct Fisher indices and no less than 54 ratios of sums of products. The formula 2m – 3 suggests that the transition from m to m + 1 only requires two additional indices because 2 = [2(m+1) – 3] – (2m – 3). With m = 7 instead of m = 6 we only have the two new factors $P^F_{06}$ and $P^F_{62}$ in

$$\text{(7b)} \qquad P^{EKS}_{02(m=7)} = \sqrt[7]{\left(P^F_{02}\right)^2 P^F_{01}P^F_{12}P^F_{03}P^F_{32}P^F_{04}P^F_{42}P^F_{05}P^F_{52}P^F_{06}P^F_{62}}$$

compared to (7). However, when we move from $P^{GEKS}_{02(m=6)}$ to $P^{GEKS}_{03(m=6)}$ there are some (more precisely m = 6) more "new" indices in the root $P^{EKS}_{03(m=6)} = \sqrt[6]{\left(P^F_{03}\right)^2 P^F_{01}P^F_{13}P^F_{02}P^F_{23}P^F_{04}P^F_{43}P^F_{05}P^F_{53}}$, viz. $P^F_{03}$, $P^F_{13}$. $P^F_{02}$, $P^F_{23}$, $P^F_{43}$, and $P^F_{53}$ .

More interesting is of course to get an idea of the extra information needed for a GEKS index compared to the direct and the chained Fisher index ($P^F_{0t}$ and $\overline{P}^F_{0t}$ respectively). Tab. 1 shows that the number of additional indices required by $P^{GEKS}$ depends of course on m, and we need two more indices when m increases by one. By contrast in all cases we have the same index $P^F_{02}$ (direct) and $\overline{P}^F_{02} = P^F_{01}P^F_{12}$ (chain index), and these are indices which are among other indices also included in the formula of $P^{GEKS}_{02(m)}$.

Table 1

| | $P_{02(m)}$ as GEKS index | additional indices compared to | |
|---|---|---|---|
| | | direct Fisher | chain Fisher |
| (1) | (2) | (3) | (4) |
| m= 3 periods 0, 1, 2 | $P^{EKS}_{02(3)} = \sqrt[3]{\left(P^F_{02}\right)^2 P^F_{01}P^F_{12}}$ | two: $P^F_{01}, P^F_{12}$ | one: $P^F_{02}$ |
| m= 4 periods 0, 1, 2, 3 | $P^{EKS}_{02(4)} = \sqrt[4]{\left(P^F_{02}\right)^2 P^F_{01}P^F_{12}P^F_{03}P^F_{32}}$ | four: $P^F_{01}, P^F_{12}, P^F_{03}, P^F_{32}$ | three: $P^F_{02}, P^F_{03}, P^F_{32}$ |
| m = 5 periods 0, 1, 2, 3, 4 | $P^{EKS}_{02(5)} = \sqrt[5]{\left(P^F_{02}\right)^2 P^F_{01}P^F_{12}P^F_{03}P^F_{32}P^F_{04}P^F_{42}}$ | $P^F_{01}, P^F_{12}, P^F_{03}, P^F_{32}\ P^F_{04}, P^F_{42}$ | $P^F_{02}, P^F_{03}, P^F_{32}, P^F_{04}, P^F_{42}$ |

The relationship between $P_{0k(m+1)}$ and $P_{0k(m)}$ is quite straightforward: given that t is the additional $(m+1)^{th}$ period we simply have to multiply by $P_{0t}P_{tk}$ in the $(m+1)^{th}$ root.

### 4.2. From $P_{0t}$ to $P_{0,t+1}$

As mentioned already to proceed for example from the index $P_{02}$, calculated with m = 3 periods (0, 1, 2) to $P_{03}$, with m = 4 periods (0, 1, 2, 3) a number of additional indices must be compiled (it is assumed that m grows accordingly by 1 with every new period t+1). Again it is quite obvious that as m increases the GEKS index will become ever more complicated. This is demonstrated in table 2, where the general principle of growing complexity can easily be seen.

Table 2

| | GEKS index (m continually increasing) | direct Fisher | chained Fisher |
|---|---|---|---|
| (1) | (2) | (3) | (4) |
| 1 | $P^{GEKS}_{01} = \sqrt[2]{\left(P^F_{01}\right)^2} = P^F_{01}$ | $P^F_{01}$ | $\bar{P}^F_{01} = P^F_{01}$ |
| 2 | $P^{GEKS}_{02(m=3)} = \sqrt[3]{\left(P^F_{02}\right)^2 P^F_{01}P^F_{12}} = P^F_{02}\sqrt[3]{P^F_{01}P^F_{12}/P^F_{02}} = P^F_{02}\sqrt[3]{\frac{\bar{P}^F_{02}}{P^F_{02}}}$ | $P^F_{02}$ | $\bar{P}^F_{02} = P^F_{01}P^F_{12}$ |
| 3 | $P^{GEKS}_{03(m=4)} = \sqrt[4]{\left(P^F_{03}\right)^2 P^F_{01}P^F_{13}P^F_{02}P^F_{23}} = P^F_{03}\sqrt[4]{\frac{P^F_{01}P^F_{13}}{P^F_{02}}\frac{P^F_{02}P^F_{23}}{P^F_{02}}}$ | $P^F_{03}$ | $\bar{P}^F_{03} = P^F_{01}P^F_{12}P^F_{23}$ |
| 4 | $P^{GEKS}_{04(m=5)} = P^F_{04}\sqrt[5]{\frac{P^F_{01}P^F_{14}}{P^F_{04}}\frac{P^F_{02}P^F_{24}}{P^F_{04}}\frac{P^F_{03}P^F_{34}}{P^F_{04}}}$ | $P^F_{04}$ | $\bar{P}^F_{04} = P^F_{01}P^F_{12}P^F_{23}P^F_{34}$ |

Table 2 ctd.

| $t-1 \to t$ | for $P^{GEKS}_{0t}$ as compared to $P^F_{0t}$ | for $P^{GEKS}_{0t}$ as compared to $\bar{P}^F_{0t}$ |
|---|---|---|
| $1 \to 2$ | two, viz. $P^F_{01}$, $P^F_{12}$ | one only, viz. $P^F_{02}$, |
| $2 \to 3$ | four, $P^F_{01}$, $P^F_{13}$, $P^F_{02}$, | three: $P^F_{03}$, $P^F_{02}$, $P^F_{13}$ |
| $3 \to 4$ | six, $P^F_{01}$, $P^F_{14}$, $P^F_{02}$, $P^F_{24}$, $P^F_{03}$, $P^F_{34}$ | five: $P^F_{04}$, $P^F_{02}$, $P^F_{03}$, $P^F_{14}$, $P^F_{24}$ |
| general $t-1 \to t$ | t-1 of the sort $P^F_{01}$, $P^F_{02}$, …, $P^F_{0,t-1}$,<br>t-1 of the sort $P^F_{1t}$, $P^F_{2t}$, …, $P^F_{t-1,t}$,<br>altogether 2t -2 | one of the type $P^F_{0t}$ plus t-2 of the sort<br>$P^F_{02}$, $P^F_{03}$, …, $P^F_{0,t-1}$, and t-2 of the sort<br>$P^F_{1t}$. $P^F_{2t}$, …, $P^F_{t-2,t}$; in sum: 2t - 3 |

Columns 2 and 4 of tab. 2 show that "updating" of a GEKS index (using a recursive formula) is considerably more complicated than the same updating procedure in the case of a chain index. Tab. 3 again makes this quite clear.

The second part of tab. 2 shows how many additional direct Fisher indices are needed to update a (standard) GEKS index (where m increases when a new period is added to the time series).[28]

Table 3 (Updating of indices)

| standard GEKS | chain index |
|---|---|
| $P_{03(4)}^{GEKS} = \sqrt[4]{\left(P_{02(3)}^{EKS}\right)^3 \cdot \left(P_{03}^{F}\right)^2 \cdot \frac{P_{13}^F P_{23}^F}{P_{01}^F P_{02}^F}}$ | $\overline{P}_{03}^F = \overline{P}_{02}^F P_{23}^F$ |
| $P_{04(5)}^{GEKS} = \sqrt[5]{\left(P_{03(4)}^{EKS}\right)^4 \cdot \left(P_{04}^{F}\right)^2 \cdot \frac{P_{14}^F P_{24}^F P_{34}^F}{P_{03}^F P_{13}^F P_{23}^F}}$ | $\overline{P}_{04}^F = \overline{P}_{03}^F P_{34}^F$ |
| $P_{05(6)}^{EKS} = \sqrt[6]{\left(P_{04(5)}^{EKS}\right)^5 \cdot \left(P_{05}^{F}\right)^2 \cdot \frac{P_{15}^F P_{25}^F P_{35}^F P_{45}^F}{P_{04}^F P_{14}^F P_{24}^F P_{34}^F}}$ | $\overline{P}_{05}^F = \overline{P}_{04}^F P_{45}^F$ |

Hence the extra-requirements (in terms of additional index compilations) are considerable when we proceed from $P_{03}$, to $P_{04}$, etc. with GEKS indices (as opposed to chain indices).

### 4.3. Revision of formerly compiled indices $P_{0k}$ (k < t-1) with a new ($t^{th}$) period

We now come to what we called the second task "…when a new period of data becomes available all of the previous period parities must be recomputed" (IFD, p. 22) which is not necessary with direct and chained Fisher indices and gave rise to suggesting the "rolling" method. Assume we had a series of $P_{0t(5)}^{GEKS}$ indices with m = 5 and t ≤ 4. With t = 5 we now can provide the index $P_{05(6)}^{GEKS}$ which entails, however, that all indices $P_{01(5)}^{GEKS}$, …, $P_{04(5)}^{EKS}$, should be re-worked to get $P_{01(6)}^{EKS}$, …, $P_{04(6)}^{EKS}$. Tab. 4 (overleaf) shows that the relationship between these indices is quite simple, and only requires two more indices.[29]

Table 4 also shows that we have two equally valid series which will as a rule not coincide. In order to have $P_{01(6)}^{GEKS} = P_{01(5)}^{GEKS}$ the following relation must hold

(7) $$P_{05}^F P_{51}^F = \frac{(P_{01(6)}^{GEKS})^6}{(P_{01(5)}^{GEKS})^5}.$$

It is most unlikely that such a relation holds true. In what follows we try to show this by looking at some implications of (7). Assume $P_{01(5)}^{GEKS} = P_{01(6)}^{GEKS} = X$ then eq. 7 amounts to

(7a) $$P_{05}^F P_{51}^F = \frac{X^6}{X^5} = X = P_{01(5)}^{GEKS}.$$

And this in turn means that $P_{05}^F P_{51}^F = P_{01(5)}^{GEKS} = \sqrt[5]{\left(P_{01}^F\right)^2 P_{02}^F P_{21}^F P_{03}^F P_{31}^F P_{04}^F P_{41}^F}$.

In a similar vein $P_{02(6)}^{GEKS} = P_{02(5)}^{GEKS}$ is tantamount to

(7b) $$P_{05}^F P_{52}^F = P_{02(5)}^{GEKS} = \sqrt[5]{\left(P_{02}^F\right)^2 P_{01}^F P_{12}^F P_{03}^F P_{32}^F P_{04}^F P_{42}^F}.$$

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28 By contrast in the RGEKS method where m is fixed there is no need for this sort of updating.

29 It again shows that the number of indices in the root follows the rue 2m – 3 (that is 7 in the case of m = 5) and with m+1 instead of m we have two more factors in the (m+1)th root.

Taking both equations, 7a and 7b together, we have

(8) $P^F_{52}/P^F_{51} = P^{GEKS}_{02(5)}/P^{GEKS}_{01(5)} = \sqrt[5]{\left(P^F_{12}\right)^2 P^F_{10}P^F_{02}P^F_{13}P^F_{32}P^F_{14}P^F_{42}}$ , and correspondingly

(8a) $P^F_{53}/P^F_{52} = P^{GEKS}_{03(5)}/P^{GEKS}_{02(5)} = \sqrt[5]{\left(P^F_{23}\right)^2 P^F_{20}P^F_{03}P^F_{21}P^F_{13}P^F_{24}P^F_{43}}$ etc.

The 5th roots in these equations resemble a bit terms we later - that is in the RGEKS context - will call "links" $L_{12}$, $L_{23}$ etc.

Table 4 (Re-computing of indices $P^{GEKS}_{0t(5)}$ ($t \le 4$) when t = 5)

| t | m = 5 | m = 6 |
|---|---|---|
| 1 | $P^{GEKS}_{01(5)} = \sqrt[5]{\left(P^F_{01}\right)^2 P^F_{02}P^F_{21}P^F_{03}P^F_{31}P^F_{04}P^F_{41}}$ | $P^{GEKS}_{01(6)} = \sqrt[6]{\left(P^F_{01}\right)^2 P^F_{02}P^F_{21}P^F_{03}P^F_{31}P^F_{04}P^F_{41}P^F_{05}P^F_{51}}$ |
| 2 | $P^{GEKS}_{02(5)} = \sqrt[5]{\left(P^F_{02}\right)^2 P^F_{01}P^F_{12}P^F_{03}P^F_{32}P^F_{04}P^F_{42}}$ | $P^{GEKS}_{02(6)} = \sqrt[6]{\left(P^F_{02}\right)^2 P^F_{01}P^F_{12}P^F_{03}P^F_{32}P^F_{04}P^F_{42}P^F_{05}P^F_{52}}$ |
| 3 | $P^{GEKS}_{03(5)} = \sqrt[5]{\left(P^F_{03}\right)^2 P^F_{01}P^F_{13}P^F_{02}P^F_{23}P^F_{04}P^F_{43}}$ | $P^{GEKS}_{03(6)} = \sqrt[6]{\left(P^F_{03}\right)^2 P^F_{01}P^F_{13}P^F_{02}P^F_{23}P^F_{04}P^F_{43}P^F_{05}P^F_{53}}$ |
| 4 | $P^{GEKS}_{04(5)} = \sqrt[5]{\left(P^F_{04}\right)^2 P^F_{01}P^F_{14}P^F_{02}P^F_{24}P^F_{03}P^F_{34}}$ | $P^{GEKS}_{04(6)} = \sqrt[6]{\left(P^F_{04}\right)^2 P^F_{01}P^F_{14}P^F_{02}P^F_{24}P^F_{03}P^F_{34}P^F_{05}P^F_{54}}$ |

While the re-computing does not appear to be very difficult,[30] the conceptually greater problem seems to lie in the fact that the GEKS method creates (depending on the choice of m) a multitude of index series which will, as a rule not coincide, but yet should be viewed as equally valid representations of the same price movement.

For each pair jk of periods (e.g. for j = 0 and k = 2) a number of different GEKS indices exists. The general principle can easily be seen in tab. 5. The recursive formula is

(9) $P^{EKS}_{jk(m+1)} = \sqrt[m+1]{\left(P^F_{jk(m)}\right)^m P^F_{jm}P^F_{mk}}$ .[31]

Table 5

| m | $P_{02(m)}$ as GEKS index | alternatively | |
|---|---|---|---|
| | | direct | chain |
| (1) | (2) | (3) | (4) |
| 3 | $P^{GEKS}_{02(3)} = \sqrt[3]{\left(P^F_{02(2)}\right)^2 P^F_{01}P^F_{12}} = P^F_{02}\sqrt[3]{\frac{P^F_{01}P^F_{12}}{P^F_{02}}} = P^F_{02}\sqrt[3]{\frac{\overline{P}^F_{02}}{P^F_{02}}}$ | $P^F_{02}$ | $\overline{P}^F_{02}$ |
| 4 | $P^{GEKS}_{02(4)} = \sqrt[4]{\left(P^F_{02(3)}\right)^3 P^F_{03}P^F_{32}} = P^F_{02}\sqrt[4]{\frac{P^F_{01}P^F_{12}}{P^F_{02}}\frac{P^F_{03}P^F_{32}}{P^F_{02}}}$ | $P^F_{02}$ | $\overline{P}^F_{02}$ |
| 5 | $P^{GEKS}_{02(5)} = \sqrt[5]{\left(P^F_{02(4)}\right)^4 P^F_{04}P^F_{42}} = P^F_{02}\sqrt[5]{\frac{P^F_{01}P^F_{12}}{P^F_{02}}\frac{P^F_{03}P^F_{32}}{P^F_{02}}\frac{P^F_{04}P^F_{42}}{P^F_{02}}}$ | $P^F_{02}$ | $\overline{P}^F_{02}$ |

[30] Note that the GEKS-index ought to be set against two index series, that is direct and chain indices which are not affected from a prolongation of the time series (m increasing) and where thus no re-computing is needed.

[31] Thus the relationship between $P_{jk(m+1)}$ and $P_{0k(m)}$ is quite straightforward: given that m is the additional $(m+1)^{th}$ period we simply have to multiply by $P_{jm}P_{mk}$ in the $(m+1)^{th}$ root.

### 4.4. Are series of GEKS indices less volatile than direct or chain indices?

It is not infrequently being conjectured that the time series of GEKS indices will show a smoother course because of their "drift attenuation capacity" (Ribe 2012). To my knowledge no systematic attempt is made to examine the dispersion (as measure of volatility) of the respective indices, that is $P_{0t}^{GEKS}$ by contrast to $P_{0t}^{F}$ and $\overline{P}_{0t}^{F}$. We can only present here some very elementary reflections in this direction. The reduction of the drift (in the sense of $\overline{P}_{0t}^{F}$ drifting away from $P_{0t}^{F}$ ?) is seems to be inferred from the fact that GEKS indices make use of geometric means. However, when for example the indices $P_{01(4)}^{GEKS} = \sqrt[4]{\left(P_{01}^{F}\right)^2 P_{02}^{F} P_{21}^{F} P_{03}^{F} P_{31}^{F}}$, $P_{02(4)}^{GEKS}$ and $P_{03(4)}^{GEKS}$ are compared to $P_{01}^{F}$, $P_{02}^{F}$, $P_{03}^{F}$ it should be borne in mind that the GEKS indices are not simply geometric means of the terms $P_{01}^{F}$, $P_{02}^{F}$ , and $P_{03}^{F}$ but

$$P_{01(4)}^{GEKS} = \sqrt[4]{\left(P_{01}^{F}\right)^2 P_{02}^{F} P_{03}^{F}} \cdot \sqrt[4]{P_{21}^{F} P_{31}^{F}},\ P_{02(4)}^{GEKS} = \sqrt[4]{\left(P_{02}^{F}\right)^2 P_{01}^{F} P_{03}^{F}} \cdot \sqrt[4]{P_{12}^{F} P_{32}^{F}},\ P_{03(4)}^{GEKS} = \sqrt[4]{\left(P_{03}^{F}\right)^2 P_{01}^{F} P_{02}^{F}} \cdot \sqrt[4]{P_{13}^{F} P_{23}^{F}}.$$

Obviously the product of these three terms is equal to $P_{01}^{F} P_{02}^{F} P_{03}^{F}$ so that the series $P_{0t}^{GEKS}$ and $P_{0t}^{F}$ have the same geometric mean. The product of the terms

$$\left(P_{01(4)}^{GEKS}\right)^2 = \left(P_{01}^{F}\right)^2 \cdot P_{10}^{F} \cdot \sqrt[2]{P_{02}^{F} P_{03}^{F}} \cdot \sqrt[2]{P_{21}^{F} P_{31}^{F}}$$, and $\left(P_{02(4)}^{GEKS}\right)^2$ and $\left(P_{03(4)}^{GEKS}\right)^2$ analogously

amounts to $\left(P_{01}^{F} P_{02}^{F} P_{03}^{F}\right)^2 \cdot P_{10}^{F} P_{20}^{F} P_{30}^{F} \cdot \sqrt{P_{02}^{F} P_{03}^{F} P_{01}^{F} P_{03}^{F} P_{01}^{F} P_{02}^{F}} \cdot \sqrt{P_{21}^{F} P_{31}^{F} P_{12}^{F} P_{32}^{F} P_{13}^{F} P_{21}^{F}} = \left(P_{01}^{F} P_{02}^{F} P_{03}^{F}\right)^2$

because the fourth factor as well as the product of the second and third factor clearly yields unity. So not only the terms $P_{0t}^{GEKS}$ and $P_{0t}^{F}$, but also the squared terms, $\left(P_{0t}^{GEKS}\right)^2$ and $\left(P_{0t}^{F}\right)^2$ respectively have the same geometric mean. Hence it is far from clear that $P_{0t}^{GEKS}$ indices are less volatile than $P_{0t}^{F}$ indices.

We now examine chain indices. Their product is $\overline{P}_{01}^{F} \overline{P}_{02}^{F} \overline{P}_{03}^{F} = \left(P_{01}^{F}\right)^2 P_{12}^{F} \cdot \left(P_{01}^{F} P_{12}^{F} P_{23}^{F}\right) = A \cdot \left(P_{01}^{F} P_{12}^{F} P_{23}^{F}\right)$, which differs from the product of the $P_{0t}^{GEKS}$ and $P_{0t}^{F}$ terms by the first factor (A). In the same manner the product of squared terms $\overline{P}_{01}^{F}, \overline{P}_{02}^{F}, \overline{P}_{03}^{F}$ is given by

$$\left(\left(P_{01}^{F}\right)^2 P_{12}^{F} P_{20}^{F} P_{23}^{F} P_{30}^{F}\right)^2 \cdot \left(P_{01}^{F} P_{12}^{F} P_{23}^{F}\right)^2 = B \cdot \left(P_{01}^{F} P_{12}^{F} P_{23}^{F}\right)^2,$$

and differs from the respective term of the other two series by the factor B. This again does not clearly indicate that the variance of the chain index exceeds the variance of the GEKS indices.

With m = 3 the situation is a bit simpler because with two indices only the variance is given by $\frac{1}{4}\left(P_{01} - P_{02}\right)^2$, and this amounts to $\frac{1}{4}\left(P_{01}^{F} \sqrt[3]{\frac{P_{02}^{F} P_{21}^{F}}{P_{01}^{F}}} - P_{02}^{F} \sqrt[3]{\frac{P_{01}^{F} P_{12}^{F}}{P_{12}^{F}}}\right)^2$ in the case of the GEKS indices and to $\left(\frac{1}{2} P_{01}^{F}\left(1 - P_{12}^{F}\right)\right)^2$ in the case of the chain indices $\overline{P}_{01}^{F} = P_{01}^{F}$ and $\overline{P}_{02}^{F} = P_{01}^{F} P_{12}^{F}$.

To sum up: it appears not so easy to arrive at some general conclusions concerning the relative volatility of GEKS indices as compared to direct or chained Fisher indices. This does not invalidate the assumption that it might be quite likely that "normally" (with "normal" data) a "drift attenuation capacity" in fact exists.

## 5. The "rolling" approach in combination with the GEKS index formulas

The method discussed so far is called "standard GEKS" by IFD. The problem with this method is that with new data previous price indices have to be recomputed. To avoid such revisions IFD recommend a "rolling" (R) or "rolling window" (RW) approach with a fixed predetermined window of length m. Besides the choice of m, the number of periods in the moving "window" of the GEKS-formula,[32] this RGEKS method raises two new questions:

1. how can the sequence of rolling estimates be combined to a seamless chain, and
2. can RWGEKS indices still maintain the advantage of transitivity intended by GEKS indices and are they (in which sense?) "drift-free"?

We start with the first point, that is the method of chain-linking proposed by IFD and we will then discuss the properties of this linking design called RGEKS.[33]

### 5.1. Chain-linking of successive RWGEKS indices

We demonstrate the linking method and its rationale with the simple situation of a small m with only m = 4 (we do so because with greater values for m as for example m = 13 or m = 15 things will become much more complicated). The first window then covers the periods 0, 1, 2, and 3. The formulas for the first four index numbers are given in tab. 6.

How to compute $\hat{P}_{04}^{GEKS}$ ? IFD proposed to use the link

$$(10) \qquad L_{34} = \frac{P_{14}^{GEKS}}{P_{13}^{GEKS}} = \frac{\sqrt[4]{\left(P_{14}^F\right)^2 P_{12}^F P_{24}^F P_{13}^F P_{34}^F}}{\sqrt[4]{\left(P_{13}^F\right)^2 P_{12}^F P_{23}^F P_{14}^F P_{43}^F}} = \sqrt[4]{\left(P_{34}^F\right)^2 P_{14}^F P_{24}^F P_{31}^F P_{32}^F},$$

and multiply $P_{03}^{GEKS}$ by this link, to get

$$(10a) \qquad \hat{P}_{04}^{GEKS} = P_{03}^{GEKS} \cdot L_{34} = \sqrt[4]{\left(P_{03}^F P_{34}^F\right)^2 P_{01}^F P_{02}^F P_{14}^F P_{24}^F}.$$

As tab. 6 shows an alternative to $L_{34}$ defined by (10) the change in the prices from 3 to 4 could also be measured as

$$(10b) \qquad L_{34}^* = \frac{P_{24}^{GEKS}}{P_{23}^{GEKS}} = \frac{\sqrt[4]{\left(P_{24}^F\right)^2 P_{23}^F P_{34}^F P_{25}^F P_{54}^F}}{\sqrt[4]{\left(P_{23}^F\right)^2 P_{24}^F P_{43}^F P_{25}^F P_{53}^F}} = \sqrt[4]{\left(P_{34}^F\right)^2 P_{32}^F P_{24}^F P_{35}^F P_{54}^F}.$$

The problem with this kind of link, however is that in period 4 when an update of $P_{03}^{GEKS}$ to $\hat{P}_{04}$ has to be made we cannot yet dispose of the required indices $P_{35}^F$ and $P_{54}^F$. Hence there does not exist a viable alternative to the specific sequence of windows displayed in tab. 6 with overlaps of m-1 periods. [34] Analogously to (10) we get

[32] IFD recommend 13 months as a "natural choice … as it allows strongly seasonal commodities to be compared" (IFD, 22).

[33] One might also think of alternatives concerning the sequence of windows, that is to start (continually) a new period not just one period after the previous window but m-1 periods after. But such ideas did not prove useful.

[34] Hence there is with m = 4 windows no alternative of the sequence (0 – 3), (1 – 4), etc. of windows. For example the sequence (0 – 3), (2 – 5), etc. or (0 – 3), (3 – 6) etc. would not make sense for the practical index computation.

(11) $L_{45} = \frac{P_{25}^{GEKS}}{P_{24}^{GEKS}} = \sqrt[4]{\left(P_{45}^F\right)^2 P_{25}^F P_{35}^F P_{42}^F P_{43}^F}$ and

(11a) $\hat{P}_{05}^{GEKS} = \hat{P}_{04}^{GEKS} L_{45} = P_{03}^{GEKS} L_{34} L_{45}$.

Table 6: Original (not linked) and linked* rolling GEKS indices (m = 4)

| first window (0 – 3) | second window (1 – 4) | third window (2 – 5) |
|---|---|---|
| $P_{00}^{GEKS} = 1$ | | |
| $P_{01}^{GEKS} = \sqrt[4]{\left(P_{01}^F\right)^2 P_{02}^F P_{21}^F P_{03}^F P_{31}^F}$ | $P_{11}^{GEKS} = 1$ | |
| $P_{02}^{GEKS} = \sqrt[4]{\left(P_{02}^F\right)^2 P_{01}^F P_{12}^F P_{03}^F P_{32}^F}$ | $P_{12}^{GEKS} = \sqrt[4]{\left(P_{12}^F\right)^2 P_{13}^F P_{32}^F P_{14}^F P_{42}^F}$ | $P_{22}^{GEKS} = 1$ |
| $P_{03}^{GEKS} = \sqrt[4]{\left(P_{03}^F\right)^2 P_{01}^F P_{13}^F P_{02}^F P_{23}^F}$ | $P_{13}^{GEKS} = \sqrt[4]{\left(P_{13}^F\right)^2 P_{12}^F P_{23}^F P_{14}^F P_{43}^F}$ | $P_{23}^{GEKS} = \sqrt[4]{\left(P_{23}^F\right)^2 P_{24}^F P_{43}^F P_{25}^F P_{53}^F}$ |
| $\hat{P}_{04}^{GEKS} = P_{03}^{GEKS} \cdot \left(P_{14}^{GEKS} / P_{13}^{GEKS}\right)$ | $P_{14}^{GEKS} = \sqrt[4]{\left(P_{14}^F\right)^2 P_{12}^F P_{24}^F P_{13}^F P_{34}^F}$ | $P_{24}^{GEKS} = \sqrt[4]{\left(P_{24}^F\right)^2 P_{23}^F P_{34}^F P_{25}^F P_{54}^F}$ |
| $\hat{P}_{05}^{GEKS} = P_{04}^{GEKS} \cdot \left(P_{25}^{GEKS} / P_{24}^{GEKS}\right)$ | | $P_{25}^{GEKS} = \sqrt[4]{\left(P_{25}^F\right)^2 P_{23}^F P_{35}^F P_{24}^F P_{45}^F}$ |

* green fields

In a similar vein we define $L_{56}$ and $\hat{P}_{06}^{GEKS} = \hat{P}_{05}^{GEKS} L_{56} = P_{03}^{GEKS} L_{34} L_{45} L_{56}$ etc.

Characteristic for chain indices is that they are independent of the base, so that $\frac{\overline{P}_{03}}{\overline{P}_{02}} = \frac{\overline{P}_{13}}{P_{01}} = P_{23}$ holds by construction. This, however, does not hold true for the GEKS indices (m > 2). It can easily be verified that we have with m = 4 two options for $L_{34}$, either (10) or

(12) $L_{34}^* = \frac{P_{24}^{GEKS}}{P_{23}^{GEKS}} = \sqrt[4]{\left(P_{34}^F\right)^2 P_{24}^F P_{32}^F P_{35}^F P_{54}^F} \neq L_{34} = \frac{P_{14}^{GEKS}}{P_{13}^{GEKS}} = \sqrt[4]{\left(P_{34}^F\right)^2 P_{14}^F P_{24}^F P_{31}^F P_{32}^F}$.

We have an inequality because the terms $L_{34}$ and $L_{34}^*$ refer to different windows. With a standard GEKS approach and m = 5 we have of course $\frac{P_{24(m=5)}^{GEKS}}{P_{23(m=5)}^{GEKS}} = \frac{P_{14(m=5)}^{GEKS}}{P_{33(m=5)}^{GEKS}}$ which amounts to $\sqrt[5]{\left(P_{34}^F\right)^2 P_{30}^F P_{04}^F P_{31}^F P_{14}^F P_{32}^F P_{24}^F}$. Also because of the existing overlap we have three different ways to express the change in prices from period 2 to period 3, viz. $\frac{P_{03}^{GEKS}}{P_{02}^{GEKS}} = \sqrt[4]{\left(P_{23}^F\right)^2 P_{03}^F P_{13}^F P_{20}^F P_{21}^F}$, and the two indices $\frac{P_{13}^{GEKS}}{P_{12}^{GEKS}} = \sqrt[4]{\left(P_{23}^F\right)^2 P_{13}^F P_{21}^F P_{24}^F P_{43}^F}$, and $P_{23}^{GEKS} = \sqrt[4]{\left(P_{23}^F\right)^2 P_{24}^F P_{43}^F P_{25}^F P_{53}^F}$ which, however, are available only in retrospect when t = 4 and t = 5 respectively.

In order to better make clear the difference between these ratios of indices (and the rule behind the formulas) it may be useful to write the formulas in fashion which looks a bit more complicated than necessary, and without making use of $P_{22}^F = P_{33}^F = 1$ and the time reversibility:

$\frac{P^{GEKS}_{03}}{P^{GEKS}_{02}} = \sqrt[4]{\frac{P^F_{03}P^F_{13}P^F_{23}P^F_{33}}{P^F_{02}P^F_{12}P^F_{22}P^F_{32}}}$, $\frac{P^{GEKS}_{13}}{P^{GEKS}_{12}} = \sqrt[4]{\frac{P^F_{13}P^F_{23}P^F_{33}P^F_{43}}{P^F_{12}P^F_{22}P^F_{32}P^F_{42}}}$, and $P^{GEKS}_{23} = \frac{P^{GEKS}_{23}}{P^{GEKS}_{22}} = \sqrt[4]{\frac{P^F_{23}P^F_{33}P^F_{43}P^F_{53}}{P^F_{22}P^F_{32}P^F_{42}P^F_{52}}}$ (here of course $P^{GEKS}_{22} = 1$). The equivalent terms in the case of chain indices (m = 2) are

$\frac{P^{GEKS}_{03(m=2)}}{P^{GEKS}_{02(m=2)}} = \frac{\overline{P}^F_{03}}{\overline{P}^F_{02}} = \frac{P^F_{01}P^F_{12}P^F_{23}}{P^F_{01}P^F_{12}} = \frac{P^F_{12}P^F_{23}}{P^F_{12}} = \frac{\overline{P}^F_{13}}{\overline{P}^F_{12}} = P^F_{23}$, hence obviously all three ratios are equal which evidently is not true in the case of RGEKS indices.

It can easily be seen in tab. 7 that the "normal" chain index is simply the limiting case of a rolling GEKS index with m = 2, where also the GEKS-index and the corresponding direct Fisher index coincide.

Table 7: Original (not linked) and linked rolling GEKS indices (m = 2)

| window no. 1 (periods 0 and 1) | no. 2 (1 – 2) | no. 3 (2 – 3) | no. 4 (3 – 4) |
|---|---|---|---|
| $P^{GEKS}_{00(m=2)} = 1$ | | | |
| $P^{GEKS}_{01} = \sqrt[2]{\left(P^F_{01}\right)^2} = P^F_{01}$ | 1 | | |
| $\hat{P}^{GEKS}_{02} = P^F_{01}\frac{P^F_{12}}{P^F_{11}} = P^F_{01}L_{12} = P^F_{01}P^F_{12} = \overline{P}^F_{02}$ | $P^{GEKS}_{12} = P^F_{12}$ | 1 | |
| $\hat{P}^{GEKS}_{03} = \overline{P}^F_{02}\frac{P^F_{23}}{P^F_{22}} = \overline{P}^F_{02}P^F_{23} = \overline{P}^F_{03}$ | | $P^{GEKS}_{23} = P^F_{23}$ | 1 |
| $\hat{P}^{GEKS}_{04} = \overline{P}^F_{03}P^F_{34} = \overline{P}^F_{04}$ | | | $P^F_{34}$ |

The following tab. 8 once more shows that the GEKS, and RGEKS method offers a variety of indices. It lists the various index series we get with the GEKS, RGEKS and chain index method.

Table 8: Alternative time series created with (R)GEKS method
(Note, we also have the series $P^F_{0t}$, and $\overline{P}^F_{0t}$)*

| | rolling (m= 3) | rolling (m= 4) | standard GEKS (m = 6) available only in retrospect at t = 5 |
|---|---|---|---|
| | (1) | (2) | (3) |
| $P_{01}$ | $\sqrt[3]{\left(P^F_{01}\right)^2 P^F_{02}P^F_{21}}$ | $\sqrt[4]{\left(P^F_{01}\right)^2 P^F_{02}P^F_{21}P^F_{03}P^F_{31}}$ | $\sqrt[6]{\left(P^F_{01}\right)^2 P^F_{02}P^F_{21}P^F_{03}P^F_{31}P^F_{04}P^F_{41}P^F_{05}P^F_{51}}$ |
| $P_{02}$ | $\sqrt[3]{\left(P^F_{02}\right)^2 P^F_{01}P^F_{12}}$ | $\sqrt[4]{\left(P^F_{02}\right)^2 P^F_{01}P^F_{12}P^F_{03}P^F_{32}}$ | $\sqrt[6]{\left(P^F_{02}\right)^2 P^F_{01}P^F_{12}P^F_{03}P^F_{32}P^F_{04}P^F_{42}P^F_{05}P^F_{52}}$ |
| $P_{03}$ | $\sqrt[3]{\left(P^F_{02}P^F_{23}\right)^2 P^F_{01}P^F_{13}}$ | $\sqrt[4]{\left(P^F_{03}\right)^2 P^F_{01}P^F_{13}P^F_{02}P^F_{23}}$ | $\sqrt[6]{\left(P^F_{03}\right)^2 P^F_{01}P^F_{13}P^F_{02}P^F_{23}P^F_{04}P^F_{43}P^F_{05}P^F_{53}}$ |
| $P_{04}$ | $\sqrt[3]{\left(P^F_{02}P^F_{34}\right)^2 P^F_{01}P^F_{13}P^F_{23}P^F_{24}}$ | $\sqrt[4]{\left(P^F_{03}P^F_{34}\right)^2 P^F_{01}P^F_{02}P^F_{14}P^F_{24}}$ | $\sqrt[6]{\left(P^F_{04}\right)^2 P^F_{01}P^F_{14}P^F_{02}P^F_{24}P^F_{03}P^F_{34}P^F_{05}P^F_{54}}$ |
| $P_{05}$ | $\sqrt[3]{\left(P^F_{02}P^F_{45}\right)^2 P^F_{01}P^F_{13}P^F_{23}P^F_{24}P^F_{35}}$ | $\sqrt[4]{\left(P^F_{03}P^F_{45}\right)^2 P^F_{01}P^F_{02}P^F_{14}P^F_{25}P^F_{34}P^F_{35}}$ | $\sqrt[6]{\left(P^F_{05}\right)^2 P^F_{01}P^F_{15}P^F_{02}P^F_{25}P^F_{03}P^F_{35}P^F_{04}P^F_{45}}$ |
| $P_{06}$ | $\sqrt[3]{\left(P^F_{02}P^F_{56}\right)^2 P^F_{01}P^F_{13}P^F_{23}P^F_{24}P^F_{34}P^F_{35}P^F_{45}P^F_{46}}$ | $\sqrt[4]{\left(P^F_{03}P^F_{56}\right)^2 P^F_{01}P^F_{14}P^F_{02}P^F_{25}P^F_{34}P^F_{45}P^F_{36}P^F_{46}}$ | |

* green colour indicates that this index is estimated using links $L_{23}$, $L_{34}$, $L_{45}$

And tab. 9 shows that while the formulas for the (linked) indices become longer and longer, the formulas for the links are simple and follow a straightforward principle.

Table 9: Links in RWGEKS indices (m = 2 is the chain index)

| | m = 2 | m = 3 | m = 4 |
|---|---|---|---|
| $1 \to 2$ | $P^F_{12}$ | | |
| $2 \to 3$ | $P^F_{23}$ | $\sqrt[3]{\left(P^F_{23}\right)^2 P^F_{21}P^F_{13}} = P^F_{23}\sqrt[3]{\frac{P^F_{21}P^F_{13}}{P^F_{23}}}$ | |
| $3 \to 4$ | $P^F_{34}$ | $\sqrt[3]{\left(P^F_{34}\right)^2 P^F_{32}P^F_{24}} = P^F_{34}\sqrt[3]{\frac{P^F_{32}P^F_{24}}{P^F_{34}}}$ | $\sqrt[4]{\left(P^F_{34}\right)^2 P^F_{31}P^F_{14}P^F_{32}P^F_{24}} = P^F_{34}\sqrt[4]{\frac{P^F_{31}P^F_{14}}{P^F_{34}}\frac{P^F_{32}P^F_{24}}{P^F_{34}}}$ |
| $4 \to 5$ | $P^F_{45}$ | $\sqrt[3]{\left(P^F_{45}\right)^2 P^F_{43}P^F_{35}} = P^F_{45}\sqrt[3]{\frac{P^F_{43}P^F_{35}}{P^F_{45}}}$ | $\sqrt[4]{\left(P^F_{45}\right)^2 P^F_{42}P^F_{25}P^F_{43}P^F_{35}} = P^F_{34}\sqrt[4]{\frac{P^F_{31}P^F_{14}}{P^F_{34}}\frac{P^F_{32}P^F_{24}}{P^F_{34}}}$ |
| $5 \to 6$ | $P^F_{56}$ | $\sqrt[3]{\left(P^F_{56}\right)^2 P^F_{54}P^F_{46}} = P^F_{56}\sqrt[3]{\frac{P^F_{54}P^F_{46}}{P^F_{56}}}$ | $\sqrt[4]{\left(P^F_{56}\right)^2 P^F_{53}P^F_{36}P^F_{54}P^F_{46}} = P^F_{34}\sqrt[4]{\frac{P^F_{31}P^F_{14}}{P^F_{34}}\frac{P^F_{32}P^F_{24}}{P^F_{34}}}$ |
| $6 \to 7$ | $P^F_{67}$ | $\sqrt[3]{\left(P^F_{67}\right)^2 P^F_{65}P^F_{57}} = P^F_{23}\sqrt[3]{\frac{P^F_{21}P^F_{13}}{P^F_{23}}}$ | $\sqrt[4]{\left(P^F_{67}\right)^2 P^F_{64}P^F_{47}P^F_{65}P^F_{57}} = P^F_{34}\sqrt[4]{\frac{P^F_{31}P^F_{14}}{P^F_{34}}\frac{P^F_{32}P^F_{24}}{P^F_{34}}}$ |

## 5.2. Properties of RGEKS indices

### a) No longer transitivity of standard GEKS indices

There are good reasons to raise doubts against RGEKS indices. Unlike standard GEKS indices (with the correct m) they are not transitive, fail the test of multi-period proportionality and provide various values for an index $P_{st}$ comparing of the same periods s and t.

These indices are no longer able to satisfy the transitivity axiom for the simple reason that a series with indices $\hat{P}^{GEKS}_{0t(m\leq t)}$ differs from the series of standard GEKS indices with a suitably chosen m which is greater than in the rolling approach.

We show this in tab. 10 where the formulas of tab. 8 are re-written in a way which more easily reveals the general principle of (R)GEKS indices although they might look a bit awkward at first glance. As mentioned above these formulas can be viewed as ratios of price levels, and an index function that possesses such a presentation is transitive.

The formulas for $\hat{P}^{GEKS}_{03(m=3)}$ etc. are given in tab. 8. It can easily be seen that

(13) $$\hat{P}^{GEKS}_{03(m=3)} = \frac{\sqrt[3]{P^F_{03}P^F_{13}P^F_{23}\left(\frac{P^F_{02}P^F_{23}}{P^F_{03}}\right)}}{\sqrt[3]{P^F_{00}P^F_{10}P^F_{20}}} \neq \frac{\sqrt[3]{P^F_{03}P^F_{13}P^F_{23}}}{\sqrt[3]{P^F_{00}P^F_{10}P^F_{20}}}$$ and quite similarly

(14) $$\hat{P}^{GEKS}_{04(m=4)} = \frac{\sqrt[4]{P^F_{04}P^F_{14}P^F_{24}P^F_{34}\left(\frac{P^F_{03}P^F_{34}}{P^F_{04}}\right)}}{\sqrt[4]{P^F_{00}P^F_{10}P^F_{20}P^F_{30}}} \neq \frac{\sqrt[4]{P^F_{04}P^F_{14}P^F_{24}P^F_{34}}}{\sqrt[4]{P^F_{00}P^F_{10}P^F_{20}P^F_{30}}}.$$

However $\hat{P}^{GEKS}_{03(m=3)}$ can nonetheless satisfy multi-period identity. Assume $p_{i3} = p_{i0}$ and $q_{i3} = q_{i0}$ for all i = 1, …, n commodities. Then obviously $P^F_{03} = P^F_{01}P^F_{13} = P^F_{02}P^F_{23} = 1$, and therefore also $\frac{\sqrt[3]{P^F_{03}P^F_{13}P^F_{23}}}{\sqrt[3]{P^F_{00}P^F_{10}P^F_{20}}} = 1$, but $\hat{P}^{GEKS}_{03(m=3)} = \sqrt[3]{\frac{P^F_{03}P^F_{13}P^F_{23}}{P^F_{00}P^F_{10}P^F_{20}}\left(\frac{P^F_{02}P^F_{23}}{P^F_{03}}\right)} = 1$ because also $\left(\frac{P^F_{02}P^F_{23}}{P^F_{03}}\right) = 1$.

Table 10: RWGEKS indices and standard GEKS indices
(Compare this table to tab. 8)

| | rolling (m= 3) | rolling (m= 4) | standard GEKS (m = 5) available only at t = 4 |
|---|---|---|---|
| | (1) | (2) | (3) |
| $P_{01}$ | $P^{GEKS}_{01(m=3)} = \frac{\sqrt[3]{P^F_{01}P^F_{11}P^F_{21}}}{\sqrt[3]{P^F_{00}P^F_{10}P^F_{20}}}$ | $P^{GEKS}_{01(m=4)} = \frac{\sqrt[4]{P^F_{01}P^F_{11}P^F_{21}P^F_{31}}}{\sqrt[4]{P^F_{00}P^F_{10}P^F_{20}P^F_{30}}}$ | $P^{GEKS}_{01(m=5)} = \frac{\sqrt[5]{P^F_{01}P^F_{11}P^F_{21}P^F_{31}P^F_{41}}}{\sqrt[5]{P^F_{00}P^F_{10}P^F_{20}P^F_{30}P^F_{40}}}$ |
| $P_{02}$ | $P^{GEKS}_{02(m=3)} = \frac{\sqrt[3]{P^F_{02}P^F_{12}P^F_{22}}}{\sqrt[3]{P^F_{00}P^F_{10}P^F_{20}}}$ | $P^{GEKS}_{02(m=4)} = \frac{\sqrt[4]{P^F_{02}P^F_{12}P^F_{22}P^F_{32}}}{\sqrt[4]{P^F_{00}P^F_{10}P^F_{20}P^F_{30}}}$ | $P^{GEKS}_{02(m=5)} = \frac{\sqrt[5]{P^F_{02}P^F_{12}P^F_{22}P^F_{32}P^F_{42}}}{\sqrt[5]{P^F_{00}P^F_{10}P^F_{20}P^F_{30}P^F_{40}}}$ |
| $P_{03}$ | $\hat{P}^{GEKS}_{03(m=3)}$ | $P^{GEKS}_{03(m=4)} = \frac{\sqrt[4]{P^F_{03}P^F_{13}P^F_{23}P^F_{33}}}{\sqrt[4]{P^F_{00}P^F_{10}P^F_{20}P^F_{30}}}$ | $P^{GEKS}_{03(m=5)} = \frac{\sqrt[5]{P^F_{03}P^F_{13}P^F_{23}P^F_{33}P^F_{43}}}{\sqrt[5]{P^F_{00}P^F_{10}P^F_{20}P^F_{30}P^F_{40}}}$ |
| $P_{04}$ | $\hat{P}^{GEKS}_{04(m=3)}$ | $\hat{P}^{GEKS}_{04(m=4)}$ | $P^{GEKS}_{04(m=5)} = \frac{\sqrt[5]{P^F_{04}P^F_{14}P^F_{24}P^F_{34}P^F_{44}}}{\sqrt[5]{P^F_{00}P^F_{10}P^F_{20}P^F_{30}P^F_{40}}}$ |

The formulas for $\hat{P}^{GEKS}_{04(m=3)}$ and $\hat{P}^{GEKS}_{05(m=4)}$ are more complicated and they will become ever more and more complicated when t (and thus m) increases:

$$(14a)\quad \hat{P}^{GEKS}_{04(m=3)} = \frac{\sqrt[3]{P^F_{04}P^F_{14}P^F_{24}\left(\left(P^F_{34}\right)^2 \frac{P^F_{02}P^F_{13}P^F_{23}}{P^F_{04}P^F_{14}}\right)}}{\sqrt[3]{P^F_{00}P^F_{10}P^F_{20}}} \neq \frac{\sqrt[3]{P^F_{04}P^F_{14}P^F_{24}}}{\sqrt[3]{P^F_{00}P^F_{10}P^F_{20}}} = \sqrt[3]{P^F_{04}P^F_{01}P^F_{14}P^F_{02}P^F_{24}} \quad \text{etc.}$$

We may again examine a case of identity and can see that with $\hat{P}^{GEKS}_{04(m=3)}$ identity may well be violated. Assume analogously now $p_{i4} = p_{i0}$ and $q_{i4} = q_{i0}$. This obviously implies $\sqrt[3]{P^F_{04}\left(P^F_{01}P^F_{14}\right)\left(P^F_{02}P^F_{24}\right)} = \sqrt[3]{1 \cdot 1 \cdot 1} = 1$ but there is no reason to expect that also $\left(P^F_{34}\right)^2 \frac{P^F_{02}P^F_{13}P^F_{23}}{P^F_{04}P^F_{14}} = 1$ and thus $\hat{P}^{GEKS}_{04(m=3)} = 1$.

More generally, tab. 10 and eqs. 14 and 14a show that RGEKS indices may violate multi-period- proportionality. Assume that all prices in 4 are λ-fold prices of period 0 such that $p_{i4} = p_{i0}$ for all i, and identical quantities $q_{i4} = q_{i0}$ we then have because $P^F_{0k}P^F_{k4} = \lambda$

$$(14^*)\quad \hat{P}^{GEKS}_{04(m=4)} = \sqrt[4]{P^F_{04}P^F_{01}P^F_{14}P^F_{02}P^F_{24}P^F_{03}P^F_{34}\left(\frac{P^F_{03}P^F_{34}}{P^F_{04}}\right)} = \sqrt[4]{\lambda^4\left(\frac{P^F_{03}P^F_{34}}{\lambda}\right)} = \lambda \;\; (\text{as } P^F_{03}P^F_{34} = \lambda), \text{ however,}$$

(14a*) $\hat{P}^{GEKS}_{04(m=3)} = \sqrt[3]{P^F_{04}P^F_{01}P^F_{14}P^F_{02}P^F_{24}\left(\left(P^F_{34}\right)^2 \frac{P^F_{02}P^F_{13}P^F_{23}}{P^F_{04}P^F_{14}}\right)} = \sqrt[3]{\lambda\left(P^F_{34}\right)^2 P^F_{01}P^F_{13}P^F_{02}P^F_{23}}$, which equals λ only if $\left(P^F_{34}\right)^2 P^F_{01}P^F_{13}P^F_{02}P^F_{23} = \lambda^2$. On the other hand the standard GEKS formula with m = 5 passes the test because $P^{GEKS}_{04(m=5)} = \sqrt[5]{\left(P^F_{04}\right)^2 P^F_{01}P^F_{14}P^F_{02}P^F_{24}P^F_{03}P^F_{34}} = \sqrt[5]{\lambda^2\lambda^3} = \lambda$.

Moreover the obvious rule behind column 3 in tab. 10 suggests that

$$P^{GEKS}_{23(m=5)} = \frac{\sqrt[5]{P^F_{03}P^F_{13}P^F_{23}P^F_{33}P^F_{43}}}{\sqrt[5]{P^F_{02}P^F_{12}P^F_{22}P^F_{32}P^F_{42}}}$$ which is in fact equal to $\sqrt[5]{\left(P^F_{23}\right)^2 P^F_{20}P^F_{03}P^F_{21}P^F_{13}P^F_{24}P^F_{43}}$.

However $\frac{\hat{P}^{GEKS}_{03(m=3)}}{P^{GEKS}_{02(m=3)}} = L_{23(m=3)} = \sqrt[3]{\left(P^F_{23}\right)^2 P^F_{21}P^F_{13}} = \frac{\sqrt[3]{P^F_{13}P^F_{23}}}{\sqrt[3]{P^F_{12}P^F_{32}}} \neq \frac{\sqrt[3]{P^F_{03}P^F_{13}P^F_{23}}}{\sqrt[3]{P^F_{02}P^F_{12}P^F_{22}}}$. Analogously while

$P^{GEKS}_{34(m=5)} = \frac{\sqrt[5]{P^F_{04}P^F_{14}P^F_{24}P^F_{34}P^F_{44}}}{\sqrt[5]{P^F_{03}P^F_{13}P^F_{23}P^F_{33}P^F_{43}}}$ we have $\frac{\hat{P}^{GEKS}_{04(m=3)}}{\hat{P}^{GEKS}_{03(m=3)}} = L_{34(m=3)} = \sqrt[3]{\left(P^F_{34}\right)^2 P^F_{32}P^F_{24}}$. It goes without saying that this is different from $\frac{\sqrt[3]{P^F_{04}P^F_{14}P^F_{24}}}{\sqrt[3]{P^F_{03}P^F_{13}P^F_{23}}}$ and $\frac{\sqrt[4]{P^F_{04}P^F_{14}P^F_{24}P^F_{34}}}{\sqrt[4]{P^F_{03}P^F_{13}P^F_{23}P^F_{33}}}$ which in turn differs from $\frac{\hat{P}^{GEKS}_{04(m=4)}}{P^{GEKS}_{03(m=4)}}$.

Note that $\frac{\hat{P}^{GEKS}_{04(m=4)}}{P^{GEKS}_{03(m=4)}}$ and $\frac{\sqrt[4]{P^F_{04}P^F_{14}P^F_{24}P^F_{34}}}{\sqrt[4]{P^F_{03}P^F_{13}P^F_{23}P^F_{33}}}$ cannot be written in a GEKS formula form. However $\frac{P^{GEKS}_{03(m=4)}}{P^{GEKS}_{02(m=4)}}$ is equal to $\frac{\sqrt[4]{P^F_{03}P^F_{13}P^F_{23}P^F_{33}}}{\sqrt[4]{P^F_{02}P^F_{12}P^F_{22}P^F_{32}}}$ and this in turn is equal to $\sqrt[4]{\left(P^F_{23}\right)^2 P^F_{20}P^F_{03}P^F_{21}P^F_{13}} = P^{GEKS}_{23(m=4)}$, a standard GEKS formula.

**b) RGEKS indices not independent of the base period (unlike chain indices)**

Chain indices are not transitive but they are in a way quite convenient in that they are independent of the base in the sense of $\frac{\overline{P}^F_{04}}{\overline{P}^F_{02}} = \frac{\overline{P}^F_{14}}{\overline{P}^F_{12}} = \overline{P}^F_{24} = P^F_{23}P^F_{34}$. This, however, does not apply to RGEKS indices when a "linked" index $\hat{P}^{GEKS}_{0t(m)}$ is implied and compared to $\hat{P}^{GEKS}_{0k(m)}$ where $k \leq t-2$. We demonstrate this with m = 3 and the first three windows (tab. 11):

Table 11

| first window (periods 0 – 2) | second window (1 – 3) | third window (2 – 4) |
|---|---|---|
| $P^{GEKS}_{00} = 1$ | | |
| $P^{GEKS}_{01}$ | 1 | |
| $P^{GEKS}_{02}$ | $P^{GEKS}_{12}$ | 1 |
| $\hat{P}^{GEKS}_{03}$ | $P^{GEKS}_{13}$ | $P^{GEKS}_{23}$ |
| $\hat{P}^{GEKS}_{04}$ | $\hat{P}^{GEKS}_{14}$ | $P^{GEKS}_{24}$ |

In the ratio $R_1 = \frac{\hat{P}^{GEKS}_{04(m=3)}}{P^{GEKS}_{02(m=3)}} = L_{23(m=3)}L_{34(m=3)}$ (first window) we have a product of the two links $L_{23} = \frac{P^{GEKS}_{13(m=3)}}{P^{GEKS}_{12(m=3)}}$ and $L_{34} = \frac{P^{GEKS}_{24(m=3)}}{P^{GEKS}_{23(m=3)}}$. The ratio $R_2 = \frac{\hat{P}^{GEKS}_{14(m=3)}}{P^{GEKS}_{12(m=3)}} = R_1 = \sqrt[3]{\left(P^F_{34}\right)^2 P^F_{13}P^F_{21}P^F_{23}P^F_{24}}$, because $\hat{P}^{GEKS}_{14(m=3)} = P^{GEKS}_{13(m=3)}$ $L_{34}$ yields the same result $R_2 = R_1$. In each ratio, at least one link is involved. This is so because with m = 3 we have no window covering all 5 periods, 0 through 4. On the other hand we have from the third window $P^{GEKS}_{24(m=3)} = \sqrt[3]{\left(P^F_{24}\right)^2 P^F_{23}P^F_{34}} = R_1 \cdot \sqrt[3]{\frac{P^F_{12}P^F_{24}}{P^F_{13}P^F_{34}}}$. In addition to this ambiguity of ratios due to different window (all with the same m) involved, we have of course also ratios on the basis of different parameters m. So $R_2 = \hat{P}^{GEKS}_{14(m=3)} \big/ P^{GEKS}_{12(m=3)}$ will as a rule not coincide with $P^{GEKS}_{14(m=4)} \big/ P^{GEKS}_{12(m=4)} = P^{GEKS}_{24(m=4)} = \sqrt[4]{\left(P^F_{24}\right)^2 P^F_{21}P^F_{14}P^F_{23}P^F_{34}}$.

Another example where now three links are involved is $\frac{\hat{P}^{GEKS}_{06(m=3)}}{\hat{P}^{GEKS}_{03(m=3)}} = L_{34}L_{45}L_{56}$ or alternatively $P^{GEKS}_{35(m=3)}L_{56}$. The product $L_{34}L_{45}L_{56}$ (see tab. 9) amounts to $P^F_{34}P^F_{45}P^F_{56}\sqrt[3]{\frac{P^F_{32}P^F_{24}}{P^F_{34}}\frac{P^F_{43}P^F_{35}}{P^F_{45}}\frac{P^F_{54}P^F_{46}}{P^F_{56}}}$ (the first factor $P^F_{34}P^F_{45}P^F_{56} = \overline{P}^F_{36}$). On the other hand $P^{GEKS}_{35(m=3)}L_{56} = \sqrt[3]{\left(P^F_{35}\right)^2 P^F_{34}P^F_{45}} \cdot \sqrt[3]{\left(P^F_{56}\right)^2 P^F_{46}P^F_{54}} = \sqrt[3]{\left(P^F_{35}P^F_{56}\right)^2 P^F_{34}P^F_{46}} = \sqrt[3]{P^F_{35}P^F_{56}\Phi}$ is clearly different from the product of the three links or from a standard GEKS index where m must be ≥ 4 in order to encompass an interval from 3 to 6, for example $P^{GEKS}_{36(m=4)} = \sqrt[4]{\left(P^F_{36}\right)^2 P^F_{34}P^F_{46}P^F_{35}P^F_{56}} = \sqrt[4]{\left(P^F_{36}\right)^2 \Phi}$. As indicated by the factor Φ there is some resemblance to the approach with the computing of $P^{GEKS}_{35(m=3)}L_{56}$.

To sum up: The RGEKS approach has the advantage over the standard GEKS approach in that there is no longer "the need to revise parities for previous periods" (IFD). However a series of RWGEKS indices $P^{GEKS}_{01(m)},\dots,P^{GEKS}_{0,m-1(m)},\hat{P}^{GEKS}_{0,m},\hat{P}^{GEKS}_{0,m+1},\dots\hat{P}^{GEKS}_{0,M-1}$ is necessarily different from the corresponding standard GEKS index series $P^{GEKS}_{01(M)},\dots,P^{GEKS}_{0,m-1(M)},P^{GEKS}_{0,m(M)},\dots P^{GEKS}_{0,M-1(M)}$ when M > m. The standard GEKS indices are the only indices which are transitive for an interval from 0 to m-1 (or for m countries). So the RGEKS index cannot be transitive simply because they are different from standard GEKS indices. RGEKS index share with the usual chain indices

- the advantage (over standard GEKS indices) that no revision of previously computed indices are necessary, and
- the disadvantage of intransitivity in the sense that not all ways to compare s and t in $P_{st}$ indirectly are consistent,[35]

[35] However, as mentioned above, when periods in time are involved the set of meaningful comparisons may reasonably (in view of practical purposes of index computations) be reduced to comparisons over adjacent subintervals, for example $P_{0t}$ to $P_{0s}$ when 0 < s < t. And in this limited sense all comparisons made with chain indices are consistent. It is clear that $P_{st} = P_{0t}/P_{0s} = P_{1t}/P_{1s}$ etc.

- however, while chain indices are "base independent" in that for example (by construction) $\frac{\overline{P}_{04}^F}{\overline{P}_{02}^F} = \frac{\overline{P}_{14}^F}{\overline{P}_{12}^F} = \overline{P}_{24}^F$ equivalent ratios using RGEKS indices (in particular when in numerator and or denominator links are involved) are not necessarily equal.

Furthermore with chain indices (equivalent to RGEKS when m = 2) there is always one and only one index $P_{t,t+1}$ comparing two adjacent periods.

### c) Further ambiguities with the rolling method

As mentioned above there is no unique GEKS-index for any two periods compared, say j and k. This also applies to RGEKS indices even with a given m. Tab. 12 shows that the rolling method provides two indices $P_{12}$, $P_{23}$, … when m = 3, three indices indices $P_{23}$, $P_{34}$, … when m = 4 etc., depending on which window of width m you take.

Table 12 ($P_{st}$ for rolling with m-1 periods overlap; the recommended method)

| index | m = 3 | | m = 4 | | m = 5 |
|---|---|---|---|---|---|
| | window (periods) | index | window (periods) | index | index |
| $P_{12}$ | 1 (0 – 2) | $\sqrt[3]{(P_{12}^F)^2 P_{10}^F P_{02}^F}$ | | | |
| | 2 (1 – 3) | $\sqrt[3]{(P_{12}^F)^2 P_{13}^F P_{32}^F}$ | | | |
| $P_{23}$ | 2 (1 – 3)<br>3 (2 – 4) | $\sqrt[3]{(P_{23}^F)^2 P_{21}^F P_{13}^F}$<br>$\sqrt[3]{(P_{23}^F)^2 P_{24}^F P_{43}^F}$ | 1 (0 – 3) | $\sqrt[4]{(P_{23}^F)^2 P_{20}^F P_{03}^F P_{21}^F P_{13}^F}$ | |
| | | | 2 (1 – 4) | $\sqrt[4]{(P_{23}^F)^2 P_{21}^F P_{13}^F P_{24}^F P_{43}^F}$ | |
| | | | 3 (2 – 5) | $\sqrt[4]{(P_{23}^F)^2 P_{24}^F P_{43}^F P_{25}^F P_{53}^F}$ | |
| $P_{34}$ | 3 (2 – 4)<br>and<br>4 (3 – 5) | $\sqrt[3]{(P_{34}^F)^2 P_{32}^F P_{24}^F}$<br>$\sqrt[3]{(P_{34}^F)^2 P_{35}^F P_{54}^F}$ | 2 (1 – 4)<br>3 (2 – 5)<br>and<br>4 (3 – 6) | $\sqrt[4]{(P_{34}^F)^2 P_{31}^F P_{14}^F P_{32}^F P_{24}^F}$<br>$\sqrt[4]{(P_{34}^F)^2 P_{32}^F P_{24}^F P_{35}^F P_{54}^F}$<br>$\sqrt[4]{(P_{34}^F)^2 P_{35}^F P_{54}^F P_{36}^F P_{64}^F}$ | $\sqrt[5]{(P_{34}^F)^2 P_{30}^F P_{04}^F P_{31}^F P_{14}^F P_{32}^F P_{24}^F}$<br>$\sqrt[5]{(P_{34}^F)^2 P_{31}^F P_{14}^F P_{32}^F P_{24}^F P_{35}^F P_{54}^F}$<br>$\sqrt[5]{(P_{34}^F)^2 P_{32}^F P_{24}^F P_{35}^F P_{54}^F P_{36}^F P_{64}^F}$<br>$\sqrt[5]{(P_{34}^F)^2 P_{35}^F P_{54}^F P_{36}^F P_{64}^F P_{37}^F P_{74}^F}$ |

Hence ambiguities (alternative estimates for the same index) not only arise from the choice of m. In the RGEKS method as opposed to the GEKS method, even a fixed and definite m can yield up to m-1 different results for an index $P_{jk}$ due to the overlap of windows of the same length m > 2. Interestingly there is no such ambiguity with a chain index. The scheme of tab. 12 also explains why we have only one unique index $P_{12}$, $P_{23}$, … when m = 2, that is in the case of a chain index. In this sense (vanishing ambiguity) the chain index is clearly preferable to a RGEKS index (with m ≥ 3). There is also no room for ambiguities with the standard GEKS index once there is a decision made about a

- fixed suitable m, say m = 5, and
- the specific number in the sequence of 5-periods windows.

Assume, we took the first of these (m = 5) windows, covering periods 0 to 4, then $P_{34}$ the GEKS index is given by $\sqrt[5]{(P_{34}^F)^2 P_{30}^F P_{04}^F P_{31}^F P_{14}^F P_{32}^F P_{24}^F}$ (with the second window [periods 1 through 5] it would be of course $\sqrt[5]{(P_{34}^F)^2 P_{31}^F P_{14}^F P_{32}^F P_{24}^F P_{35}^F P_{54}^F}$ .

The standard direct Fisher index is of course simply $P_{34}^{F}$, clearly a definite and unique index as opposed to the nine (R)GEKS indices listed in tab. 12. As a rule these nine indices will yield different results for the same pair of periods, 3 and 4 in our example. Hence we may ask: Can we think of reasonable alternative rolling method where this ambiguity no longer exists?

**d) Are there alternative methods of linking?**

The rolling method is working in such a way that we proceed with each new link only one step forward. What also might appear quite meaningful at first glance only, however, is to link the second window of a standard GEKS index with m = 4 (covering the periods 3, 4, 5, and 6) to the first window (periods 0, 1, 2, and 3). Now we have an overlap of one period (rather than m – 1 periods) This means to consider the product $P_{03(m=4)}^{GEKS}\ P_{36(m=4)}^{GEKS}$ as a sort of measure for the price change between 0 and 6. This then might be continued using the factors $P_{69(m=4)}^{GEKS}$, $P_{9,12(m=4)}^{GEKS}$ etc. It is interesting to compare the product $P_{03(m=4)}^{GEKS}P_{36(m=4)}^{EKS}$ to $P_{06(m=7)}^{GEKS}$, and to $P_{06}^{F}$ as well as to $\overline{P}_{06}^{F}$.

Table 13: Rolling with only one period overlap ( $P_{0t(m,w)}^{GEKS*}$ with w = number of the window)

| m = 3 | | m = 4 | |
|---|---|---|---|
| window (periods) | index | window (periods) | index |
| 1 (0 – 2) | $P_{01(3,1)}^{GEKS*}=\sqrt[3]{\left(P_{01}^{F}\right)^2 P_{02}^{F}P_{21}^{F}}$<br>$P_{12(3,1)}^{GEKS*}=\sqrt[3]{\left(P_{12}^{F}\right)^2 P_{10}^{F}P_{02}^{F}}$ | 1 (0 – 3) | $P_{01(4,1)}^{GEKS*}=\sqrt[4]{\left(P_{01}^{F}\right)^2 P_{02}^{F}P_{21}^{F}P_{03}^{F}P_{31}^{F}}$<br>$P_{12(4,1)}^{GEKS*}=\sqrt[4]{\left(P_{12}^{F}\right)^2 P_{10}^{F}P_{02}^{F}P_{13}^{F}P_{32}^{F}}$<br>$P_{23(4,1)}^{GEKS*}=\sqrt[4]{\left(P_{23}^{F}\right)^2 P_{20}^{F}P_{03}^{F}P_{21}^{F}P_{13}^{F}}$ |
| 2 (2 – 4) | $P_{23(3,2)}^{GEKS*}=\sqrt[3]{\left(P_{23}^{F}\right)^2 P_{24}^{F}P_{43}^{F}}$<br>$P_{34(3,2)}^{GEKS*}=\sqrt[3]{\left(P_{34}^{F}\right)^2 P_{32}^{F}P_{24}^{F}}$ | 2 (3 – 6) | $P_{34(4,2)}^{GEKS*}=\sqrt[4]{\left(P_{34}^{F}\right)^2 P_{35}^{F}P_{54}^{F}P_{36}^{F}P_{64}^{F}}$<br>$P_{45(4,2)}^{GEKS*}=\sqrt[4]{\left(P_{34}^{F}\right)^2 P_{35}^{F}P_{54}^{F}P_{36}^{F}P_{64}^{F}}$<br>$P_{56(4,2)}^{GEKS*}=\sqrt[4]{\left(P_{34}^{F}\right)^2 P_{35}^{F}P_{54}^{F}P_{36}^{F}P_{64}^{F}}$ |

The index $P_{04(3)}^{GEKS*}$ then would be the product $P_{01(3,1)}^{GEKS*}\ P_{12(3,1)}^{GEKS*}\ P_{23(3,2)}^{GEKS*}\ P_{34(3,2)}^{GEKS*}$ just analogously to the usual chain index method, but the result $\sqrt[3]{\left(P_{02}^{F}P_{24}^{F}\right)^2 \overline{P}_{04}^{F}}$ does not seem to make much sense. The product of the first two factors (both referring to the same window) is $\sqrt[3]{\left(P_{01}^{F}\right)^2 P_{02}^{F}P_{21}^{F}}$, which in fact is the standard GEKS index $P_{02(m=3)}^{GEKS}$. However, when multiplied by $P_{23(3,2)}^{GEKS*}$ this gives the rather meaningless expression $\sqrt[3]{\left(P_{02}^{F}\right)^2 \overline{P}_{03}^{F}P_{23}^{F}P_{24}^{F}P_{43}^{F}}$. By the same token, with m = 4 the product of indices of the same window, e.g. the three indices in the first window makes sense. It coincides of course with $P_{03(m=4)}^{GEKS}=\sqrt[4]{\left(P_{03}^{F}\right)^2 P_{01}^{F}P_{13}^{F}P_{02}^{F}P_{23}^{F}}$, the standard GEKS index. But once multiplied by $P_{34(4,2)}^{GEKS*}=\sqrt[4]{\left(P_{34}^{F}\right)^2 P_{35}^{F}P_{54}^{F}P_{36}^{F}P_{64}^{F}}$ (and a fortiori with the next indices of the second window as further factors) we get an obviously senseless result. Alternatively one might think of the product $P_{03(m=4)}^{GEKS}P_{36(m=4)}^{GEKS}$ by analogy to $\overline{P}_{06}^{F}=(P_{01}^{F}P_{12}^{F}P_{23}^{F})\left(P_{34}^{F}P_{45}^{F}P_{56}^{F}\right)=\overline{P}_{03}^{F}\overline{P}_{36}^{F}$ (a product of indices which may serve here as a model).

The result $P_{03(m=4)}^{GEKS}P_{36(m=4)}^{GEKS}=\sqrt[4]{(P_{03}^{F}P_{36}^{F})^2 P_{01}^{F}P_{13}^{F}P_{02}^{F}P_{23}^{F}P_{34}^{F}P_{46}^{F}P_{35}^{F}P_{56}^{F}}$, however, is quite different from $\overline{P}_{06}^{F}$, and both, $P_{06}^{F}$ and $P_{06(m=7)}^{GEKS}=\sqrt[7]{(P_{06}^{F})^2 P_{01}^{F}P_{16}^{F}P_{02}^{F}P_{26}^{F}P_{03}^{F}P_{36}^{F}P_{04}^{F}P_{46}^{F}P_{05}^{F}P_{56}^{F}}$ which has 11 factors, and the product $P_{03(m=4)}^{GEKS} P_{36(m=4)}^{GEKS}$ only 10 factors (and only six of them in common with $P_{06(m=7)}^{GEKS}$, viz. $P_{01}^{F}$, $P_{02}^{F}$, $P_{03}^{F}$, and $P_{36}^{F}$, $P_{46}^{F}$, $P_{56}^{F}$).

**e) Do (R)GEKS indices satisfy identity?**

When the assumption is made that in two periods, s and t not only the same prices but also the same quantities exist, the standard GEKS index with m periods (comprising periods s and t) cannot differ from unity. Take for example the periods 1 and 2 and m = 4 (covering the periods 0, 1, 2, and 3), then of course $P_{12}^{F} = 1$, because no price changed (and thus $\overline{P}_{02}^{F} = \overline{P}_{01}^{F}$), this also applies to $P_{12(m=4)}^{GEKS}$ which is given by

(15) $$\sqrt[4]{\left(P_{12}^{F}\right)^2 P_{10}^{F} P_{02}^{F} P_{13}^{F} P_{32}^{F}} = \sqrt[4]{P_{10}^{F} P_{01}^{F} P_{13}^{F} P_{31}^{F}} = 1 .$$

With RGEKS indices this is not necessarily true. For example when two links L are involved, as in $\hat{P}_{04(m=3)}^{GEKS} = P_{02(m=3)}^{GEKS} L_{23} L_{34}$ the result is $\sqrt[3]{\left(P_{02}^{F} P_{34}^{F}\right)^2 P_{01}^{F} P_{13}^{F} P_{23}^{F} P_{24}^{F}}$. Then equality $p_{i0} = p_{i4}$ *and* $q_{i0} = q_{i4}$ for all commodities i and therefore $P_{04}^{F} = 1$ is not sufficient to infer $\hat{P}_{04(m=3)}^{GEKS} = 1$. The formula reduces to $\sqrt[3]{\left(P_{30}^{F}\right)^2 P_{01}^{F} P_{02}^{F} P_{13}^{F} P_{23}^{F}}$ which well can differ from unity. This will be demonstrated by ways of a numerical example in the appendix. Despite identical prices and quantities in 0 and 4 respectively a rolling approach with m = 3 resulted in $\hat{P}_{04(m=3)}^{GEKS} = 0.913348$ (see tab. A.14).

Another numerical example ("scenario"), this one we owe to M. Ribe (2012) – and also depicted in detail in the appendix – revealed another interesting point: eq. 15 rests on the assumption that prices *and* quantities are identical in the two periods compared. Ribe introduced different quantities as follows:

| t | $p_{1t}$ | $p_{2t}$ | $q_{1t}$ | $q_{2t}$ |
|---|---|---|---|---|
| 1 | 30 | 100 | 100 | 10 |
| 2 | 30 | 100 | 20 | 10 |

Of course a reasonable index should yield unity as $P_{12}^{F} = 1$. However $P_{12(m=4)}^{GEKS} = 1.100482$ indicating a rise of prices by 10% although both prices remain unchanged. The reason is that we can no longer assume $P_{10}^{F} P_{02}^{F} = P_{13}^{F} P_{32}^{F} = 1$. Instead we have

$P_{10}^{F} P_{02}^{F} = \sqrt{\frac{\mathbf{p}_0'\mathbf{q}_1}{\mathbf{p}_1'\mathbf{q}_1} \frac{\mathbf{p}_1'\mathbf{q}_2}{\mathbf{p}_0'\mathbf{q}_2}}$ and $P_{13}^{F} P_{32}^{F} = \sqrt{\frac{\mathbf{p}_3'\mathbf{q}_1}{\mathbf{p}_1'\mathbf{q}_1} \frac{\mathbf{p}_1'\mathbf{q}_2}{\mathbf{p}_3'\mathbf{q}_2}}$ which both amount to 1.211, and $\sqrt{1.211}$ is just the result 1.10048 above. The equality $P_{10}^{F} P_{02}^{F} = P_{13}^{F} P_{32}^{F}$ is due to the fact, that also prices in 0 and 3 are the same. The fact that $P_{10}^{F} P_{02}^{F} = 1.211 \neq 1$ also entails that $P_{12(m=4)}^{GEKS} = \sqrt[3]{1.211} = 1.0659 \neq 1$. Furthermore $P_{12(m=5)}^{GEKS} = 1.0796$, so standard GEKS index compilations indicate a rise by 6.6%, 10%, and 8% (for m = 3, 4 and 5) although prices did not rise at all between the two adjacent periods. It is useful to keep two situations distinct,

1. no change in prices between two adjacent periods (this takes place twice in Ribe's example, from 1 to 2 and also from 3 to 4, so that the direct index is rightly $P_{12}^{F} = P_{34}^{F} = 1$), and
2. the same prices (but not quantities) in two non-adjacent periods (therefore in Ribe's example $P_{03}^{F} = P_{04}^{F}$).

In both situations (R)GEKS indices can yield counterintuitive results (yield ≠ 1) as demonstrated in tab. 14:

- a direct index can do so in the first but not the second situation (for example $P_{12}^{F} = 1$ does of course not imply that $P_{01}^{F} = P_{02}^{F}$), and
- a chain index will reflect the first situation correctly ($\overline{P}_{01}^{F} = \overline{P}_{02}^{F}$) but not the second one ($\overline{P}_{03}^{F} \neq 1$ despite $P_{03}^{F}$=1).

Table 14

| t | $P_{0t(m=3)}^{GEKS}$ * | $P_{0t(m=4)}^{GEKS}$ | $P_{0t}^{F}$ | $\overline{P}_{0t}^{F}$ |
|---|---|---|---|---|
| 1 | 0.518218 | 0.529948 | 0.486172 | 0.486172 |
| 2 | 0.552371 | 0.583198 | 0.588784 | 0.486172 |
| 3 | 1.023471 | 0.926184 | 1 | 0.708319 |
| 4 | 1.077153 | 1 | 1 | 0.708319 |

* rolling GEKS $P_{03}$ estimated with link $L_{23}$ and $P_{04}$ with two links $L_{23}$ and $L_{34}$

The results concerning standard and rolling GEKS index compilations $P_{st}$ look strange for a price index which ideally should reflect only price changes regarding the periods compared, that is s and t respectively (principle of "pure" price comparison). The reason seems to be that these indices are affected by changing prices, and also quantities in periods other than s and t.

The result $P_{03(m=4)}^{GEKS}$ = 0.926184 between 0.583198 and 1 may be viewed as a product of smoothing. It is, however, simply wrong because 1 is the only reasonable result for $P_{03}$. Moreover it can hardly be seen as a "drift attenuation" relative to $\overline{P}_{03}^{F}$. Another obviously nonsensical result is $P_{03(m=3)}^{GEKS} > 1$ (and at the same time $P_{03(m=4)}^{GEKS} < 1$), and $P_{04(m=3)}^{GEKS} > P_{03(m=3)}^{GEKS}$, in spite of $P_{34}^{F} = 1$. We will come back to Ribe's example at the end of the appendix.

## 6. Empirical findings and conclusions

As mentioned above (in sec. 4.4) M. Ribe, and possibly others as well, spoke of the "drift attenuation capacity" of the (R)GEKS approach.[36] Ribe demonstrated this with his numerical example above mentioned (and dealt with in the appendix) where one price was fluctuating (over four periods). He seems to have concluded "attenuation" of "chain drift" by $P_{0t}^{GEKS}$ (relative to $\overline{P}_{0t}^{F}$) from the result $\overline{P}_{03}^{F} < P_{03(m=4)}^{GEKS} < 1$ of his example.[37] In our view, however, chain indices and attenuation of their drift is not the issue when dealing with the GEKS method. The problem here is rather a direct index, which like $P_{0t}^{F}$ is usually lacking transitivity. It was this situation (with direct indices of Fisher), the GEKS method was made for.

36 The idea seems to be that one might expect a smoother and less volatile time series of such indices relative to the direct Fisher indices or chain Fisher indices. We mentioned (in sec. 4.4) that to our knowledge no rigorous and general proof of this contention exists for GEKS indices. Nor do we know of such considerations in the case of RGEKS indices.

37 We could confirm Ribes result concerning the chain index (which was by 29% short of the drift free result 1), but we were unable to see how arrived at - 2.6% for the GEKS index. The fact that 2.6% is less than 29%, or in other words, the GEKS index comes closer to unity than the chain index is understood as evidence for drift attenuation.

Besides the frequently discussed "drift attenuation" which may be understood as smoothing of the time series of chain indices (as a possible alternative to direct and chain indices) multi-period proportionality (and thereby also identity) is perhaps the most interesting and most discussed issue. We gave some thought to this just in the preceding paragraph (sec. 5.2.e) as well as in the appendix. We saw that when prices in t =3 and t = 0 are identical, then for example $P^{GEKS}_{03(m=4)}$ is not necessarily equal to $P^F_{03} = 1$ unless also quantities of 0 and 3 are identical, so that $P^F_{01}P^F_{13} = P^F_{02}P^F_{23} = 1$.

Some further observations illustrated by the numerical example in the appendix are worth being mentioned:

- As with chain indices we also get different GEKS indices depending on how the interval under consideration is partitioned into subintervals. With five points in time, 0, 1, 2, 3, and 4 we have

  $P^{GEKS}_{04(m=5)} = \sqrt[5]{\left(P^F_{04}\right)^2 P^F_{01}P^F_{14}P^F_{02}P^F_{24}P^F_{03}P^F_{34}}$ , as opposed to $P^{GEKS}_{04(m=3)} = \sqrt[3]{\left(P^F_{04}\right)^2 P^F_{02}P^F_{24}}$

  with only two subintervals and three points in time, 0, 2, and 4.

- In the RGEKS method with m = 4 the first window ends with $P^{GEKS}_{03(m=4,w=1)}$. The next value $\hat{P}^{GEKS}_{04(m=4)}$ is given by multiplying with the link $L_{34} = P^{GEKS}_{14(m=4)} / P^{GEKS}_{13(m=4)}$. This procedure is equivalent to $P^{GEKS}_{03(m=4,w=1)}$ by $P^{GEKS}_{34(m=4,w=2)}$ from the second window. Alternative methods of linking (which will not coincide with this result) are, however, the products $P^{GEKS}_{01(m=4,w=1)}$ $P^{GEKS}_{14(m=4,w=1)}$ and $P^{GEKS}_{02(m=4,w=1)}$ $P^{GEKS}_{24(m=4,w=1)}$. The example shows that the difference can be quite substantial (or remember the example above $P^{GEKS}_{03(m=4)}P^{GEKS}_{36(m=4)} \neq P^{GEKS}_{06(m=7)}$).

- Most importantly: in the appendix to this paper we will demonstrate by means of a small numerical example that when prices show a cyclical movement of k periods but no trend, RGEKS indices (m ≠ λk, λ = 1,2,…) may well (just like chain indices) fluctuate around a positively or negatively sloped trend (although the underlying price data don't show a trend).[38]

- While chain indices are in way independent of the base period so that – by construction – we have $\frac{\overline{P}^F_{0t}}{\overline{P}^F_{0s}} = \frac{\overline{P}^F_{1t}}{\overline{P}^F_{1s}} = \frac{\overline{P}^F_{2t}}{\overline{P}^F_{2s}} = \ldots$, this does not apply to the GEKS method (in particular its rolling variant). In the standard GEKS method we not even have a unique estimate for one of such ratios and get (depending on m) different measures for the rate of change of the prices between two periods, for example from one period to the next

  $P^{GEKS}_{03(m=5)} / P^{GEKS}_{02(m=5)} = \sqrt[5]{\left(P^F_{23}\right)^2 P^F_{20}P^F_{03}P^F_{21}P^F_{13}P^F_{24}P^F_{43}}$ and $P^{GEKS}_{03(m=4)} / P^{GEKS}_{02(m=4)} = \sqrt[4]{\left(P^F_{23}\right)^2 P^F_{20}P^F_{03}P^F_{21}P^F_{13}}$

  will in general be different, and this will again differ from $\hat{P}^{GEKS}_{03(m=3)} / P^{GEKS}_{02(m=3)}$ where one RGEKS index is involved; and

- While $P^{GEKS}_{03(m=5)} / P^{GEKS}_{02(m=5)} = P^{GEKS}_{23(m=5)}$ and $P^{GEKS}_{03(m=4)} / P^{GEKS}_{02(m=4)} = P^{GEKS}_{23(m=4)}$ are indices within a (transitive) system of standard GEKS indices (with m = 5 and m = 4 respectively) this

[38] The more general question seems to be: Do (R)GEKS indices properly reflect or distort a trend and/or a cycle in the price data? Is, for example a cycle in RGEKS less pronounced (volatile) than in the series of chain indices or direct indices, and how does the volatility (if there is any) of RGEKS indices depend on the choice of m?

no longer applies to ratios with one or two $\hat{P}^{GEKS}$ terms indicating a rolling approach. In this case we have *a number of different estimates* for the same price change, for example

$$\hat{P}^{GEKS}_{04(m=4)} / P^{GEKS}_{03(m=4)} \neq \hat{P}^{GEKS}_{04(m=3)} / \hat{P}^{GEKS}_{03(m=3)} \neq P^{GEKS}_{04(m=5)} / P^{GEKS}_{03(m=5)} = P^{GEKS}_{34(m=5)}$$

just like ratios $P^F_{04}/P^F_{03}$, $P^F_{14}/P^F_{13}$, $P^F_{24}/P^F_{23}$ and $P^F_{34}$ of the (intransitive) direct Fisher indices will be different. Hence transitivity of the standard GEKS indices (for a given m) is lost when we make use the rolling method.

- Chain indices can be shown to be just a special (limiting) case of RGEKS (the case m = 2). They provide a definite and unequivocal result for an index $P_{st}$ comparing t to s, or for a ratio So as regards avoiding ambiguity Chain indices may be preferred over (R)GEKS indices
- With GEKS indices it is possible that $P^{GEKS}_{t,t+1(m)} \neq 1$ although prices (but not quantities) remained constant so that every reasonable direct index will amount to $P_{t,t+1} = 1$ and also chain indices correctly show $\overline{P}_{0,t+1} = \overline{P}_{0t}$. As we have various standard GEKS indices for the same pair of periods, depending on m the existence of different quantities in the periods under considerations is more likely when is large. To indicate e.g. no change between 2 and 3 as in $P^F_{23} = 1$ with GEKS indices some restrictions are to be observed. To get $P^{GEKS}_{23(m=4)} = 1$ the condition $\sqrt[4]{P^F_{20}P^F_{03}P^F_{21}P^F_{13}} = 1$ or preferably $P^F_{20}P^F_{03} = P^F_{31}P^F_{12}$ or equivalently $P^F_{20}/P^F_{30} = P^F_{31}/P^F_{21}$ should be met. Likewise to get $P^{GEKS}_{23(m=5)}$ requires $P^F_{20}P^F_{21}P^F_{24}$ = $P^F_{30}P^F_{31}P^F_{34}$ and not just $P^F_{20}P^F_{21} = P^F_{30}P^F_{31}$. To see the system behind these formulas remember that $P^F_{22} = P^F_{33} = 1$ and $P^F_{23} = P^F_{32} = 1$, so $P^{GEKS}_{23(m)}$ requires

  $$P^F_{20}P^F_{21}P^F_{22}P^F_{23}P^F_{24}...P^F_{2,m-1} = P^F_{30}P^F_{31}P^F_{32}P^F_{33}P^F_{34}...P^F_{3,m-1}$$

  which poses a lot of restrictions prices and quantities of other periods when m is large. Hence that GEKS indices (unlike direct indices $P^F_{st}$) indicate a rise or fall of prices although the respective prices remained constant in the periods under consideration is anything but unlikely.[39]
- A point worthy of further consideration is a possibly existing relationship between m and the smoothness of a time series of GEKS indices $P^{GEKS}_{01(m)}$, $P^{GEKS}_{02(m)}$,… .[40] The frequently met conjecture of a smoothing effect of the GEKS method appears to be inferred from the fact that the RGEKS index is a geometric mean of indices, and taking a mean will as a rule result in smoothing. However, $P^{GEKS}_{03(m=4)}$ for example is not just a mean of $P^F_{01}$ $P^F_{02}$, and $P^F_{03}$ or of the components of the chain index $\overline{P}^F_{03}$, that is $P^F_{01}$, $P^F_{12}$, and $P^F_{23}$ but of $P^F_{01}$. $P^F_{02}$, $P^F_{03}$, $P^F_{13}$ and $P^F_{23}$, and in $P^{GEKS}_{03(m=5)}$, $P^{GEKS}_{03(m=6)}$ etc. many more indices are included in the geometric mean than just $P^F_{01}$ $P^F_{02}$, and $P^F_{03}$.

39 This can be seen in the numerical example of Ribe referred to in our appendix. In Ribe's example no less than 3 out of 4 GEKS indices which should yield unity (as the corresponding direct indices) fail to do so.

40 Our example (in the appendix) with a regular cycle in the prices did not support this suggestion.

# Appendix

The numerical example in this appendix is designed to demonstrate

- that chain indices are path dependent, and that with chain indices the frequency of updating (i.e. the number of sub-intervals of an interval in time) matters,[41]
- identity (and proportionality) is met by standard GEKS but not necessarily RGEKS indices (depending on the choice of m, the number of periods involved),
- how standard GEKS and (rolling) RGEKS indices provide a number of different (though theoretically equally justified) indices $P_{0t}$ and consequently also of changes (growth rates) in the price level ($P_{0s}/P_{0r}$) depending on the choice of m,
- when prices show a cyclical movement of k periods but no trend, RGEKS indices ($m \neq \lambda k$, $\lambda = 1,2,\ldots$) may well (just like chain indices) fluctuate around a positively or negatively sloped trend, and
- that the not infrequently expressed conjecture that (R)GEKS indices may be less volatile (or a cycle will appear less pronounced with them) than direct or chain indices can well be called in question.[42]

In another numerical example (one of the two scenarios Martin Ribe 2012 presented) we will see that a characteristic development (in this case a constancy of the price level) may not be properly reflected by GEKS (and a fortiori possibly by RGEKS) indices.

## 1. Identity and path dependence ("chain drift")

### *1.1. Chain indices are path dependent*

The following numerical example may serve as an illustration of the identity axiom and path dependence (regarding chain indices) and it will later be used again for some other demonstrations:

Tab. A.1

| t = 0 | | t = 1 | | t = 2 | | t = 3 | | t = 4 | |
|---|---|---|---|---|---|---|---|---|---|
| p | q | p | q | p | q | p | q | p | q |
| 2 | 10 | 4 | 12 | 3 | 20 | 1 | 16 | 2 | 10 |
| 5 | 20 | 3 | 15 | 4 | 10 | 4 | 12 | 5 | 20 |

The direct Laspeyres price index $P_{04}^{L}$ (and also the direct Fisher price index $P_{04}^{F}$ is of course $P_{04}^{L} = P_{04}^{F} = 1$ because all prices (and also quantities) in 4 and 0 are equal (indicated by shadows). The chain Laspeyres index, however, not only violates identity but also yields different results depending on the frequency of chainlinking:

(a) $\overline{P}_{04}^{L} = P_{1}^{L} P_{2}^{L} P_{3}^{L} P_{4}^{L} = 0.7419 \neq 1,$

given a partition of the interval into *four* subintervals (0, 1), ... , (3, 4), and

(b) $\overline{P}_{04}^{L*} = P_{02}^{L} P_{24}^{L} = 11/12 \cdot 9/10 = 0.825,$

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41 We will see later, that this applies also to GEKS indices.

42 In particular there does not seem to be a straightforward relationship between m and the volatility of RGEKS indices. A more general question might be: Do (R)GEKS indices properly reflect or distort a trend and/or a cycle in price data?

with only *two* sub-intervals (0, 2) and (2, 4) of the *same* interval.

Obviously both indices $\overline{P}_{04}^{L} \neq 1$ , and $\overline{P}_{04}^{L*} \neq 1$, though equally valid have a "chain drift" which is different at that, *depending on how the interval is subdivided (or in other words: on the frequency of chaining)*. Hence chain indices are not able to provide a definite (unequivocal) comparison of the prices in 0 and 4. Their result is *"path dependent" (not consistent in temporal aggregation)* which is the very opposite of transitivity.

The corresponding results applying the Paasche and Fisher formula are $\overline{P}_{04}^{P} =$ 0.7591, $\overline{P}_{04}^{F} = 0.7505$, and with two subintervals only $\overline{P}_{04}^{P*} = 1/0.825 = 1.2121$, $\overline{P}_{04}^{F*} = 1$.

By contrast with a truly *transitive* index: for example the unweighted Jevons' indices we get

$\overline{P}_{04}^{J} = P_1^J P_2^J P_3^J P_4^J = \sqrt{\frac{4}{2} \cdot \frac{3}{5}} \cdot \sqrt{\frac{3}{4} \cdot \frac{4}{3}} \cdot \sqrt{\frac{1}{3} \cdot \frac{4}{4}} \cdot \sqrt{\frac{2}{1} \cdot \frac{5}{4}} = 1$ with four sub-intervals, and

$\overline{P}_{04}^{J*} = P_{02}^J P_{24}^J = \sqrt{\frac{3}{2} \cdot \frac{4}{5}} \cdot \sqrt{\frac{2}{3} \cdot \frac{5}{4}} = 1$ with two intervals only

### *1.2. The standard GEKS-index (m = 5) satisfies identity*

It may be useful to present all elements $P_{ij}^F$ used to calculate various indices like $\overline{P}_{04}^F$, and GEKS indices in a table (see tab. A.2). As we are going to make some modifications of this example concerning prices and quantities in period 4 the column four (indices $P_{i4}$) is set apart from the other columns with a grey colour.

Tab. A.2: Fisher indices for the (initial) numerical example

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | $P_{00}^F = 1$ | $P_{01}^F = \sqrt{\frac{155}{198}}$ | $P_{02}^F = \sqrt{\frac{55}{54}}$ | $P_{03}^F = \sqrt{\frac{12}{23}}$ | 1 |
| 1 | $P_{10}^F = \sqrt{\frac{198}{155}}$ | $P_{11}^F = 1$ | $P_{12}^F = \sqrt{\frac{320}{341}}$ | $P_{13}^F = \sqrt{\frac{1152}{2325}}$ | $\sqrt{\frac{198}{155}}$ |
| 2 | $P_{20}^F = \sqrt{\frac{54}{55}}$ | $P_{21}^F = \sqrt{\frac{341}{320}}$ | $P_{22}^F = 1$ | $P_{23}^F = \sqrt{0.4}$ | $\sqrt{\frac{54}{55}}$ |
| 3 | $P_{30}^F = \sqrt{\frac{23}{12}}$ | $P_{31}^F = \sqrt{\frac{2325}{1152}}$ | $P_{32}^F = \sqrt{2.5}$ | $P_{33}^F = 1$ | $\sqrt{\frac{23}{12}}$ |

It can easily be seen that the standard GEKS-method satisfies identity. Not only $P_{04}^F = 1$, but also $P_{04(m=5)}^{GEKS} = 1$ so that $P_{04(m=5)}^{GEKS} = \sqrt[5]{\left(P_{04}^F\right)^2 P_{01}^F P_{14}^F P_{02}^F P_{24}^F P_{03}^F P_{34}^F} = 1$. The reason is that the components of the formula are $\left(P_{04}^F\right)^2 = 1^2 = 1$, $P_{01}^F P_{14}^F = P_{02}^F P_{24}^F = P_{03}^F P_{34}^F = 1$ due to time reversibility of the Fisher index and identical prices in 0 and 4 so that $P_{0k}^L = \frac{1}{P_{k0}^P} = \frac{1}{P_{k4}^P}$, and $P_{0k}^P = \frac{1}{P_{k4}^L}$ for all k = 1, 2, 3.

### *1.3. The RGEKS-index (rolling method) can violate identity*

The first window of RGEKS (m = 4) ends with $P_{03(m=4)}^{GEKS} = \sqrt[4]{\left(P_{03}^F\right)^2 P_{01}^F P_{13}^F P_{02}^F P_{23}^F} = 0.674845$ while the standard GEKS method delivers $\sqrt[5]{\left(P_{03}^F\right)^2 P_{01}^F P_{13}^F P_{02}^F P_{23}^F P_{04}^F P_{43}^F} = 0.684083$.

In order to continue the procedure by estimating $P_{04} = \hat{P}^{GEKS}_{04(m=4)}$ with the rolling method we may multiply $P^{GEKS}_{03(m=4)}$ by $L_{34} = \sqrt[4]{\left(P^F_{34}\right)^2 P^F_{31}P^F_{14}P^F_{32}P^F_{24}}$ or compute $\hat{P}^{GEKS}_{04} = \sqrt[4]{\left(P^F_{03}P^F_{34}\right)^2 P^F_{01}P^F_{14}P^F_{02}P^F_{24}}$ directly. The last formula is particularly useful in that it makes clear why also this RGEKS index satisfies identity (just like $P^{GEKS}_{04(m=5)}$) . Given that prices and quantities in 4 are the same as in 0 the formula amounts to $\hat{P}^{GEKS}_{04(m=4)} = \sqrt[4]{1} = 1$. Evidently $\hat{P}^{GEKS}_{04(m=4)}$ in fact equals $P^{GEKS}_{04(m=5)}$ = $\sqrt[5]{\left(P^F_{04}\right)^2 P^F_{01}P^F_{14}P^F_{02}P^F_{24}P^F_{03}P^F_{34}}$ because $P^F_{04}$ = $P^F_{03}P^F_{34}$ = 1.

However, with m = 3 we get two links, $L_{23}$ and $L_{34}$ and have to compute $P^{GEKS}_{02(m=3)}L_{23}L_{34}$ with $P^{GEKS}_{02(m=3)}$ = $\sqrt[3]{\left(P^F_{02}\right)^2 P^F_{01}P^F_{12}}$ , $L_{23} = \sqrt[3]{\left(P^F_{23}\right)^2 P^F_{21}P^F_{13}}$ , and $L_{34}$ = $\sqrt[3]{\left(P^F_{34}\right)^2 P^F_{32}P^F_{24}}$ . The product of these three factors as well as the computation directly using (14a) gives $\sqrt[3]{\left(P^F_{34}\right)^2 P^F_{02}P^F_{24}P^F_{02}P^F_{23}P^F_{01}P^F_{13}}$ , or equivalently $\sqrt[3]{P^F_{02}P^F_{24}(P^F_{02}P^F_{23}P^F_{34})(P^F_{01}P^F_{13}P^F_{34})}$ . It is only the first product $P^F_{02}P^F_{24}$ that yields unity, the other two factors are $P^F_{02}P^F_{23}P^F_{34} = \sqrt{506/648}$ , and $P^F_{01}P^F_{13}P^F_{34} = \sqrt{102671/138105}$ respectively, so that we end up with $\hat{P}^{GEKS}_{04(m=3)} = 0.913348 \neq 1$.

By the same token with $p_{i5} = p_{i0}$ and $q_{i5} = q_{i0}$ $P^{GEKS}_{05(m=6)} = \sqrt[6]{\left(P^F_{05}\right)^2 P^F_{01}P^F_{15}P^F_{02}P^F_{25}P^F_{03}P^F_{35}P^F_{04}P^F_{45}}$ = 1, because $P^F_{0k}P^F_{k5} = 1$. However, for the rolling indices with m = 4 and m = 3 we get

$\hat{P}^{GEKS}_{05(m=4)}$ = $\sqrt[4]{\left(P^F_{03}P^F_{45}\right)^2 P^F_{01}P^F_{02}P^F_{14}P^F_{25}P^F_{34}P^F_{35}} = \sqrt[4]{\left(P^F_{45}\right)^2 P^F_{01}P^F_{14}P^F_{03}P^F_{34}}$ = $\sqrt[4]{\left(P^F_{01}P^F_{14}P^F_{45}\right)\left(P^F_{03}P^F_{34}P^F_{45}\right)}$, and

$\hat{P}^{GEKS}_{05(m=3)}$ = $\sqrt[3]{\left(P^F_{02}P^F_{45}\right)^2 P^F_{01}P^F_{13}P^F_{23}P^F_{24}P^F_{35}} = \sqrt[3]{P^F_{45}\left(P^F_{02}P^F_{23}\right)\left(P^F_{02}P^F_{24}P^F_{45}\right)\left(P^F_{01}P^F_{13}P^F_{35}\right)}$ respectively, and there is no reason why we should expect $\hat{P}^{GEKS}_{05(m=4)}$ = $\hat{P}^{GEKS}_{05(m=3)}$ = 1.

## *1.4. Variety of GEKS-indices (standard method with different values for m)*

As the example covers the periods 0 to 4 (so that m = 5) it may be quite interesting to compare the result $P^{GEKS}_{0k(m=5)}$ with m = 5 periods for a period k < 4 to the corresponding standard GEKS indices $P^{GEKS}_{0k(m<5)}$. The results are given in tab. A.3:

Tab. A.3 (see also tab A.10)

| | m = 3 | m = 4 | m = 5 |
|---|---|---|---|
| $P^{GEKS}_{02(m)}$ | $\sqrt[3]{\left(P^F_{02}\right)^2 P^F_{01}P^F_{12}}$ = A = 0.955726 | $\sqrt[4]{\left(P^F_{02}\right)^2 P^F_{01}P^F_{12}P^F_{03}P^F_{32}}$ = B = $\sqrt[4]{A^3P^F_{03}P^F_{32}}$ = 0.999250 | $\sqrt[5]{\left(P^F_{02}\right)^2 P^F_{01}P^F_{12}P^F_{03}P^F_{32}P^F_{04}P^F_{42}}$ = C = $\sqrt[5]{B^4P^F_{04}P^F_{42}}$ = 1.001235 |
| $P^{GEKS}_{03(m)}$ | | $\sqrt[4]{\left(P^F_{03}\right)^2 P^F_{01}P^F_{13}P^F_{02}P^F_{23}}$ = D = 0.674845 | $\sqrt[5]{\left(P^F_{03}\right)^2 P^F_{01}P^F_{13}P^F_{02}P^F_{23}P^F_{03}P^F_{32}}$ = $\sqrt[5]{D^4P^F_{03}P^F_{32}}$ = E = 0.684083 |
| $P^{GEKS}_{04(m)}$ | | | 1 |

In addition to $P^{GEKS}_{03(m=4)}$ and $P^{GEKS}_{03(m=5)}$ we have two other estimates of the price change for the same interval, viz. $P^F_{03} = \sqrt{12/23}$ = 0.722315 and $\overline{P}^F_{03}$ = 0.542077. So we have altogether four equally valid estimates for the same change in prices, and they are ranging between 0.54 up to 0.72.

All indices are within the interval between the smallest price relative $p_{13}/p_{10}$ = ½ and the largest, $p_{23}/p_{20}$ = 4/5 = 0.8. This applies in particular to $P_{03}^F$ as the geometric mean of $P_{03}^P$ = 0.69565 and $P_{03}^L$ = 0.75. Note that not only $\overline{P}_{03}^F$ but also both indices, D = $P_{03(m=4)}^{EKS}$ as well as E = $P_{03(m=5)}^{EKS}$ are smaller than $P_{03}^P$.

To see how unlikely it is that all In a similar vein we have three estimates for $P_{02}$, viz A, B, and C coincide (or even more than three if m > 5) note that B = A requires $A = B = P_{03}^F P_{32}^F$, or $P_{02}^F = \sqrt{\frac{P_{03}^F P_{32}^F}{P_{01}^F P_{12}^F}}$, and for A = B = C also $P_{02}^F = \sqrt{\frac{P_{04}^F P_{42}^F}{P_{01}^F P_{12}^F}} = \sqrt{\frac{P_{03}^F P_{32}^F}{P_{01}^F P_{12}^F}}$ must hold.

## 2. Two modifications of the example (standard method m = 5, rolling *m = 4 and m = 3)*

### *2.1 Proportionality (modification 1)*

In what follows we examine two modifications of the numerical example and compute again $P_{04(m=5)}^{EKS}$ or simply $P_{04(5)}^{GEKS} = \sqrt[5]{\left(P_{04}^F\right)^2 P_{01}^F P_{14}^F P_{02}^F P_{24}^F P_{03}^F P_{34}^F}$.

Tab. A.4

| Modification 1 | | | | | |
|---|---|---|---|---|---|
| t = 0 | | t = 3 | | t = 4 | |
| p | q | p | q | p | q |
| 2 | 10 | 1 | 16 | 4 | 10 |
| 5 | 20 | 4 | 12 | 10 | 20 |

| Modification 2 | | | | | |
|---|---|---|---|---|---|
| t = 0 | | t = 3 | | t = 4 | |
| p | q | p | q | p | q |
| 2 | 10 | 1 | 16 | 3 | 15 |
| 5 | 20 | 4 | 12 | 4 | 25 |

Evidently the modifications of the example only concern period 4.

*a) <u>Proportionality with standard GEKS</u>* The first modification refers to proportionality.[43] It is assumed that $p_{i4} = \lambda p_{i0}$ ($\lambda$ = 2 in our numerical example modification 1) and $q_{i4} = q_{i0}$ holds for each commodity i = 1, … , n and periods k = 1, 2, 3 in time. A reasonable index should yield $\lambda$ = 2 as all prices in t = 4 are exactly redoubled prices of t = 0. It turns out that not only $P_{04}^F$ but also all products $P_{01}^F P_{14}^F$, $P_{02}^F P_{24}^F$, and $P_{03}^F P_{34}^F = \sqrt{(12/23)(23/3)}$ uniformly amount to 2, so that $P_{04(m=5)}^{GEKS} = \sqrt[5]{\left(P_{04}^F\right)^2 P_{01}^F P_{14}^F P_{02}^F P_{24}^F P_{03}^F P_{34}^F} = \sqrt[5]{\lambda^5}$ = = $\lambda$ = 2, because $P_{0k}^F P_{k4}^F = \lambda$ and of course $P_{jk}^F P_{kj}^F = 1$. Note that all indices $P_{st}^F$ remain unchanged except $P_{34}^F$. However, for the chain index we now get $\overline{P}_{04}^F$= 1.500944 which is exactly twice the result we had before, viz. 0.750472 but still much less than the required value 2.[44] And as to RGEKS we see what we found regarding identity also applies to the more general property of proportionality: RGEKS passes the test only with m = 4, but it fails with m = 3.

*b) <u>Proportionality and RGEKS (m = 4)</u>* Interestingly with m = 4 (one link $L_{34}$ only) RGEKS and GEKS indices again coincide and yield $\lambda$ = 2. We see that not only the standard GEKS method results in $P_{04(m=5)}^{GEKS}$= 2 as required by proportionality, but also the rolling GEKS index $\hat{P}_{04(m=4)}^{GEKS}$, which is not at all a matter of course. In $\hat{P}_{04(m=4)}^{GEKS}$ only one link $L_{34}$ is involved, and

[43] Proportionality with $p_{it}/p_{i0} = \lambda = 2$ for all i (note that identity is the special case $\lambda$ = 1).

[44] This is not surprising as it is well known that chain indices will as a rule violate proportionality.

it can easily be seen that proportionality (and therefore also identity) is preserved. Multiplying $P^{GEKS}_{03(m=4)}$ by $L_{34}$ gives $\hat{P}^{GEKS}_{04(m=4)}$ as

$$\sqrt[4]{\left(P^F_{03}\right)^2 P^F_{01}P^F_{13}P^F_{02}P^F_{23}\left(P^F_{34}\right)^2 P^F_{31}P^F_{14}P^F_{32}P^F_{24}} = \sqrt[4]{\left(P^F_{03}P^F_{34}\right)^2 P^F_{01}P^F_{14}P^F_{02}P^F_{24}P^F_{13}P^F_{31}P^F_{23}P^F_{32}} = \sqrt[4]{(\lambda)^2\lambda\cdot\lambda\cdot 1\cdot 1} = \lambda$$

So in the case of proportionality all indices coincide $P^{GEKS}_{04(m=5)} = \hat{P}^{GEKS}_{04} = P^F_{04} = \lambda$, and again $\overline{P}^F_{04} \neq \lambda$ unless calculated with two sub-intervals [0, 2] and [2, 4] only. But let's try now $m \neq 4$.

*RGEKS (m = 3)*

The rolling procedure with m = 3 requires two links, $L_{23}$ and $L_{34}$ and results in $\hat{P}^{GEKS}_{04(m=3)}$ = $\sqrt[3]{\left(P^F_{34}\right)^2 P^F_{02}P^F_{24}P^F_{02}P^F_{23}P^F_{01}P^F_{13}} = \sqrt[3]{2\left(P^F_{34}\right)^2 P^F_{02}P^F_{23}P^F_{01}P^F_{13}} = 1.826696 < 2$ (using $P^F_{02}P^F_{24} = 2$). As with the chain index we fall short of 2 and have here the result of sec. 1.3 (where $\hat{P}^{GEKS}_{04(m=3)}$= 0.913348) exactly redoubled.

Tab. A.5: Summary of results of modification 1 (proportionality λ = 2)

| direct | chain* | standard GEKS | rolling GEKS |
|---|---|---|---|
| $P^F_{04}=2$ | $\overline{P}^F_{04} = P^F_{01}P^F_{12}P^F_{23}P^F_{34} = 1.501$<br>$\overline{P}^{F*}_{04} = P^F_{02}P^F_{24} = 2$ | $P^{GEKS}_{04(m=5)} = \sqrt[5]{\left(P^F_{04}\right)^2 P^F_{01}P^F_{14}\dots} = 2$ | $\hat{P}^{GEKS}_{04(m=4)} = 2$<br>$\hat{P}^{EKS}_{04(m=3)} = 1.8267$ |

* = rolling, m = 2

## *2.2. The second modification: GEKS and RGEKS*

*a) <u>Standard GEKS</u>*

In what follows we concentrate on the more interesting unrestricted modification 2 where prices in 4 are not just proportional (or equal) to those in 0. Here we have $P^F_{04} = \sqrt{319/372}$ = 0.926, $P^F_{14} = \sqrt{928/837} = 1.053$, $P^F_{24} = 1$, and $P^F_{34} = \sqrt{87/46}$. With these figures $P^{GEKS}_{04(m=5)}$ can be computed. The result is 0.956570 as opposed to $P^F_{04} = 0.926027$ and $\overline{P}^F_{04} = 0.745490$.

Table A.6

| t | standard m = 4 | standard m = 5 | $P^F_{04}P^F_{4t}$ | $P^F_{0t}$ | $\overline{P}^F_{0t}$ |
|---|---|---|---|---|---|
| 1 | $P^{GEKS}_{01(4)} = 0.95646$ | $P^{GEKS}_{01(5)} = 0.94053$ | 0.879453 | 0.88478 | 0.88478 |
| 2 | $P^{GEKS}_{02(4)} = 0.99925$ | $P^{GEKS}_{02(5)} = 0.98416$ | 0.926027 | 1.00922 | 0,857099 |
| 3 | $P^{GEKS}_{03(4)} = 0.67485$ | $P^{GEKS}_{03(5)} = 0.67455$ | 0.673354 | 0.722315 | 0.542077 |
| 4 | | $P^{GEKS}_{04(5)} = 0.95657$ | | 0.926027 | 0.745490 |

So again it turns out that the chain index yields a considerably smaller result than the direct or the GEKS index. Tab. A.6 permits calculations similar to those made in sec. 4.3 of the main text. We can for example study the conditions that should be fulfilled for $P^{GEKS}_{01(4)} = P^{GEKS}_{01(5)} = X$. In this case $X^5 = \left(P^F_{01}\right)^2 P^F_{02}P^F_{21}P^F_{03}P^F_{31}P^F_{04}P^F_{41}$, and $X^4 = \left(P^F_{01}\right)^2 P^F_{02}P^F_{21}P^F_{03}P^F_{31}$ so that $X^5/X^4 = X$ requires that $P^F_{04}P^F_{41} = P^{GEKS}_{01(4)} = P^{GEKS}_{01(5)} = X$. However $P^F_{04}P^F_{41}$ = 0.879 differs quite a bit from both $P^{GEKS}_{01(4)}$ and $P^{GEKS}_{01(5)}$ (see tab. A.6). In the same manner for $P^{GEKS}_{02(4)} = P^{GEKS}_{02(5)} = Y$ to hold Y should be equal to $P^F_{04}P^F_{42}$ = 0.926. Likewise $P^{EKS}_{03(4)} = P^{EKS}_{32(5)} = Z$ requires $P^F_{04}P^F_{43} = Z$ (however, in tab. A.6 $P^F_{04}P^F_{43}$ = 0.673 and different from both $P^{GEKS}_{03(4)}$ and $P^{GEKS}_{03(5)}$). Hence as a rule we will have different estimates for the same price change depending on m,[45] and will get even more estimates when we consider also RGEKS indices in addition to standard GEKS indices.

*b)* *RGEKS (rolling window m = 4)* The results are displayed in table A.7.

Tab. A.7: Rolling GEKS-indices m = 4

| first window (0 – 3) | | second window (1 – 4) | |
|---|---|---|---|
| GEKS | direct Fisher | GEKS | direct Fisher |
| $P^{GEKS}_{01} = 0.956459$ | $P^F_{01} = 0.884776$ | | |
| $P^{GEKS}_{02} = 0.999250$ | $P^F_{02} = 1.009217$ | $P^{GEKS}_{12} = 1.024054$ | $P^F_{12} = 0.968719$ |
| $P^{GEKS}_{03} = 0.674845$ | $P^F_{03} = 0.722315$ | $P^{GEKS}_{13} = 0.694337$ | $P^F_{13} = 0.703906$ |
| $\hat{P}^{GEKS}_{04} = 0.981434$* | $P^F_{04} = 0.926027$ | $P^{GEKS}_{14} = 1.009785$ | $P^F_{14} = 1.052958$ |

*The link $L_{34}$ is $\sqrt[4]{\left(P^F_{34}\right)^2 P^F_{31}P^F_{14}P^F_{32}P^F_{24}} = P^{GEKS}_{14(m=4)} / P^{GEKS}_{13(m=4)}$, so that $\hat{P}^{GEKS}_{04(m=4)} = P^{GEKS}_{03(m=4)}L_{34}$.

*RGEKS and GEKS*

It is worth being noted that $\hat{P}^{GEKS}_{04}$ (rolling) is different from $P^{GEKS}_{04(m=5)}$ (standard m = 5), $P^F_{04}$, and $\overline{P}^F_{0t}$ in tab. A.6. Also results of RGEKS with m = 3, that is $\hat{P}^{GEKS}_{04(m=3)} = \sqrt[3]{\left(P^F_{02}P^F_{34}\right)^2 P^F_{01}P^F_{13}P^F_{23}P^F_{24}}$ and with m = 4, $\hat{P}^{GEKS}_{04(m=4)} = \sqrt[4]{\left(P^F_{03}P^F_{34}\right)^2 P^F_{01}P^F_{14}P^F_{02}P^F_{24}}$ differ. See tab. A.8 for all results so far.

Tab. A.8: Summary of results of modification 2

| direct and chain | standard and rolling GEKS |
|---|---|
| $P^F_{04} = \sqrt{319/372} = 0.926$<br>$\overline{P}^F_{04} = P^F_{01}P^F_{12}P^F_{23}P^F_{34} = 0.7455$<br>$\overline{P}^{F*}_{04} = P^F_{02}P^F_{24} = P^F_{02} = \sqrt{55/54} = 1.0092$*<br>because $P^F_{24} = 1$ | standard<br>$P^{GEKS}_{04(m=5)} = \sqrt[5]{\left(P^F_{04}\right)^2 P^F_{01}P^F_{14}\ldots} = 0.95657$<br>rolling<br>$\hat{P}^{GEKS}_{04(m=4)} = 0.981434$, $\hat{P}^{EKS}_{04(m=3)} = 0.933313$ |

* We will soon see that standard GEKS indices will also differ with different partitions of the interval.

Interestingly the chain index provides both, the smallest as well as the largest figure depending on the frequency of updating (linking). With two subintervals $\overline{P}^{F*}_{04}$ even exceeds unity. Also standard GEKS indices depend on the partitioning of the interval: we have $\sqrt[3]{\left(P^F_{04}\right)^2 P^F_{02}P^F_{24}}$ = 0.95297 with two sub-intervals, and $\sqrt[5]{\left(P^F_{04}\right)^2 P^F_{01}P^F_{14}P^F_{02}P^F_{24}P^F_{03}P^F_{34}}$ = 0.95667 with the usual four sub-intervals, just like $\overline{P}^{F*}_{04} = P^F_{02}P^F_{24}$ differs from $\overline{P}^F_{04} = P^F_{01}P^F_{12}P^F_{23}P^F_{34}$.

[45] Note also that we have *two GEKS estimates* (as shown in the following table A. 7), *but only one unique series of the direct Fisher index and the (usual) Fisher chain index* (with links $P_{01}$, $P_{12}$, $P_{21}$, …) on the other hand.

### *2.3. Second modification: Some more analysis*

### *Growth rates (change of price level) and links*

As there is a multitude of results regarding the index numbers (levels) there is also no unique figure for the corresponding growth rates. For the change in the prices from 2 to 3 and from 3 to 4 we find a number of different results which ideally should coincide (Tab. A.9). Note that chain indices provide the lowest rate of change for the change 2 → 3, but not for 3 → 4.[46]

Tab. A.9

| from 2 to 3 | from 3 to 4 |
|---|---|
| standard GEKS m = 4* = $\sqrt[4]{\left(P^F_{23}\right)^2 P^F_{20}P^F_{03}P^F_{21}P^F_{13}}$<br>RGEKS <u>m = 4</u><br>first window (0 – 3)*: 0.675351 ($=P^{GEKS}_{03}/P^{GEKS}_{02}$)<br>second window (1 – 4): 0.678032 ($=P^{GEKS}_{13}/P^{GEKS}_{12}$)<br><u>m = 3</u>: second window (1 – 3) $P^{EKS}_{23(m=3)} = \sqrt[3]{\left(P^F_{23}\right)^2 P^F_{21}P^F_{13}}$ = 0.662408 | RGEKS<br><u>m = 4</u>: $L_{34} = \hat{P}^{GEKS}_{04(m=4)}/P^{GEKS}_{03(m=4)} = P^{GEKS}_{14(m=4)}/P^{GEKS}_{13(m=4)}$<br>$= \sqrt[4]{\left(P^F_{34}\right)^2 P^F_{31}P^F_{14}P^F_{32}P^F_{24}}$ = 1.454310<br><u>m = 3</u>: (two links $L_{23}$, $L_{34}$) $\hat{P}^{GEKS}_{04(m=3)}/\hat{P}^{GEKS}_{03(m=3)} =$<br>$\sqrt[3]{\left(P^F_{34}\right)^2 P^F_{32}P^F_{24}}$ = 1.440712 |
| $P^F_{03}/P^F_{02}$ = = 0.715718 or $P^F_{13}/P^F_{12}$ = 0.726636<br>$\overline{P}^F_{03}/\overline{P}^F_{02} = P^F_{23} = \sqrt{0.4}$ = 0.632456 | $P^F_{04}/P^F_{03}$ = 1.282027, $P^F_{14}/P^F_{13}$ = 1.495879, $P^F_{24}/P^F_{23}$ = 1.581138 and $\overline{P}^F_{04}/\overline{P}^F_{03} = P^F_{34}$ = 1.375247 |

* periods 0 to 3 (the standard GEKS m = 4 is here equivalent to RGEKS m = 4, first window )

Note that $\overline{P}^F_{24}/\overline{P}^F_{23} = \overline{P}^F_{04}/\overline{P}^F_{03} = P^F_{34} = \sqrt{\frac{\mathbf{p}_4'\mathbf{q}_4}{\mathbf{p}_3'\mathbf{q}_3}\frac{\mathbf{p}_4'\mathbf{q}_3}{\mathbf{p}_3'\mathbf{q}_4}}$ = 1.375 has to be kept distinct from $P^F_{24}/P^F_{23}$ $= \sqrt{\frac{\mathbf{p}_4'\mathbf{q}_4}{\mathbf{p}_3'\mathbf{q}_3}\frac{\mathbf{p}_4'\mathbf{q}_2}{\mathbf{p}_3'\mathbf{q}_2}\frac{\mathbf{p}_2'\mathbf{q}_3}{\mathbf{p}_2'\mathbf{q}_4}}$ = 1.581, and which in turn is not equal to $P^F_{04}/P^F_{03}$.

In addition to the index-changes listed in tab. A.9 one could of course also compute $P^{GEKS}_{04(m=5)}/P^{GEKS}_{03(m=5)} = P^{GEKS}_{34(m=5)} = \sqrt[5]{\left(P^F_{34}\right)^2 P^F_{30}P^F_{04}P^F_{31}P^F_{14}P^F_{32}P^F_{24}}$ = 1.423, and[47] various GEKS indices with m > 5 (they are in this case of $P_{04}/P_{03}$ identical to the RGEKS indices with the same m), for example $P^{GEKS}_{04(m=6)}/P^{GEKS}_{03(m=6)} = \sqrt[6]{\left(P^F_{34}\right)^2 P^F_{30}P^F_{04}P^F_{31}P^F_{14}P^F_{32}P^F_{24}P^F_{35}P^F_{54}}$ which as a rule will not coincide.

Note that both indices, $P^{GEKS}_{02(m=4)} < 1$ and $P^{GEKS}_{02(m=5)}$ are < 1 while $P^F_{02}$ is $\sqrt{55/54}$ = 1.0092 > 1 and that the result of RGEKS $\hat{P}^{GEKS}_{04}/P^{GEKS}_{03(m=4)}$ = 1.4543 differs from $P^{GEKS}_{04(m=5)}/P^{GEKS}_{03(m=5)}$ = 1.4181 the standard GEKS result (which in turn differs from $P^F_{04}/P^F_{03}$ and $P^F_{34}$).[48] Interestingly in our example we have the same prices (but not quantities) in t = 2 and t = 4 so that $P^F_{24}$ = 1. However,

$P^{GEKS}_{04(m=5)}/P^{GEKS}_{02(m=5)} = \sqrt[5]{\left(P^F_{24}\right)^2 P^F_{20}P^F_{04}P^F_{21}P^F_{14}P^F_{23}P^F_{34}}$ = 0.97197 (or 0.95657/0.98416)[49] (0.971969673), and

46 Interestingly, on the one hand the change is particularly high when judged using direct indices.

47 only when our numerical example would include also periods 5, 6, …

48 So both, the rolling and the standard GEKS method imply a price rise between t = 3 and t = 4 of well over 40% as opposed to less than 40% or even 30% when judged by the direct Fisher index or chained Fisher index respectively.

49 Cf. tab. A.6.

$\hat{P}^{GEKS}_{04}\big/P^{GEKS}_{02(m=4)} = \sqrt[5]{\left(P^F_{24}\right)^2 P^F_{20}P^F_{04}P^F_{21}P^F_{14}P^F_{23}P^F_{34}} = 0.98217_{0258}$ (or 0.981434/0.999250),[50] and the results of $P^{GEKS}_{04(m^*)}\big/P^{GEKS}_{02(m^*}$ with m = m* ≥ 5 will again be different

***Transitivity***

The fact that the RGEKS series $P^{GEKS}_{01(m=4)}$, $P^{GEKS}_{02(m=4)}$, $P^{GEKS}_{03(m=4)}$, $\hat{P}^{GEKS}_{04}$ clearly differs from the standard GEKS system $P^{GEKS}_{01(m=5)}$, $P^{GEKS}_{02(m=5)}$, $P^{GEKS}_{03(m=5)}$, $P^{GEKS}_{04(m=5)}$, and this implies that the rolling system no longer has favorable property of transitivity. This can be seen by the fact that $\hat{P}^{GEKS}_{04(m=4)} = \sqrt[4]{\left(P^F_{03}P^F_{34}\right)^2 P^F_{01}P^F_{14}P^F_{02}P^F_{24}} = 0.9814$ differs from $P^{GEKS}_{04(m=5)} = \sqrt[5]{\left(P^F_{04}\right)^2 P^F_{01}P^F_{14}P^F_{02}P^F_{24}P^F_{03}P^F_{34}}$.[51]

Moreover we see that $\hat{P}^{GEKS}_{04(m=4)}$ can be viewed as product of $P^{GEKS}_{03(m=4,w=1)}$ from the first window and $P^{GEKS}_{34(m=4,w=2)}$ from the second window. However the product of $P^{GEKS}_{02(m=4,w=1)}$ and $P^{GEKS}_{24(m=4,w=1)}$ is different; it is $\sqrt[4]{\left(P^F_{23}P^F_{24}\right)^2 P^F_{01}P^F_{14}P^F_{03}P^F_{34}} = 0.773277$. And the product $P^{GEKS}_{01(m=4,w=1)}$ $P^{GEKS}_{14(m=4,w=1)}$ gives yet another index $\sqrt[4]{\left(P^F_{01}P^F_{14}\right)^2 P^F_{02}P^F_{24}P^F_{03}P^F_{34}} = 0.960693$. It is interesting to see that the difference is quite sizeable, ranging from 0.773277 to 0.981434.

## 3. Cyclical movement in the prices

### *3.1 The numerical example: cycles in the prices and in the RGEKS indices*

In what follows we examine the consequences of a four-periods regular cycle, where for example the price of the first commodity develops as follows: 2, 4, 3, 1, 2, 4, 3, 1, 2, … The assumptions are laid down in the following table:

Tab. A.10

| t = 0, 4, … | | t = 1, 5, … | | t = 2, 6, … | | t = 3, 7, … | |
|---|---|---|---|---|---|---|---|
| p | q | p | q | p | q | p | q |
| 2 | 10 | 4 | 12 | 3 | 20 | 1 | 16 |
| 5 | 20 | 3 | 15 | 4 | 10 | 4 | 12 |

This gives rise to a table (tab. A.10) of binary Fisher indices which reveals a regular repetitive pattern.

For the numerical example we only have to consider the six different indices (marked in yellow colour) which form the first 4 columns of matrix of tab. A.11.

For a rolling GEKS method using m = 4 we see that the same links will appear repeatedly, and we get for example for the first link (because the first window covers periods 0, .. , 3 so that a link is needed for the first time to arrive at $\hat{P}^{GEKS}_{04(m=4)}$ from $P^{GEKS}_{03(m=4)}$):

[50] Cf. tab. A.6. The correct result should be unity (as all reasonable direct indices are $P_{24} = 1$).

[51] Cf. tab A.8. and text below tab. A.9.

Tab. A.11: Fisher price indices in a four-period regular cycle
(The superscript F is dropped for convenience of presentation)

| | 0 | 1 | 2 | 3 | 4 = 0 | 5 = 1 | 6 = 2 | 7 = 3 |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | $P_{01}$ | $P_{02}$ | $P_{03}$ | 1 | $P_{01}$ | $P_{02}$ | $P_{03}$ |
| 1 | $1/P_{01}$ | 1 | $P_{12}$ | $P_{13}$ | $1/P_{01}$ | 1 | $P_{12}$ | $P_{13}$ |
| 2 | $1/P_{02}$ | $1/P_{12}$ | 1 | $P_{23}$ | $1/P_{02}$ | $1/P_{12}$ | 1 | $P_{23}$ |
| 3 | $1/P_{03}$ | $1/P_{13}$ | $1/P_{23}$ | 1 | $1/P_{03}$ | $1/P_{13}$ | $1/P_{23}$ | 1 |
| 4 = 0 | 1 | $P_{01}$ | $P_{02}$ | $P_{03}$ | 1 | $P_{01}$ | $P_{02}$ | $P_{03}$ |
| 5 = 1 | $1/P_{01}$ | 1 | $P_{12}$ | $P_{13}$ | $1/P_{01}$ | 1 | $P_{12}$ | $P_{13}$ |
| 6 = 2 | $1/P_{02}$ | $1/P_{12}$ | 1 | $P_{23}$ | $1/P_{02}$ | $1/P_{12}$ | 1 | $P_{23}$ |
| 7 = 3 | $1/P_{03}$ | $1/P_{13}$ | $1/P_{23}$ | 1 | $1/P_{03}$ | $1/P_{13}$ | $1/P_{23}$ | 1 |

$$L_{34} = L_{78} = L_{11,12} = \ldots = \sqrt[4]{\left(P_{30}^{F}\right)^{2} \frac{P_{32}^{F}P_{31}^{F}}{P_{02}^{F}P_{01}^{F}}} = \sqrt[4]{\left(P_{30}^{F}\right)^{2} P_{31}^{F}P_{10}^{F}P_{32}^{F}P_{20}^{F}} = P_{30(m=4)}^{GEKS}.$$

Likewise the second link $L_{45} = P_{01(m=4)}^{GEKS}$ serves also as $L_{89}$, $L_{12,13}$ etc. in a cycle as follows:

m = 4 (the first link is $L_{34}$)

| $L_{34} = L_{78} = \ldots$ | $L_{45} = L_{89} = \ldots$ | $L_{56} = L_{9,10} = \ldots$ | $L_{67} = L_{10,11} = \ldots$ |
|---|---|---|---|
| ① $P_{30(m=4)}^{GEKS}$ | ② $P_{01(m=4)}^{GEKS}$ | ③ $P_{12(m=4)}^{GEKS}$ | ④ $P_{23(m=4)}^{GEKS}$ |
| 1.481822 | 0.956459 | 1.044739 | 0.675351 |

It turns out that the product of these links amounts to unity. Hence just like the direct Fisher index the series of RGEKS indices will show a cycle provided that m exactly coincides with the length of the completely regular cycle in the price movement.

Before examining the series of RGEKS indices with various lengths $m \neq 4$ it seems useful to see how the links $L_{23}$, $L_{34}$ etc. in the rolling method are in fact simply GEKS indices for various windows, referred to as $P_{st(m,w)}^{GEKS}$, where m is again the length of the window and w the number of the window in the sequence of windows. This will be demonstrated with the formulas in the case of m = 3, 6, and 12 in table A.12.

Also for a rolling GEKS method using m = 3 the same links again reappear after 4 periods, however, now (unlike in the case of m = 4)

- the product of the links (POL) differs from unity, and
- the links are also no longer simply standard GEKS indices of the first window only.

Tab. A.12: Relation between links Lt,t+1 and GEKS-indices
(notation $P^{GEKS}_{23(m,w)}$ where w = length of window, and m = number of the window)

| m = 3 | m = 6 | m = 12 |
|---|---|---|
| w \| periods involved<br>1 \| 0 1 2<br>2 \| 1 2 3<br>3 \| 2 3 4<br>4 \| …….3 4 5 | 1 \| 0 1 2 3 4 5<br>2 \| 1 2 3 4 5 6<br>3 \| 2 3 4 5 6 7 | 1 \| 0 1 2 3 … 10 11<br>2 \| 1 2 3 … 10 11 12<br>3 \| 2 3 … 10 11 12 13 |
| $L_{23} = \sqrt[3]{\left(P^F_{23}\right)^2 P^F_{21}P^F_{13}} = P^{GEKS}_{23(3,2)}$<br>$L_{34} = \sqrt[3]{\left(P^F_{34}\right)^2 P^F_{32}P^F_{24}} = P^{GEKS}_{34(3,3)}$ *)<br>$L_{45} = \sqrt[3]{\left(P^F_{45}\right)^2 P^F_{43}P^F_{35}} = P^{GEKS}_{45(3,4)}$<br>$L_{56} = \sqrt[3]{\left(P^F_{56}\right)^2 P^F_{54}P^F_{46}} = P^{GEKS}_{56(3,5)}$<br>$= P^{GEKS}_{12(3,1)}$ | $L_{56} = \sqrt[6]{\left(P^F_{56}\right)^2 P^F_{51}P^F_{16}\ldots P^F_{51}P^F_{16}}$<br>$= P^{GEKS}_{56(6,2)}$<br>$L_{67} = \sqrt[6]{\left(P^F_{67}\right)^2 P^F_{62}P^F_{27}\ldots P^F_{65}P^F_{57}}$<br>$= P^{GEKS}_{67(6,3)}$<br>$L_{78} = P^{GEKS}_{78(6,4)}$<br>$L_{89} = P^{GEKS}_{89(6,5)}$ | $L_{11,12} =$<br>$\sqrt[12]{\left(P^F_{11,12}\right)^2 P^F_{11.2}P^F_{2,12}\ldots P^F_{11,102}P^F_{10,12}} =$<br>$\sqrt[12]{\left(\left(P^F_{30}\right)^2 P^F_{31}P^F_{10}P^F_{32}P^F_{20}\right)^3}$<br>$= P^{GEKS}_{11,12(12,2)} = P^{GEKS}_{78(8,2)} = P^{GEKS}_{34(4,2)}$<br>$L_{12,13} = P^{GEKS}_{45(4,3)}$<br>$L_{13,14} = P^{GEKS}_{56(4,4)}$<br>$L_{14,15} = P^{GEKS}_{67(4,4)}$ |

*) note that $P_{34} = P_{30}$ and $P_{24} = P_{20}$

Tab. A.13: Links for the rolling method and GEKS indices
The numbers ①, ②,… indicate the sequence in which new links are needed for obtaining the RGEKS results (for m = 4 see above)

| | m = 3 | m = 4 | m = 5 |
|---|---|---|---|
| $L_{23}$ | ① $P^{GEKS}_{23(3,2)} = \sqrt[3]{\left(P^F_{23}\right)^2 P^F_{21}P^F_{13}} = 0.662408$ | | |
| $L_{34}$ | ② $P^{GEKS}_{34(3,3)} = \sqrt[3]{\left(P^F_{30}\right)^2 P^F_{32}P^F_{20}} = 1.442704$ * | ① $P^{GEKS}_{30(4,1)} = P^{GEKS}_{34(4,2)}$ | |
| $L_{45}$ | ③ $P^{GEKS}_{45(3,4)} = \sqrt[3]{\left(P^F_{01}\right)^2 P^F_{03}P^F_{31}} = 0.929593$ * | ② $P^{GEKS}_{01(4,1)} = P^{GEKS}_{45(4,3)}$ | ① $P^{GEKS}_{45(5,2)} = 0.941672$ |
| $L_{56}$ | ④ $P^{GEKS}_{56(3,5)} = \sqrt[3]{\left(P^F_{12}\right)^2 P^F_{10}P^F_{02}} = 1.022937$ * | ③ $P^{GEKS}_{12(4,1)} = P^{GEKS}_{56(4,4)}$ | ② $P^{GEKS}_{56(5,3)} = 1.029072$ |
| $L_{67}$ | | ④ $P^{GEKS}_{23(4,1)} = P^{GEKS}_{67(4,5)}$ | ③ $P^{GEKS}_{67(5,4)} = 0.666546$ |
| $L_{78}$ | | | ④ $P^{GEKS}_{78(5,5)} = 1.461812$ |

* cf. tab. A12 for the formulas of the GEKS indices. These link indices can also be written as indicated above in tab. A.12 (because the price indices for period 4, 5 and 6 equal those of periods 0, 1, and 2), but there is no m = 3 window which covers the four periods 0, 1, 2, 3.

In the following table A.14 (displayed in fig. A.1 and A.2) the results of the RGEKS indices for m = 4 are reported together with some other indices, in particular with those where m < 4 and m > 4. The striking difference is that indices $P^{GEKS}_{04(m=4)} = P^{GEKS}_{08(m=4)} = \ldots = 1$ just like the $P^F_{0t}$ indices, whereas RGEKS indices with m = 3 and m = 5 don't follow this pattern and show a (negatively sloped) trend much like the chain index $P^F_{0t}$.

Table A.14

| t | rolling m = 4 | rolling m = 3 | rolling m = 5 | $\overline{P}^F_{0t}$ * | $P^F_{0t}$ |
|---|---|---|---|---|---|
| 1 | 0.956459 | 0.934296 | 0.941672 | 0.884776 | 0.884776 |
| 2 | 0.999250 | 0.955726 | 1.001235 | 0.857099 | 1.009217 |
| 3 | 0.674845 | 0.633081 | 0.684083 | 0.542077 | 0.722315 |
| 4 | 1 | 0.913348 | 1 | 0.750472 = $\overline{P}^F_{04}$ | 1 |
| 5 | 0.956459 | 0.849042 | 0.941672 | 0.663999 | 0.884776 |
| 6 | 0.999250 | 0.868517 | 0.969049 | 0.643229 | 1.009217 |
| 7 | 0.674845 | 0.575312 | 0.645915 | 0.406814 | 0.722315 |
| 8 | 1 | 0.830006 | 0.944206 | 0.563208 = $\left(\overline{P}^F_{04}\right)^2$ | 1 |
| 9 | 0.956459 | 0.771568 | 0.889133 | 0.498313 | 0.884776 |
| 10 | 0.999250 | 0.789265 | 0.914982 | 0.482725 | 1.009217 |
| 11 | 0.674845 | 0.522816 | 0.609877 | 0.305302 | 0.722315 |
| 12 | 1 | 0.754268 | 0.891525 | 0.422672 = $\left(\overline{P}^F_{04}\right)^3$ | 1 |
| 13 | 0.956459 | 0.701163 | 0.839525 | 0.373970 | 0.884776 |
| 14 | 0.999250 | 0.717245 | 0.863931 | 0.362271 | 1.009217 |
| 15 | 0.674845 | 0.475109 | 0.575850 | 0.229121 | 0.722315 |
| 16 | 1 | 0.685442 | 0.841784 | 0.317203 = $\left(\overline{P}^F_{04}\right)^4$ | 1 |
| 17 | 0.956459 | 0.637182 | 0.792684 | 0.280654 | 0.884776 |
| 18 | 0.999250 | 0.651797 | 0.815729 | 0.271874 | 1.009217 |
| 19 | 0.674845 | 0.431756 | 0.543721 | 0.171949 | 0.722315 |
| 20 | 1 | 0.622896 | 0.794818 | 0.238052 = $\left(\overline{P}^F_{04}\right)^5$ | 1 |

* It is interesting to see that in this numerical example after each cycle the chain index is only three quarter of what it was before, because the chain index is ≈ ¾ = 0.75. After two full cycles the level is reduced to about $(¾)^2$ = 9/16 = 0.5625 (The exact value according to the table is 0.5632).

In the case of m = 3 the product of the links (POL)[52] amounts to $\sqrt[3]{P^F_{01}P^F_{12}P^F_{23}P^F_{30}} = \sqrt[3]{\overline{P}^F_{04}} =$ 0.908751 (note that $P^F_{30} = P^F_{34}$). Hence each cycle ends up with a value 9.12% less (1 - 0.90876 = 0.09124): 0.830 is 9.12% less than 0.9113, and 0.754 is in turn 9.12% less than 0.830 etc.

The RGEKS procedure with m = 3 for example of linking starts with $P^{GEKS}_{02(m=3)} = \sqrt[3]{\left(P^F_{02}\right)^2 P^F_{01}P^F_{12}} =$ 0.955726. Multiplying $P^{GEKS}_{02(m=3)}$ by $L_{23}$ = $P^{GEKS}_{23(3,1)}$ = 0.662408 gives $\hat{P}^{GEKS}_{03(m=3)}$ = 0.633081 as opposed to $P^{EKS}_{03(m=4)} = \sqrt[4]{\left(P^F_{03}\right)^2 P^F_{01}P^F_{13}P^F_{02}P^F_{23}} = 0.674845$.

In the situation of m = 5 the first window covers periods 0, 1, 2, 3, and 4, so that the first link needed is $L_{45} = \sqrt[5]{P^F_{01}\left(P^F_{01}\right)^2 P^F_{02}P^F_{21}P^F_{03}P^F_{31}} = \sqrt[5]{P^F_{01}C}$ where C is equal to $\left(P^{GEKS}_{01(m=4)}\right)^4$, and $L_{45}$ may be written as $P^{GEKS}_{45(5,2)}$. In the same manner $L_{56}$ = $P^{GEKS}_{56(5,3)}$. Again the product of the links (POL) is no longer unity but $\sqrt[5]{P^F_{01}P^F_{12}P^F_{23}P^F_{30}} = \sqrt[5]{\overline{P}^F_{04}} = 0.944206$, so that the index declines over each cycle by

52 Cf. tab. A.16 for details about the regular pattern of the four links and their product for various values of m.

5.6%. Hence after four full cycles from t = 4 up to t = 20 the value is only $(0.944206)^4$ = 0.7948, that is 20.52% less than in t = 4.

### *3.2 Trend and smoothing*

Such observations suggest that the time series of the index numbers will possess an ever less negatively sloped trend, i.e. is becoming more and more horizontal, as m increases. However, the values m = 4, m = 8, …. in the sequence of increasing values of m seem to form an exception in which no negative slope exists.

Figure A.1 (time series of price indices, data of tab. A.14)

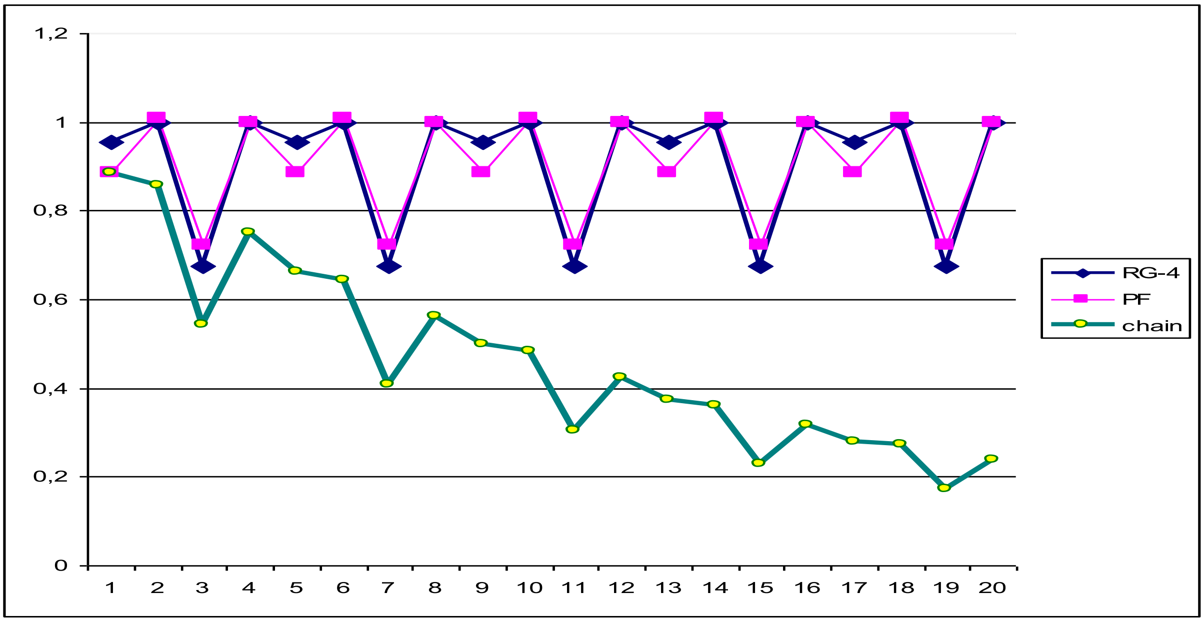



Fig. A.1 shows how PF = $\overline{P}^F_{0t}$ and RG-4 = $P^{GEKS}_{0t(m=4)}$ oscillate regularly around the mean with no trend (or a horizontal trend) while chain = $\overline{P}^F_{0t}$ is clearly characterized by a negatively sloped trend (see fig. A.2). Obviously the amplitude in $\overline{P}^F_{0t}$ is continually decreasing and tab. A.15 indeed confirms that the standard deviation is decreasing over four adjacent cycles of four periods (or "steps") in which the level of the index is constantly decreasing.

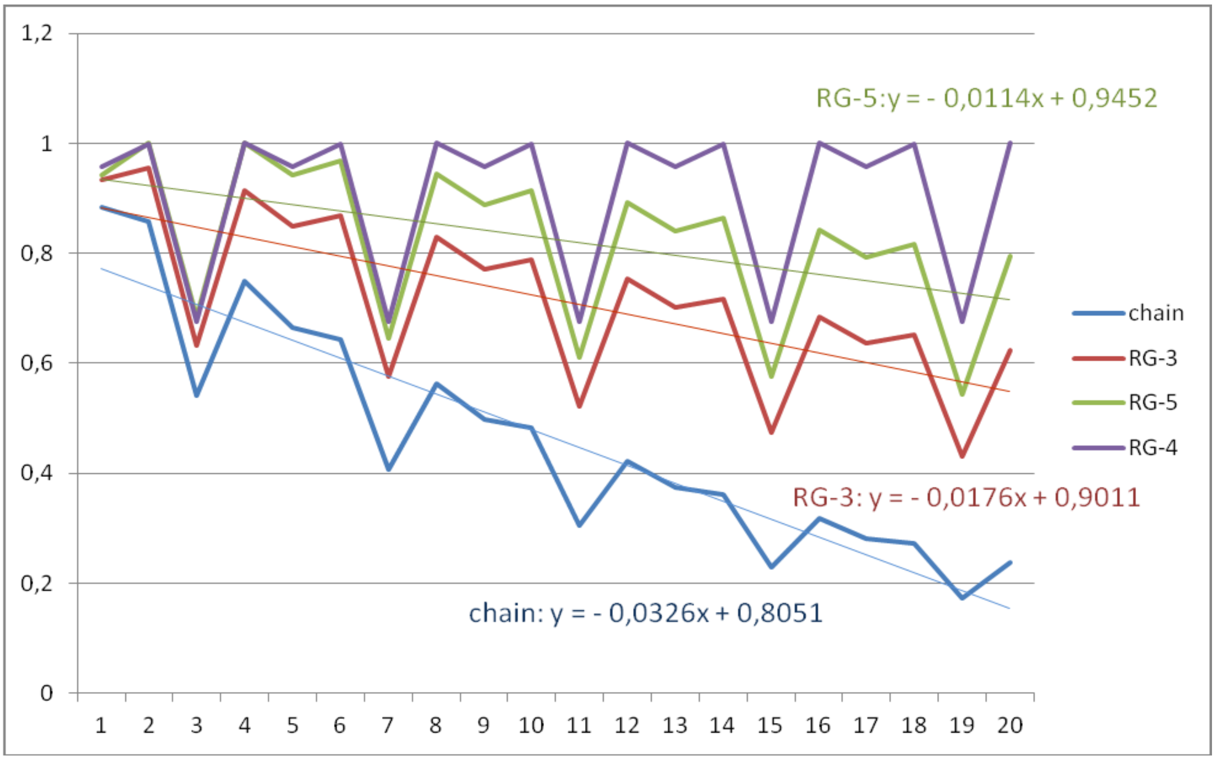
Tab. A.15 (cycles of $\overline{P}^F_{0t}$)

| | t = 0 to t = 3 | 4 to 7 | 8 to 11 | 12 to 15 | 16 to 19 |
|---|---|---|---|---|---|
| 1 | 1 | 0.750472 | 0.563208 | 0.422672 | 0.317203 |
| 2 | 0.884776 | 0.663999 | 0.498313 | 0.373970 | 0.280654 |
| 3 | 0.857099 | 0.643229 | 0.482725 | 0.362271 | 0.271874 |
| 4 | 0.542077 | 0.406814 | 0.305302 | 0.229121 | 0.171949 |
| mean | 0.820988 | 0.616129 | 0.326774 | 0.347008 | 0.253675 |
| std.dev. | 0.906084 | 0.784939 | 0.571641 | 0.589074 | 0.503661 |

Fig. A.2 shows in perhaps an even more pronounced way that the trend seems to be a function of m (except for those m's that are multiples of 4 and where the time series of RGEKS index numbers looks quite similar to the direct Fisher price index as displayed in fig. A.1).

In fig. A.2 also the trend-functions are being reported and it can be seen that the slope changes from – 0.0326 in the case of the chain index $P^F_{0t}$ (or equivalently the $P^{GEKS}_{0t(m=2)}$ index) to – 0.0114 which confirms the conjecture above that in principle a greater m causes a less negatively sloped trend. This is also in line with the results concerning the standard deviations (see table below fig. A.2).

Figure A.2: Trends in RGEKS indices




| | $\overline{P}^{F}_{0t}$ | $P^{GEKS}_{0t(m=3)}$ | $P^{GEKS}_{0t(m=5)}$ | $P^{GEKS}_{0t(m=4)}$ | $P^{F}_{0t}$ |
|---|---|---|---|---|---|
| std.dev. | 0.204381 | 0.146785 | 0.138556 | 0.135554 | 0.115829 |
| slope (trend) | - 0.032556 | - 0.017626 | - 0.011444 | - 0.000728 | 0.000221 |

Also the pattern followed by the links (in particular the product of the links, POL) is quite similar to what we found for the trends. With the exception of m = 4 and m = 8 the POL seems to be continually increasing (and tending to 1) as m increases from m = 3 to m = 9.

Tab. A.16: The regular pattern of the four links for GEKS indices of various m

| | m = 3 | m = 4 | m = 5 | m = 6 | m = 7 | m 8 | m = 9 [b)] |
|---|---|---|---|---|---|---|---|
| $L_{23}$ | 0.662408 | | | | | | |
| $L_{34}$ | 1.442704 | 1.481822 | | | | | |
| $L_{45}$ | 0.929593 | 0.956459 | 0.941672 | | | | |
| $L_{56}$ | 1.022937 | 1.044739 | 1.029072 | 1.018758 | | | |
| $L_{67}$ | 0.662408 | 0.675351 | 0.666546 | 0.660739 | 0.669774 | | |
| $L_{78}$ | 1.442704 | 1.481822 | 1.461812 | 1.448622 | 1.464929 | 1.481822 | |
| $L_{89}$ | 0.929593 | 0.956459 | 0.941672 | 0.931941 | 0.991093 | 0.956459 | 0.948216 |
| $L_{9,10}$ | 1.022937 | 1.044739 | 1.029072 | 1.018758 | 1.035339 | 1.044739 | 1.036006 |
| $L_{10,11}$ | 0.662408 | 0.675351 | 0.666546 | 0.660739 | 0.669774 | 0.675351 | 0.670445 |
| $L_{11,12}$ | 1.442704 | 1.481822 | 1.461812 | 1.448622 | 1.464929 | 1.481822 | 1.470672 |
| POL [a)] | $\sqrt[3]{P^{F}_{04}} =$ 0.908751 | 1 | $\sqrt[5]{P^{F}_{04}} =$ 0.944206 | $\sqrt[3]{P^{F}_{04}} =$ 0.908751 | $\sqrt[7]{P^{F}_{04}} =$ 0.959822 | 1 | $\sqrt[9]{P^{F}_{04}} =$ 0.968608 |

a) POL = product of the four links
b) m = 10 generates exactly the same pattern of links as does m = 5 (much like m = 8 and m=4)

With m = 2 the POL is of course $\overline{P}^F_{04}$ = 0.750472. Note that the POL is the same in m = 6 and m = 3 although the links themselves are a bit different. For all values of m we find the same periods of a rather high growth of the price level (this applies to the transitions 3 → 4, 7 → 8, 11 → 12 etc. indicated by orange colour) and the same periods with a rather large decline of the price level (this applies to the transitions 2 → 3, 6 → 7, 10 → 11 etc., blue coloured fields).

As tab. A.16 already showed, the links in the RGEKS approach with m = 6 (for the $\hat{P}^{GEKS}_{0t(m=6)}$ indices) differ from those of m = 3 (for the $\hat{P}^{GEKS}_{0t(m=3)}$ indices). For example the first link needed for $\hat{P}^{GEKS}_{06(m=6)}$ is $L_{56(m=6)} = \sqrt[6]{\left(P^F_{56}\right)^2 \frac{P^F_{16}P^F_{26}P^F_{36}P^F_{46}}{P^F_{15}P^F_{25}P^F_{35}P^F_{45}}}$ = 1.018758. In the case of m = 3, however, the equivalent link $L_{56(m=3)}$ is given by $\sqrt[3]{\left(P^F_{56}\right)^2 \frac{P^F_{46}}{P^F_{45}}}$ = = 1.022937 (in the case of m = 3 it is the fourth link after having already gained $\hat{P}^{GEKS}_{03(m=3)}$, $\hat{P}^{GEKS}_{04(m=3)}$, and $\hat{P}^{GEKS}_{05(m=3)}$ in the rolling [chainlinking] manner).

So evidently the expressions $L_{56(m=6)}$ and $L_{56(m=3)}$ are a bit different, although the product of all four links $L_{56}$, …, $L_{89}$ is identically $\sqrt[3]{P^F_{04}}$ . Consequently also the resulting series of the indices (gained in the standard GEKS manner up to $P_{0,m-1}$, or gained by linking for t > m-1) are different, as can be seen in tab. A. 17:

Tab. A.17:
RGEKS indices for various values of m (in grey fields for t ≥ m index is gained by linking)

| | m = 4 | m = 8 |
|---|---|---|
| $P_{01}$ | 0.956459 | 0.956459 |
| $P_{02}$ | 0.999250 | 0.999250 |
| $P_{03}$ | 0.674845 | 0.674845 |
| $P_{04}$ | 1 | 1 |
| $P_{05}$ | 0.956459 | 0.956459 |
| $P_{06}$ | 0.999250 | 0.999250 |
| $P_{07}$ | 0.674845 | 0.674845 |
| st.dev. | 0,151245 | |
| slope | 0,140025 | |

| m = 3 | m = 6 | m = 7 |
|---|---|---|
| 0.934296 | 0.931941 | 0.946897 |
| 0.955726 | 0.975630 | 0.965897 |
| 0.633081 | 0.696626 | 0.668323 |
| 0.913348 | 1 | 1 |
| 0.849042 | 0.931941 | 0.946897 |
| 0.868517 | 0.94923 | 0.980339 |
| 0.575312 | 0.627321 | 0.656619 |
| 0,151762 | 0,147697 | 0,150277 |
| - 0,036979 | - 0,026120 | - 0,021206 |

It can easily be seen why the rolling method with m = 4 and m = 8 provides the same indices irrespective of whether gained from the standard approach ($P_{01}$ through $P_{07}$ in the case of m = 8) or by linking (as for example $P_{04}$ through $P_{07}$ in the case of m = 4).[53]

The standard GEKS index m = 8 for $P_{01}$ reads as follows

$P^{GEKS}_{01(m=8)} = \sqrt[8]{\left(P^F_{01}\right)^2 P^F_{02}P^F_{21}P^F_{03}P^F_{31}P^F_{01} \cdot \left(P^F_{04}P^F_{41}P^F_{05}P^F_{51}P^F_{06}P^F_{61}\right)}$, and due to the circularity, the second factor (in brackets), that is $P^F_{04}P^F_{41}P^F_{05}P^F_{51}P^F_{06}P^F_{61}$ can be written as

[53] A similar situation is given with m = 10 relative to m = 5, but not – as just mentioned – in the case m = 6 as compared to m = 3.

$1\cdot P^F_{01}P^F_{01}\cdot 1\cdot P^F_{02}P^F_{21}P^F_{03}P^F_{31} = \left(P^F_{01}\right)^2 P^F_{02}P^F_{21}P^F_{03}P^F_{31}$, so that the whole expression boils down to $P^{GEKS}_{01(m=8)} = \sqrt[8]{\left(P^F_{01}\right)^2 P^F_{02}P^F_{21}P^F_{03}P^F_{31}P^F_{01}\cdot\left(P^F_{01}\right)^2 P^F_{02}P^F_{21}P^F_{03}P^F_{31}P^F_{01}} = \sqrt[8]{\left(P^{GEKS}_{01(m=4)}\right)^4\left(P^{GEKS}_{01(m=4)}\right)^4} = P^{GEKS}_{01(m=4)}$. And by the same token we have $P^{GEKS}_{02(m=8)} = P^{GEKS}_{02(m=4)}$ etc.

## 4. Another look at the numerical example from M. Ribe

We finally come to some interesting observations from a numerical example presented in a paper for Ottawa Group Meeting 2011 by Martin Ribe (Statistics Sweden)[54]. His example is characterized by some sharp price changes between two periods and also constant prices over two periods. Also Ribe made the assumption that there is a repetitive (cyclical) movement of prices and quantities in that way that prices and quantities in t = 4, 5, … are the same as those in t = 0,1, …. (see tab. A.18).

Tab. A.18 Numerical example of Ribe $P^F_{0t}$, $\overline{P}^F_{01} = \sqrt{13/55}$

| t | $p_1$ | $p_2$ | $q_1$ | $q_2$ |
|---|---|---|---|---|
| 0 | 100 | 100 | 10 | 10 |
| 1 | 30 | 100 | 100 | 10 |
| 2 | 30 | 100 | 20 | 10 |
| 3 | 100 | 100 | 2 | 10 |

| | 1 | 2 | 3 | 4 (= 0) |
|---|---|---|---|---|
| 0 | $\sqrt{13/55}$ | $\sqrt{26/75}$ | 1 | 1 |
| 1 | | 1 | $\sqrt{165/53}$ | $\sqrt{55/13}$ |
| 2 | | | $\sqrt{225/106}$ | $\sqrt{75/26}$ |
| 3 | | | | 1 |

Because of the equality of 4 and 0 we have $P_{14} = (P_{41})^{-1} = (P_{01})^{-1}$ etc.

The noteworthy feature of this example is that prices remain constant between 1 and 2 so that $P^F_{12} = 1$ and likewise $P^F_{03} = 1$, as well as $P^F_{04} = P^F_{34} = 1$

On the other hand we have standard GEKS indices (m = 4 and m = 5) as follows:

| $P_{0t}$ | GEKS (m = 4) | GEKS (m = 5) |
|---|---|---|
| $P^{GEKS}_{12}$ | 1.100482 | 1.121759 |
| $P^{GEKS}_{03}$ | 0.926184 | 0.940498 |
| $P^{GEKS}_{34}$ | $L_{34}$ = 1.0848 | 1.063267 |

and all these figures in the table ($P^{GEKS}_{12}$, $P^{GEKS}_{03}$, and $P^{GEKS}_{34}$) should amount to unity (as the corresponding $P^F_{st}$ correctly does). The situation is different, however, as regards $P^{GEKS}_{04}$. Note that $L_{34} = \left(P^{GEKS}_{03}\right)^{-1} = \sqrt[4]{\left(P^F_{30}\right)^2 P^F_{31}P^F_{10}P^F_{32}P^F_{20}}$, and for this reason $\hat{P}^{EKS}_{04(m=4)} = P^{EKS}_{03(m=4)}\ L_{34} = 1$ (as in the table below). Hence for $\hat{P}^{EKS}_{04(m=4)}$, or $P^{GEKS}_{04(m=5)}$ we in fact end up with unity as it should be, and the oddity now lies in $\overline{P}^F_{0t}$ because $\overline{P}^F_{04} \neq 1$ although $P^F_{04} = 1$:

| t | $P^{GEKS}_{0t(m=3}$ | $P^{GEKS}_{0t(m=4)}$ | $P^{GEKS}_{0t(m=5)}$ | $P^F_{0t}$ | $\overline{P}^F_{0t}$ |
|---|---|---|---|---|---|
| 1 | 0.518218 | 0.529948 | 0.520888 | 0.486172 | 0.486172 |
| 2 | 0.552371 | 0.583198 | 0.584311 | 0.588784 | 0.486172 |
| 3 | 1.023471 | 0.926184 | 0.940498 | 1 | 0.708319 |
| 4 | 1.077153 | 1 | 1 | 1 | 0.708319 |

Note that $\overline{P}^F_{0t}$ rightly remains constant so that $\overline{P}^F_{02}$ = $\overline{P}^F_{01}$ = 0.486 because $P_{12}$, = 1; also $\overline{P}^F_{04} = \overline{P}^F_{03} = 0.708$.

See fig. A.3 for a graphical representation of this result (where G3 = $P^{GEKS}_{0t(m=3}$, PF = $P^F_{0t}$ etc.).

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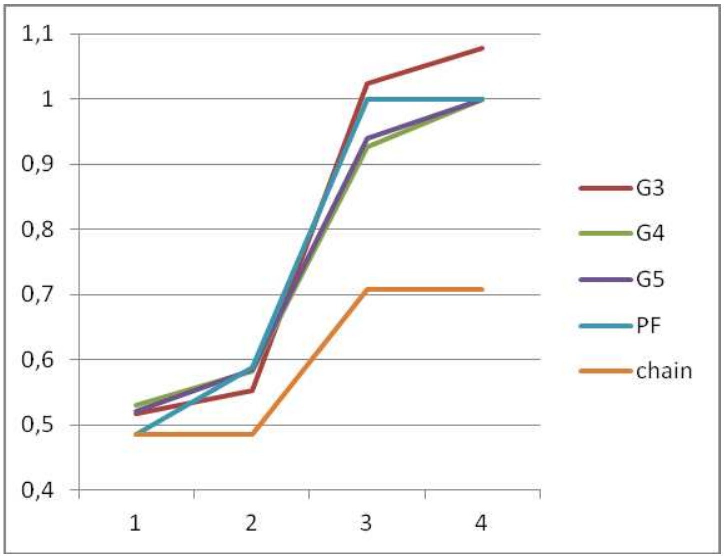
54 available in the internet (paper of 2012), and quoted in the references above.

Fig. A.3




Interestingly GEKS indices indicate a change (where constancy prevails for example $P^{GEKS}_{02(m=4)} > P^{GEKS}_{01(m=4)}$ despite $P_{12} = 1$). In summary, what makes Ribe's example most interesting is the fact that (R)GEKS indices can be > 1 or < 1 although prices did not change.

This raises the question: Can the opposite situation ($P^{GEKS}_{st(m)} = 1$ despite different prices in s and t, say 0 and 1, so that $P^F_{st} = P^F_{01} \neq 1$) occur?

With m = 2 (the chain index) this is definitely not possible, because as $P^{GEKS}_{01(m=2)} = \overline{P}^F_{01} = P^F_{01}$ holds by definition $P^{GEKS}_{01(m=2)} = \overline{P}^F_{01}$ =1 is incompatible with $P^F_{01} \neq 1$. However, with m = 3 $P^{GEKS}_{01(m=3)} = \sqrt[3]{\left(P^F_{01}\right)^2 P^F_{02} P^F_{21}}$ = 1 requires $\left(P^F_{01}\right)^2 = P^F_{12} P^F_{20} \neq 1$ because $P^F_{01} \neq 1$ by assumption and from $P^{GEKS}_{01(m=4)} = \sqrt[4]{\left(P^F_{01}\right)^2 P^F_{02} P^F_{21} P^F_{03} P^F_{31}}$ =1 follows $\left(P^F_{01}\right)^2 = P^F_{12} P^F_{20} P^F_{13} P^F_{30} \neq$ 1 and in the same manner $P^{GEKS}_{01(m=5)} = 1$ is tantamount to $\left(P^F_{01}\right)^2 = P^F_{12} P^F_{20} P^F_{13} P^F_{30} P^F_{14} P^F_{40} \neq 1$ etc., and there seems to be no reason why such constellations of index numbers should not occur. Perhaps they tend to be more likely - in principle at least - when m is getting larger.

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