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Abstract

In the current paper, we study the asymmetric normal-form game between two heterogeneous groups of populations by employing the stochastic replicator dynamics driven by Lévy process. A new game equilibrium, i.e., the game equilibrium of a stochastic differential cooperative game on time, is derived by introducing optimal-stopping technique into evolutionary game theory, which combines with the Pareto optimal standard leads us to the existence of Pareto optimal endogenous matching.

Keywords: Stochastic differential cooperative game on time; Endogenous matching; Fair matching; Pareto optimality; Adaptive learning.

JEL classification: C62; C70; C78.

1. Introduction

[1] I am very grateful for helpful comments and suggestions from one anonymous referee. And I wish to thank the anonymous referee for the careful reading. Any remaining errors are, of course, my own responsibility.

Our purpose of the paper is to supply a general framework for studying Pareto-optimal endogenous matching in any given normal-form game situations with two groups of heterogeneous populations. Firstly, the existence of the endogenous matching is confirmed. Secondly, the derived endogenous matching exhibits the following good properties: Pareto efficiency, individual rationality, and also fairness. Furthermore, random matching as an extreme case of the endogenous matching under consideration yields economic-welfare intuitions and implications. Indeed, the present study provides conditions under which the well-known random matching is *asymptotically Pareto efficient*. And in this sense, we can further argue that this investigation has illustrated the existence of Pareto-optimal social structure or social network in given game situations. In other words, it is confirmed that there exists a matching mechanism such that any given social structure can be led to the Pareto-optimal social structure. This hence deepens our understanding of matching mechanism in game theory.[2]

It is convincing to argue that people live in a highly *structured*[3] society (see, Schelling, 1969, 1971; Bowles and Gintis, 1998; Pollicott and Weiss, 2001; Zhang, 2004; Pacheco et al., 2006; Pacheco et al., 2008) consists of groups rather than individuals, which implies that random matching will not always provide us with compelling approximation to reality when we are concerned with the interactions among the players. In fact, Ellison (1993) shows that local interaction[4] will have very important and also different implications in equilibrium selection relative to that of uniform interaction or random matching (e.g., Gilboa and Matsui, 1992). So, given the importance of non-random matching in equilibrium selection, we express the motivation of the present paper as follows, i.e., can we directly prove the existence of certain non-random matching that is Pareto optimal and also endogenously determined in a given game situation? If we can, what are the conditions we will rely on? In other words, the major goal of the present exploration is not to study any exogenously given matching mechanism but to find out the optimal matching mechanism in a given game situation.[5]

In two pioneering papers, Kandori et al. (1993) and Young (1993) prove that the trial-and-error learning processes of the players will definitely converge to one particular pure-strategy Nash equilibrium, which is named as the long run equilibrium by Kandori et al. and the convention by Young. From the perspective of multiple-equilibrium problem, they provide us with an equilibrium

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2 Noting that Haag and Lagunoff (2006) address the question of the optimal spatial or neighborhood design when free-rider problems are localized, they indeed share the similar basic idea as the current paper. Nevertheless, there are two obvious differences between the both. Firstly, they focus on free-rider problems when social spillovers exist, that is, their problem is much more explicit than that of the current study. And hence the present paper supplies us a much more general framework. Secondly, they employ graph theory and focus on specific spatial-structure while the current investigation uses evolutionary game theory and optimal stopping theory. All in all, our endogenous matching does (to some extent) include the considerations of the above spatial structure.

3 For example, it is induced or determined by the following factors: institutional segregation, market division, spatial structure, informational distribution, reputation, preference, emotion and motive, and so on. In particular, individuals usually have motives to sort themselves into matches with like agents, for example, better-qualified workers match with better jobs, more handsome men marry more beautiful women, that is, only "similar" agents match, as is emphasized by assortative-matching theory (see, Shimer and Smith, 2000; Atakan, 2006; Hoppe et al., 2009; Eeckhout and Kircher, 2010, and among others).

4 These local interaction settings share the characteristic that each person interacts with only a subset of the relevant population. One economic intuition of this aspect is to capture in a simple abstract way a socioeconomic environment in which markets do not exist to mediate all of agents' choices (see, Bisin et al., 2006). As noticed and stated by Bisin et al. (2006), local interactions represent an important aspect of several socioeconomic phenomena.

5 That is to say, in an artificial world, we can employ the matching mechanism to lead the players to play the Pareto optimal Nash equilibrium regardless of the enforcement cost. And in this sense, matching mechanism plays the role of equilibrium selection device.

selection device, under which the players are correctly predicted to play a particular Nash equilibrium. However, we can also evaluate their contribution from the following viewpoint, i.e., provided a particular Nash equilibrium, they prove that there exists a pattern of learning mechanism that will definitely lead the players to play the given Nash equilibrium. To summarize, they confirm the existence of certain type of learning mechanism, based upon which the players' behavior will be uniquely predicted in the long run. Instead of emphasizing *micro-strategy*, we focus on *macro-structure* and it is confirmed that there exists certain macro-structure or social network (e.g., Skyrms and Pemantle, 2000; Bala and Goyal, 2000; Galeotti et al., 2006, and among others) under which one particular Pareto optimal Nash equilibrium will be definitely played by the players.

In the paper, we are encouraged to study the asymmetric normal-form games between two heterogeneous groups of populations under the modified framework of evolutionary game theory. Each of the two groups is assumed to have countably many pure strategies. Hyper-rational assumptions (see, Aumann, 1976; Andreoni and Samuelson, 2006) about the players broadly used in classical non-cooperative game theory will be dropped in the present model, instead, the players or individuals play the game following certain adaptive learning processes arising from the stochastic replicator dynamics driven by Lévy processes (for the first time). On the contrary, the strategies themselves are supposed to be smart and rational enough to optimize their fitness, which directly depends on the stochastic replicator dynamics or the learning processes of the players, following the classical *as if* methodology from the perspective of posteriori. And the corresponding control variables of these fitness-optimization problems are chosen to be stochastic stopping times or stopping rules, which reasonably reflects the fact that strategies themselves are no longer suitable for the roles of control variables (as in the best-response correspondences of Nash equilibria) because "strategies" of the players' strategies will not be well-defined through the traditional approach. Luckily, noting that the optimal stopping rules are partially determined and completely characterized by the learning processes of the players, the optimal stopping rules as a vector may be exactly one of the Nash equilibra, no matter it is mixed-strategy Nash equilibrium or pure-strategy Nash equilibrium, of the original normal-form games.

Generally speaking, the optimal stopping rules as a vector will not be equal to anyone of the Nash equilibria, that is, there exists certain difference between the both. However, it is confirmed that it is just the difference between the optimal stopping rules as a vector and the Pareto optimal Nash equilibrium of the original normal-form game that established our Pareto optimal endogenous matching. We, hence, to the best of our knowledge, enrich the matching rule widely used in evolutionary game theory by naturally adding into economic-welfare implications for the first time.

Moreover, it is shown that the well-known random matching (e.g., Maynard Smith, 1982; Fudenberg and Levine, 1993; Ellison, 1994; Okuno-Fujiwara and Postlewaite, 1995; Weibull, 1995; Hofbauer and Sigmund, 2003; Benaïm and Weibull, 2003; Aliprantis et al., 2007; Duffie and Sun, 2007; Takahashi, 2010; Podczeck and Puzzello, 2012, and among others) just represents one special and extreme case of the current endogenous matching and we supply the conditions under which the random matching will be *asymptotically Pareto efficient*. Thus, proving the existence of Pareto optimal endogenous matching would be regarded as one innovation of the present paper by noticing the above facts.

In the next section, we will construct the formal model, introduce some basic concepts and prove the key theorem of the present paper. There is a brief concluding section. All proofs appear in the Appendix.

2. Formulation

2.1. Set-up and Assumptions

Let $A_{I_1 \times I_2}$ be the payoff matrix for row players and $B_{I_1 \times I_2}$ be the payoff matrix for column players with $A_{I_1 \times I_2}$, $B_{I_1 \times I_2} \in \mathbb{R}^{I_1 \times I_2}$, and I_1, $I_2 \geq 1$. Here, and throughout the current paper, we study the replicator dynamics of $I_1 \times I_2$ normal-form games between two groups of populations. Put $\sum_{i_1=1}^{I_1} M^{i_1}(t) \triangleq M(t)$, where $M^{i_1}(t)$ denotes the number of strategy-i_1 players at period t. Similarly, let $\sum_{i_2=1}^{I_2} N^{i_2}(t) \triangleq N(t)$, where $N^{i_2}(t)$ denotes the number of strategy-i_2 players at period t.

We let $X^{i_1}(t) \triangleq M^{i_1}(t)/M(t)$, $Y^{i_2}(t) \triangleq N^{i_2}(t)/N(t)$ denote the frequencies of strategies i_1 and i_2, respectively, with $i_1 = 1, 2, ..., I_1$ and $i_2 = 1, 2, ..., I_2$. Therefore, the average payoffs of strategy i_1 and strategy i_2 are given by $u\left(i_1, Y(t)\right) \triangleq \overline{e}_{i_1}^T A Y(t)$ and $u\left(i_2, X(t)\right) \triangleq \tilde{e}_{i_2}^T B^T X(t)$, respectively, with the superscript "T" denoting transpose, and $X(t) \triangleq \left(X^1(t), ..., X^{i_1}(t), ..., X^{I_1}(t)\right)^T$, $Y(t) \triangleq \left(Y^1(t), ..., Y^{i_2}(t), ..., Y^{I_2}(t)\right)^T$, and also $\overline{e}_{i_1} = (0, ..., 1, ..., 0)^T$, $\tilde{e}_{i_2} = (0, ..., 1, ..., 0)^T$, where the i_1-th entry and i_2-th entry are ones, respectively, for $i_1 = 1, 2, ..., I_1$ and $i_2 = 1, 2, ..., I_2$.

Specifically, in the current paper, we employ the following *endogenous matching* mechanism by incorporating two vectors, i.e., $\bar{\rho} \triangleq \left(\bar{\rho}^1, ..., \bar{\rho}^{i_1}, ..., \bar{\rho}^{I_1}\right)^T \in \mathbb{R}^{I_1}$ and $\tilde{\rho} \triangleq (\tilde{\rho}^1, ..., \tilde{\rho}^{i_2}, ..., \tilde{\rho}^{I_2})^T \in \mathbb{R}^{I_2}$ with $\sum_{i_1=1}^{I_1} \bar{\rho}^{i_1} = 0$ and $\sum_{i_2=1}^{I_2} \tilde{\rho}^{i_2} = 0$, into the present model. Now, the *generalized* average payoffs of strategies i_1 and i_2 are rewritten as $u\left(i_1, Y(t) + \tilde{\rho}\right) \triangleq \bar{e}_{i_1}^T A\left(Y(t) + \tilde{\rho}\right) = \bar{e}_{i_1}^T AY(t) + \bar{e}_{i_1}^T A\tilde{\rho}$ and $u\left(i_2, X(t) + \bar{\rho}\right) \triangleq \tilde{e}_{i_2}^T B^T \left(X(t) + \bar{\rho}\right) = \tilde{e}_{i_2}^T B^T X(t) + \tilde{e}_{i_2}^T B^T \bar{\rho}$, respectively, for $i_1 = 1, 2, ..., I_1$ and $i_2 = 1, 2, ..., I_2$. In other words, $u\left(i_1, Y(t) + \tilde{\rho}\right)$ and $u\left(i_2, X(t) + \bar{\rho}\right)$ can be seen as $\bar{e}_{i_1}^T A\tilde{\rho}$ -perturbation and $\tilde{e}_{i_2}^T B^T \bar{\rho}$ -perturbation of $u\left(i_1, Y(t)\right)$ and $u\left(i_2, X(t)\right)$, respectively.

We now denote by $\left(\Omega^{(W^{i_\beta})}, \mathcal{F}^{(W^{i_\beta})}, \left\{\mathcal{F}_t^{(W^{i_\beta})}\right\}_{0 \le t \le \tau^{i_\beta}(\omega)}, \mathbb{P}^{(W^{i_\beta})}\right)$ the filtered probability space with $\mathbb{F}^{(W^{i_\beta})} \triangleq \left\{\mathcal{F}_t^{(W^{i_\beta})}\right\}_{0 \le t \le \tau^{i_\beta}(\omega)}$ the $\mathbb{P}^{(W^{i_\beta})}-$ augmented filtration generated by $d_\beta-$ dimensional standard Brownian motion $\left(W^{i_\beta}(t), 0 \le t \le \tau^{i_\beta}(\omega)\right)$ with $\mathcal{F}^{(W^{i_\beta})} \triangleq \mathcal{F}_{\tau^{i_\beta}(\omega)}^{(W^{i_\beta})}$, $\omega \in \Omega^{(W^{i_\beta})}$ and $\tau^{i_\beta}(\omega)$ a stopping time, to be endogenously determined. Moreover, we define

$$\begin{aligned} \tilde{N}^{i_\beta}\left(dt, dz^{i_\beta}\right) &\triangleq \left(\tilde{N}_1^{i_\beta}\left(dt, dz_1^{i_\beta}\right), ..., \tilde{N}_{n_\beta}^{i_\beta}\left(dt, dz_{n_\beta}^{i_\beta}\right)\right)^T \\ &\triangleq \left(N_1^{i_\beta}\left(dt, dz_1^{i_\beta}\right) - \nu_1^{i_\beta}\left(dz_1^{i_\beta}\right)dt, ..., N_{n_\beta}^{i_\beta}\left(dt, dz_{n_\beta}^{i_\beta}\right) - \nu_{n_\beta}^{i_\beta}\left(dz_{n_\beta}^{i_\beta}\right)dt\right)^T, \end{aligned}$$

in which $\left\{N_{l_\beta}^{i_\beta}\right\}_{l_\beta=1}^{n_\beta}$ are independent Poisson random measures with Lévy measures $\nu_{l_\beta}^{i_\beta}$ coming from n_β independent (one-dimensional) Lévy processes $\eta_1^{i_\beta}(t) \triangleq \int_0^t \int_{\mathbb{R}_0} z_1^{i_\beta} \tilde{N}_1^{i_\beta}\left(ds, dz_1^{i_\beta}\right)$, …, $\eta_{n_\beta}^{i_\beta}(t) \triangleq \int_0^t \int_{\mathbb{R}_0} z_{n_\beta}^{i_\beta} \tilde{N}_{n_\beta}^{i_\beta}\left(ds, dz_{n_\beta}^{i_\beta}\right)$ with $\mathbb{R}_0 \triangleq \mathbb{R} - \{0\}$, and then the corresponding stochastic basis is given by $\left(\Omega^{(\tilde{N}^{i_\beta})}, \mathcal{F}^{(\tilde{N}^{i_\beta})}, \left\{\mathcal{F}_t^{(\tilde{N}^{i_\beta})}\right\}_{0 \le t \le \tau^{i_\beta}(\omega)}, \mathbb{P}^{(\tilde{N}^{i_\beta})}\right)$ with $\mathbb{F}^{(\tilde{N}^{i_\beta})} \triangleq \left\{\mathcal{F}_t^{(\tilde{N}^{i_\beta})}\right\}_{0 \le t \le \tau^{i_\beta}(\omega)}$ the $\mathbb{P}^{(\tilde{N}^{i_\beta})}-$ augmented filtration and $\mathcal{F}^{(\tilde{N}^{i_\beta})} \triangleq \mathcal{F}_{\tau^{i_\beta}(\omega)}^{(\tilde{N}^{i_\beta})}$, $\omega \in \Omega^{(\tilde{N}^{i_\beta})}$ and $\tau^{i_\beta}(\omega)$ a stopping time, to be endogenously determined. Thus, we are provided with a new stochastic basis $\left(\Omega^{i_\beta}, \mathcal{F}^{i_\beta}, \left\{\mathcal{F}_t^{i_\beta}\right\}_{0 \le t \le \tau^{i_\beta}(\omega)}, \mathbb{P}^{i_\beta}\right)$, where $\Omega^{i_\beta} \triangleq \Omega^{(W^{i_\beta})} \times \Omega^{(\tilde{N}^{i_\beta})}$, $\mathcal{F}^{i_\beta} \triangleq \mathcal{F}^{(W^{i_\beta})} \otimes \mathcal{F}^{(\tilde{N}^{i_\beta})}$, $\mathcal{F}_t^{i_\beta} \triangleq \mathcal{F}_t^{(W^{i_\beta})} \otimes \mathcal{F}_t^{(\tilde{N}^{i_\beta})}$,

$\mathbb{P}^{i_\beta} \triangleq \mathbb{P}^{(W^{i_\beta})} \otimes \mathbb{P}^{(\tilde{N}^{i_\beta})}$ and $\mathbb{F}^{i_\beta} \triangleq \left\{\mathcal{F}_t^{i_\beta}\right\}_{0\le t\le \tau^{i_\beta}(\omega)}$ denotes the corresponding filtration satisfying the well-known "usual conditions". Here, and throughout the current paper, $\mathbb{E}^{i_\beta}$ is used to denote the expectation operator with respect to (w. r. t.) the probability law $\mathbb{P}^{i_\beta}$ for $\forall\ i_\beta = 1,2,...,I_\beta$ and for $\beta = 1,2$. Naturally, we have stochastic basis $\left(\Omega^\beta, \mathcal{F}^\beta,\ \left\{\mathcal{F}_t^\beta\right\}_{0\le t\le \tau^\beta(\omega)}, \mathbb{P}^\beta\right)$ with $\Omega^\beta \triangleq \times_{i_\beta}^{I_\beta} \Omega^{i_\beta}$, $\mathcal{F}^\beta \triangleq \otimes_{i_\beta}^{I_\beta} \mathcal{F}^{i_\beta}$, $\mathcal{F}_t^\beta \triangleq \otimes_{i_\beta}^{I_\beta} \mathcal{F}_t^{i_\beta}$, $\mathbb{P}^\beta \triangleq \otimes_{i_\beta}^{I_\beta} \mathbb{P}^{i_\beta}$, $\tau^\beta(\omega) \triangleq \vee_{i_\beta}^{I_\beta} \tau^{i_\beta}(\omega) \triangleq \vee_{i_\beta}^{I_\beta} \bar{\tau}^{i_\beta}(\omega)$ if $\beta = 1$, and $\tau^\beta(\omega) \triangleq \vee_{i_\beta}^{I_\beta} \tau^{i_\beta}(\omega) \triangleq \vee_{i_\beta}^{I_\beta} \tilde{\tau}^{i_\beta}(\omega)$ if $\beta = 2$ with $\omega \in \Omega^\beta$, $\mathbb{F}^\beta \triangleq \left\{\mathcal{F}_t^\beta\right\}_{0\le t\le \tau^\beta(\omega)}$ denoting the corresponding filtration satisfying the usual conditions, and $\mathbb{E}^\beta$ is used to denote the expectation operator w. r. t. the probability law $\mathbb{P}^\beta$ for $\beta = 1,2$. Furthermore, we are led to the following probability space $\left(\Omega, \mathcal{F},\ \left\{\mathcal{F}_t\right\}_{0\le t\le \tau(\omega)},\ \mathbb{P}\right)$ with $\Omega \triangleq \times_{\beta=1}^2 \Omega^\beta$, $\mathcal{F} \triangleq \otimes_{\beta=1}^2 \mathcal{F}^\beta$, $\mathcal{F}_t \triangleq \otimes_{\beta=1}^2 \mathcal{F}_t^\beta$, $\mathbb{P} \triangleq \otimes_{\beta=1}^2 \mathbb{P}^\beta$, $\tau(\omega) \triangleq \vee_{\beta=1}^2 \tau^\beta(\omega)$ with $\omega \in \Omega$, $\mathbb{F} \triangleq \left\{\mathcal{F}_t\right\}_{0\le t\le \tau(\omega)}$ denoting the corresponding filtration satisfying the usual conditions, and $\mathbb{E}$ is used to denote the expectation operator w. r. t. the probability law $\mathbb{P}$.

We now define the canonical Lebesgue measure μ on measure space $\left(\mathbb{R}_+,\ \mathfrak{B}\left(\mathbb{R}_+\right)\right)$ with $\mathbb{R}_+ \triangleq [0,\infty)$, $\mathbb{R}_{++} \triangleq (0,\infty)$ and $\mathfrak{B}\left(\mathbb{R}_+\right)$ the Borel sigma-algebra, and also the corresponding regular properties about Lebesgue measure are supposed to be fulfilled. Thus, we can define the following product measure spaces $\left(\Omega^{i_\beta} \times \mathbb{R}_+, \mathbb{F}^{i_\beta} \otimes \mathfrak{B}\left(\mathbb{R}_+\right)\right)$ and $\left(\Omega^\beta \times \mathbb{R}_+, \mathbb{F}^\beta \otimes \mathfrak{B}\left(\mathbb{R}_+\right)\right)$ with corresponding product measures $\mu \otimes \mathbb{P}^{i_\beta}$ and $\mu \otimes \mathbb{P}^\beta$, respectively, for $\forall\ i_\beta = 1,2,...,I_\beta$ and for $\beta = 1,2$.

Now, based upon the probability space $\left(\Omega^{i_\beta}, \mathcal{F}^{i_\beta},\ \mathbb{F}^{i_\beta}, \mathbb{P}^{i_\beta}\right)$ for $\beta = 1,2$, and following Fudenberg and Harris (1992), Cabrales (2000), Imhof (2005), Benaïm et al (2008), Hofbauer and Imhof (2009), the stochastic replicator dynamics[6] of the two groups of populations can be

[6] Throughout, the stochastic replicator dynamics will help us to construct adaptive learning processes for the players following the

respectively given as follows,

$$dM^{i_1}(t)=M^{i_1}(t)\left[\bar{e}_{i_1}^T AY(t)dt+\sum_{k_1=1}^{d_1}\bar{\sigma}_{i_1k_1}(t)dW_{k_1}^{i_1}(t)+\sum_{l_1=1}^{n_1}\int_{\mathbb{R}_0}\bar{\gamma}_{i_1l_1}\left(t,z_{l_1}^{i_1}\right)\tilde{N}_{l_1}^{i_1}\left(dt,dz_{l_1}^{i_1}\right)\right],$$

$$dN^{i_2}(t)=N^{i_2}(t)\left[\tilde{e}_{i_2}^T B^T X(t)dt+\sum_{k_2=1}^{d_2}\tilde{\sigma}_{i_2k_2}(t)dW_{k_2}^{i_2}(t)+\sum_{l_2=1}^{n_2}\int_{\mathbb{R}_0}\tilde{\gamma}_{i_2l_2}\left(t,z_{l_2}^{i_2}\right)\tilde{N}_{l_2}^{i_2}\left(dt,dz_{l_2}^{i_2}\right)\right].$$

where $M^{i_1}(t)$ is assumed to be $\mathbb{F}^{i_1}\otimes\mathfrak{B}(\mathbb{R}_+)-$adapted, $N^{i_2}(t)$ is $\mathbb{F}^{i_2}\otimes\mathfrak{B}(\mathbb{R}_+)-$adapted, $Y(t)$ is also assumed to be $\mathbb{F}^2\otimes\mathfrak{B}(\mathbb{R}_+)-$ adapted, $X(t)$ is $\mathbb{F}^1\otimes\mathfrak{B}(\mathbb{R}_+)-$ adapted, $\bar{\sigma}_{i_1k_1}(t)$ and $\bar{\gamma}_{i_1l_1}\left(t,z_{l_1}^{i_1}\right)$ are $\mathbb{F}^{i_1}\otimes\mathfrak{B}(\mathbb{R}_+)-$progressively measurable, and $\tilde{\sigma}_{i_2k_2}(t)$ and $\tilde{\gamma}_{i_2l_2}\left(t,z_{l_2}^{i_2}\right)$ are $\mathbb{F}^{i_2}\otimes$ $\mathfrak{B}\left(\mathbb{R}_+\right)-$progressively measurable, for $\forall\ i_1=1,2,...,I_1$, $\forall\ i_2=1,2,...,I_2$, $\forall\ k_1=1,2,...,d_1$, $\forall$ $k_2=1,2,...,d_2$, $\forall\ l_1=1,2,...,n_1$ and $\forall\ l_2=1,2,...,n_2$.

ASSUMPTION 1: *Throughout the current paper, both* $M(t)$ *and* $N(t)$*, sufficiently large, are assumed to be finite constants.*

Notice from Assumption 1 that the sizes of the two populations are finite constants, based on Itô's rule one can easily obtain,

$$\begin{aligned}dX^{i_1}(t)&=X^{i_1}(t)\left[\bar{e}_{i_1}^T AY(t)dt+\sum_{k_1=1}^{d_1}\bar{\sigma}_{i_1k_1}(t)dW_{k_1}^{i_1}(t)+\sum_{l_1=1}^{n_1}\int_{\mathbb{R}_0}\bar{\gamma}_{i_1l_1}\left(t,z_{l_1}^{i_1}\right)\tilde{N}_{l_1}^{i_1}\left(dt,dz_{l_1}^{i_1}\right)\right]\\&\triangleq X^{i_1}(t)\left[\bar{e}_{i_1}^T AY(t)dt+\bar{\sigma}^{i_1}(t)dW^{i_1}(t)+\int_{\mathbb{R}_0^{n_1}}\bar{\gamma}^{i_1}\left(t,z^{i_1}\right)\tilde{N}^{i_1}\left(dt,dz^{i_1}\right)\right],\\dY^{i_2}(t)&=Y^{i_2}(t)\left[\tilde{e}_{i_2}^T B^T X(t)dt+\sum_{k_2=1}^{d_2}\tilde{\sigma}_{i_2k_2}(t)dW_{k_2}^{i_2}(t)+\sum_{l_2=1}^{n_2}\int_{\mathbb{R}_0}\tilde{\gamma}_{i_2l_2}\left(t,z_{l_2}^{i_2}\right)\tilde{N}_{l_2}^{i_2}\left(dt,dz_{l_2}^{i_2}\right)\right]\\&\triangleq Y^{i_2}(t)\left[\tilde{e}_{i_2}^T B^T X(t)dt+\tilde{\sigma}^{i_2}(t)dW^{i_2}(t)+\int_{\mathbb{R}_0^{n_2}}\tilde{\gamma}^{i_2}\left(t,z^{i_2}\right)\tilde{N}^{i_2}\left(dt,dz^{i_2}\right)\right].\end{aligned}$$

subject to the initial conditions, i.e., $W^{i_1}(0)=(0,...,0)^T$ $\mathbb{P}^{i_1}-$a.s., $W^{i_2}(0)=(0,...,0)^T$ $\mathbb{P}^{i_2}-$a.s., $X(0)=\left(X^1(0),...,X^{i_1}(0),...,X^{I_1}(0)\right)^T\triangleq\left(x^1,...,x^{i_1},...,x^{I_1}\right)^T\triangleq x>0$ $\mathbb{P}^1-$ a.s., $Y(0)=\left(Y^1(0),\right.$

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argument of Gale et al. (1995), Binmore et al. (1995), Börgers and Sarin (1997), Cabrales (2000), and Beggs (2002). Thus, we will take indifference between the stochastic replicator dynamics and the adaptive learning processes.

$\ldots, Y^{i_2}(0), \ldots, Y^{I_2}(0)\big)^T \triangleq \big(y^1, \ldots, y^{i_2}, \ldots, y^{I_2}\big)^T \triangleq y > 0$ $\mathbb{P}^2-$ a.s., $X^{i_1}(t)$ is assumed to be $\mathbb{F}^{i_1} \otimes \mathfrak{B}(\mathbb{R}_+)-$adapted, and $Y^{i_2}(t)$ is assumed to be $\mathbb{F}^{i_2} \otimes \mathfrak{B}(\mathbb{R}_+)-$adapted, for $\forall\ i_1 = 1, 2, \ldots, I_1$ and $\forall\ i_2 = 1, 2, \ldots, I_2$. Moreover, with a little abuse of notations, we put $\bar{\sigma}^{i_1}(0) = \big(\bar{\sigma}_{i_1 1}(0), \ldots, \bar{\sigma}_{i_1 k_1}(0), \ldots, \bar{\sigma}_{i_1 d_1}(0)\big)^T \triangleq \big(\bar{\sigma}_{i_1 1}, \ldots, \bar{\sigma}_{i_1 k_1}, \ldots, \bar{\sigma}_{i_1 d_1}\big)^T \triangleq \bar{\sigma}^{i_1}$, $\bar{\gamma}^{i_1}\big(0, z^{i_1}\big) = \big(\bar{\gamma}_{i_1 1}\big(0, z_1^{i_1}\big), \ldots, \bar{\gamma}_{i_1 l_1}\big(0, z_{l_1}^{i_1}\big), \ldots, \bar{\gamma}_{i_1 n_1}\big(0, z_{n_1}^{i_1}\big)\big)^T \triangleq \big(\bar{\gamma}_{i_1 1}\big(z_1^{i_1}\big), \ldots, \bar{\gamma}_{i_1 l_1}\big(z_{l_1}^{i_1}\big), \ldots, \bar{\gamma}_{i_1 n_1}\big(z_{n_1}^{i_1}\big)\big)^T \triangleq \bar{\gamma}^{i_1}\big(z^{i_1}\big)$, $\tilde{\sigma}^{i_2}(0) = \big(\tilde{\sigma}_{i_2 1}(0), \ldots, \tilde{\sigma}_{i_2 k_2}(0), \ldots, \tilde{\sigma}_{i_2 d_2}(0)\big)^T \triangleq \big(\tilde{\sigma}_{i_2 1}, \ldots, \tilde{\sigma}_{i_2 k_2}, \ldots, \tilde{\sigma}_{i_2 d_2}\big)^T \triangleq \tilde{\sigma}^{i_2}$, and $\tilde{\gamma}^{i_2}\big(0, z^{i_2}\big) = \big(\tilde{\gamma}_{i_2 1}\big(0, z_1^{i_2}\big), \ldots, \tilde{\gamma}_{i_2 l_2}\big(0, z_{l_2}^{i_2}\big), \ldots, \tilde{\gamma}_{i_2 n_2}\big(0, z_{n_2}^{i_2}\big)\big)^T \triangleq \big(\tilde{\gamma}_{i_2 1}\big(z_1^{i_2}\big), \ldots, \tilde{\gamma}_{i_2 l_2}\big(z_{l_2}^{i_2}\big), \ldots, \tilde{\gamma}_{i_2 n_2}\big(z_{n_2}^{i_2}\big)\big)^T \triangleq \tilde{\gamma}^{i_2}\big(z^{i_2}\big)$, for $\forall\ i_1 = 1, 2, \ldots, I_1$ and $\forall\ i_2 = 1, 2, \ldots, I_2$. And also we set the following technical assumption,

ASSUMPTION 2: *The initial conditions* $X^{i_1}(0) = x^{i_1} > 0$, $Y^{i_2}(0) = y^{i_2} > 0$, $X(0) = x > 0$ *and* $Y(0) = y > 0$ *are all supposed to be deterministic and bounded for* $\forall\ i_1 = 1, 2, \ldots, I_1$ *and* $\forall\ i_2 = 1, 2, \ldots, I_2$. *Furthermore,* $\bar{\sigma}^{i_1} \neq 0$ $\mathbb{P}^{i_1}-$ *a.s.,* $\tilde{\sigma}^{i_2} \neq 0$ $\mathbb{P}^{i_2}-$ *a.s.,* $\bar{\gamma}_{i_1 l_1}\big(t, z_{l_1}^{i_1}\big) > -1 + \varepsilon_{l_1}^{i_1}$ $\mu \otimes \mathbb{P}^{i_1}-$ *a.e., and* $\tilde{\gamma}_{i_2 l_2}\big(t, z_{l_2}^{i_2}\big) > -1 + \varepsilon_{l_2}^{i_2}$ $\mu \otimes \mathbb{P}^{i_2}-$ *a.e., for* $\forall\ \varepsilon_{l_1}^{i_1} > 0$, $\varepsilon_{l_2}^{i_2} > 0$ *and for* $\forall\ i_1 = 1, 2, \ldots, I_1$; $i_2 = 1, 2, \ldots, I_2$; $l_1 = 1, 2, \ldots, n_1$ *and* $l_2 = 1, 2, \ldots, n_2$.

2.2. Stochastic Differential Cooperative Game on Time

Now, as in the model of Nowak et al (2004), and Imhof and Nowak (2006), we define the following *generalized* expected discounted fitness functions,

$$\bar{f}_{i_1}\big(t, Y(t)\big) \triangleq \mathbb{E}^2_{(s,y)}\Big[\exp\big(-\bar{\theta}^{i_1} t\big)\Big\{1 - \bar{w}^{i_1} + \bar{w}^{i_1}\Big[\bar{e}_{i_1}^T A\big(Y(t) + \tilde{\rho}\big)\Big]\Big\}\Big],$$

$$\tilde{f}_{i_2}\big(t, X(t)\big) \triangleq \mathbb{E}^1_{(s,x)}\Big[\exp\big(-\tilde{\theta}^{i_2} t\big)\Big\{1 - \tilde{w}^{i_2} + \tilde{w}^{i_2}\Big[\tilde{e}_{i_2}^T B^T\big(X(t) + \bar{\rho}\big)\Big]\Big\}\Big].$$

with $\bar{\theta}^{i_1}$, $\tilde{\theta}^{i_2} \in [0,1]$ ($\forall\ i_1 = 1, 2, \ldots, I_1$; $i_2 = 1, 2, \ldots, I_2$) denoting the discounted factors, $\bar{w}^{i_1}$, $\tilde{w}^{i_2}$

$\in [0,1]$ ($\forall\ i_1 = 1, 2, ..., I_1;\ i_2 = 1, 2, ..., I_2$) the parameters that measure the contributions of the matrix payoffs of the game to the fitness of the corresponding strategies, and $\mathbb{E}^2_{(s,y)}$, $\mathbb{E}^1_{(s,x)}$ representing the expectation operators w. r. t. the complete probability law $\mathbb{P}^2$, $\mathbb{P}^1$ with depending on initial conditions $(s, y) \in \mathbb{R}_+ \times [0,1]^{I_2}$ and $(s, x) \in \mathbb{R}_+ \times [0,1]^{I_1}$, respectively. Thus, the problem, after technically modifying the above generalized expected discounted fitness functions, facing us can be expressed as follows,

PROBLEM 1 (*Stochastic Differential Cooperative Game on Time*): We need to demonstrate that there exist two vectors of $\mathcal{F}-$stopping times $\overline{\tau}^*(\omega) \triangleq \left(\overline{\tau}^{1*}(\omega), ..., \overline{\tau}^{i_1*}(\omega), ..., \overline{\tau}^{I_1*}(\omega)\right)^T$ and $\tilde{\tau}^*(\omega) \triangleq \left(\tilde{\tau}^{1*}(\omega), ..., \tilde{\tau}^{i_2*}(\omega), ..., \tilde{\tau}^{I_2*}(\omega)\right)^T$ with $\omega \in \Omega$ such that,

$$\overline{f}^*_{i_1}\left(\overline{\tau}^{i_1*}(\omega), Y\left(\overline{\tau}^{i_1*}(\omega)\right)\right)$$

$$\triangleq \sup_{\overline{\tau}^{i_1}(\omega)\le\infty} \mathbb{E}_{(s,y)}\left[\exp\left(-\overline{\theta}^{i_1}\overline{\tau}^{i_1}(\omega)\right)\left\{1-\overline{w}^{i_1}+\overline{w}^{i_1}\left[\overline{e}^T_{i_1}A\left(Y\left(\overline{\tau}^{i_1}(\omega)\right)+\tilde{\rho}\right)\right]\right\}\right],$$

$$\triangleq \mathbb{E}_{(s,y)}\left[\exp\left(-\overline{\theta}^{i_1}\overline{\tau}^{i_1*}(\omega)\right)\left\{1-\overline{w}^{i_1}+\overline{w}^{i_1}\left[\overline{e}^T_{i_1}A\left(Y\left(\overline{\tau}^{i_1*}(\omega)\right)+\tilde{\rho}\right)\right]\right\}\right].$$

And simultaneously,

$$\tilde{f}^*_{i_2}\left(\tilde{\tau}^{i_2*}(\omega), X\left(\tilde{\tau}^{i_2*}(\omega)\right)\right)$$

$$\triangleq \sup_{\tilde{\tau}^{i_2}(\omega)\le\infty} \mathbb{E}_{(s,x)}\left[\exp\left(-\tilde{\theta}^{i_2}\tilde{\tau}^{i_2}(\omega)\right)\left\{1-\tilde{w}^{i_2}+\tilde{w}^{i_2}\left[\tilde{e}^T_{i_2}B^T\left(X\left(\tilde{\tau}^{i_2}(\omega)\right)+\overline{\rho}\right)\right]\right\}\right],$$

$$\triangleq \mathbb{E}_{(s,x)}\left[\exp\left(-\tilde{\theta}^{i_2}\tilde{\tau}^{i_2*}(\omega)\right)\left\{1-\tilde{w}^{i_2}+\tilde{w}^{i_2}\left[\tilde{e}^T_{i_2}B^T\left(X\left(\tilde{\tau}^{i_2*}(\omega)\right)+\overline{\rho}\right)\right]\right\}\right].$$

with $\overline{\tau}^{i_1*}(\omega) = \overline{\tau}^{k_1*}(\omega)$ ($\forall\ i_1 \ne k_1, i_1, k_1 = 1, 2, ..., I_1$) $\mathbb{P}-$a.s., $\tilde{\tau}^{i_2*}(\omega) = \tilde{\tau}^{k_2*}(\omega)$ ($\forall\ i_2 \ne k_2$, i_2, $k_2 = 1, 2, ..., I_2$) $\mathbb{P}-$a.s., $\mathbb{E}_{(s,y)}$ and $\mathbb{E}_{(s,x)}$ standing for the expectation operators depending on initial conditions or information (s, y) and (s, x), respectively.

REMARK 2.1: By applying Girsanov Theorem under comparatively weak conditions, the game between different strategies will become a *fair-game* after the martingale-payoffs being incorporated into the game. In this sense, we argue that the corresponding endogenous matching is *fair*. Moreover, it is easily seen that the endogenous matching fulfills Pareto efficiency as well as individual rationality (from the viewpoint of strategies).

DEFINITION 1 (*Pareto Optimal Endogenous Matching and Induced Nash Equilibrium*): The solution, if it exists, to Problem 1 defines a game equilibrium, denoted

$$\Big(x^*(y^*,\bar{\rho}) \triangleq \left(x^{1*}(y^*,\bar{\rho}),...,x^{i_1*}(y^*,\bar{\rho}),...,x^{I_1*}(y^*,\bar{\rho})\right)^T,$$

$$y^*(x^*,\tilde{\rho}) \triangleq \left(y^{1*}(x^*,\tilde{\rho}),...,y^{i_2*}(x^*,\tilde{\rho}),...,y^{I_2*}(x^*,\tilde{\rho})\right)^T\Big).$$

with $\sum_{i_1}^{I_1} x^{i_1*}(y^*,\bar{\rho})=1$ and $\sum_{i_2}^{I_2} y^{i_2*}(x^*,\tilde{\rho})=1$, induced by stochastic group evolution and rational individual choice corresponding to very general normal form game situations. Suppose that we are provided with a Pareto optimal Nash equilibrium denoted by $\big(\hat{x} \triangleq (\hat{x}^1,...,\hat{x}^{i_1},...,\hat{x}^{I_1})^T$, $\hat{y} \triangleq (\hat{y}^1,...,\hat{y}^{i_2},...,\hat{y}^{I_2})^T\big)$ with $\sum_{i_1}^{I_1}\hat{x}^{i_1}=1$ and $\sum_{i_2}^{I_2}\hat{y}^{i_2}=1$ in the original normal form game, then we arrive at the *Pareto optimal endogenous matching* by solving the following equations, i.e., $x^*(y^*,\bar{\rho})=\hat{x}$ and $y^*(x^*,\tilde{\rho})=\hat{y}$, and we represent the corresponding Pareto optimal endogenous matching by $(\bar{\rho}^*,\tilde{\rho}^*)$. Moreover, we call the Pareto optimal Nash equilibrium $(\hat{x},\hat{y})$ *induced Nash equilibrium* (in some sense) in the current game situations.

REMARK 2.2: Specifically, it is worth noting that there exists intrinsic relationship between the endogenous matching and the broadly applied random matching (see, Ellison, 1994; Weibull, 1995; Aliprantis et al., 2007; Duffie and Sun, 2007; Takahashi, 2010, and among others). Notice that the present endogenous matching could be naturally (to some extent and in some sense) viewed as certain perturbation of the perfect world with well-mixed population, random matching indeed represents a special case of the endogenous matching studied in the paper. In other words, if we suppose that individuals or players play the game in a perfect world rather than a *structured society*, random-matching hypothesis is quite appropriate and also random matching itself would be regarded as endogenously determined, i.e., determined by the corresponding game environment. Generally speaking, and to the best of our knowledge, random matching is just employed as an exogenous matching mechanism which does not imply any welfare standard (or implied by any welfare standard) in existing studies (e.g., Ellison, 1994; Weibull, 1995; Hofbauer and Sigmund, 2003; Benaïm and Weibull, 2003; and Takahashi, 2010, and among others). Nevertheless, as an extreme case of the endogenous matching studied here, random matching itself indeed yields economic-welfare intuitions and implications. For example, if we can establish that $\lim_{\bar{\rho}\to 0} x^*(y^*,\bar{\rho})=\hat{x}$ and $\lim_{\tilde{\rho}\to 0} y^*(x^*,\tilde{\rho})=\hat{y}$, we can definitely call the corresponding random matching *asymptotically Pareto* *efficient* (or *Pareto optimal*). As is well known, people live in a structured society and thus random matching only works as certain limit of the endogenous matching. And random matching will be supplied with much richer economic intuitions and implications as long as it is studied in a way intimately related to the present endogenous matching. All in all, game rule is implied by the society structure[7] in some sense and meanwhile the society structure rather implies certain economic-welfare implication, so our study of endogenous matching indeed deepens existing studies of matching theory.

2.3. Existence of Pareto Optimal Endogenous Matching

To do this, we now define $\tilde{Z}(t) \triangleq \left(s+t, X(t)\right)$ for $\forall\ t \in \mathbb{R}_+$ with $\tilde{Z}(0) \triangleq (s,x) \in \mathbb{R}_+ \times \left[0,1\right]^{I_1}$,

and $\overline{Z}(t) \triangleq \left(s+t, Y(t)\right)$ for $\forall\ t \in \mathbb{R}_+$ with $\overline{Z}(0) \triangleq (s,y) \in \mathbb{R}_+ \times \left[0,1\right]^{I_2}$. And also we let,

$$\nabla \tilde{f}(s,x) \triangleq \left(\tfrac{\partial \tilde{f}}{\partial x^1}(s,x),\ldots,\tfrac{\partial \tilde{f}}{\partial x^{i_1}}(s,x),\ldots,\tfrac{\partial \tilde{f}}{\partial x^{I_1}}(s,x)\right)^T,$$

$$\overline{\gamma}_{l_1}(x) \triangleq \left(x^1\overline{\gamma}_{1l_1}\left(z_{l_1}^1\right), \ldots, x^{i_1}\overline{\gamma}_{i_1 l_1}\left(z_{l_1}^{i_1}\right),\ldots, x^{I_1}\overline{\gamma}_{I_1 l_1}\left(z_{l_1}^{I_1}\right)\right)^T,$$

$$\nabla \overline{f}(s,y) \triangleq \left(\tfrac{\partial \overline{f}}{\partial y^1}(s,y),\ldots,\tfrac{\partial \overline{f}}{\partial y^{i_2}}(s,y),\ldots,\tfrac{\partial \overline{f}}{\partial y^{I_2}}(s,y)\right)^T,$$

And,

$$\tilde{\gamma}_{l_2}(y) \triangleq \left(y^1\tilde{\gamma}_{1l_2}\left(z_{l_2}^1\right),\ldots, y^{i_2}\tilde{\gamma}_{i_2 l_2}\left(z_{l_2}^{i_2}\right),\ldots, y^{I_2}\tilde{\gamma}_{I_2 l_2}\left(z_{l_2}^{I_2}\right)\right)^T.$$

Then the characteristic operators of $\tilde{Z}(t)$ and $\overline{Z}(t)$ can be respectively given by (and $\langle \cdot,\cdot \rangle$ is used to denote the scalar product),

$$\begin{aligned}\mathcal{A}\tilde{f}(s,x) = {} & \frac{\partial \tilde{f}}{\partial s}(s,x) + \sum_{i_1=1}^{I_1} x^{i_1}\left(\overline{e}_{i_1}^T A y\right)\frac{\partial \tilde{f}}{\partial x^{i_1}}(s,x) + \frac{1}{2}\sum_{i_1=1}^{I_1}\left(x^{i_1}\right)^2\left(\overline{\sigma}^{i_1}\right)^T\overline{\sigma}^{i_1}\frac{\partial^2 \tilde{f}}{\partial (x^{i_1})^2}(s,x) \\ & + \sum_{k_1=1}^{I_1}\int_{\mathbb{R}_0}\sum_{l_1=1}^{n_1}\left\{\tilde{f}\left(s, x+\overline{\gamma}_{l_1}(x)\right) - \tilde{f}(s,x) - \left\langle \nabla\tilde{f}(s,x), \overline{\gamma}_{l_1}(x)\right\rangle\right\}\nu_{l_1}^{k_1}\left(dz_{l_1}^{k_1}\right), \\ & \qquad \forall\ \tilde{f} \in C^2\left(\mathbb{R}^{I_1+1}\right).\end{aligned}$$

And,

[7] It should include both spatial structure and division structure of any given mature market.

$$\mathcal{A}\overline{f}(s,y)=\frac{\partial\overline{f}}{\partial s}(s,y)+\sum_{i_2=1}^{I_2}y^{i_2}\left(\tilde{e}_{i_2}^T B^T x\right)\frac{\partial\overline{f}}{\partial y^{i_2}}(s,y)+\frac{1}{2}\sum_{i_2=1}^{I_2}\left(y^{i_2}\right)^2\left(\tilde{\sigma}^{i_2}\right)^T\tilde{\sigma}^{i_2}\frac{\partial^2\overline{f}}{\partial(y^{i_2})^2}(s,y)$$

$$+\sum_{k_2=1}^{I_2}\int_{\mathbb{R}_0}\sum_{l_2=1}^{n_2}\left\{\overline{f}\left(s,y+\tilde{\gamma}_{l_2}(y)\right)-\overline{f}(s,y)-\left\langle\nabla\overline{f}(s,y),\tilde{\gamma}_{l_2}(y)\right\rangle\right\}\nu_{l_2}^{k_2}\left(dz_{l_2}^{k_2}\right),$$

$$\forall\ \overline{f}\in C^2\left(\mathbb{R}^{I_2+1}\right).$$

Furthermore, we let $\sum_{i_1=1}^{I_1-1}x^{i_1}=\overline{\delta}_1$, then $x^{I_1}=1-\overline{\delta}_1$ with $0\le\overline{\delta}_1\le1$ by noting that $\sum_{i_1=1}^{I_1}x^{i_1}=1$. Let $\sum_{i_1=1}^{I_1-2}x^{i_1}=\overline{\delta}_2$, then we can get $x^{I_1-1}=\overline{\delta}_1-\overline{\delta}_2$ with $0\le\overline{\delta}_2\le\overline{\delta}_1\le1$. Inductively, we let $\sum_{i_1=1}^{I_1-(I_1-2)}x^{i_1}=\overline{\delta}_{I_1-2}$, then we have $x^3=x^{I_1-(I_1-3)}=\overline{\delta}_{I_1-3}-\overline{\delta}_{I_1-2}$ with $0\le\overline{\delta}_{I_1-2}\le\overline{\delta}_{I_1-3}\le\ldots\le\overline{\delta}_1\le1$; let $\sum_{i_1=1}^{I_1-(I_1-1)}x^{i_1}=\overline{\delta}_{I_1-1}$, i.e., $x^1=\overline{\delta}_{I_1-1}$, then we get $x^2=x^{I_1-(I_1-2)}=\overline{\delta}_{I_1-2}-\overline{\delta}_{I_1-1}$ with $0\le\overline{\delta}_{I_1-1}\le\overline{\delta}_{I_1-2}\le\overline{\delta}_{I_1-3}\le\ldots\le\overline{\delta}_1\le1$. And without loss of any generality, we put $\overline{\delta}_0\equiv1$. Then we obtain,

$$u\left(i_2,x+\overline{\rho}\right)=\tilde{e}_{i_2}^T B^T(x+\overline{\rho})=\left(b_{i_2 1}-b_{i_2 2}\right)x^1+b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1}b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)+\tilde{e}_{i_2}^T B^T\overline{\rho}.$$

Similarly, notice that $\sum_{i_2=1}^{I_2}y^{i_2}\equiv1$ and let $\sum_{i_2=1}^{I_2-1}y^{i_2}=\tilde{\delta}_1$, then we have $y^{I_2}=1-\tilde{\delta}_1$ with $0\le\tilde{\delta}_1\le1$. Let $\sum_{i_2=1}^{I_2-2}y^{i_2}=\tilde{\delta}_2$, then we see that $y^{I_2-1}=\tilde{\delta}_1-\tilde{\delta}_2$ with $0\le\tilde{\delta}_2\le\tilde{\delta}_1\le1$. Inductively, let $\sum_{i_2=1}^{I_2-(I_2-2)}y^{i_2}=\tilde{\delta}_{I_2-2}$, then we have $y^3=y^{I_2-(I_2-3)}=\tilde{\delta}_{I_2-3}-\tilde{\delta}_{I_2-2}$ with $0\le\tilde{\delta}_{I_2-2}\le\tilde{\delta}_{I_2-3}\le\ldots\le\tilde{\delta}_1\le1$; let $\sum_{i_2=1}^{I_2-(I_2-1)}y^{i_2}=\tilde{\delta}_{I_2-1}$, i.e., $y^1=\tilde{\delta}_{I_2-1}$, then it follows that $y^2=y^{I_2-(I_2-2)}=\tilde{\delta}_{I_2-2}-\tilde{\delta}_{I_2-1}$ with $0\le\tilde{\delta}_{I_2-1}\le\tilde{\delta}_{I_2-2}\le\tilde{\delta}_{I_2-3}\le\ldots\le\tilde{\delta}_1\le1$. And we, without loss of any generality, put $\tilde{\delta}_0\equiv1$. Then we get,

$$u\left(i_1,y+\tilde{\rho}\right)=\overline{e}_{i_1}^T A(y+\tilde{\rho})=\left(a_{i_1 1}-a_{i_1 2}\right)y^1+a_{i_1 2}\tilde{\delta}_{I_2-2}+\sum_{i_2=3}^{I_2}a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2}-\tilde{\delta}_{I_2-i_2+1}\right)+\overline{e}_{i_1}^T A\tilde{\rho}.$$

Therefore, the discounted fitness functions in Problem 1 can be rewritten as,

$$\overline{f}_{i_1}\left(s,y^1\right)=\exp\left(-\overline{\theta}^{i_1}s\right)$$

$$\times\left\{1-\bar{w}^{i_1}+\bar{w}^{i_1}\left[\left(a_{i_1 1}-a_{i_1 2}\right)y^1+a_{i_1 2}\tilde{\delta}_{I_2-2}+\sum_{i_2=3}^{I_2}a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2}-\tilde{\delta}_{I_2-i_2+1}\right)+\bar{e}_{i_1}^T A\tilde{\rho}\right]\right\},$$

$$\forall\ i_1=1,\,2,\,...,I_1.$$

$$\tilde{f}_{i_2}\left(s,x^1\right)=\exp\left(-\tilde{\theta}^{i_2}s\right)$$

$$\times\left\{1-\tilde{w}^{i_2}+\tilde{w}^{i_2}\left[\left(b_{i_2 1}-b_{i_2 2}\right)x^1+b_{i_2 2}\bar{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1}b_{i_2 i_1}\left(\bar{\delta}_{I_1-i_1}-\bar{\delta}_{I_1-i_1+1}\right)+\tilde{e}_{i_2}^T B^T\bar{\rho}\right]\right\},$$

$$\forall\ i_2=1,\,2,\,...,I_2. \quad (1)$$

with $0\le\tilde{\delta}_{I_2-1}\le\tilde{\delta}_{I_2-2}\le\tilde{\delta}_{I_2-3}\le...\le\tilde{\delta}_1\le\tilde{\delta}_0\equiv 1$ and $0\le\bar{\delta}_{I_1-1}\le\bar{\delta}_{I_1-2}\le\bar{\delta}_{I_1-3}\le...\le\bar{\delta}_1\le\bar{\delta}_0\equiv 1$. And inspection of the fitness functions given in (1) reveals that one can just define $\tilde{Z}(t)\triangleq\left(s+t,X^1(t)\right)$ for $\forall\ t\in\mathbb{R}_+$ with $\tilde{Z}(0)\triangleq\left(s,x^1\right)\in\mathbb{R}_+\times[0,1]$, and $\bar{Z}(t)\triangleq\left(s+t,Y^1(t)\right)$ for $\forall\ t\in\mathbb{R}_+$ with $\bar{Z}(0)\triangleq\left(s,y^1\right)\in\mathbb{R}_+\times[0,1]$. And hence the corresponding characteristic operators of $\tilde{Z}(t)$ and $\bar{Z}(t)$ are respectively given by,

$$\mathcal{A}\tilde{f}\left(s,x^1\right)=\frac{\partial\tilde{f}}{\partial s}\left(s,x^1\right)+x^1\left(\bar{e}_1^T Ay\right)\frac{\partial\tilde{f}}{\partial x^1}\left(s,x^1\right)+\frac{1}{2}\left(x^1\right)^2\left(\bar{\sigma}^1\right)^T\bar{\sigma}^1\frac{\partial^2\tilde{f}}{\partial(x^1)^2}\left(s,x^1\right)$$

$$+\int_{\mathbb{R}_0}\sum_{l_1=1}^{n_1}\left\{\tilde{f}\left(s,x^1+x^1\bar{\gamma}_{1l_1}\left(z_{l_1}^1\right)\right)-\tilde{f}\left(s,x^1\right)-x^1\bar{\gamma}_{1l_1}\left(z_{l_1}^1\right)\tfrac{\partial\tilde{f}}{\partial x^1}\left(s,x^1\right)\right\}\nu_{l_1}^1\left(dz_{l_1}^1\right),$$

$$\forall\ \tilde{f}\in C^2\left(\mathbb{R}^2\right).$$

And,

$$\mathcal{A}\bar{f}\left(s,y^1\right)=\frac{\partial\bar{f}}{\partial s}\left(s,y^1\right)+y^1\left(\tilde{e}_1^T B^T x\right)\frac{\partial\bar{f}}{\partial y^1}\left(s,y^1\right)+\frac{1}{2}\left(y^1\right)^2\left(\tilde{\sigma}^1\right)^T\tilde{\sigma}^1\frac{\partial^2\bar{f}}{\partial(y^1)^2}\left(s,y^1\right)$$

$$+\int_{\mathbb{R}_0}\sum_{l_2=1}^{n_2}\left\{\bar{f}\left(s,y^1+y^1\tilde{\gamma}_{1l_2}\left(z_{l_2}^1\right)\right)-\bar{f}\left(s,y^1\right)-y^1\tilde{\gamma}_{1l_2}\left(z_{l_2}^1\right)\tfrac{\partial\bar{f}}{\partial y^1}\left(s,y^1\right)\right\}\nu_{l_2}^1\left(dz_{l_2}^1\right)$$

$$\forall\ \bar{f}\in C^2\left(\mathbb{R}^2\right).$$

Therefore, based upon the above assumptions and specifications, the following theorem is derived,

THEOREM 1: *There exists a unique solution to Problem* 1 *under very weak conditions, and accordingly the existence of the Pareto optimal endogenous matching is confirmed provided that in the original normal-form game we are given a Pareto optimal Nash equilibrium* $(\hat{x}, \hat{y})$*, which is given in Definition* 1.

PROOF: See Appendix A. $\blacksquare$

REMARK 2.3: It is especially worth noting that Theorem 1 not only shows the existence of the Pareto optimal endogenous matching and induced Nash equilibrium given by Definition 1 but also provides us with the *explicit time length* needed so that the Pareto optimal endogenous matching and also the induced Nash equilibrium can be achieved by decentralized players. Moreover, it is also worth emphasizing that our conclusion holds true for any Pareto optimal *strategy combination* of very general normal-form games although we have only considered Pareto optimal *Nash equilibrium* in Theorem 1. For instance, (cooperation, cooperation) is a Pareto-optimal strategy combination in PD games although it is generally not Nash equilibrium at all. Obviously, our endogenous matching rule can lead us to cooperation in PD games.

3. Conclusion

In this study, the players follow certain adaptive learning processes while the strategies themselves are assumed to be rational by applying the classical *as if* methodology. Based upon this methodology, it is argued that optimal stopping theory is very useful in establishing the game equilibrium. The major innovations can be summarized as follows. Firstly, we provide a very *general* framework for studying endogenous matching, and those explicit matching mechanisms developed by Haag and Lagunoff (2006), and Bisin et al. (2006) can be regarded as special realizations of the present model. Secondly, the well-known random matching (e.g., Gilboa and Matsui, 1992; Ellison, 1994; Weibull, 1995; Aliprantis et al., 2007; Duffie and Sun, 2007; Takahashi, 2010; Podczeck and Puzzello, 2012, and among others), as an extreme case of the endogenous matching under consideration, yields economic-welfare intuitions and implications, and the present study provides conditions under which the random matching is *asymptotically Pareto efficient*. Thirdly, our study shows a reasonable approach to combine optimal stopping theory and evolutionary game theory, thereby throwing new insights into the classical evolutionary game theory (see, Weibull, 1995; Hofbauer and Sigmund, 2003; Benaïm and Weibull, 2003, and among others). Last but not least, noting that certain matching mechanism implies certain macro-social structure, the present study thus reveals a general existence of the Pareto-optimal social structure in any given normal-form game situations. In particular, one can interpret this result from the following viewpoint, i.e., if the purpose of institutional segmentation (or segregation) (e.g., Schelling, 1969, 1971; Bowles and Gintis, 1998) is to improve people's welfare in the same community, then we confirm that there exists an *optimal level of segmentation* so that the welfare of the entire community is maximized.

The current paper can be naturally extended in the following ways: first, asymmetric information can be introduced into the present model to capture much more economic implications, for instance, one can explore the value of information with respect to the endogenous matching; second, the classical stochastic differential cooperative game can be explored based upon the present framework; third, specific mechanism, say, reputation mechanism (for example, see, Anderson and Smith, 2010) or searching mechanism (e.g., Eeckhout and Kircher, 2010, and references therein), can be incorporated into the model to support any other pattern of endogenous matching; forth, our approach can be easily extended to include multiple priors (see, Riedel, 2009, for instance) and also to explore the evolutionary equilibria on graphs (see, Ohtsuki et al., 2007, and among others).

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Appendix

A. Proof of Theorem 1

We proceed the proof as follows. In Step 1, we derive the optimal stopping rules (and also the corresponding supporting conditions) for strategies i_1, $i_1 = 1,2,...,I_1$; in Step 2, we similarly establish the optimal stopping rules for strategies i_2, $i_2 = 1,2,...,I_2$; and in Step 3, we show (by solving a group of equations) that strategies i_1, $i_1 = 1,2,...,I_1$, can cooperate to the same optimal stopping rule given the optimal stopping rule of strategies i_2, $i_2 = 1,2,...,I_2$, and vice versa. About the optimal stopping theory used here, one can refer to Øksendal and Sulem (2005) for much more details.

STEP 1: For strategy i_1, $\forall\, i_1 = 1,2,...,I_1$. Notice that,

$$\begin{aligned}
\mathcal{A}\overline{f}_{i_1}\left(s,y^1\right) &= -\overline{\theta}^{i_1}\exp\left(-\overline{\theta}^{i_1}s\right) \\
&\quad\times\left\{1-\overline{w}^{i_1}+\overline{w}^{i_1}\left[\left(a_{i_1 1}-a_{i_1 2}\right)y^1+a_{i_1 2}\tilde{\delta}_{I_2-2}+\sum_{i_2=3}^{I_2}a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2}-\tilde{\delta}_{I_2-i_2+1}\right)+\overline{e}_{i_1}^T A\tilde{\rho}\right]\right\} \\
&\quad+y^1\left(\tilde{e}_1^T B^T x\right)\exp\left(-\overline{\theta}^{i_1}s\right)\overline{w}^{i_1}\left(a_{i_1 1}-a_{i_1 2}\right)\geq 0 \\
&\Leftrightarrow \left(\tilde{e}_1^T B^T x-\overline{\theta}^{i_1}\right)\overline{w}^{i_1}\left(a_{i_1 1}-a_{i_1 2}\right)y^1 \\
&\quad\geq \overline{\theta}^{i_1}\left(1-\overline{w}^{i_1}\right)+\overline{\theta}^{i_1}\overline{w}^{i_1}\left[a_{i_1 2}\tilde{\delta}_{I_2-2}+\sum_{i_2=3}^{I_2}a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2}-\tilde{\delta}_{I_2-i_2+1}\right)+\overline{e}_{i_1}^T A\tilde{\rho}\right].
\end{aligned}$$

Case 1.1: $\begin{cases}\overline{\theta}^{i_1}\left(1-\overline{w}^{i_1}\right)+\overline{\theta}^{i_1}\overline{w}^{i_1}\left[a_{i_1 2}\tilde{\delta}_{I_2-2}+\sum_{i_2=3}^{I_2}a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2}-\tilde{\delta}_{I_2-i_2+1}\right)+\overline{e}_{i_1}^T A\tilde{\rho}\right]<0 \\ \operatorname{sgn}\left(\tilde{e}_1^T B^T x-\overline{\theta}^{i_1}\right)=\operatorname{sgn}\left(a_{i_1 2}-a_{i_1 1}\right)\end{cases}$

Then,

$$\mathcal{A}\overline{f}_{i_1}\left(s,y^1\right)\geq 0$$

$$\Leftrightarrow y^1 \leq \frac{\overline{\theta}^{i_1}\left(1-\overline{w}^{i_1}\right)+\overline{\theta}^{i_1}\overline{w}^{i_1}\left[a_{i_1 2}\tilde{\delta}_{I_2-2}+\sum_{i_2=3}^{I_2}a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2}-\tilde{\delta}_{I_2-i_2+1}\right)+\overline{e}_{i_1}^T A\tilde{\rho}\right]}{\left(\tilde{e}_1^T B^T x-\overline{\theta}^{i_1}\right)\overline{w}^{i_1}\left(a_{i_1 1}-a_{i_1 2}\right)}.$$

Hence, we have,

$$U^{i_1}=\left\{\left(s,y^1\right);y^1\le\frac{\overline{\theta}^{i_1}\left(1-\overline{w}^{i_1}\right)+\overline{\theta}^{i_1}\overline{w}^{i_1}\left[a_{i_12}\tilde{\delta}_{I_2-2}+\sum_{i_2=3}^{I_2}a_{i_1i_2}\left(\tilde{\delta}_{I_2-i_2}-\tilde{\delta}_{I_2-i_2+1}\right)+\overline{e}_{i_1}^T A\tilde{\rho}\right]}{\left(\tilde{e}_1^T B^T x-\overline{\theta}^{i_1}\right)\overline{w}^{i_1}\left(a_{i_11}-a_{i_12}\right)}\right\}. \quad \text{(A.1)}$$

And it is natural to guess that the continuation region D^{i_1} has the following form,

$$D^{i_1}\left(y_{i_1}^{1*}\right)=\left\{\left(s,y^1\right);0\le y^1\le y_{i_1}^{1*}\right\}.$$

where,

$$y_{i_1}^{1*}\ge\frac{\overline{\theta}^{i_1}\left(1-\overline{w}^{i_1}\right)+\overline{\theta}^{i_1}\overline{w}^{i_1}\left[a_{i_12}\tilde{\delta}_{I_2-2}+\sum_{i_2=3}^{I_2}a_{i_1i_2}\left(\tilde{\delta}_{I_2-i_2}-\tilde{\delta}_{I_2-i_2+1}\right)+\overline{e}_{i_1}^T A\tilde{\rho}\right]}{\left(\tilde{e}_1^T B^T x-\overline{\theta}^{i_1}\right)\overline{w}^{i_1}\left(a_{i_11}-a_{i_12}\right)}. \quad \text{(A.2)}$$

Notice that the generator of $\overline{Z}(t)$ is given by,

$$\mathcal{A}\overline{\phi}_{i_1}\left(s,y^1\right)=\frac{\partial\overline{\phi}_{i_1}}{\partial s}+y^1\left(\tilde{e}_1^T B^T x\right)\frac{\partial\overline{\phi}_{i_1}}{\partial y^1}+\frac{1}{2}\left(y^1\right)^2\left(\tilde{\sigma}^1\right)^T\tilde{\sigma}^1\frac{\partial^2\overline{\phi}_{i_1}}{\partial(y^1)^2}$$

$$+\int_{\mathbb{R}_0}\sum_{l_2=1}^{n_2}\left\{\overline{\phi}_{i_1}\left(s,y^1+y^1\tilde{\gamma}_{1l_2}\left(z_{l_2}^1\right)\right)-\overline{\phi}_{i_1}\left(s,y^1\right)-y^1\tilde{\gamma}_{1l_2}\left(z_{l_2}^1\right)\frac{\partial\overline{\phi}_{i_1}}{\partial y^1}\left(s,y^1\right)\right\}\nu_{l_2}^1\left(dz_{l_2}^1\right)$$

for $\forall\overline{\phi}_{i_1}\left(s,y^1\right)\in C^2\left(\mathbb{R}^2\right)$. If we try a function $\overline{\phi}_{i_1}$ of the following form,

$$\overline{\phi}_{i_1}\left(s,y^1\right)=\exp\left(-\overline{\theta}^{i_1}s\right)\left(y^1\right)^{\overline{\lambda}^{i_1}} \quad \text{for some constant } \overline{\lambda}^{i_1}\in\mathbb{R}.$$

We then get,

$$\mathcal{A}\overline{\phi}_{i_1}\left(s,y^1\right)=\exp\left(-\overline{\theta}^{i_1}s\right)\Bigg[-\overline{\theta}^{i_1}\left(y^1\right)^{\overline{\lambda}^{i_1}}+\left(\tilde{e}_1^T B^T x\right)y^1\overline{\lambda}^{i_1}\left(y^1\right)^{\overline{\lambda}^{i_1}-1}$$

$$+\frac{1}{2}\left(\tilde{\sigma}^1\right)^T\tilde{\sigma}^1\left(y^1\right)^2\overline{\lambda}^{i_1}\left(\overline{\lambda}^{i_1}-1\right)\left(y^1\right)^{\overline{\lambda}^{i_1}-2}$$

$$+\int_{\mathbb{R}_0}\sum_{l_2=1}^{n_2}\left\{\left[y^1+y^1\tilde{\gamma}_{1l_2}\left(z_{l_2}^1\right)\right]^{\overline{\lambda}^{i_1}}-\left(y^1\right)^{\overline{\lambda}^{i_1}}-\tilde{\gamma}_{1l_2}\left(z_{l_2}^1\right)y^1\overline{\lambda}^{i_1}\left(y^1\right)^{\overline{\lambda}^{i_1}-1}\right\}\nu_{l_2}^1\left(dz_{l_2}^1\right)\Bigg]$$

$$=\exp\left(-\overline{\theta}^{i_1}s\right)\left(y^1\right)^{\overline{\lambda}^{i_1}}\overline{h}_{i_1}\left(\overline{\lambda}^{i_1}\right).$$

where,

$$\overline{h}_{i_1}\left(\overline{\lambda}^{i_1}\right) \triangleq -\overline{\theta}^{i_1} + \left(\tilde{e}_1^T B^T x\right)\overline{\lambda}^{i_1} + \frac{1}{2}\left(\tilde{\sigma}^1\right)^T \tilde{\sigma}^1 \overline{\lambda}^{i_1}\left(\overline{\lambda}^{i_1} - 1\right)$$
$$+\int_{\mathbb{R}_0} \sum_{l_2=1}^{n_2} \left\{ \left[1 + \tilde{\gamma}_{1l_2}\left(z_{l_2}^1\right)\right]^{\overline{\lambda}^{i_1}} - 1 - \tilde{\gamma}_{1l_2}\left(z_{l_2}^1\right)\overline{\lambda}^{i_1} \right\} \nu_{l_2}^1\left(dz_{l_2}^1\right).$$

Note that,

$$\overline{h}_{i_1}(1) = \tilde{e}_1^T B^T x - \overline{\theta}^{i_1} \quad \text{and} \quad \lim_{\overline{\lambda}^{i_1} \to \infty} \overline{h}_{i_1}\left(\overline{\lambda}^{i_1}\right) = \infty .$$

Therefore, if we assume that,

$$\tilde{e}_1^T B^T x < \overline{\theta}^{i_1}, \tag{A.3}$$

Then we find that there exists $\overline{\lambda}^{i_1} > 1$ such that,

$$\overline{h}_{i_1}\left(\overline{\lambda}^{i_1}\right) = 0. \tag{A.4}$$

with this value of $\overline{\lambda}^{i_1}$ we put,

$$\overline{\phi}_{i_1}\left(s, y^1\right) = \begin{cases} e^{-\overline{\theta}^{i_1} s}\overline{C}^{i_1}\left(y^1\right)^{\overline{\lambda}^{i_1}}, 0 \le y^1 \le y_{i_1}^{1*} \\ e^{-\overline{\theta}^{i_1} s}\left\{1 - \overline{w}^{i_1} + \overline{w}^{i_1}\left[\left(a_{i_1 1} - a_{i_1 2}\right)y^1 + a_{i_1 2}\tilde{\delta}_{I_2-2} + \sum_{i_2=3}^{I_2} a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2} - \tilde{\delta}_{I_2-i_2+1}\right) + \overline{e}_{i_1}^T A\tilde{\rho}\right]\right\}, y_{i_1}^{1*} \le y^1 \le 1 \end{cases}$$

for some constant $\overline{C}^{i_1} > 0$, to be determined. We, without loss of any generality, guess that the value function is C^1 at $y^1 = y_{i_1}^{1*}$ and this leads us to the following "high contact" conditions,

$$\overline{C}^{i_1}\left(y_{i_1}^{1*}\right)^{\overline{\lambda}^{i_1}} = 1 - \overline{w}^{i_1} + \overline{w}^{i_1}\left[\left(a_{i_1 1} - a_{i_1 2}\right)y_{i_1}^{1*} + a_{i_1 2}\tilde{\delta}_{I_2-2} + \sum_{i_2=3}^{I_2} a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2} - \tilde{\delta}_{I_2-i_2+1}\right) + \overline{e}_{i_1}^T A\tilde{\rho}\right]$$

(continuity at $y^1 = y_{i_1}^{1*}$)

$$\overline{C}^{i_1}\overline{\lambda}^{i_1}\left(y_{i_1}^{1*}\right)^{\overline{\lambda}^{i_1}-1} = \overline{w}^{i_1}\left(a_{i_1 1} - a_{i_1 2}\right) \quad \text{(differentiability at } y^1 = y_{i_1}^{1*}\text{)}$$

Combining the above equations shows that,

$$\frac{\overline{C}^{i_1}\left(y_{i_1}^{1*}\right)^{\overline{\lambda}^{i_1}}}{\overline{C}^{i_1}\overline{\lambda}^{i_1}\left(y_{i_1}^{1*}\right)^{\overline{\lambda}^{i_1}-1}} = \frac{1 - \overline{w}^{i_1} + \overline{w}^{i_1}\left[\left(a_{i_1 1} - a_{i_1 2}\right)y_{i_1}^{1*} + a_{i_1 2}\tilde{\delta}_{I_2-2} + \sum_{i_2=3}^{I_2} a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2} - \tilde{\delta}_{I_2-i_2+1}\right) + \overline{e}_{i_1}^T A\tilde{\rho}\right]}{\overline{w}^{i_1}\left(a_{i_1 1} - a_{i_1 2}\right)}$$

$$\Leftrightarrow y_{i_1}^{1*} = \frac{\bar{\lambda}^{i_1}\left\{1-\bar{w}^{i_1}+\bar{w}^{i_1}\left[a_{i_1 2}\tilde{\delta}_{I_2-2}+\sum_{i_2=3}^{I_2} a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2}-\tilde{\delta}_{I_2-i_2+1}\right)+\bar{e}_{i_1}^T A\tilde{\rho}\right]\right\}}{\left(1-\bar{\lambda}^{i_1}\right)\bar{w}^{i_1}\left(a_{i_1 1}-a_{i_1 2}\right)}. \tag{A.5}$$

And this gives,

$$\bar{C}^{i_1} = \frac{\bar{w}^{i_1}\left(a_{i_1 1}-a_{i_1 2}\right)}{\bar{\lambda}^{i_1}\left(y_{i_1}^{1*}\right)^{\bar{\lambda}^{i_1}-1}}. \tag{A.6}$$

Hence, by (A.4), (A.5) and (A.6), we can define,

$$\bar{\bar{f}}_{i_1}\left(s,y^1\right) \triangleq \exp\left(-\bar{\theta}^{i_1}s\right)\bar{C}^{i_1}\left(y^1\right)^{\bar{\lambda}^{i_1}}.$$

And then we are in the position to prove that,

$$\bar{\bar{f}}_{i_1}\left(s,y^1\right) \triangleq \exp\left(-\bar{\theta}^{i_1}s\right)\bar{C}^{i_1}\left(y^1\right)^{\bar{\lambda}^{i_1}} = \bar{f}_{i_1}^*\left(s,y^1\right).$$

in which $\bar{f}_{i_1}^*\left(s,y^1\right)$ is a supermeanvalued majorant of $\bar{f}_{i_1}\left(s,y^1\right)$. Firstly, noting that,

$$\mathcal{A}\bar{f}_{i_1}\left(s,y^1\right) = -\bar{\theta}^{i_1}\exp\left(-\bar{\theta}^{i_1}s\right)$$

$$\times\left\{1-\bar{w}^{i_1}+\bar{w}^{i_1}\left[\left(a_{i_1 1}-a_{i_1 2}\right)y^1+a_{i_1 2}\tilde{\delta}_{I_2-2}+\sum_{i_2=3}^{I_2} a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2}-\tilde{\delta}_{I_2-i_2+1}\right)+\bar{e}_{i_1}^T A\tilde{\rho}\right]\right\}$$

$$+y^1\left(\tilde{e}_1^T B^T x\right)\exp\left(-\bar{\theta}^{i_1}s\right)\bar{w}^{i_1}\left(a_{i_1 1}-a_{i_1 2}\right) \le 0,\quad \forall y^1 \ge y_{i_1}^{1*}.$$

$$\Leftrightarrow \left(\tilde{e}_1^T B^T x-\bar{\theta}^{i_1}\right)\bar{w}^{i_1}\left(a_{i_1 1}-a_{i_1 2}\right)y^1$$

$$\le \bar{\theta}^{i_1}\left(1-\bar{w}^{i_1}\right)+\bar{\theta}^{i_1}\bar{w}^{i_1}\left[a_{i_1 2}\tilde{\delta}_{I_2-2}+\sum_{i_2=3}^{I_2} a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2}-\tilde{\delta}_{I_2-i_2+1}\right)+\bar{e}_{i_1}^T A\tilde{\rho}\right],\quad \forall y^1 \ge y_{i_1}^{1*}.$$

$$\Leftrightarrow y_{i_1}^{1*} \ge \frac{\bar{\theta}^{i_1}\left(1-\bar{w}^{i_1}\right)+\bar{\theta}^{i_1}\bar{w}^{i_1}\left[a_{i_1 2}\tilde{\delta}_{I_2-2}+\sum_{i_2=3}^{I_2} a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2}-\tilde{\delta}_{I_2-i_2+1}\right)+\bar{e}_{i_1}^T A\tilde{\rho}\right]}{\left(\tilde{e}_1^T B^T x-\bar{\theta}^{i_1}\right)\bar{w}^{i_1}\left(a_{i_1 1}-a_{i_1 2}\right)}.$$

which holds by (A.2). Secondly, to prove,

$$\bar{C}^{i_1}\left(y^1\right)^{\bar{\lambda}^{i_1}} \ge 1-\bar{w}^{i_1}+\bar{w}^{i_1}\left[\left(a_{i_1 1}-a_{i_1 2}\right)y^1+a_{i_1 2}\tilde{\delta}_{I_2-2}+\sum_{i_2=3}^{I_2} a_{i_1 i_2}\left(\tilde{\delta}_{I_2-i_2}-\tilde{\delta}_{I_2-i_2+1}\right)+\bar{e}_{i_1}^T A\tilde{\rho}\right],$$

for $\forall 0 \le y^1 \le y_{i_1}^{1*}$. Define

$$\overline{\xi}^{i_1}\left(y^1\right) \triangleq \overline{C}^{i_1}\left(y^1\right)^{\overline{\lambda}^{i_1}} - 1 + \overline{w}^{i_1}$$

$$-\overline{w}^{i_1}\left[\left(a_{i_1 1} - a_{i_1 2}\right)y^1 + a_{i_1 2}\tilde{\delta}_{I_2 - 2} + \sum_{i_2=3}^{I_2} a_{i_1 i_2}\left(\tilde{\delta}_{I_2 - i_2} - \tilde{\delta}_{I_2 - i_2 + 1}\right) + \overline{e}_{i_1}^T A\tilde{\rho}\right].$$

Then with our chosen values of $\overline{C}^{i_1}$ and $\overline{\lambda}^{i_1}$, we see that $\overline{\xi}^{i_1}\left(y_{i_1}^{1*}\right) = \overline{\xi}^{i_1\prime}\left(y_{i_1}^{1*}\right) = 0$. Furthermore, noting that $\overline{\xi}^{i_1\prime\prime}\left(y^1\right) = \overline{C}^{i_1}\overline{\lambda}^{i_1}\left(\overline{\lambda}^{i_1} - 1\right)\left(y^1\right)^{\overline{\lambda}^{i_1} - 2}$, and hence $\overline{\xi}^{i_1\prime\prime}\left(y^1\right) > 0$ holds for $\forall 0 \le y^1 \le y_{i_1}^{1*}$ given $\overline{\lambda}^{i_1} > 1$ in (A.4), that is, $\overline{\xi}^{i_1}\left(y^1\right) > 0$ follows for $\forall\ 0 \le y^1 \le y_{i_1}^{1*}$. And this completes the short proof.

Case 1.2: $\begin{cases} \overline{\theta}^{i_1}\left(1 - \overline{w}^{i_1}\right) + \overline{\theta}^{i_1}\overline{w}^{i_1}\left[a_{i_1 2}\tilde{\delta}_{I_2 - 2} + \sum_{i_2=3}^{I_2} a_{i_1 i_2}\left(\tilde{\delta}_{I_2 - i_2} - \tilde{\delta}_{I_2 - i_2 + 1}\right) + \overline{e}_{i_1}^T A\tilde{\rho}\right] > 0 \\ \operatorname{sgn}\left(\tilde{e}_1^T B^T x - \overline{\theta}^{i_1}\right) = \operatorname{sgn}\left(a_{i_1 1} - a_{i_1 2}\right) \end{cases}$

It is easy to see that the proof is quite similar to that of case 1.1, so we take it omitted.

STEP 2: For strategy i_2, $\forall\ i_2 = 1, 2, \ldots, I_2$. Notice that,

$$\mathcal{A}\tilde{f}_{i_2}\left(s, x^1\right) = -\tilde{\theta}^{i_2}\exp\left(-\tilde{\theta}^{i_2}s\right)$$

$$\times\left\{1 - \tilde{w}^{i_2} + \tilde{w}^{i_2}\left[\left(b_{i_2 1} - b_{i_2 2}\right)x^1 + b_{i_2 2}\overline{\delta}_{I_1 - 2} + \sum_{i_1=3}^{I_1} b_{i_2 i_1}\left(\overline{\delta}_{I_1 - i_1} - \overline{\delta}_{I_1 - i_1 + 1}\right) + \tilde{e}_{i_2}^T B^T\overline{\rho}\right]\right\}$$

$$+x^1\left(\overline{e}_1^T Ay\right)\exp\left(-\tilde{\theta}^{i_2}s\right)\tilde{w}^{i_2}\left(b_{i_2 1} - b_{i_2 2}\right) \ge 0$$

$$\Leftrightarrow \left(\overline{e}_1^T Ay - \tilde{\theta}^{i_2}\right)\tilde{w}^{i_2}\left(b_{i_2 1} - b_{i_2 2}\right)x^1$$

$$\ge \tilde{\theta}^{i_2}\left(1 - \tilde{w}^{i_2}\right) + \tilde{\theta}^{i_2}\tilde{w}^{i_2}\left[b_{i_2 2}\overline{\delta}_{I_1 - 2} + \sum_{i_1=3}^{I_1} b_{i_2 i_1}\left(\overline{\delta}_{I_1 - i_1} - \overline{\delta}_{I_1 - i_1 + 1}\right) + \tilde{e}_{i_2}^T B^T\overline{\rho}\right].$$

$$\textit{Case } 2.1\text{:}\quad \begin{cases} \tilde{\theta}^{i_2}\left(1-\tilde{w}^{i_2}\right)+\tilde{\theta}^{i_2}\tilde{w}^{i_2}\left[b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1}b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)+\tilde{e}_{i_2}^T B^T\overline{\rho}\right]<0 \\ \operatorname{sgn}\left(\overline{e}_1^T Ay-\tilde{\theta}^{i_2}\right)=\operatorname{sgn}\left(b_{i_2 2}-b_{i_2 1}\right) \end{cases}$$

Hence,

$$\mathcal{A}\tilde{f}_{i_2}\left(s,x^1\right)\geq 0$$

$$\Leftrightarrow x^1\leq\frac{\tilde{\theta}^{i_2}\left(1-\tilde{w}^{i_2}\right)+\tilde{\theta}^{i_2}\tilde{w}^{i_2}\left[b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1}b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)+\tilde{e}_{i_2}^T B^T\overline{\rho}\right]}{\left(\overline{e}_1^T Ay-\tilde{\theta}^{i_2}\right)\tilde{w}^{i_2}\left(b_{i_2 1}-b_{i_2 2}\right)}.$$

Then, we have,

$$U^{i_2}=\left\{\left(s,x^1\right);x^1\leq\frac{\tilde{\theta}^{i_2}\left(1-\tilde{w}^{i_2}\right)+\tilde{\theta}^{i_2}\tilde{w}^{i_2}\left[b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1}b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)+\tilde{e}_{i_2}^T B^T\overline{\rho}\right]}{\left(\overline{e}_1^T Ay-\tilde{\theta}^{i_2}\right)\tilde{w}^{i_2}\left(b_{i_2 1}-b_{i_2 2}\right)}\right\}. \quad \text{(A.7)}$$

So it is natural to guess that the continuation region D^{i_2} has the following form,

$$D^{i_2}\left(x_{i_2}^{1*}\right)=\left\{\left(s,x^1\right);0\leq x^1\leq x_{i_2}^{1*}\right\}.$$

where,

$$x_{i_2}^{1*}\geq\frac{\tilde{\theta}^{i_2}\left(1-\tilde{w}^{i_2}\right)+\tilde{\theta}^{i_2}\tilde{w}^{i_2}\left[b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1}b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)+\tilde{e}_{i_2}^T B^T\overline{\rho}\right]}{\left(\overline{e}_1^T Ay-\tilde{\theta}^{i_2}\right)\tilde{w}^{i_2}\left(b_{i_2 1}-b_{i_2 2}\right)}. \quad \text{(A.8)}$$

Notice that the generator of $\tilde{Z}(t)$ is given by,

$$\mathcal{A}\tilde{\phi}_{i_2}\left(s,x^1\right)=\frac{\partial\tilde{\phi}_{i_2}}{\partial s}+x^1\left(\overline{e}_1^T Ay\right)\frac{\partial\tilde{\phi}_{i_2}}{\partial x^1}+\frac{1}{2}\left(x^1\right)^2\left(\overline{\sigma}^1\right)^T\overline{\sigma}^1\frac{\partial^2\tilde{\phi}_{i_2}}{\partial(x^1)^2}$$

$$+\int_{\mathbb{R}_0}\sum_{l_1=1}^{n_1}\left\{\tilde{\phi}_{i_2}\left(s,x^1+x^1\overline{\gamma}_{1l_1}\left(z_{l_1}^1\right)\right)-\tilde{\phi}_{i_2}\left(s,x^1\right)-x^1\overline{\gamma}_{1l_1}\left(z_{l_1}^1\right)\tfrac{\partial\tilde{\phi}_{i_2}}{\partial x^1}\left(s,x^1\right)\right\}\nu_{l_1}^1\left(dz_{l_1}^1\right)$$

for $\forall\tilde{\phi}_{i_2}\left(s,x^1\right)\in C^2\left(\mathbb{R}^2\right)$. If we try to choose $\tilde{\phi}_{i_2}\left(s,x^1\right)=\exp\left(-\tilde{\theta}^{i_2}s\right)\left(x^1\right)^{\tilde{\lambda}^{i_2}}$ for some constant $\tilde{\lambda}^{i_2}\in\mathbb{R}$. Then we get,

$$\mathcal{A}\tilde{\phi}_{i_2}\left(s,x^1\right)=\exp\left(-\tilde{\theta}^{i_2}s\right)\Big[-\tilde{\theta}^{i_2}\left(x^1\right)^{\tilde{\lambda}^{i_2}}+\left(\overline{e}_1^T Ay\right)x^1\tilde{\lambda}^{i_2}\left(x^1\right)^{\tilde{\lambda}^{i_2}-1}$$

$$+\frac{1}{2}\left(\overline{\sigma}^1\right)^T\overline{\sigma}^1\left(x^1\right)^2\tilde{\lambda}^{i_2}\left(\tilde{\lambda}^{i_2}-1\right)\left(x^1\right)^{\tilde{\lambda}^{i_2}-2}$$

$$+\int_{\mathbb{R}_0}\sum_{l_1=1}^{n_1}\left\{\left[x^1+x^1\overline{\gamma}_{1l_1}\left(z_{l_1}^1\right)\right]^{\tilde{\lambda}^{i_2}}-\left(x^1\right)^{\tilde{\lambda}^{i_2}}-\overline{\gamma}_{1l_1}\left(z_{l_1}^1\right)x^1\tilde{\lambda}^{i_2}\left(x^1\right)^{\tilde{\lambda}^{i_2}-1}\right\}\nu_{l_1}^1\left(dz_{l_1}^1\right)\Big]$$

$$=\exp\left(-\tilde{\theta}^{i_2}s\right)\left(x^1\right)^{\tilde{\lambda}^{i_2}}\tilde{h}_{i_2}\left(\tilde{\lambda}^{i_2}\right).$$

where,

$$\tilde{h}_{i_2}\left(\tilde{\lambda}^{i_2}\right)\triangleq-\tilde{\theta}^{i_2}+\left(\overline{e}_1^T Ay\right)\tilde{\lambda}^{i_2}+\frac{1}{2}\left(\overline{\sigma}^1\right)^T\overline{\sigma}^1\tilde{\lambda}^{i_2}\left(\tilde{\lambda}^{i_2}-1\right)$$

$$+\int_{\mathbb{R}_0}\sum_{l_1=1}^{n_1}\left\{\left[1+\overline{\gamma}_{1l_1}\left(z_{l_1}^1\right)\right]^{\tilde{\lambda}^{i_2}}-1-\overline{\gamma}_{1l_1}\left(z_{l_1}^1\right)\tilde{\lambda}^{i_2}\right\}\nu_{l_1}^1\left(dz_{l_1}^1\right).$$

Noting that,

$$\tilde{h}_{i_2}(1)=\overline{e}_1^T Ay-\tilde{\theta}^{i_2}\quad\text{and}\quad\lim_{\tilde{\lambda}^{i_2}\to\infty}\tilde{h}_{i_2}\left(\tilde{\lambda}^{i_2}\right)=\infty.$$

Consequently, if we suppose that,

$$\overline{e}_1^T Ay<\tilde{\theta}^{i_2},\tag{A.9}$$

Thus, it is easily seen that there exists $\tilde{\lambda}^{i_2}>1$ such that,

$$\tilde{h}_{i_2}\left(\tilde{\lambda}^{i_2}\right)=0.\tag{A.10}$$

with this value of $\tilde{\lambda}^{i_2}$ we put,

$$\tilde{\phi}_{i_2}\left(s,x^1\right)=\begin{cases}e^{-\tilde{\theta}^{i_2}s}\tilde{C}^{i_2}\left(x^1\right)^{\tilde{\lambda}^{i_2}},0\le x^1\le x_{i_2}^{1*}\\ e^{-\tilde{\theta}^{i_2}s}\left\{1-\tilde{w}^{i_2}+\tilde{w}^{i_2}\left[\left(b_{i_2 1}-b_{i_2 2}\right)x^1+b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1}b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)+\tilde{e}_{i_2}^T B^T\overline{\rho}\right]\right\},x_{i_2}^{1*}\le x^1\le 1\end{cases}$$

in which $\tilde{C}^{i_2}>0$ is some constant that remains to be determined. If we require that $\tilde{\phi}_{i_2}$ is continuous at $x^1=x_{i_2}^{1*}$ we get the following equation,

$$\tilde{C}^{i_2}\left(x_{i_2}^{1*}\right)^{\tilde{\lambda}^{i_2}}=1-\tilde{w}^{i_2}+\tilde{w}^{i_2}\left[\left(b_{i_2 1}-b_{i_2 2}\right)x_{i_2}^{1*}+b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1}b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)+\tilde{e}_{i_2}^T B^T\overline{\rho}\right],\tag{A.11}$$

If we require that $\tilde{\phi}_{i_2}$ is differentiable at $x^1 = x^{1*}_{i_2}$ we get the additional equation,

$$\tilde{C}^{i_2}\tilde{\lambda}^{i_2}\left(x^{1*}_{i_2}\right)^{\tilde{\lambda}^{i_2}-1} = \tilde{w}^{i_2}\left(b_{i_2 1} - b_{i_2 2}\right). \tag{A.12}$$

So, combining equation (A.11) and equation (A.12) yields,

$$\frac{\tilde{C}^{i_2}\left(x^{1*}_{i_2}\right)^{\tilde{\lambda}^{i_2}}}{\tilde{C}^{i_2}\tilde{\lambda}^{i_2}\left(x^{1*}_{i_2}\right)^{\tilde{\lambda}^{i_2}-1}} = \frac{1-\tilde{w}^{i_2} + \tilde{w}^{i_2}\left[\left(b_{i_2 1} - b_{i_2 2}\right)x^{1*}_{i_2} + b_{i_2 2}\bar{\delta}_{I_1-2} + \sum_{i_1=3}^{I_1} b_{i_2 i_1}\left(\bar{\delta}_{I_1-i_1} - \bar{\delta}_{I_1-i_1+1}\right) + \tilde{e}^T_{i_2}B^T\bar{\rho}\right]}{\tilde{w}^{i_2}\left(b_{i_2 1} - b_{i_2 2}\right)}$$

$$\Leftrightarrow x^{1*}_{i_2} = \frac{\tilde{\lambda}^{i_2}\left\{1-\tilde{w}^{i_2} + \tilde{w}^{i_2}\left[b_{i_2 2}\bar{\delta}_{I_1-2} + \sum_{i_1=3}^{I_1} b_{i_2 i_1}\left(\bar{\delta}_{I_1-i_1} - \bar{\delta}_{I_1-i_1+1}\right) + \tilde{e}^T_{i_2}B^T\bar{\rho}\right]\right\}}{\left(1-\tilde{\lambda}^{i_2}\right)\tilde{w}^{i_2}\left(b_{i_2 1} - b_{i_2 2}\right)}. \tag{A.13}$$

And this produces,

$$\tilde{C}^{i_2} = \frac{\tilde{w}^{i_2}\left(b_{i_2 1} - b_{i_2 2}\right)}{\tilde{\lambda}^{i_2}\left(x^{1*}_{i_2}\right)^{\tilde{\lambda}^{i_2}-1}}. \tag{A.14}$$

Then, by applying equation (A.10), equation (A.13) and equation (A.14), we are in the position to prove that $\tilde{f}^*_{i_2}\left(s, x^1\right) = \exp\left(-\tilde{\theta}^{i_2}s\right)\tilde{C}^{i_2}\left(x^1\right)^{\tilde{\lambda}^{i_2}}$ is a supermeanvalued majorant of $\tilde{f}_{i_2}\left(s, x^1\right)$. Firstly, noting that,

$$\begin{aligned}
\mathcal{A}\tilde{f}_{i_2}\left(s, x^1\right) = {} & -\tilde{\theta}^{i_2}\exp\left(-\tilde{\theta}^{i_2}s\right) \\
& \times\left\{1-\tilde{w}^{i_2} + \tilde{w}^{i_2}\left[\left(b_{i_2 1} - b_{i_2 2}\right)x^1 + b_{i_2 2}\bar{\delta}_{I_1-2} + \sum_{i_1=3}^{I_1} b_{i_2 i_1}\left(\bar{\delta}_{I_1-i_1} - \bar{\delta}_{I_1-i_1+1}\right) + \tilde{e}^T_{i_2}B^T\bar{\rho}\right]\right\} \\
& + x^1\left(\bar{e}^T_1 Ay\right)\exp\left(-\tilde{\theta}^{i_2}s\right)\tilde{w}^{i_2}\left(b_{i_2 1} - b_{i_2 2}\right) \le 0, \quad \forall x^1 \ge x^{1*}_{i_2} \\
& \Leftrightarrow \left(\bar{e}^T_1 Ay - \tilde{\theta}^{i_2}\right)\tilde{w}^{i_2}\left(b_{i_2 1} - b_{i_2 2}\right)x^1 \\
& \le \tilde{\theta}^{i_2}\left(1-\tilde{w}^{i_2}\right) + \tilde{\theta}^{i_2}\tilde{w}^{i_2}\left[b_{i_2 2}\bar{\delta}_{I_1-2} + \sum_{i_1=3}^{I_1} b_{i_2 i_1}\left(\bar{\delta}_{I_1-i_1} - \bar{\delta}_{I_1-i_1+1}\right) + \tilde{e}^T_{i_2}B^T\bar{\rho}\right], \quad \forall x^1 \ge x^{1*}_{i_2} \\
& \Leftrightarrow x^1 \ge \frac{\tilde{\theta}^{i_2}\left(1-\tilde{w}^{i_2}\right) + \tilde{\theta}^{i_2}\tilde{w}^{i_2}\left[b_{i_2 2}\bar{\delta}_{I_1-2} + \sum_{i_1=3}^{I_1} b_{i_2 i_1}\left(\bar{\delta}_{I_1-i_1} - \bar{\delta}_{I_1-i_1+1}\right) + \tilde{e}^T_{i_2}B^T\bar{\rho}\right]}{\left(\bar{e}^T_1 Ay - \tilde{\theta}^{i_2}\right)\tilde{w}^{i_2}\left(b_{i_2 1} - b_{i_2 2}\right)}, \quad \forall x^1 \ge x^{1*}_{i_2}
\end{aligned}$$

$$\Leftrightarrow x_{i_2}^{1*} \geq \frac{\tilde{\theta}^{i_2}\left(1-\tilde{w}^{i_2}\right)+\tilde{\theta}^{i_2}\tilde{w}^{i_2}\left[b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1} b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)+\tilde{e}_{i_2}^T B^T \overline{\rho}\right]}{\left(\overline{e}_1^T Ay-\tilde{\theta}^{i_2}\right)\tilde{w}^{i_2}\left(b_{i_2 1}-b_{i_2 2}\right)}$$

which holds by (A.8). Secondly, to show that,

$$\tilde{C}^{i_2}\left(x^1\right)^{\tilde{\lambda}^{i_2}} \geq 1-\tilde{w}^{i_2}+\tilde{w}^{i_2}\left[\left(b_{i_2 1}-b_{i_2 2}\right)x^1+b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1} b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)+\tilde{e}_{i_2}^T B^T \overline{\rho}\right],$$

for $\forall 0 \leq x^1 \leq x_{i_2}^{1*}$. Define

$$\tilde{\xi}^{i_2}\left(x^1\right) \triangleq \tilde{C}^{i_2}\left(x^1\right)^{\tilde{\lambda}^{i_2}} -1+\tilde{w}^{i_2}$$
$$-\tilde{w}^{i_2}\left[\left(b_{i_2 1}-b_{i_2 2}\right)x^1+b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1} b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)+\tilde{e}_{i_2}^T B^T \overline{\rho}\right].$$

Then with our chosen values of $\tilde{C}^{i_2}$ and $\tilde{\lambda}^{i_2}$, we see that $\tilde{\xi}^{i_2}\left(x_{i_2}^{1*}\right)=\tilde{\xi}^{i_2\prime}\left(x_{i_2}^{1*}\right)=0$. Furthermore, noting that $\tilde{\xi}^{i_2\prime\prime}\left(x^1\right)=\tilde{C}^{i_2}\tilde{\lambda}^{i_2}\left(\tilde{\lambda}^{i_2}-1\right)\left(x^1\right)^{\tilde{\lambda}^{i_2}-2}$, and hence $\tilde{\xi}^{i_2\prime\prime}\left(x^1\right)>0$ holds for $\forall 0 \leq x^1 \leq x_{i_2}^{1*}$ given $\tilde{\lambda}^{i_2}>1$ in (A.10), that is, $\tilde{\xi}^{i_2}\left(x^1\right)>0$ follows for $\forall 0 \leq x^1 \leq x_{i_2}^{1*}$. And hence the desired result is established.

Case 2.2: $\begin{cases} \tilde{\theta}^{i_2}\left(1-\tilde{w}^{i_2}\right)+\tilde{\theta}^{i_2}\tilde{w}^{i_2}\left[b_{i_2 2}\overline{\delta}_{I_1-2}+\sum_{i_1=3}^{I_1} b_{i_2 i_1}\left(\overline{\delta}_{I_1-i_1}-\overline{\delta}_{I_1-i_1+1}\right)+\tilde{e}_{i_2}^T B^T \overline{\rho}\right]>0 \\ \operatorname{sgn}\left(\overline{e}_1^T Ay-\tilde{\theta}^{i_2}\right)=\operatorname{sgn}\left(b_{i_2 1}-b_{i_2 2}\right) \end{cases}$

We take the proof of case 2.2, which is quite similar to that of case 2.1, omitted.

STEP 3: The existence of the Pareto optimal endogenous matching.

It follows from the requirements of Problem 1 that $y_1^{1*}=y_2^{1*}=\ldots=y_{i_1}^{1*}=\ldots=y_{I_1}^{1*}$ with $y_{i_1}^{1*}$ defined in (A.5). Let $y_{i_1}^{1*}=y_{k_1}^{1*}$ $(\forall i_1 \neq k_1, i_1, k_1=1,2,\ldots,I_1)$, then one can easily see that,

$$\overline{\Sigma}_{i_1k_1,23}\tilde{\delta}_{I_2-2}+\overline{\Sigma}_{i_1k_1,34}\tilde{\delta}_{I_2-3}+\ldots+\overline{\Sigma}_{i_1k_1,I_2-1,I_2}\tilde{\delta}_1=\overline{\Gamma}_{i_1k_1}.$$

where,

$$\bar{\Sigma}_{i_1k_1,j_2,j_2+1} \triangleq \frac{\bar{\lambda}^{i_1}\left(a_{i_1j_2}-a_{i_1,j_2+1}\right)}{\left(1-\bar{\lambda}^{i_1}\right)\left(a_{i_11}-a_{i_12}\right)} - \frac{\bar{\lambda}^{k_1}\left(a_{k_1j_2}-a_{k_1,j_2+1}\right)}{\left(1-\bar{\lambda}^{k_1}\right)\left(a_{k_11}-a_{k_12}\right)}.$$

$$\bar{\Gamma}_{i_1k_1} \triangleq \frac{\bar{\lambda}^{k_1}\left[\left(1-\bar{w}^{k_1}\right)+\bar{w}^{k_1}\left(a_{k_1I_2}+\bar{e}_{k_1}^T A\tilde{\rho}\right)\right]}{\left(1-\bar{\lambda}^{k_1}\right)\left(a_{k_11}-a_{k_12}\right)\bar{w}^{k_1}} - \frac{\bar{\lambda}^{i_1}\left[\left(1-\bar{w}^{i_1}\right)+\bar{w}^{i_1}\left(a_{i_1I_2}+\bar{e}_{i_1}^T A\tilde{\rho}\right)\right]}{\left(1-\bar{\lambda}^{i_1}\right)\left(a_{i_11}-a_{i_12}\right)\bar{w}^{i_1}}.$$

$$\forall i_1 \neq k_1,\ \ i_1,k_1 = 1,2,\ldots,I_1;\ \ j_2 = 2,3,\ldots,I_2-1.$$

Accordingly, we have,

$$\begin{bmatrix} \bar{\Sigma}_{12,23} & \bar{\Sigma}_{12,34} & \cdots & \bar{\Sigma}_{12,I_2-1,I_2} \\ \bar{\Sigma}_{23,23} & \bar{\Sigma}_{23,34} & \cdots & \bar{\Sigma}_{23,I_2-1,I_2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\Sigma}_{I_1-1,I_1,23} & \bar{\Sigma}_{I_1-1,I_1,34} & \cdots & \bar{\Sigma}_{I_1-1,I_1,I_2-1,I_2} \end{bmatrix}_{(I_1-1)\times(I_2-2)} \begin{bmatrix} \tilde{\delta}_{I_2-2} \\ \tilde{\delta}_{I_2-3} \\ \vdots \\ \tilde{\delta}_1 \end{bmatrix}_{(I_2-2)\times 1} = \begin{bmatrix} \bar{\Gamma}_{12} \\ \bar{\Gamma}_{23} \\ \vdots \\ \bar{\Gamma}_{I_1-1,I_1} \end{bmatrix}_{(I_1-1)\times 1}.$$

which implies that,

$$\tilde{\delta} = \bar{\Sigma}^{+}\bar{\Gamma}. \tag{A.15}$$

where " $+$ " denotes Moore-Penrose generalized inverse.

Similarly, we obtain $x_1^{1*} = x_2^{1*} = \ldots = x_{i_2}^{1*} = \ldots = x_{I_2}^{1*}$ with $x_{i_2}^{1*}$ defined in (A.13) according to Problem 1. Now, let $x_{i_2}^{1*} = x_{k_2}^{1*}$ ($\forall i_2 \neq k_2, i_2, k_2 = 1,2,\ldots,I_2$), then we get,

$$\tilde{\Sigma}_{i_2k_2,23}\bar{\delta}_{I_1-2} + \tilde{\Sigma}_{i_2k_2,34}\bar{\delta}_{I_1-3} + \ldots + \tilde{\Sigma}_{i_2k_2,I_1-1,I_1}\bar{\delta}_1 = \tilde{\Gamma}_{i_2k_2}.$$

where,

$$\tilde{\Sigma}_{i_2k_2,j_1,j_1+1} \triangleq \frac{\tilde{\lambda}^{i_2}\left(b_{i_2j_1}-b_{i_2,j_1+1}\right)}{\left(1-\tilde{\lambda}^{i_2}\right)\left(b_{i_21}-b_{i_22}\right)} - \frac{\tilde{\lambda}^{k_2}\left(b_{k_2j_1}-b_{k_2,j_1+1}\right)}{\left(1-\tilde{\lambda}^{k_2}\right)\left(b_{k_21}-b_{k_22}\right)}.$$

$$\tilde{\Gamma}_{i_2k_2} \triangleq \frac{\tilde{\lambda}^{k_2}\left[\left(1-\tilde{w}^{k_2}\right)+\tilde{w}^{k_2}\left(b_{k_2I_1}+\tilde{e}_{k_2}^T B^T\bar{\rho}\right)\right]}{\left(1-\tilde{\lambda}^{k_2}\right)\left(b_{k_21}-b_{k_22}\right)\tilde{w}^{k_2}} - \frac{\tilde{\lambda}^{i_2}\left[\left(1-\tilde{w}^{i_2}\right)+\tilde{w}^{i_2}\left(b_{i_2I_1}+\tilde{e}_{i_2}^T B^T\bar{\rho}\right)\right]}{\left(1-\tilde{\lambda}^{i_2}\right)\left(b_{i_21}-b_{i_22}\right)\tilde{w}^{i_2}}.$$

$$\forall i_2 \neq k_2,\ \ i_2,k_2 = 1,2,\ldots,I_2;\ \ j_1 = 2,3,\ldots,I_1-1.$$

Consequently, we obtain,

$$\begin{bmatrix} \tilde{\Sigma}_{12,23} & \tilde{\Sigma}_{12,34} & \cdots & \tilde{\Sigma}_{12,I_1-1,I_1} \\ \tilde{\Sigma}_{23,23} & \tilde{\Sigma}_{23,34} & \cdots & \tilde{\Sigma}_{23,I_1-1,I_1} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\Sigma}_{I_2-1,I_2,23} & \tilde{\Sigma}_{I_2-1,I_2,34} & \cdots & \tilde{\Sigma}_{I_2-1,I_2,I_1-1,I_1} \end{bmatrix}_{(I_2-1)\times(I_1-2)} \begin{bmatrix} \overline{\delta}_{I_1-2} \\ \overline{\delta}_{I_1-3} \\ \vdots \\ \overline{\delta}_1 \end{bmatrix}_{(I_1-2)\times 1} = \begin{bmatrix} \tilde{\Gamma}_{12} \\ \tilde{\Gamma}_{23} \\ \vdots \\ \tilde{\Gamma}_{I_2-1,I_2} \end{bmatrix}_{(I_2-1)\times 1}.$$

which leads us to the following equation,

$$\overline{\delta} = \tilde{\Sigma}^{+}\tilde{\Gamma}. \tag{A.16}$$

where " + " stands for the Moore-Penrose generalized inverse.

Consequently, by equations in (A.16) and (A.15), we get $y^{2*} = \tilde{\delta}_{I_2-2} - y^{1*}$, $y^{3*} = \tilde{\delta}_{I_2-3} - \tilde{\delta}_{I_2-2}$, $\ldots$, $y^{I_2*} = 1 - \tilde{\delta}_1$ and $x^{2*} = \overline{\delta}_{I_1-2} - x^{1*}, x^{3*} = \overline{\delta}_{I_1-3} - \overline{\delta}_{I_1-2}, \ldots, x^{I_1*} = 1 - \overline{\delta}_1$ with $y^{1*} \equiv y_1^{1*} = y_2^{1*} = \ldots = y_{i_1}^{1*} = \ldots = y_{I_1}^{1*}$ and $x^{1*} \equiv x_1^{1*} = x_2^{1*} = \ldots = x_{i_2}^{1*} = \ldots = x_{I_2}^{1*}$. So, we obtain the corresponding game equilibrium, denoted by

$$\Big(x^*(y^*, \overline{\rho}) \triangleq \big(x^{1*}(y^*, \overline{\rho}), \ldots, x^{i_1*}(y^*, \overline{\rho}), \ldots, x^{I_1*}(y^*, \overline{\rho}) \big)^T,$$

$$y^*(x^*, \tilde{\rho}) \triangleq \big(y^{1*}(x^*, \tilde{\rho}), \ldots, y^{i_2*}(x^*, \tilde{\rho}), \ldots, y^{I_2*}(x^*, \tilde{\rho}) \big)^T \Big)$$

with $\sum_{i_1}^{I_1} x^{i_1*}(y^*, \overline{\rho}) = 1$ and $\sum_{i_2}^{I_2} y^{i_2*}(x^*, \tilde{\rho}) = 1$, but noting that this game equilibrium may not be the Pareto optimal equilibrium of the original normal form games thanks to the stochastic factors, and this is why we need to choose appropriate values of $\overline{\rho}$ and $\tilde{\rho}$ such that the original Pareto optimal Nash equilibrium $(\hat{x}, \hat{y})$ will be definitely chosen by the players.

To summarize, we get the following theorem,

THEOREM 1': *If we are provided that the following inequalities hold, that is,* $\tilde{e}_1^T B^T x < \overline{\theta}^{i_1}$ *in* (A.3) *and* $\overline{e}_1^T A y < \tilde{\theta}^{i_2}$ *in* (A.9)*, then Problem* 1 *is solved as long as we have* $\tilde{\delta} = \overline{\Sigma}^{+}\overline{\Gamma}$ *in* (A.15) *and* $\overline{\delta} = \tilde{\Sigma}^{+}\tilde{\Gamma}$ *in* (A.16). *That is to say, the existence of the Pareto optimal endogenous matching is confirmed just via putting* $x^*(y^*, \overline{\rho}) = \hat{x}$ *and* $y^*(x^*, \tilde{\rho}) = \hat{y}$*, in which* $(\hat{x}, \hat{y})$ *is the given Pareto optimal Nash equilibrium in the corresponding normal form games.*

Therefore, Theorem 1 is established thanks to Theorem 1'. ▌