Asymptotic relations in Cournot's game

Guerrazzi, Marco

Department of Economics, University of Genoa

November 2012

ASYMPTOTIC RELATIONS IN COURNOT'S GAME

Marco Guerrazzi*
Department of Economics
University of Genoa

Abstract

In this note, I derive the asymptotic relation verified by oligopolists' iso-profit curves within Cournot's game. Thereafter, I provide an economic rationale for such a mathematical relation. The results of this exploration suggest that for each firm the asymptotes of the iso-profit curves convey the boundaries beyond which output competitors become net purchasers of the good supplied in the market.

JEL Classification: C72.
Keywords: Cournot's Game; Nash Equilibrium; Asymptotes.

1 Introduction

Cournot's (1838) output game is one of the most influential model of oligopoly. As it is well known, in the standard version of this framework a limited set of firms compete one another for the quantity of output supplied in the market of a given good without knowing the production decisions of their competitors.[1] The model predicts that firms will not coordinate their output decisions by achieving an output array that in terms of profits results in being inferior with respect to the one that would be achieved in the case of (monopolistic) coordination. Extensive reviews can be found (inter alia) in Varian (1992) and Mas-Colell et al. (1995).

*Assistant Professor, Department of Economics, University of Genoa, Via F. Vivaldi n. 5, 56126 Genoa (Italy), Phone +39 010 2095702, Fax +39 010 2095269, e-mail guerrazzi@economia.unige.it

[1]Hahn (1987) and, more recently, Guerrazzi (2012) use this theoretical setting to frame a problem of efficiency-wage competition.

This simple framework has been developed in different directions. For example, under product differentiation, Sing and Vives (1984) show that Cournot competition with substitutes is the dual of Bertrand (1883) competition with complements and viceversa. Moreover, Kopel (1996) explores the implications of different out-of-equilibrium adjustments towards the Nash equilibrium. Furthermore, exploiting a spatial strategic framework, Mayer (2000) analyses Cournot (1838) output competition by allowing for different production costs across locations.

In this note, I develop a not-widely stressed feature of Cournot's (1838) game. Specifically, under the assumption that there are only two firms, I derive the asymptotic relation that holds for oligopolists' iso-profit curves. Thereafter, I provide an economic rationale for such a mathematical relation. The results of this theoretical exploration reveal that for each firm iso-profit curves' asymptotes convey the boundaries beyond which output competitors become net purchasers of the good supplied in the market.

The note is arranged as follows. Section 2 reviews the mathematics of Cournot's (1838) game. Section 3 provides an economic rationale for the asymptotic relation that bounds iso-profit curves. Finally, section 4 concludes.

2 Mathematics

Suppose that in the market of the Q good takes place a quantity-competition between two firms indexed by $i = 1, 2$. In this market, the function of (inverse) total demand is given by

$$p = M - Q \tag{1}$$

where Q is given by the sum of the output produced by two firms, i.e., $q_1 + q_2$, while M is a positive constant that conveys the reservation price of the concerned good.

2.1 Firm 1's problem

Firm 1's reaction curve can be derived by solving the following problem:

$$\max_{q_1} \pi_1 \equiv (M - q_1 - q_2)\, q_1 - Cq_1 \tag{2}$$

where $C > 0$ is the value of the average and marginal cost that I assume equal for both firms.

The first-order condition (FOC) for problem in (2) is given by

$$\frac{\partial \pi_1}{\partial q_1} = M - C - 2q_1 - q_2 = 0 \tag{3}$$

As a consequence, the reaction curve of firm 1 has the following (linear) specification:

$$q_1 = \frac{1}{2}\left(M - C - q_2\right) \quad \text{or} \quad q_2 = M - C - 2q_1 \tag{4}$$

In other words, the reaction function provides the output best response of firm 1 for each possible level of output produced by firm 2. Its negative slope suggests that for each firm the output produced by the competitor is a strategic substitute. This means that whenever firm 2 increases (decreasesI its output offer, it is in the best interst of firm 1 to reduce (increase) its production level.

Taking into account the result in (2), firm 1's iso-profit curves are parametric equations like the one that follows:

$$q_2\left(\overline{\pi}_1, q_1\right) = M - C - q_1\left(1 + \frac{\overline{\pi}_1}{q_1^2}\right) \tag{5}$$

where $\overline{\pi}_1$ is a positive constant.

Considering different levels of $\overline{\pi}_1$, i.e., different firm 1's level of profit, the parametric equation in (5) represents a set of hyperbola that can be traced out in a (q_1, q_2) diagram whose domain of definition is given by $\{q_1 \in \Re^+ : q_1 \neq 0\}$.

The asymptotes of (5) can be derived as follows. First, the limit

$$\lim_{q_1 \to 0^+} M - C - q_1\left(1 + \frac{\overline{\pi}_1}{q_1^2}\right) = -\infty \tag{6}$$

conveys that

$$q_1 = 0 \tag{7}$$

is a vertical asymptote for the set of hyperbola in (5). Second, the limit

$$\lim_{q_1 \to +\infty} M - C - q_1\left(1 + \frac{\overline{\pi}_1}{q_1^2}\right) = -\infty \tag{8}$$

suggests that (5) might have an oblique asymptote of the following linear form:

$$q_2 = n + mq_1 \tag{9}$$

where n is the vertical intercept while m is the slope of the asymptote itself.

The parameters of the linear function in (9) can be found as follows:

$$m = \lim_{q_1 \to +\infty} \frac{M - C - q_1 \left(1 + \frac{\overline{\pi}_1}{q_1^2}\right)}{q_1} = -1 \tag{10}$$

$$n = \lim_{q_1 \to +\infty} M - C - q_1 \left(1 + \frac{\overline{\pi}_1}{q_1^2}\right) - m q_1 = M - C \tag{11}$$

Taking into account the result in (10) and (11), (9) becomes

$$q_2 = M - C - q_1 \tag{12}$$

An illustration is given in figure 1.

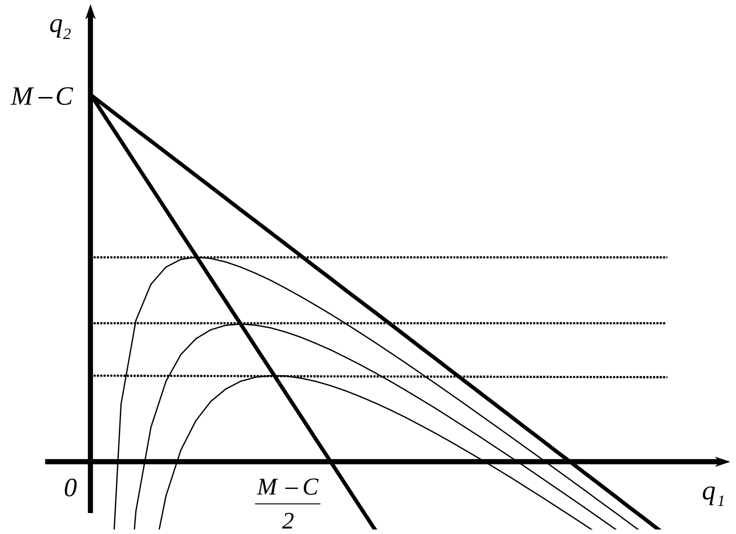

Figure 1: Asymptotes of firm 1's iso-profit curves.

It is worth noting that higher (lower) values of $\overline{\pi}_1$ lead to lower (higher) iso-profit curves. Moreover, the diagram in figure 1 shows that each iso-profit curve reaches its maximum at the intersection point with the reaction curve in (4). This result can be formally proved as follows. First, taking into account that

$$\partial \frac{M - C - q_1 \left(1 + \frac{\overline{\pi}_1}{q_1^2}\right)}{\partial q_1} = 0 \Leftrightarrow q_1 = \pm\sqrt{\overline{\pi}_1} \tag{13}$$

Thereafter, the equality

$$M - C - q_1 \left(1 + \frac{\overline{\pi}_1}{q_1^2}\right) = M - C - 2q_1 \tag{14}$$

holds for the same values of q_1 conveyed in (13).

The same conclusion can be also achieved by exploiting a graphical argument. Consider the diagram in figure 2.

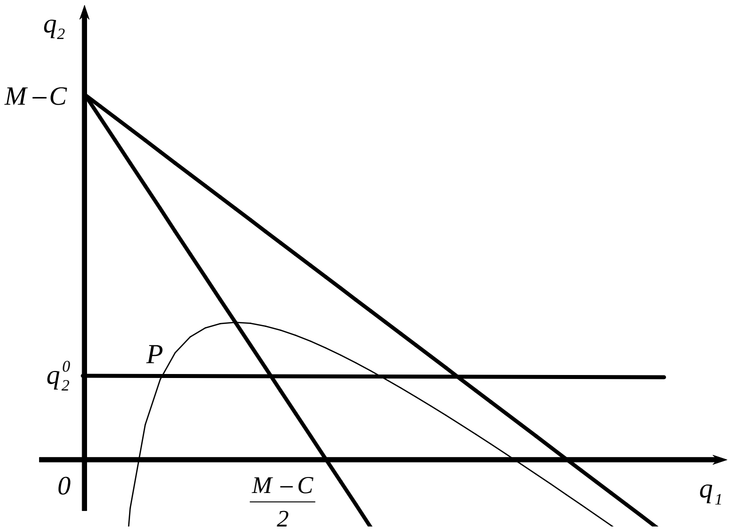

Figure 2: Intersection between iso-profit and reaction curves.

The situation illustrated in figure 2 is inconsistent with the definition of reaction curve detailed in (4). Suppose that firm 2's output is equal to q_2^0. Thereafter, if the iso-profit curve does not reach its maximum level in P, then the reaction curve does not provide either the output best response of firm 1.

2.2 Firm 2's problem

Firm 2's reaction curve can be derived by solving the following problem:

$$\max_{q_2} \pi_2 \equiv (M - q_1 - q_2)\, q_2 - Cq_2 \tag{15}$$

The FOC for problem in (15) is given by

$$\frac{\partial \pi_2}{\partial q_2} = M - C - 2q_2 - q_1 = 0 \tag{16}$$

As a consequence, the reaction curve of firm 2 has the following specification:

$$q_2 = \frac{1}{2}(M - C - q_1) \quad \text{or} \quad q_1 = M - C - 2q_2 \tag{17}$$

Taking into account the result in (14), firm 2's iso-profit curves are as follows:

$$q_1(\overline{\pi}_2, q_2) = M - C - q_2\left(1 + \frac{\overline{\pi}_2}{q_2^2}\right) \tag{18}$$

where $\overline{\pi}_2$ is a positive constant.

Rotating the (q_1, q_2) diagram with respect to the vertical axis and following the procedure implemented for firm 1, it is possible to show that

$$q_2 = 0 \tag{19}$$

is a horizontal asymptote for the set of hyperbola in (18) while

$$q_2 = M - C - q_1 \tag{20}$$

is its oblique asymptote.[2]

An illustration is given in figure 3.

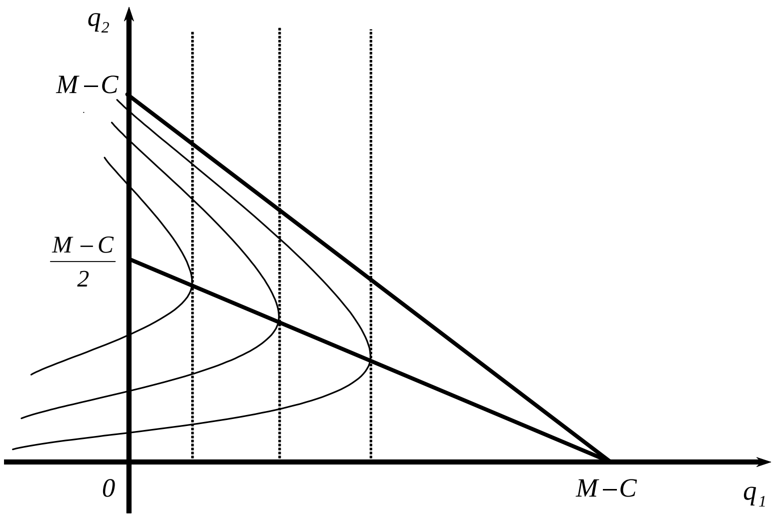

Figure 3: Asymptotes of firm 2's iso-profit curves.

[2]As far as a (q_1, q_2) diagram is concerned, the line such that $q_2 = 0$ is a horizontal asymptote for (18).

It is worth noting that higher (lower) values of $\overline{\pi}_2$, lead the iso-profit curves to shift towards left (right).

2.3 Nash equilibrium

The Nash equilibrium of Cournot's (1838) game is found in the point of intersection between the reaction curves of the 2 firms. See figure 4.

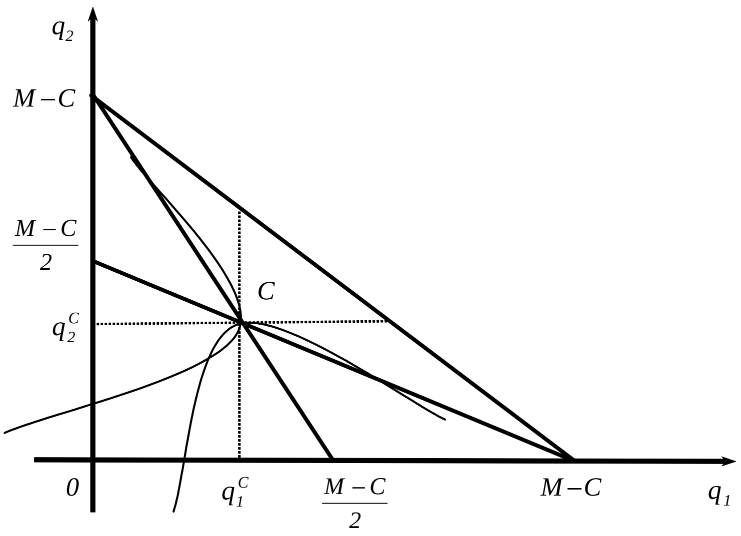

Figure 4: Nash equilibrium of Cournot's game.

It is worth noting that in the Nash equilibrium of Cournot's (1838) game there are two iso-profit curves that intersect each other. Obviously, this means that the Nash equilibrium is non-cooperative.[3] Moreover, under the assumption that each firm reacts to the output of the other with a lag of one period, i.e., assuming that the two firms play Cournot's(1838) game as a game of alternate output offer, it is possible to show that the Nash equilibrium is stable.[4] In other words, starting from an arbitrary allocation different from $\left\{q_1^C, q_2^C\right\}$, the output supply of the 2 firms will converge towards the point where the reaction functions intersect each other.

Taking into account the results in (4) and (17), the (identical) output level produced by the two firms in the Nash equilibrium is given by

[3]Indeed, cooperation would require tangency of isoprofit curves.

[4]Formally, $\left|\frac{\partial q_1}{\partial q_2}\right|\left|\frac{\partial q_2}{\partial q1}\right| = \left|-\frac{1}{2}\right|\left|-\frac{1}{2}\right| = \frac{1}{4} < 1$ (e.g. Kopel 1996).

$$q_i^C = \frac{M - C}{3} \qquad i = 1, 2 \tag{21}$$

Furthermore, taking into account the results in (2), (15) and (21), the (identical) profit level achieved by the two firms in the Nash equilibrium is given by

$$\pi_i^C = \left(\frac{M - C}{3}\right)^2 \qquad i = 1, 2 \tag{22}$$

3 An economic rationale

The asymptotic relationships derived in section 2 can be also rationalized from an economic point of view. In what follows, I consider the situation of firm 1. Obviously, the argument for firm 2 is a specular reflection.

First, consider the vertical asymptote in (7). Take a given level of profit - say $\overline{\pi}_1^0$ - with its iso-profit curve. Whenever firm 1 produces a very low output level (e.g. near to zero), $\overline{\pi}_1^0$ can be actually achieved if and only if firm 2 produces the output level conveyed by (5). For a given level of q_1, firm 1's profit increases if and only if the output level produced by firm 2 decreases. In the limit, for values of $\overline{\pi}_1^0$ above a certain threshold - and taking q_1 as constant, it becomes necessary for firm 2 to put forward a negative output supply (i.e. become a net purchaser of the good) in order to actually achieve those profit levels.

Furthermore, consider the oblique asymptote in (12). Taking into account the result in (1), it can be easily verified that that (12) traces out all the pairs (q_1, q_2) such that the price of the good is equal to the marginal and average cost. In the theoretical framework developed in this note, price values in the neighbourhood of the average and marginal cost allow to reach positive profits if and only if firm 2 makes a negative output supply, i.e., if and only if firm 2 becomes a net purchaser of the good. Obviously, the higher $\overline{\pi}_1$, the higher the output demand from firm 2.

4 Concluding remarks

In this note, I derive the asymptotic relation that holds for oligopolists' iso-profit curves within Cournot's (1938) game. Thereafter, I provide an economic rationale for such a mathematical relation. The results of this theoretical exploration suggest that for each firm iso-profit curves' asymptotes convey the boundaries beyond which output competitors become net purchasers of the good supplied in the market.

References

[1] Bertrand, J. (1883), Revue de la Theorie Mathematique de la Richesse Sociale et des Recherches sur les Principles Mathematiques de la Theorie des Richesses, *Journal des Savants*, pp. 499-508.

[2] Cournot A. (1838), *Recherces sur les Principes Mathematiques de la Theorie de la Richesse.* Paris: L. Hachette.

[3] Guerrazzi M. (2012), On Involuntary Unemployment: Notes on Efficiency-Wage Competition, *MPRA Paper*, No. 38140.

[4] Hahn F.H. (1987), On Involuntary Unemployment, *Economic Journal*, Supplement: Conference Papers, pp. 1-16.

[5] Kopel M. (1996), Simple and Complex Adjustment Dynamics in Cournot Duopoly Models, *Chaos, Solitons & Fractals*, Vol. 7, No. 12, pp. 2031-2048.

[6] Mas-Colell A., M. D. Whinston and J.R. Green (1995), *Microeconomic Theory*, New York: Oxford University Press.

[7] Mayer T. (2000), Spatial Cournot Competition and Heterogeneous Production Costs Across Locations, *Regional Science and Urban Economics*, Vol. 30, No. 3, pp. 325–352.

[8] Sing N. and X. Vives (1984), Price and Quantity Competition in a Differentiated Duopoly, *RAND Journal of Economics*, Vol. 15, No. 4, pp. 546-554

[9] Varian H.R. (1992), *Microeconomic Analysis,* 3rd Edition, New York: W.W. Norton & Company.