BSFTDWithMultiJump Model and Pricing of Quanto FTD with FX Devaluation Risk

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Abstract
We present a new model for pricing Quanto FTD where the FX could be strongly dependent to some or all credit names. The model assumes lognormal hazard rate and deterministic FX local volatility where the FX spot can jump at time of first to default and where the jump size depends on credit name reference. We present the model, the calibration algorithm, and the Quanto FTD pricing. This model is an extension of the model BSWithJump[1] for pricing Quanto CDS with FX devaluation risk.

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Introduction

In this paper, we present a new model for pricing quanto FTD and walk away (in case of FTD) cross currency swaps. Example: pricing of FTD on (BRAZIL, MEXICO, and Microsoft) with USD as natural currency but paid in real (Brazil currency). Ideally, we would like the model to address two issues. Firstly, we would like the model to take into account the different dependencies between the constituents of the portfolio and the FX. In the example above, we expect the FX spot USDBRL to jump more at time of default if the credit name who defaulted first is Brazil than if the FTD name is Microsoft. Secondly, we would like the model to be compatible with the standard pricing model used for FTD. In other words, the model assumes that the FTD prices are an input.

The model assumes lognormal dynamic for FTD spreads and lognormal FX but with a jump at time of default that depends on the first to default credit name. This model is a natural extension of BSWithJump model to the multi case where the credit reference is FTD instead of CDS. In particular, when the jump size does not depends on the FTD name, the model is equivalent to BSWithJump model.

In the first section, we describe briefly the pricing of FTD under the Gaussian copula model with single correlation, which is assumed to imply the FTD market.

In the second section, we present the main results of the recent work of T.R.Bielecki, M.Jeanblanc, and M.Rutkowski where they extend the hazard process approach to multi-names in their recent paper [4]. The main results that we will be using in BSFTDWithMultiJump are the properties of what they call the FTD intensities or what we call the joint intensities of single credit names.

In the third section, we present the dynamic of the FTD intensity, the joint intensities, and the FX spot. We have chosen a lognormal dynamic for the FTD intensity and deterministic conditional intensities (defined as the ratios of the joint intensities to the FTD intensity). Also, the joint intensities are lognormal processes proportional to the FTD intensity. However, The FX spot has a lognormal dynamic with a jump at time of first to default that depends on the credit name who defaulted first.

In the fourth section, we present the pricing formulas for FTD within this model, and we show that the FTD pricing in this model is equivalent to the pricing with LN model (single name model) of a synthetic CDS with time dependent recovery.

In the fifth section, we present the pricing formulas for FX options.

In the sixth section, we present the calibration algorithms of BSFTDWithMultiJump to the term structure of FTD and to the term structure of FX implied volatilities. We show that the calibration of the model to FTD is similar to the calibration of LN model to a synthetic CDS with time dependent recovery. However, the calibration of the implied volatility term structure is different from the calibration of BSWithJump to implied volatilities because the pricing formulas are more complex. In addition, the calibration algorithm is fast, very accurate and, based on an iterative algorithm that allows us to achieve very small calibration errors.

In the seventh section, we describe how we can price a quanto FTD within this model and show that the pricing is equivalent to the pricing of a quanto synthetic CDS with time dependent recovery. Also, similarly to the model BSWithJump, we can use one forward PDE to calculate the term structure of quanto FTD survival probabilities and quanto FTD.

In the last section, we show some examples where we see the calibration accuracy and robustness for different FTD baskets and market data.

1. Pricing FTD under Gaussian copula model

We assume that τ_i is the default time of the name *i* and, g and g_i are Gaussian variables.

$$\tau_i = \inf\left\{t, U^i = N\left(h_i\right) \le q^i\left(t\right)\right\}$$

$$h_i = \alpha g + \sqrt{1-\alpha^2}\, g_i$$

$q^i\left(t\right)$ is the survival probability of the credit name *i*.

The correlation between h_i and h_j is $\rho = \alpha^2$

1.1. The FTD time distribution

Let us calculate the distribution of the FTD time using the fact that the default times are conditionally independent:

$$Q\left(\tau^{ftd} > T\right) = Q\left(\tau_1 > T, ..., \tau_n > T\right) = E\left(\prod_{i=1}^{n} Q\left(\tau_i > T \mid g\right)\right)$$

$$= \int_{-\infty}^{\infty} \prod_{i=1}^{n} \phi^i\left(t, x\right) n\left(x\right) dx$$

$$\phi^i\left(t, x\right) = N\left(\frac{N^{-1}\left(q^i\left(t\right)\right) - \alpha x}{\sqrt{1-\alpha^2}}\right)$$

$$n\left(x\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

1.2. The joint distribution of the FTD time and the FTD indicator

The joint distribution of the FTD time and the FTD indicator is given by:

$$Q\left(I = i, t < \tau^{ftd} \le t + dt\right) = Q\left(\tau_1 > t + dt, ..., t < \tau_i \le t + dt, ..., \tau_n > t + dt\right)$$

$$= Q\left(\tau_1 > t + dt, ..., \tau_i > t, ..., \tau_n > t + dt\right)$$

$$-Q\left(\tau_1 > t + dt, ..., \tau_i > t + dt, ..., \tau_n > t + dt\right)$$

$$= \int_{-\infty}^{\infty} \left(\phi^i\left(t, x\right) - \phi^i\left(t + dt, x\right)\right) \prod_{j \ne i} \phi^j\left(t + dt, x\right) n\left(x\right) dx$$

1.3. Pricing the recovery leg

The recovery leg of a first to default pays $\left(1 - R_I\right)$ at the first to default time (if it occurs before the maturity T) where I is the index of the name who defaulted.

The price of the recovery leg is given by:

$$RFTD(T) = E^Q\left(e^{-\int_0^{\tau^{ftd}} r_u du}\,(1-R_I)1_{\{\tau^{ftd}\le T\}}\right)$$
$$= \int_0^T e^{-\int_0^t r_s ds}\sum_{i=1}^n (1-R_i)Q\left(I=i, t<\tau^{ftd}\le t+dt\right)dt$$
$$= \sum_{i=1}^n (1-R_i)\int_0^T e^{-\int_0^t r_s ds}Q\left(I=i, t<\tau^{ftd}\le t+dt\right)dt$$
$$= \sum_{i=1}^n RFTD_i(T)$$

The price of the recovery leg of FTD_i is given by

$$RFTD_i(T) = (1-R_i)\int_0^T e^{-\int_0^t r_s ds}\int_{-\infty}^{\infty}\left(\phi^i(t,x)-\phi^i(t+dt,x)\right)\prod_{j\ne i}\phi^j(t+dt,x)\,n(x)\,dx\,dt$$

1.4. Pricing the riskyBPV

Given a schedule $\{T_0=0, T_1,\dots,T_N=T\}$, the FTD riskybpv price is given by:

$$RiskyBPV(T) = \sum_{j=1}^N (T_j-T_{j-1})B(0,T_j)Q(\tau^{ftd}>T_j) + \int_0^T (t-T_{\kappa(t)})e^{-\int_0^t r_s ds}Q(t<\tau^{ftd}\le t+dt)$$
$$= \sum_{j=1}^N (T_j-T_{j-1})B(0,T_j)\int_{-\infty}^{\infty}\prod_{i=1}^n\phi^i(T_j,x)n(x)\,dx$$
$$+\int_0^T (t-T_{\kappa(t)})e^{-\int_0^t r_s ds}\left[\int_{-\infty}^{\infty}\prod_{i=1}^n\phi^i(t,x)n(x)\,dx - \int_{-\infty}^{\infty}\prod_{i=1}^n\phi^i(t+dt,x)n(x)\,dx\right]$$

2. Hazard Process Approach

In this section, we present the main results of the multi-names hazard process approach as presented by T.R.Bielecki, M.Jeanblanc, and M.Rutkowski in their recent paper [4].

2.1. Definitions and notations

τ_i is the default time of the name *i* which is a strictly positive random variable.

$N_t^i = 1_{\{\tau_i\le t\}}$ is the default indicator of the name *i*.

H_t^i is the filtration generated by the process N_t^i.

H_t is the filtration generated by all the processes N_t^i.

G_t is the filtration generated by all the processes N_t^i and the Brownian filtration F_t.

We introduce the conditional joint survival process $G(u_1,\dots,u_n;t)$:

$$G(u_1,\dots,u_n;t) = Q(\tau_1>u_1,\dots,\tau_n>u_n \mid F_t)$$

2.2. First To Default Intensity

Let us set $\tau^{ftd} = \tau_1\wedge\dots\wedge\tau_n$ the first to default time.

Let us define the process $G_{ftd}(t;t)$ by setting:

$$G_{ftd}(t;t) = G(t,\dots,t;t) = Q(\tau_1 > t,\dots,\tau_n > t \mid F_t) = Q(\tau^{ftd} > t \mid F_t)$$

We define λ_t^{ftd} the intensity of the FTD time

$$\lambda_t^{ftd} = \lim_{h\to 0} \frac{1}{h} \frac{Q(t < \tau^{ftd} \le t+h \mid F_t)}{Q(\tau^{ftd} > t \mid F_t)}$$

The process $\hat{M}$ defined by:

$$\hat{M}_t = N_t^{ftd} - \Lambda_{t\wedge\tau^{ftd}} = N_t^{ftd} - \int_0^{t\wedge\tau^{ftd}} \lambda_u^{ftd} du = N_t^{ftd} - \int_0^t \left(1 - N_u^{ftd}\right) \lambda_u^{ftd} du$$

is a G_t martingale.

2.3. Joint Intensities

We have the following result, in which we introduce the first-to-default intensity (or the joint-intensity of the name credit name *i*) h_t^i and the associated martingale $\hat{M}_t^i$ for each credit name $i = 1,\dots,n$.

Lemma1

For any $i = 1,\dots,n$, the process h_t^i given by

$$h_t^i = \lim_{h\to 0} \frac{1}{h} \frac{Q(t < \tau_i \le t+h, \tau^{ftd} > t \mid F_t)}{Q(\tau^{ftd} > t \mid F_t)}$$ **is well defined and the process**

$\hat{M}_t^i = N_{t\wedge\tau^{fftd}}^i - \int_0^{t\wedge\tau^{ftd}} h_t^i du$ **is a G_t martingale.**

We have, the equalities $\sum_{i=1}^n h_t^i = \lambda_t^{ftd}$ and $\sum_{i=1}^n \hat{M}_t^i = \hat{M}_t$.

Lemma2

We define the F_t -predictable vector of processes $\left(Y_{\tau^{ftd}}^i\right)_{1\le i\le n}$, real valued processes, where $Y_{\tau^{ftd}}^i$ represent the discounted value of recovery received at time τ^{ftd} id default occurs prior to or at T and the ith name is the first defaulted name, that is, on the event $\{\tau_i = \tau^{ftd} \le T\}$

We have the following equality:

$$E^Q\left(\sum_{i=1}^n 1_{\{t<\tau^{ftd}=\tau_i\le T\}} Y_{\tau^{ftd}}^i \mid G_t\right) = 1_{\{\tau^{ftd}>t\}} \frac{1}{G_{ftd}(t;t)} E^Q\left(\int_t^T \sum_{i=1}^n Y_u^i h_u^i G_{ftd}(u;u) du \mid F_t\right)$$

$$= 1_{\{\tau^{ftd}>t\}} E^Q\left(\int_t^T \sum_{i=1}^n Y_u^i h_u^i e^{-\int_t^u \lambda_s^{ftd}} du \mid F_t\right)$$

The proof on these two lemmas could be found in [4].

The intensity h_t^i is different from the default intensity λ_t^i , which satisfies the property $M_t^i = N_{t\wedge\tau_i}^i - \int_0^{t\wedge\tau_i} \lambda_t^i du$ is a G_t -martingale.

2.4. Conditional Intensities

We define the conditional intensity of the name *i*:

$$x_t^i = \frac{h_t^i}{\sum_{j=1}^n h_t^j} = \frac{h_t^i}{\lambda_t^{ftd}}$$

In single name case, there is one conditional intensity, which is deterministic and equal to 1.

3. BSFTDWithMultiJump model

3.1. Definitions and notations

$S_t^{d/loc}$ is the FX spot where d is the domestic currency (typically a G7) and loc is the foreign currency (typically: an emerging market currency).
In all the paper, we assume that the interest rates are deterministic**.**
We denote $B^d\left(0,T\right)$, and $B^{loc}\left(0,T\right)$ respectively the domestic zero coupon and local currency zero coupon respectively with maturity *T*.
We denote $F_t^{d/loc} = \dfrac{S_0^{d/loc} B^{loc}\left(0,t\right)}{B^d\left(0,t\right)}$ the FX forward.
We assume that we are given the term structures FTD survival probabilities and the term structure of FTD prices (the prices could be given by a Gaussian copula or by any other pricing model).
$N_t^i = 1_{\{\tau_i \le t\}}$ is the default indicator of the credit name *i*.
h_t^i is the joint-intensity of the credit name *i* as defined in the previous section. The intensity h_t^i is such as the process $\hat{M}_t^i = N^i_{t\wedge\tau^{ftd}} - \int_0^{t\wedge\tau^{ftd}} h_t^i du$ is a G_t-martingale.
x_t^i is the conditional intensity of the credit name *i* defined as $x_t^i = \dfrac{h_t^i}{\lambda_t^{ftd}}$

3.2. Modelling the default intensity: LNFTD model

We suppose that the FTD intensity follows a lognormal dynamic and the conditional intensities are deterministic.

$$\boxed{\lambda_t^{ftd} = \Gamma^{ftd}\left(t\right) e^{\sigma^\lambda e^{-\kappa t}\int_0^t e^{\kappa s} dW_s^\lambda} = \Gamma^{ftd}\left(t\right) e^{\sigma_t^\lambda Z_t^\lambda}}$$

The joint intensities follow lognormal dynamics with constant volatility and constant mean reversion.

$$\boxed{h_t^i = x_t^i \lambda_t^{ftd} = \Gamma^i\left(t\right) e^{\sigma_t^\lambda Z_t^\lambda}}$$

All the joint intensities share the same Gaussian driver with the same volatility and mean reversion but with different drifts.
The FTD intensity is the sum of the joint intensities:

$$\boxed{\lambda_t^{ftd} = \left(\sum_{i=1}^n \Gamma^i\left(t\right)\right) e^{\sigma_t^\lambda Z_t^\lambda} = \Gamma^{ftd}\left(t\right) e^{\sigma_t^\lambda Z_t^\lambda}}$$

The conditional intensity of the name *i* is given by:

$$\boxed{x_t^i = \frac{\Gamma^i(t)}{\sum_{i=1}^{n} \Gamma^i(t)}}$$

3.3. Modelling Emerging market FX with jump at FTD time

We assume that the FX spot has the dynamic:

$$\frac{dS_t^{d/loc}}{S_{t^-}^{d/loc}} = \left(r_t^d - r_t^{loc}\right)dt + \sigma_t^{fx} dW_t^{fx} - \sum_{i=1}^{n} J^i \left(dN_{t\wedge\tau^{ftd}}^i - h_t^i 1_{\{\tau^{ftd} > t\}} dt\right)$$

$$h_t^i = \Gamma^i(t) e^{\sigma_t^\lambda Z_t^\lambda}$$

$$d\left\langle W_t^{fx}, W_t^\lambda \right\rangle = \rho dt$$

The process Z_t^λ is a Gaussian process described in the previous section.

J^i are constants between 0 and 1.

In case of deterministic credit, the spot process follows a lognormal dynamic before and after the default time.

The FX spot can jump only once: at time of FTD.

Let us calculate the dynamic of $\ln S_t^{d/loc}$

$$d\ln S_t^{d/loc} = \left(r_t^d - r_t^{loc} - \frac{\left(\sigma_t^{fx}\right)^2}{2} + \left(\sum_{i=1}^{n} J^i h_t^i\right) 1_{\{\tau^{ftd} > t\}}\right)dt + \sigma_t^{fx} dW_t^{d/loc} + \sum_{i=1}^{n} \ln\left(1 - J^i\right) dN_{t\wedge\tau^{ftd}}^i$$

It follows that the FX spot process is given by:

$$\begin{aligned} S_t^{d/loc} &= X_t^{d/loc} \exp\left(\sum_{i=1}^{n} \ln\left(1 - J^i\right) \int_0^t dN_{s\wedge\tau^{ftd}}^i + \int_0^t \left(\sum_{i=1}^{n} J^i h_s^i\right) 1_{\{\tau^{ftd} > s\}} ds\right) \\ &= X_t^{d/loc} \prod_{i=1}^{n} \left[\left(1 - J^i\right)^{N_{t\wedge\tau^{ftd}}^i} \exp\left(J^i \int_0^{t\wedge\tau^{ftd}} h_s^i ds\right)\right] \\ &= X_t^{d/loc} Z_t^{d/loc} \end{aligned}$$

Where

$$X_t^{d/loc} = S_0^{d/loc} \frac{B^{loc}(0,t)}{B^d(0,t)} \exp\left(\int_0^t \sigma_u^{fx} dW_u^{fx} - \frac{1}{2}\int_0^t \left(\sigma_u^{fx}\right)^2 du\right)$$

The FX spot process is the product of the forward, a continuous martingale, and a discontinuous martingale.

If the jump sizes are the same then the FX spot dynamic is similar to the dynamic of the FX spot in BSWithJump dynamic with intensity λ_t^{ftd} :

$$\begin{aligned} S_t^{d/loc} &= X_t^{d/loc} \exp\left(\ln(1 - J) N_{s\wedge\tau^{ftd}}^{ftd} + J\int_0^t \lambda_s^{ftd} 1_{\{\tau^{ftd} > s\}} ds\right) \\ &= X_t^{d/loc} \left[(1 - J)^{N_{t\wedge\tau^{ftd}}^{ftd}} \exp\left(J\int_0^{t\wedge\tau^{ftd}} \lambda_s^{ftd} ds\right)\right] \end{aligned}$$

4. Pricing FTD and FTD_i

4.1. Pricing the recovery leg under LNFTD

The price of the recovery leg of a FTD with notionals 1 is given by:

$$RFTD(T)=\sum_{i=1}^{n}(1-R_i)E^Q\left(1_{\{\tau^{ftd}\le T,I=i\}}\right)=\sum_{i=1}^{n}RFTD_i$$

Using lemma2 defined in the section 2, we have:

$$\boxed{RFTD_i(T)=(1-R_i)E^Q\left(\int_0^T h_t^i e^{-\int_0^t\left(r_u+\lambda_u^{ftd}\right)du}dt\right)}$$

$RFTD_i$ is the price of the recovery leg of a FTD with notional 0 except the credit name *i* where the notional is 1 or, the recovery leg price of a CDS on the credit name *i* which knock out in case of FTD.

We have:

$$RFTD_i(T+dT)-RFTD_i(T)=(1-R_i)B^d(0,T)\frac{\Gamma^i(T)}{\Gamma^{ftd}(T)}E\left(\int_T^{T+dT}\lambda_t^{ftd}e^{-\int_0^t\lambda_u^{ftd}du}dt\right)$$

$$=(1-R_i)B^d(0,T)\frac{\Gamma^i(T)}{\Gamma^{ftd}(T)}\left(Q^{ftd}(T+dT)-Q^{ftd}(T)\right)$$

We conclude that:

$$\boxed{\frac{\Gamma^i(T)}{\Gamma^{ftd}(T)}=\frac{RFTD_i(T+dT)-RFTD_i(T)}{\left(Q^{ftd}(T+dT)-Q^{ftd}(T)\right)(1-R_i)B^d(0,T)}}$$

This relationship means that the ratio $\frac{\Gamma^i(T)}{\Gamma^{ftd}(T)}$ does not depend on the LN parameters (lognormal intensity volatility and mean reversion) and depends only on the credit market data (FTD prices)[3]. Therefore, we need only to calibrate the FTD drift ($\Gamma^{ftd}(T)$) to calibrate the LNFTD model as the $\Gamma^i(T)$ can be deduced from the relationship above.

Let us calculate the price of the FTD recovery leg:

$$RFTD(T)=\sum_{i=1}^{n}(1-R_i)E\left(\int_0^T h_t^i e^{-\int_0^t\left(r_u+\lambda_u^{ftd}\right)du}dt\right)$$

$$=\sum_{i=1}^{n}E\left(\int_0^T(1-R_i)\frac{\Gamma_t^i}{\Gamma_t^{ftd}}\lambda_t^{ftd}e^{-\int_0^t\left(r_u+\lambda_u^{ftd}\right)du}dt\right)$$

$$=E\left(\int_0^T\left(\sum_{i=1}^{n}\frac{\Gamma_t^i}{\Gamma_t^{ftd}}(1-R_i)\right)\lambda_t^{ftd}e^{-\int_0^t\left(r_u+\lambda_u^{ftd}\right)du}dt\right)$$

$$=E\left(\int_0^T\left(1-R_t^{ftd}\right)\lambda_t^{ftd}e^{-\int_0^t\left(r_u+\lambda_u^{ftd}\right)du}dt\right)$$

$$=RCDS^{ftd}(T)$$

[3] $RFTD_i$ and $Q^{ftd}(T)$ are given by the Gaussian copula model for example.

We conclude that the recovery leg of the FTD is equivalent to recovery leg of a CDS with time dependent recovery R_t^{ftd} , which is function of the individual recoveries and the Gaussian copula correlation[4]:

$$\boxed{R_t^{ftd} = 1 - \sum_{i=1}^{n} \frac{\Gamma_t^i}{\Gamma_t^{ftd}} \left(1 - R_i\right) = \sum_{i=1}^{n} \frac{\Gamma_t^i}{\Gamma_t^{ftd}} R_i}$$

4.2. Pricing the Fixed Leg

The riskybpv depends only on the interest rates and the FTD intensity:

$$RiskyBPV\left(T\right) = \sum_{i=1}^{N} \delta_i E\left(e^{-\int_0^{T_i}\left(r_u + \lambda_u^{ftd}\right)du}\right) + E\left(\int_0^T \left(t - T_{\kappa(t)}\right)\lambda_t^{ftd} e^{-\int_0^t\left(r_u + \lambda_u^{ftd}\right)du} dt\right)$$

4.3. FTD as a Synthetic CDS

In the section 4.1, we proved that the recovery leg of the FTD is equivalent to the recovery leg of a synthetic CDS with time dependent recovery. The riskybpv is the same for the FTD and the synthetic CDS. We conclude that the calibration of LNFTD model to the term structure of FTD given by the Gaussian copula model (or any other FTD pricing model) is equivalent to the calibration of a LN model to the term structure of a synthetic CDS with time dependent recovery. We note CDS^{ftd} this synthetic CDS.

5. Pricing FX call options

5.1. General case

In order to price a call option within this model we need to separate the calculations into two cases: default before maturity and no default before maturity.

$$\begin{aligned} C\left(T,K\right) &= B^d\left(0,T\right) E\left(\left(S_T^{d/loc} - K\right)_+ \left(1_{\left\{\tau^{ftd} \le T\right\}} + 1_{\left\{\tau^{ftd} > T\right\}}\right)\right) \\ &= C_{def}\left(T,K\right) + C_{sur}\left(T,K\right) \end{aligned}$$

C_{def} is the default part of the call price.

C_{sur} is the survival part of the call price.

The default part of the call price is given by (using lemma2):

$$\begin{aligned} C_{def}\left(T,K\right) &= B^d\left(0,T\right)\sum_{i=1}^{n}\int_0^T E\left[\left(S_T^{d/loc} 1_{\left\{I=i,\tau^{ftd}=u\right\}} - K\right)_+ dQ^d\left(I = i, \tau^{ftd} = u\right)\right] \\ &= B^d\left(0,T\right)\sum_{i=1}^{n}\int_0^T E\left[\left(\left(1 - J^i\right) X_T^{d/loc} \exp\left(\sum_{i=1}^{n} J^i h_t^i\right) - K\right)_+ h_t^i e^{-\int_0^t \lambda_s^{ftd} ds} dt\right] \end{aligned}$$

We define the effective intensity λ_t^{eff} :

$$\lambda_t^{eff} = \sum_{i=1}^{n}\left(1 - J^i\right) h_t^i$$

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4 $\dfrac{\Gamma_t^i}{\Gamma_t^{ftd}}$ is function only of the Gaussian copula correlation and the single name CDS.

If the jump sizes are equal to J then the effective intensity is proportional to the FTD intensity.
The survival part of the call price is given by:

$$C_{sur}\left(T,K\right)=B^{d}\left(0,T\right)E^{Q^{d}}\left(e^{-\int_{0}^{T}\lambda_{s}^{ftd}ds}\left(X_{T}^{d/loc}e^{\int_{0}^{T}\left(\lambda_{s}^{ftd}-\lambda_{s}^{eff}\right)ds}-K\right)_{+}\right)$$

The call price is given by:

$$\boxed{\begin{aligned}C\left(T,K\right)=B^{d}\left(0,T\right)\sum_{i=1}^{n}\int_{0}^{T}E\left[\left(\left(1-J^{i}\right)X_{T}^{d/loc}e^{\int_{0}^{t}\left(\lambda_{s}^{ftd}-\lambda_{s}^{eff}\right)ds}-K\right)_{+}h_{t}^{i}e^{-\int_{0}^{t}\lambda_{s}^{ftd}ds}dt\right]\\+B^{d}\left(0,T\right)E^{Q^{d}}\left(e^{-\int_{0}^{T}\lambda_{s}^{ftd}ds}\left(X_{T}^{d/loc}e^{\int_{0}^{T}\left(\lambda_{s}^{ftd}-\lambda_{s}^{eff}\right)ds}-K\right)_{+}\right)\end{aligned}}$$

Unfortunately, we cannot transform easily the call price formula to a more simple formula as we did in [1] for BSWithJump model (unless if the jump sizes are the same, in this case the problem is equivalent to one synthetic credit name defined by the first to default).

5.2. Deterministic credit case

In case of deterministic credit, the call price is given by a closed form solution. The call price is easily calculated by integrating the call payoff with respect to the lognormal distribution of $X_{T}^{d/loc}$.

6. Model Calibration

6.1. LNFTD Calibration

The calibration of the model consists on calibrating a LN model to the term structure of CDS^{ftd} premiums. The calibration of LN model using forward PDE is described in [1] (section 3.1).

6.2. Calibration of BSFTDWithMultiJump to ATM FX options

The FX volatility is calibrated using an iterative calibration method based on MonteCarlo. This method is simple to implement, robust, fast, and very accurate.

1) We calibrate the BSFTDWithMultiJump by assuming that the intensity is deterministic[5]. The calibration is performed using a root finder algorithm.
2) We calculate the calibration errors of the implied volatilities using a single MonteCarlo for all the maturities (we use control variate techniques to achieve a good convergence with few paths).
3) We shift the local FX volatility with a function of the implied volatilities errors and we repeat 2 and 3 until the calibration errors are very smalls.

This calibration algorithm is very simple and need few iterations to reach very small errors even for extreme market data[6].

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5 In this case, we have closed form solution to the call option.
6 The calibration accuracy and robustness are shown in section 8.

This calibration method could be used as well for BSWithJump model instead of the forward PDE algorithm and it is more robust in case of extreme market data and model parameters.

7. Pricing Quanto FTD survival probabilities and quanto FTD

7.1. Pricing Quanto FTD survival probabilities

Let us calculate the local currency FTD survival probability or the quanto FTD survival probability:

$$
\begin{aligned}
Q^{loc}(0,T) &= E^{Q^d}\left(\frac{B^d(0,T)S_T^{d/loc}}{S_0 B^{loc}(0,T)} 1_{\{\tau^{ftd}>T\}}\right) \\
&= E^{Q^d}\left(M_T^{d/loc}\exp\left(\sum_{i=1}^{n}\ln\left(1-J^i\right)\int_0^T dN^i_{s\wedge\tau^{ftd}} + \int_0^T\left(\sum_{i=1}^{n}J^i h_s^i\right)1_{\{\tau^{ftd}>s\}}ds\right)1_{\{\tau^{ftd}>T\}}\right) \\
&= E^{Q^d}\left(M_T^{d/loc}\exp\left(-\int_0^T \lambda_u^{ftd}du\right)\exp\left(\int_0^T\left(\sum_{i=1}^{n}J^i h_s^i\right)ds\right)\right) \\
&= E^{Q^d}\left(M_T^{d/loc}\exp\left(-\int_0^T \lambda_u^{eff}du\right)\right)
\end{aligned}
$$

$M_T^{d/loc}$ is an exponential martingale $M_T^{d/loc} = \exp\left(\int_0^T \sigma_u^{fx}dW_u^{fx} - \frac{1}{2}\int_0^T\left(\sigma_u^{fx}\right)^2 du\right)$

We can see that under the local currency measure, each conditional intensity h_t^i is multiplied by $1-J^i$. By doing a change of numeraire, we conclude that the quanto survival probability is:

$$
Q^{loc}(0,T) = E^{Q^M}\left(\exp\left(-\int_0^T \lambda_u^{eff}du\right)\right)
$$

The intensity is lognormal under the domestic measure and stays lognormal under the new measure with the same volatility and mean reversion but different $\Gamma_T^{ftd,loc} = \Gamma_T^{eff,M}$ function.

The intensity of default under the local currency is a LN model with a $\Gamma_T^{ftd,loc}$ function given by the formula:

$$
\boxed{\Gamma_T^{ftd,loc} = \left(\sum_{i=1}^{n}\left(1-J^i\right)\Gamma_T^i\right)\exp\left(\rho\sigma^{\lambda}e^{-\kappa T}\int_0^T e^{\kappa u}\sigma_u^{fx}du\right)}
$$

The term structure of quanto survival probability can be easily calculated using the same forward PDE on the green function defined in the LN calibration section.

7.2. Pricing the recovery leg of quanto FTD

We recall the FTD recovery leg payoff:

$$
\begin{aligned}
RQFTD(T) &= E^{Q^d}\left[B\left(0,\tau^{ftd}\right)\left(1-R_I\right)1_{\{\tau^{ftd}\le T\}}\right] \\
&\approx \sum_{j=1}^{M} B^d\left(0,t_j\right)E^{Q^d}\left[\left(1-R_I\right)1_{\{t_{j-1}\le\tau^{ftd}\le t_j\}}\right]
\end{aligned}
$$

Where R_I is the recovery of the first to default name.

The price of the recovery leg of a quanto FTD is given by:

$$RFTD^{loc}\left(0,T\right)=\sum_{j=1}^{M}B^{loc}\left(0,t_j\right)E^{Q^d}\left[\frac{B^d\left(0,t_j\right)}{S_0B^{loc}\left(0,t_j\right)}S_{t_j}^{d/loc}\left(1-R_I\right)1_{\left\{t_{j-1}\le\tau^{ftd}\le t_j\right\}}\right]$$

$$=\sum_{j=1}^{M}B^{loc}\left(0,t_j\right)E^{Q^d}\left[M_{t_j}^{d/loc}\exp\left(\sum_{i=1}^{n}\ln\left(1-J^i\right)\int_0^t dN^i_{s\wedge\tau^{ftd}}+\int_0^t\left(\sum_{i=1}^{n}J^ih_s^i\right)1_{\left\{\tau^{ftd}>s\right\}}ds\right)\left(1-R_I\right)1_{\left\{t_{j-1}\le\tau^{ftd}\le t_j\right\}}\right]$$

We know that

$$\exp\left(\sum_{i=1}^{n}\ln\left(1-J^i\right)\int_0^t dN^i_{s\wedge\tau^{ftd}}+\int_0^t\left(\sum_{i=1}^{n}J^ih_s^i\right)1_{\left\{\tau^{ftd}>s\right\}}ds\right)1_{\left\{t_{j-1}\le\tau^{ftd}\le t_j\right\}}=\left(1-J^i\right)\exp\left(\int_0^{t_{j-1}}\left(\sum_{i=1}^{n}J^ih_s^i\right)ds\right)$$

Using the expression of the effective intensity: $\lambda_t^{eff}=\sum_{i=1}^{n}\left(1-J^i\right)h_t^i$

It follows

$$RFTD^{loc}\left(0,T\right)=\sum_{j=1}^{M}\sum_{i=1}^{n}\left(1-J^i\right)\left(1-R^i\right)B^{loc}\left(0,t_j\right)E^{Q^d}\left(M_{t_j}^{d/loc}\exp\left(\int_0^{t_{j-1}}\left(\sum_{i=1}^{n}J^ih_s^i\right)ds\right)1_{\left\{t_{j-1}\le\tau^{ftd}\le t_j,I=i\right\}}\right)$$

$$=\sum_{j=1}^{M}\sum_{i=1}^{n}\left(1-J^i\right)\left(1-R^i\right)B^{loc}\left(0,t_j\right)E^{Q^d}\left(M_{t_j}^{d/loc}h_{t_j}^ie^{-\int_0^{t_{j-1}}\lambda_s^{eff}ds}\left(t_j-t_{j-1}\right)\right)$$

$$=\sum_{j=1}^{M}B^{loc}\left(0,t_j\right)E^{Q^d}\left(M_{t_j}^{d/loc}\left(\sum_{i=1}^{n}\left(1-J^i\right)\left(1-R^i\right)h_{t_j}^i\right)e^{-\int_0^{t_{j-1}}\lambda_s^{eff}ds}\left(t_j-t_{j-1}\right)\right)$$

$$=\int_0^T B^{loc}\left(0,t\right)\left(1-R_t^{ftd,loc}\right)E^{Q^M}\left(\lambda_t^{ftd,loc}e^{-\int_0^t\lambda_s^{ftd,loc}ds}\right)dt$$

$$\boxed{\left(1-R_t^{ftd,loc}\right)=\sum_{i=1}^{n}\frac{\left(1-J^i\right)\Gamma_t^i/\Gamma_t^{ftd}}{\sum_{p=1}^{n}\left(1-J^p\right)\Gamma_t^p/\Gamma_t^{ftd}}\left(1-R^i\right)}$$

The recovery leg of a quanto FTD is equivalent to the recovery leg of a quanto CDS using BSWithJump. We conclude that the pricing of the recovery leg of a quanto FTD is similar to the pricing of the recovery leg of a single name CDS with BSWithJump model. We notice that the local currency recovery function is different from the domestic currency recovery function unless if the jump sizes are the same or the single name recoveries are the same.

7.3. Pricing the RiskyBPV of quanto FTD

The pricing of the riskyBPV of a quanto FTD is straightforward given the term structure of quanto survival probabilities.

8. Examples: FX volatility calibration accuracy

In this section, we show the FX volatility calibration quality of the iterative method.

8.1. FX volatility calibration: USDMXN and FTD1

We choose FTD1 (FTD on B and A) for this example.

Market Data

The A CDS curve (quotation currency: USD) is given by:

Mat	1y	3y	3y	5y	7y	10y
CDS	111	131	147	177	187	197

The recover is 40%
The B CDS curve (quotation currency: USD) is

Mat	1y	3y	3y	5y	7y	10y
CDS	189	224	235	240	230	215

The recover is 40%
The Gaussian copula correlation is 0.7.

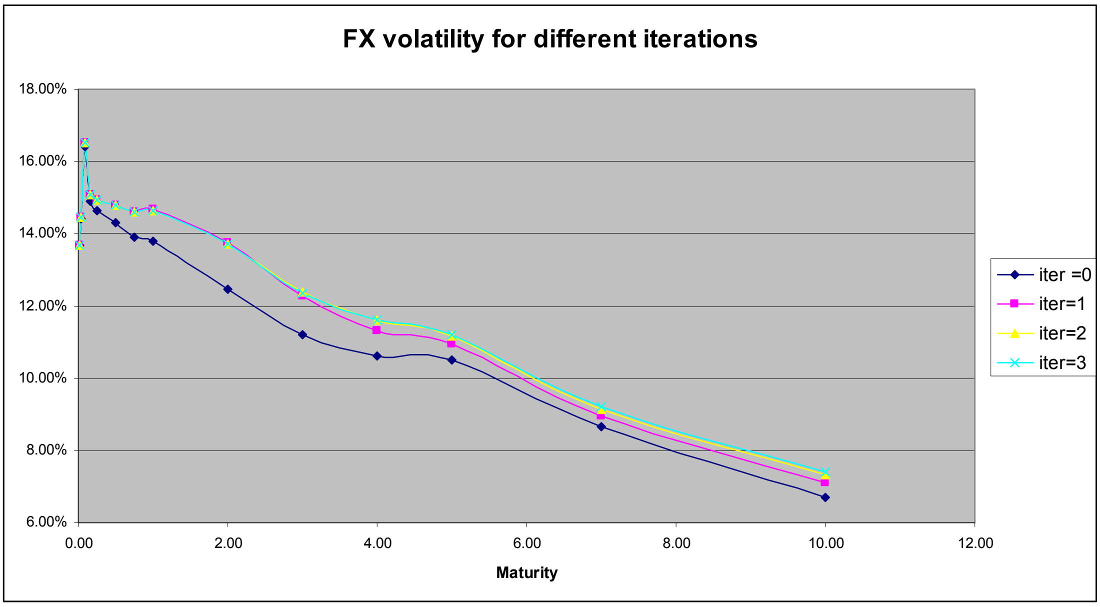
The USDMXN ATM volatility is given by:

Mat	1w	2w	1m	2m	3m	6m	9m	1y	2y	3y	4y	5y	7y	10y
volatilities	14.0%	14.5%	16.0%	16.0%	16.0%	16.0%	16.0%	16.0%	16.0%	16.0%	16.3%	16.5%	16.5%	16.5%

We choose extreme values for the jump sizes: 80% for A and 20% for B.
The volatility of the FTD intensity is 140%
The correlation between FTD1 and MXNUSD is set to -70%.

Calibration errors for different iterations

The graph below shows the FX local volatility for different values of iteration parameter. We can see that the FX local volatility converge quickly to an FX volatility function after two iterations



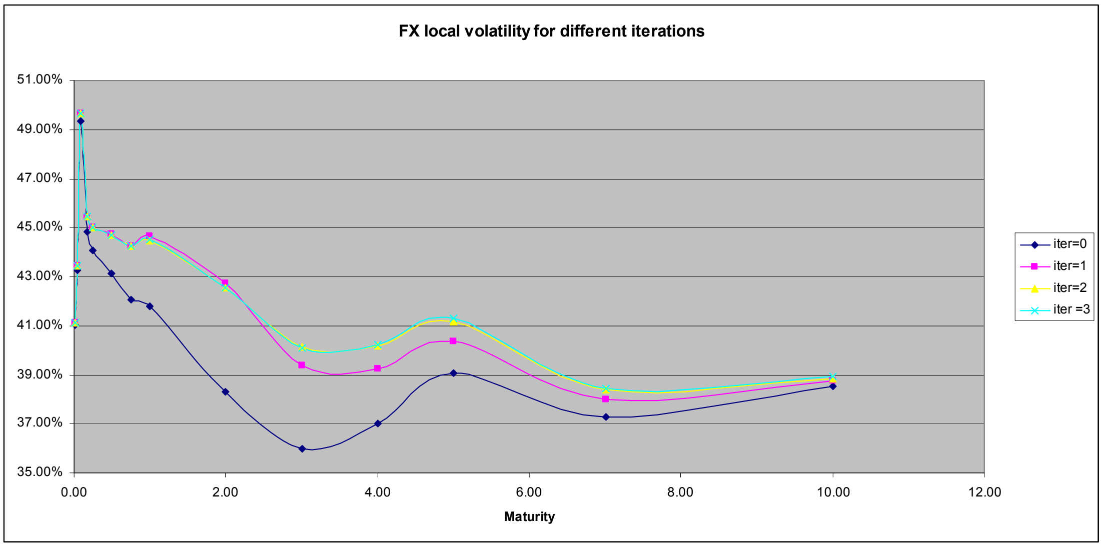
The table below shows the calibration errors (modelVol-marketVol in bp) for different values for the iteration parameter.

Maturity	iter =0	iter=1	iter=2	iter=3
0.02	3.05	0.02	0.00	0.00
0.04	4.46	-0.01	0.00	0.00
0.08	7.59	-0.04	0.00	0.00

0.17	13.61	0.22	0.00	0.00
0.25	18.64	0.13	0.00	0.00
0.50	33.36	-0.39	0.01	0.00
0.75	44.11	-0.51	0.01	0.00
1.00	54.34	-1.26	0.05	0.00
2.00	83.27	-2.81	0.14	-0.01
3.00	85.74	1.01	-0.20	0.02
4.00	80.75	6.62	-0.37	0.01
5.00	73.09	9.18	0.14	-0.03
7.01	61.98	10.84	0.93	0.02
10.00	54.98	12.73	2.44	0.43

8.2. FX volatility calibration : USDMXN and FTD2

We would like to show the FX volatility calibration quality when the market conditions are extreme. We multiply the CDS curve A by 4 and the FX implied volatility by 3. The rest of market data is the same as the first example.
The jumps sizes are 80% for both A and B.
The FTD hazard rate volatility is 140% and the correlation between FTD2 and MXNUSD is -70%
The graph below shows the FX local volatility for different values of iteration parameter. We can see that the FX local volatility converge quickly to an FX volatility function after two iterations.



The table below shows the calibration errors (modelVol-marketVol in bp) for different values for the iteration parameter.

Maturity	iter =0	iter=1	iter=2	iter=3
0.02	8.99	0.05	0.00	0.00
0.04	13.29	-0.03	0.00	0.00
0.08	22.89	-0.12	0.00	0.00
0.17	41.78	0.87	0.02	0.00
0.25	57.78	0.60	0.02	0.00
0.50	104.92	-0.71	0.00	-0.00
0.75	140.09	-1.01	0.01	-0.00
1.00	172.28	-4.77	0.28	-0.02
2.00	274.61	-9.85	0.53	-0.03
3.00	284.29	14.63	-0.52	0.03
4.00	264.43	31.71	-0.19	-0.04
5.00	235.44	39.59	2.23	-0.04

7.01	187.15	37.93	3.39	0.10
10.00	137.68	29.52	4.51	0.36

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