Cournot tatonnement and potentials

Kukushkin, Nikolai S.

Russian Academy of Sciences, Dorodnicyn Computing Center

9 December 2012

Cournot tatonnement and potentials

Nikolai S. Kukushkin*

December 9, 2012

Abstract

We study what topological assumptions should be added to the acyclicity of individual best response improvements in order to ensure the existence of a (pure strategy) Nash equilibrium as well as the possibility to reach a Nash equilibrium in the limit of a best response improvement path. *JEL* Classification Number: C 72.

Key words: Cournot tatonnement; Cournot potential; game with structured utilities; aggregative game.

1 Introduction

Cournot tâtonnement is the oldest and one of the most natural dynamic scenarios of individual myopic adaptation in strategic games. It has been studied in various contexts and from various viewpoints, see, e.g., Topkis (1979), Bernheim (1984), Moulin (1984), Vives (1990), Milgrom and Roberts (1990), Kandori and Rob (1995), and Milchtaich (1996).

The introduction of the concept of a potential game by Monderer and Shapley (1996) stimulated studies of similarities and dissimilarities between better and best response dynamics. Since most important for Monderer and Shapley was the cardinal concept of an exact potential, they defined every kind of a potential as a *real-valued* function. When Voorneveld (2000) introduced a "best-response potential," he followed their lead. For a finite game, the restriction to numeric potentials is innocuous; in the general case, it is not so. Yet, nobody has demonstrated that the possibility of a numeric representation has anything to do with dynamic properties.

Voorneveld's definition is over-exacting in another respect too, viz. it followed Monderer and Shapley's concept of an ordinal potential rather than a generalized ordinal one. Meanwhile, Monderer and Shapley (1996, Lemma 2.5) showed that it is the latter kind of a potential that is most relevant to the convergence of adaptive dynamics. Kukushkin (2004, Section 6) defined

*Russian Academy of Sciences, Dorodnicyn Computing Center, 40, Vavilova, Moscow 119333 Russia
E-mail: ququns@inbox.ru HomePage

a “Cournot potential” as a partial order on the set of strategy profiles the existence of which in a finite game is equivalent to the “finite best response property” (Milchtaich, 1996). Naturally, Voorneveld’s potential always defines a Cournot potential; the converse is generally wrong, even in a finite game (Monderer and Shapley, 1996, the example on p. 129).

When attention is turned to infinite games, the acyclicity of (either better or best response) improvements does not imply even the existence of a Nash equilibrium, to say nothing of the convergence of adaptive dynamics. Nonetheless, the main theorem of Kukushkin (2011) showed that in a game with compact strategy sets and “continuous enough” preferences, the acyclicity of *all* individual improvements ensures the existence of a (pure strategy) Nash equilibrium and the possibility to reach it (perhaps, approximately) with a unilateral improvement path. The acyclicity of the *best response* improvements, however, does not ensure even the mere existence of a Nash equilibrium even in a compact-continuous two-person game (Kukushkin, 2011, Example 1).

In this paper, we assume that the strategy sets are compact metric spaces, and study what topological restrictions should be added to the definition of a Cournot potential in order to ensure the existence of a (pure strategy) Nash equilibrium as well as the possibility to reach a Nash equilibrium in the limit (or as a cluster point) of a best response improvement path. We do not require the utility functions to be continuous, only assume that the best responses exist everywhere; a well-known sufficient condition for this is the upper semicontinuity of each utility function in own choice.

Roughly speaking, we consider two such additional requirements: “ω-transitivity” and continuity. The first ensures the existence of an equilibrium (as well as “transfinite convergence” to equilibria of all best response improvement paths). The second, the possibility to reach the set of Nash equilibria in the limit of a best response improvement path – an infinitary version of the weak FBRP (Milchtaich, 1996) – and convergence to the set of Nash equilibria of all best response improvement paths in the case of two players.

While the acyclicity of best response improvements can be shown by *reductio ad absurdum*, as in Theorem 2 of Kandori and Rob (1995) or Theorem 1 of Kukushkin (2004), it is difficult to imagine how the existence of, say, a continuous Cournot potential could be established without producing one explicitly. Fortunately, there are natural classes of strategic games where this is possible; two of them are briefly described in this paper. In the first example, “games with structured utilities,” there is even an exact (at least, an ordinal) potential as defined by Monderer and Shapley (1996), hence the behavior of *all* individual improvement paths is rather regular. In the second, “aggregative games,” arbitrary improvements may lead nowhere. Sequential Cournot tâtonnement in such games was recently considered by Jensen (2010), but his results are not directly comparable to ours, see Section 9.3.

In Section 2, the basic definitions are given. Section 3 contains the main “positive” results; Section 4, additional “positive” results which assume the uniqueness of the best responses. In Section 5, we introduce two weaker notions of a potential, which broaden the scope of applications. Sections 6 and 7 present known classes of games where the assumptions of (some of) our theorems are satisfied. Section 8 contains "negative" results, showing the impossibility of easy generalizations. A discussion of various related questions in Section 9 concludes the paper.

2 Preliminaries

Our basic model is a *strategic game with ordinal utilities*. It is defined by a finite set of players N, and strategy sets X_i and ordinal utility functions $u_i \colon X_N \to \mathbb{R}$, where $X_N = \prod_{i \in N} X_i$, for all $i \in N$. We denote $X_{-i} = \prod_{j \in N \setminus \{i\}} X_j$ for each $i \in N$. Given a strategy profile $x_N \in X_N$ and $i \in N$, we denote x_i and x_{-i} its projections to X_i and X_{-i}, respectively; a pair (x_i, x_{-i}) uniquely determines x_N. In the case of $\#N = 2$, we denote $-i$ the partner/rival of player i.

Defining the *best response correspondence* $\mathcal{R}_i \colon X_{-i} \to 2^{X_i}$ for each $i \in N$ in the usual way,

$$\mathcal{R}_i(x_{-i}) := \operatorname*{Argmax}_{x_i \in X_i} u_i(x_i, x_{-i})$$

for every $x_{-i} \in X_{-i}$, we introduce the *best response improvement* relation on X_N ($i \in N$, $y_N, x_N \in X_N$):

$$y_N \rhd_i^{\mathrm{BR}} x_N \rightleftharpoons [y_{-i} = x_{-i} \;\&\; x_i \notin \mathcal{R}_i(x_{-i}) \ni y_i]; \tag{1a}$$

$$y_N \rhd^{\mathrm{BR}} x_N \rightleftharpoons \exists i \in N \, [y_N \rhd_i^{\mathrm{BR}} x_N]. \tag{1b}$$

Every Nash equilibrium is a maximizer of $\rhd^{\mathrm{BR}}$. If $\mathcal{R}_i(x_{-i}) \neq \emptyset$ for all $i \in N$ and $x_{-i} \in X_{-i}$, then every maximizer of $\rhd^{\mathrm{BR}}$ is a Nash equilibrium.

A *Cournot path* is a finite or infinite sequence $\langle x_N^k \rangle_{k=0,1,\ldots}$ such that $x_N^{k+1} \rhd^{\mathrm{BR}} x_N^k$ whenever $k \geq 0$ and x_N^{k+1} is defined. A *Cournot potential* is an irreflexive and transitive binary relation $\succ$ on X_N such that

$$\forall x_N, y_N \in X_N \left[y_N \rhd^{\mathrm{BR}} x_N \Rightarrow y_N \succ x_N \right]. \tag{2}$$

The existence of a Cournot potential ensures the absence of *Cournot cycles*, i.e., Cournot paths $\langle x_N^0, x_N^1, \ldots, x_N^m \rangle$ such that $m > 0$ and $x_N^0 = x_N^m$. If the game is finite, this fact implies that every Cournot path, if continued whenever possible, reaches a Nash equilibrium in a finite number of steps. Example 1 from Kukushkin (2011) shows that a compact-continuous game may admit a Cournot potential and still possess no Nash equilibrium, to say nothing of the convergence of Cournot paths.

Henceforth, we assume that each X_i is a compact metric space and endow X_N with, say, the maximum metric. We do not impose any explicit continuity-style restriction on the utilities; all assumptions are formulated in terms of the best response correspondences. In particular, we assume throughout that $\mathcal{R}_i(x_{-i}) \neq \emptyset$ for every $i \in N$ and $x_{-i} \in X_{-i}$. The upper semicontinuity of u_i in own choice x_i is sufficient for that though by no means necessary.

We call a Cournot path $\langle x_N^k \rangle_k$ *inclusive* if for each player $i \in N$, there holds $x_i^k \in R_i(x_{-i}^k)$ for some k. A Cournot path $\langle x_N^k \rangle_k$ is *totally inclusive* if, whenever x_N^m is defined, the path $\langle x_N^k \rangle_{k \geq m}$ is inclusive. Thus, a totally inclusive path either is infinite or ends at a Nash equilibrium.

A binary relation $\succ$ on a metric space X is called *ω-transitive* if it is transitive and the conditions $x^\omega = \lim_{k\to\infty} x^k$ and $x^{k+1} \succ x^k$ for all $k \in \mathbb{N}$ always imply $x^\omega \succ x^0$. (It is worth noting that $x^\omega \succ x^k$ is valid for all $k \in \mathbb{N}$ under those conditions, once $\succ$ is ω-transitive.) A relation $\succ$ is *upper semicontinuous* if all its lower contours, i.e., sets $\{x \in X \mid y \succ x\}$ $(y \in X)$ are open. A relation $\succ$ is *continuous* if its "graph," $\{(x, y) \in X^2 \mid y \succ x\}$, is open. Clearly, every continuous relation is upper semicontinuous. An upper semicontinuous relation need not be ω-transitive; however, an upper semicontinuous Cournot potential can always be extended to an ω-transitive one.

A (finite or infinite) sequence $\langle x_N^k \rangle_{k=0,1,\ldots}$ in X_N *converges to a subset* $Y \subseteq X_N$ if either it is finite and ends at $x_N^m \in Y$ or it is infinite and all its cluster points belong to Y.

3 Main theorems

Theorem 3.1. *Let each X_i in a strategic game Γ be a compact metric space. Let Γ admit an ω-transitive Cournot potential. Then Γ possesses a (pure strategy) Nash equilibrium.*

Proof. By Theorem 1 from Kukushkin (2008), there exists a maximizer x_N^0 of the potential $\succ$ on X_N. By (2), x_N^0 is also a maximizer of $\rhd^{\mathrm{BR}}$ on X_N, i.e., a Nash equilibrium. $\square$

In most of the following results, we assume that each $\mathcal{R}_i$ is upper hemicontinuous. A sufficient condition for that is the upper semicontinuity of u_i in x_N and continuity in x_{-i}.

Theorem 3.2. *Let Γ be a strategic game where $\#N = 2$, each X_i is a compact metric space, and each $\mathcal{R}_i$ is upper hemicontinuous. Let Γ admit an ω-transitive Cournot potential. Let $\langle x_N^k \rangle_{k\in\mathbb{N}}$ be an infinite Cournot path. Then there is a Nash equilibrium among cluster points of the path.*

Proof. We denote $X^\omega \subseteq X_N$ the set of cluster points of $\langle x_N^k \rangle_{k\in\mathbb{N}}$ and pick a maximizer x_N^ω of $\rhd^{\mathrm{BR}}$ on X^ω; it exists by Theorem 1 from Kukushkin (2008) since X^ω is compact. If $x_i^\omega \in \mathcal{R}_i(x_{-i}^\omega)$ for both i, we are home; let $x_i^\omega \notin \mathcal{R}_i(x_{-i}^\omega)$. Since $x_N^\omega \in X^\omega$, there is a strictly increasing sequence $\langle k_h \rangle_{h\in\mathbb{N}}$ such that $x_N^{k_h} \to x_N^\omega$. We denote $y_N^h := x_N^{k_h+1}$ $(h \in \mathbb{N})$; without restricting generality, $y_N^h \to y_N^\omega \in X^\omega$. Since $\mathcal{R}_i$ is upper hemicontinuous, there holds $x_i^{k_h} \notin \mathcal{R}_i(x_{-i}^{k_h})$ for all h large enough; without restricting generality, for all h. Therefore, $x_N^{k_h} \rhd_{-i}^{\mathrm{BR}} x_N^{k_h-1}$, hence $y_N^h \rhd_i^{\mathrm{BR}} x_N^{k_h}$, hence $y_i^h \in \mathcal{R}_i(x_{-i}^{k_h})$. Since $\mathcal{R}_i$ is upper hemicontinuous, $y_i^\omega \in \mathcal{R}_i(x_{-i}^\omega)$. Thus, $y_N^\omega \rhd_i^{\mathrm{BR}} x_N^\omega$, contradicting the choice of x_N^ω. $\square$

Considering implications of the presence of a *continuous* Cournot potential, we start with a couple of auxiliary statements.

Lemma 3.3. *Let Γ be a strategic game where each X_i is a compact metric space. Let Γ admit a continuous Cournot potential. Let $\langle x_N^k \rangle_{k\in\mathbb{N}}$ be an infinite Cournot path and $X^\omega \subseteq X_N$ be the set of its cluster points. Then*

$$\forall y_N^\omega, z_N^\omega \in X^\omega \left[y_N^\omega \not\succ z_N^\omega \right]. \tag{3}$$

Proof. By (2), we have $x_N^{k+1} \succ x_N^k$ for all $k \in \mathbb{N}$, hence $y_N^\omega \succ x_N^k$ for all $y_N^\omega \in X^\omega$ and $k \in \mathbb{N}$ (the ω-transitivity of $\succ$ is sufficient here). If we supposed that $y_N^\omega, z_N^\omega \in X^\omega$ and $y_N^\omega \succ z_N^\omega$, we would have $x_N^k \succ z_N^\omega$ whenever x_N^k is close enough to y_N^ω, i.e., a contradiction. (The last step would be invalid without continuity.) $\square$

Lemma 3.4. *Let a strategic game Γ satisfy all assumptions of Lemma 3.3, and let each $\mathcal{R}_i$ be upper hemicontinuous. Let $\langle x_N^k \rangle_{k\in\mathbb{N}}$ be an infinite Cournot path, $X^\omega \subseteq X_N$ be the set of its cluster points, and $x_N^\omega \in X^\omega$. Then $x_i^\omega \in \mathcal{R}_i(x_{-i}^\omega)$ for at least two different players $i \in N$.*

Proof. As in the proof of Theorem 3.2, we pick a strictly increasing sequence $\langle k_h \rangle_{h\in\mathbb{N}}$ such that $x_N^{k_h} \to x_N^\omega$ and denote $y_N^h := x_N^{k_h+1}$ ($h \in \mathbb{N}$). Without restricting generality, $x_i^{k_h} \in \mathcal{R}_i(x_{-i}^{k_h})$ for an $i \in N$ and all h, $y_j^h \in \mathcal{R}_j(y_{-j}^h)$ for an $j \in N$ and all h, and $y_N^h \to y_N^\omega \in X^\omega$. Note that $i \neq j$ since $x_j^{k_h} \notin \mathcal{R}_j(x_{-j}^{k_h})$, and that $y_{-j}^\omega = x_{-j}^\omega$. By the upper hemicontinuity of $\mathcal{R}_i$ and $\mathcal{R}_j$, we have $x_i^\omega \in \mathcal{R}_i(x_{-i}^\omega)$ and $y_j^\omega \in \mathcal{R}_i(y_{-j}^\omega)$. Finally, an assumption that $x_j^\omega \notin \mathcal{R}_j(x_{-j}^\omega)$ would imply $y_N^\omega \rhd^{\mathrm{BR}} x_N^\omega$, hence $y_N^\omega \succ x_N^\omega$ by (2), contradicting (3). $\square$

Theorem 3.5. *Let Γ be a strategic game where $\#N = 2$, each X_i is a compact metric space, and each $\mathcal{R}_i$ is upper hemicontinuous. Let Γ admit a continuous Cournot potential. Then every Cournot path converges to the set of Nash equilibria.*

Immediately follows from Lemma 3.4.

Theorem 3.6. *Let each X_i in a strategic game Γ be a compact metric space and each $\mathcal{R}_i$ be upper hemicontinuous. Let Γ admit a continuous Cournot potential. Then for every $x_N^0 \in X_N$ there exists a Cournot path starting at x_N^0 and converging to the set of Nash equilibria.*

Proof. Given $x_N^0 \in X_N$, we recursively define a Cournot path $\langle x_N^k \rangle_k$. If x_N^k is a Nash equilibrium, the process stops, and we are home. Otherwise, we define $N^*(k) := \{i \in N \mid x_i^k \notin \mathcal{R}_i(x_{-i}^k)\}$ and $X^*(k) := \bigcup_{i\in N^*(k)} \big(X_i \times \{x_{-i}^k\}\big)$. Then we pick a maximizer $x_N^{k+1} = (x_{i^*}^{k+1}, x_{-i^*}^k)$ of $\succ$ on $X^*(k)$. By (2), we have $x_{i^*}^{k+1} \in \mathcal{R}(x_{-i^*}^k)$, hence $x_N^{k+1} \rhd^{\mathrm{BR}} x_N^k$, hence $x_N^{k+1} \succ x_N^k$.

Assuming the path infinite, we denote $X^\omega \subseteq X_N$ the set of its cluster points. Supposing, to the contrary, that $x_i^\omega \notin \mathcal{R}_i(x_{-i}^\omega)$ for $x_N^\omega \in X^\omega$ and $i \in N$, we pick $y_N \in X_N$ such that $y_N \rhd_i^{\mathrm{BR}} x_N^\omega$. Now we have $y_N \succ x_N^\omega$ by (2), hence $y_N \succ x_N^k$ for each $k \in \mathbb{N}$ by the continuity of $\succ$ (upper semicontinuity would be sufficient here). Then we pick a strictly increasing sequence $\langle k_h \rangle_{h\in\mathbb{N}}$ such that $x_N^{k_h} \to x_N^\omega$; without restricting generality, $x_i^{k_h} \notin \mathcal{R}_i(x_{-i}^{k_h})$ for all h, hence $i \in N^*(k_h)$. Since $x_{-i}^\omega = y_{-i}$, we have $(y_i, x_{-i}^{k_h}) \succ x_N^{k_h+1}$ for all $h \in \mathbb{N}$ large enough by the continuity of $\succ$, which contradicts the choice of $x_N^{k_h+1}$ since $(y_i, x_{-i}^{k_h}) \in X^*(k_h)$. $\square$

Theorem 3.7. *Let Γ be a strategic game where $\#N = 2$ and each X_i is a compact metric space. Let Γ admit a continuous Cournot potential. Then for every $x_N^0 \in X_N$ there exists a Cournot path starting at x_N^0 and converging to the set of Nash equilibria.*

Proof. Given $x_N^0 \in X_N$, we recursively define a Cournot path $\langle x_N^k \rangle_k$ in exactly the same way as in the proof of Theorem 3.6. Assuming the path infinite, we again denote $X^\omega \subseteq X_N$ the set of its cluster points.

Supposing, to the contrary, that $x_i^\omega \notin \mathcal{R}_i(x_{-i}^\omega)$ for $x_N^\omega \in X^\omega$ and $i \in N$, we pick $y_N \in X_N$ such that $y_N \rhd_i^{\mathrm{BR}} x_N^\omega$, hence $y_N \succ x_N^k$ for each $k \in \mathbb{N}$ by the continuity of $\succ$. Then we pick a strictly increasing sequence $\langle k_h \rangle_{h \in \mathbb{N}}$ such that $x_N^{k_h} \to x_N^\omega$. Without restricting generality, we may assume that either $x_i^{k_h} \notin \mathcal{R}_i(x_{-i}^{k_h})$ for all h, or $x_{-i}^{k_h} \notin \mathcal{R}_{-i}(x_i^{k_h})$ for all h. In the first case, we obtain a contradiction in exactly the same way as in the proof of Theorem 3.6.

In the second case, we notice that $x_N^{k_h} \rhd_i^{\mathrm{BR}} x_N^{k_h-1}$, hence $i \in N^*(k_h - 1)$, for each h. Since $x_{-i}^{k_h-1} = x_{-i}^{k_h}$, we have $(y_i, x_{-i}^{k_h-1}) \succ x_N^{k_h}$ for all $h \in \mathbb{N}$ large enough by the continuity of $\succ$, which contradicts the choice of $x_N^{k_h}$ since $(y_i, x_{-i}^{k_h-1}) \in X^*(k_h - 1)$. □

4 Unique best responses

Lemma 4.1. *Let Γ be a strategic game where each X_i is a compact metric space and each $\mathcal{R}_i$ is single-valued, i.e., $\mathcal{R}_i(x_{-i}) = \{r_i(x_{-i})\}$ for all $i \in N$ and $x_{-i} \in X_{-i}$. If Γ admits a continuous Cournot potential, then each r_i is continuous.*

Proof. Suppose the contrary: there are $i \in N$ and a sequence $x_N^k \in X_N$ such that $x_i^k = r_i(x_{-i}^k)$ for all $k \in \mathbb{N}$ and $x_{-i}^k \to x_{-i}^\omega$, but x_i^k does not converge to $r_i(x_{-i}^\omega)$. Since X_i is compact, we may assume $x_i^k \to x_i^\omega \neq r_i(x_{-i}^\omega)$. Denoting $y_N := (r_i(x_{-i}^\omega), x_{-i}^\omega)$, we have $y_N^\omega \rhd^{\mathrm{BR}} x_N^\omega$, hence $y_N^\omega \succ x_N^\omega$ by (2), hence $(r_i(x_{-i}^\omega), x_{-i}^k) \succ x_N^k$ for all k large enough by the continuity of $\succ$, which contradicts the assumption $x_i^k = r_i(x_{-i}^k)$. □

Theorem 4.2. *Let Γ be a strategic game where $\#N = 3$, each X_i is a compact metric space, and each $\mathcal{R}_i$ is single-valued. Let Γ admit a continuous Cournot potential. Let $\langle x_N^k \rangle_{k \in \mathbb{N}}$ be an infinite inclusive Cournot path. Then there is a Nash equilibrium among cluster points of the path.*

Proof. As usual, we denote $X^\omega \subseteq X_N$ the set of cluster points of $\langle x_N^k \rangle_{k \in \mathbb{N}}$ and assume, to the contrary, that X^ω contains no Nash equilibrium. For each pair $I \subset N$, $\#I = 2$, we denote $X^I := \{x_N \in X^\omega \mid \forall i \in I\, [x_i = r_i(x_{-i})]\}$. By Lemma 3.4, $X^\omega = \bigcup_I X^I$. Whenever $I \neq J$, $X^I \cap X^J$ consists of Nash equilibria, hence $X^I \cap X^J = \emptyset$ by our assumption. Since the path is inclusive, at least two sets X^I are nonempty. Since each X^I is closed, there are open subsets V^I such that $X^I \subseteq V^I$ for each I and $V^I \cap V^J = \emptyset$ whenever $I \neq J$.

Without restricting generality, we may assume that $x_N^k \in \bigcup_I V^I$ for all k. Therefore, there exist $I \neq J$ and a strictly increasing sequence $\langle k_h \rangle_{h\in\mathbb{N}}$ such that $x_N^{k_h} \in V^J$ and $x_N^{k_h+1} \in V^I$ for all $h \in \mathbb{N}$. Again without restricting generality, we may assume that $x_N^{k_h+1} \rhd_i^{\mathrm{BR}} x_N^{k_h}$ for all h and the same $i \in N$ while $x_N^{k_h} \to x_N^\omega \in X^J$ and $x_N^{k_h+1} \to y_N^\omega \in X^I$. By the continuity, $y_i^\omega = r_i(y_{-i}^\omega)$, hence $i \in I$. Now if $i \notin J$, we have $y_N^\omega \rhd_i^{\mathrm{BR}} x_N^\omega$, hence $y_N^\omega \succ x_N^\omega$ by (2), contradicting (3). If $i \in I \cap J$, we have $x_i^\omega = r_i(x_{-i}^\omega)$, $y_i^\omega = r_i(y_{-i}^\omega)$, and $y_{-i}^\omega = x_{-i}^\omega$, hence $y_N^\omega = x_N^\omega$, contradicting $X^I \cap X^J = \emptyset$. □

Remark. When $\#N = 2$, Theorem 3.2 gives us a stronger statement under weaker assumptions on Γ.

We call an infinite Cournot path $\langle x_N^k \rangle_{k\in\mathbb{N}}$ *uniformly inclusive* if there is a natural number $m \in \mathbb{N}$ such that for each $i \in N$ and each $k \in \mathbb{N}$, there is $h \in \mathbb{N}$ such that $k \le h \le k+m$ and $x_i^h \in R_i(x_{-i}^h)$. Every infinite Cournot path generated by the sequential tâtonnement process as defined by Moulin (1984, p. 87), see also Jensen (2010, Theorem 2), is uniformly inclusive.

Theorem 4.3. *Let Γ be a strategic game where each X_i is a compact metric space, and each $\mathcal{R}_i$ is single-valued. Let Γ admit a continuous Cournot potential. Then every uniformly inclusive Cournot path converges to the set of Nash equilibria.*

Proof. Let $\langle x_N^k \rangle_{k\in\mathbb{N}}$ be a uniformly inclusive infinite Cournot path and $X^\omega \subseteq X_N$ be the set of its cluster points. Let $x_N^\omega \in X^\omega$; we pick a strictly increasing sequence $\langle k_h \rangle_{h\in\mathbb{N}}$ such that $x_N^{k_h} \to x_N^\omega$.

Claim 4.3.1. *For each $s \in \mathbb{N}$, the sequence $\langle x_N^{k_h+s} \rangle_h$ converges to x_N^ω.*

Proof of Claim 4.3.1. We argue by induction in s. For $s = 0$, the definition of $\langle k_h \rangle_{h\in\mathbb{N}}$ suffices. The general induction step is identical with the case of $s = 1$. Let $\langle k'_h \rangle_{h\in\mathbb{N}}$ be a subsequence of $\langle k_h \rangle_{h\in\mathbb{N}}$ such that $x_N^{k'_h+1} \to y_N^\omega \in X^\omega$. Since N is finite, we may, without restricting generality, assume that $x_i^{k'_h+1} = r_i(x_{-i}^{k'_h+1})$ and $x_{-i}^{k'_h+1} = x_{-i}^{k'_h}$ for an $i \in N$ and all h. Then, $y_i^\omega = r_i(y_{-i}^\omega)$ and $y_{-i}^\omega = x_{-i}^\omega$. An assumption that $x_i^\omega \neq r_i(x_{-i}^\omega)$ would contradict (3); therefore, $y_N^\omega = x_N^\omega$. □

Let us fix $i \in N$. Since the path is uniformly inclusive, there is an $s \in \{0, \dots, m\}$ for each $h \in \mathbb{N}$ such that $x_i^{k_h+s} = r_i(x_{-i}^{k_h+s})$; without restricting generality, $x_i^{k_h+s} = r_i(x_{-i}^{k_h+s})$ for all $h \in \mathbb{N}$ and the same s. Since $\langle x_N^{k_h+s} \rangle_h$ converges to x_N^ω, we have $x_i^\omega = r_i(x_{-i}^\omega)$. Since $i \in N$ was arbitrary, x_N^ω is a Nash equilibrium. □

Remark. If $\#N = 2$, then every infinite Cournot path is uniformly inclusive (with $m = 1$). Theorem 4.3 in this case does not add anything to Theorem 3.5, stronger assumptions notwithstanding.

5 Weaker concepts

To broaden the scope of applications, we introduce two weaker notions: a "partial Cournot potential" and a "restricted Cournot potential." The first leads to virtually the same results as the basic version; the second has weaker implications, but still deserves some attention.

We call a subset $Y \subseteq X_N$ *BR-absorbing* if it satisfies the following three conditions.

1. If $y_N \rhd^{\mathrm{BR}} x_N$ and $x_N \in Y$, then $y_N \in Y$ too.
2. If a Cournot path $\langle x_N^0, \ldots, x_N^m \rangle$ is inclusive, then $x_N^m \in Y$.
3. If $\langle x_N^k \rangle_{k \in \mathbb{N}}$ is an infinite Cournot path, x_N^ω is its cluster point, and $x_N^k \in Y$ for each k, then $x_N^\omega \in Y$.

We call an irreflexive and transitive binary relation $\succ$ on X_N a *partial Cournot potential* if there is a BR-absorbing subset $Y \subseteq X_N$ such that (2) holds whenever $x_N \in Y$.

Theorem 5.1. *All theorems of Sections 3 and 4 remain valid if the "Cournot potential" in each of them is replaced with "partial Cournot potential." Lemmas 3.3 and 3.4 remain valid in this case if restricted to Cournot paths in Y.*

Remark. Lemma 4.1 needs a more serious modification in this case.

A straightforward proof is omitted. Concerning Theorem 3.6, it is easy to see that an inclusive Cournot path can be started from any strategy profile $x_N^0 \in X_N$.

Given correspondences $\mathcal{R}_i^* \colon X_{-i} \to 2^{X_i}$ such that

$$\emptyset \neq \mathcal{R}_i^*(x_{-i}) \subseteq \mathcal{R}_i(x_{-i}) \tag{4}$$

for every $i \in N$ and $x_{-i} \in X_{-i}$ ("admissible best responses"), we define the *admissible best response improvement* relation $\rhd^{\mathrm{BR}*}$ on X_N by replacing (1) with

$$y_N \rhd_i^{\mathrm{BR}*} x_N \rightleftharpoons [y_{-i} = x_{-i} \;\&\; x_i \notin \mathcal{R}_i(x_{-i}) \;\&\; y_i \in \mathcal{R}_i^*(x_{-i})]; \tag{5a}$$

$$y_N \rhd^{\mathrm{BR}*} x_N \rightleftharpoons \exists i \in N \, [y_N \rhd_i^{\mathrm{BR}*} x_N]. \tag{5b}$$

We call an irreflexive and transitive binary relation $\succ$ on X_N a *restricted Cournot potential* if there are correspondences $\mathcal{R}_i^* \colon X_{-i} \to 2^{X_i} \setminus \{\emptyset\}$ such that (2) holds for $\rhd^{\mathrm{BR}*}$. A Cournot path is *admissible* if $x_N^{k+1} \rhd^{\mathrm{BR}*} x_N^k$ for each relevant k.

Theorem 5.2. *All theorems of Sections 3 and 4 remain valid if the "Cournot potential" in each of them is replaced with "restricted Cournot potential," the assumptions on $\mathcal{R}_i$ are shifted onto $\mathcal{R}_i^*$, and only admissible Cournot paths are allowed.*

A straightforward proof is omitted.

6 Games with structured utilities

A *game with structured utilities* (and additive aggregation) may have an arbitrary finite set of players N and arbitrary sets of strategies whereas the utility functions satisfy certain structural requirements. There is a set A of *processes* and a finite subset $\Upsilon^i \subseteq \mathrm{A}$ of processes where each player $i \in N$ participates (given exogenously). With every $\alpha \in \mathrm{A}$, an *intermediate utility function* is associated, $\varphi_\alpha \colon X_{N(\alpha)} \to \mathbb{R}$, where $N(\alpha) = \{i \in N \mid \alpha \in \Upsilon^i\}$. The "ultimate" utility functions of the players are built of the intermediate utilities:

$$u_i(x_N) := \sum_{\alpha \in \Upsilon^i} \varphi_\alpha(x_{N(\alpha)}), \tag{6}$$

where $i \in N$ and $x_N \in X_N$.

Defining $P \colon X_N \to \mathbb{R}$ by

$$P(x_N) := \sum_{\alpha \in \mathrm{A}} \varphi_\alpha(x_{N(\alpha)}), \tag{7}$$

we immediately see that P is an exact potential (Monderer and Shapley, 1996): $P(x_N) = \sum_{\alpha \in \Upsilon^i} \varphi_\alpha(x_{N(\alpha)}) + \sum_{\alpha \in \mathrm{A} \setminus \Upsilon^i} \varphi_\alpha(x_{N(\alpha)}) = u_i(x_N) + Q_i(x_{-i})$ for all $i \in N$ and $x_N \in X_N$; clearly, it is a Cournot potential as well. If all functions φ_α are continuous, then P is continuous too. If we additionally assume, e.g., each set X_i to be convex and each function φ_α strictly concave, then the results of Section 4 become applicable.

Remark. A strategic game admits an exact potential if and only if it can be represented as a game with structured utilities and additive aggregation rule (6), see Kukushkin (2007, Theorem 5).

Utility functions satisfying (6) can be found in so called "network transmission games," see, e.g., Facchinei et al. (2011) and references therein, which are somewhat similar to Rosenthal's (1973) congestion games, but do not belong to the class. There is a directed graph with the set of links E; each player $i \in N$ is assigned a path $\pi_i \subseteq E$ in the graph (between a source and a target) and sends a *flow* $x_i \in [0, b_i] \subset \mathbb{R}$ along the path, getting a reward $w_i(x_i)$ depending on her flow and bearing costs $\sum_{e \in \pi_i} c_e(\sum_{j \colon e \in \pi_j} x_j)$ depending on the total flow through each link in π_i. Setting $\mathrm{A} := L \cup N$, $\Upsilon^i := \pi_i \cup \{i\}$, $\varphi_i(x_i) := w_i(x_i)$ and $\varphi_e(x_{N(e)}) := -c_e(\sum_{j \in N(e)} x_j)$ for each $e \in E$, we see that (6) holds for each player.

Another example is the Cournot oligopoly with a linear cost function (Monderer and Shapley, 1996). The structure of utilities (6) only holds where the "formal," linear price is positive, so we have to rely on Theorem 5.1.

Given continuous and strictly increasing mappings $\lambda_i, \mu_\alpha \colon \mathbb{R} \to \mathbb{R}$, we may extend this approach further, replacing (6) with

$$u_i(x_N) = \lambda_i\big(\sum_{\alpha \in \Upsilon^i} \mu_\alpha \circ \varphi_\alpha(x_{N(\alpha)})\big), \tag{8}$$

for all $i \in N$ and $x_N \in X_N$. Then $P(x_N) := \sum_{\alpha \in \mathrm{A}} \mu_\alpha \circ \varphi_\alpha(x_{N(\alpha)})$ is an ordinal potential, hence a continuous Cournot potential again. This trick works, e.g., for the Cournot oligopoly with identical linear cost functions (Monderer and Shapley, 1996; Kukushkin, 1994) or voluntary provision of a public good with Cobb-Douglas utilities.

7 Aggregative games with increasing best responses

A rather general (though not the most general imaginable) definition of an aggregative game sounds as follows: each X_i is a compact subset of $\mathbb{R}$, and there are mappings $\sigma_i \colon X_{-i} \to \mathbb{R}$ such that

$$u_i(x_N) = U_i(\sigma_i(x_{-i}), x_i) \tag{9}$$

for all $i \in N$ and $x_N \in X_N$. For each $i \in N$, we denote $S_i := \sigma_i(X_{-i}) \subset \mathbb{R}$, and redefine the best response correspondence:

$$R_i(s_i) := \operatorname*{Argmax}_{x_i \in X_i} U_i(s_i, x_i).$$

Our assumption $\mathcal{R}_i(x_{-i}) \neq \emptyset$ is equivalent to $R_i(s_i) \neq \emptyset$ for each $s_i \in S_i$.

We also assume that each player's best responses are increasing in s_i (in a rather strong sense):

$$[s'_i > s_i \ \&\ x'_i \in R_i(s'_i) \ \&\ x_i \in R_i(s_i)] \Rightarrow x'_i \geq x_i \tag{10}$$

for all $i \in N$ and $s'_i, s_i \in S_i$. The following *strict single crossing* condition (Milgrom and Shannon, 1994) is sufficient for (10):

$$[x'_i > x_i \ \&\ s'_i > s_i \ \&\ U_i(s_i, x'_i) \geq U_i(s_i, x_i)] \Rightarrow U_i(s'_i, x'_i) > U_i(s'_i, x_i) \tag{11}$$

for all $i \in N$, $x'_i, x_i \in X_i$, and $s'_i, s_i \in S_i$.

If each σ_i is increasing in each x_j, then the existence of a Nash equilibrium (but not the acyclicity of the best response improvements) immediately follows from Tarski's fixed point theorem. Novshek (1985) was the first to notice that the existence also obtains in the case of $\sigma_i(x_{-i}) = -\sum_{j \neq i} x_j$; this fact has nothing to do with Tarski's theorem. Kukushkin (2004) proved the impossibility of Cournot cycles in both Novshek' case and when $\sigma_i(x_{-i}) = \sum_{j \neq i} x_j$. Dubey et al. (2006) modified a trick developed by Huang (2002) for different purposes, providing a tool for the construction of a continuous partial Cournot potential. A rather broad class of aggregative games where the trick works is described in Jensen (2010); the class may be the broadest possible although it is unclear how such a claim could be proven. (The technical assumptions of Jensen's main theorem, however, should have been much stronger.)

We describe the trick in some details for a case of intermediate generality (Kukushkin, 2005), sufficient for many applications in economics. Let

$$\sigma_i(x_{-i}) = \sum_{j \neq i} a_{ij} x_j, \tag{12}$$

with $a_{ij} = a_{ji} \in \mathbb{R}$ for all $i \neq j$. Assuming that each best response correspondence R_i is upper hemicontinuous and satisfies (10), the approach of Huang-Dubey et al. recommends the following steps. First, we pick a selection r_i from each R_i, e.g., $r_i(s_i) := \min R_i(s_i)$ for every $s_i \in S_i$; then we extend r_i to the whole closed interval $[\min S_i, \max S_i]$ preserving its monotonicity; finally, we define

$$P(x_N) := \sum_{i \in N} \Big[-x_i \cdot \max S_i + \int_{\min S_i}^{\max S_i} \min\{x_i, r_i(s_i)\}\, ds_i \Big] + \frac{1}{2} \Big[\sum_{i,j \in N\ i \neq j} a_{ij} \cdot x_i \cdot x_j \Big].$$

Straightforward calculations show that $P(y_N) > P(x_N)$ whenever $y_N \rhd_i^{\mathrm{BR}} x_N$ and $x_i \in X_i^0 := \bigcup_{s_i \in S_i} R_i(s_i)$. Therefore, P represents a continuous partial Cournot potential satisfying (2) on $Y := \prod_{i \in N} X_i^0$.

Remark. When $a_{ij} \geq 0$ for all $j \neq i$, we have a game with strategic complementarity; when $a_{ij} \leq 0$ for all $j \neq i$, a game with strategic substitutability. A more general situation with coefficients of both signs is also possible.

The following *weak single crossing* condition (Shannon, 1995),

$$[x_i' > x_i \ \&\ s_i' > s_i \ \&\ U_i(s_i, x_i') > U_i(s_i, x_i)] \Rightarrow U_i(s_i', x_i') \geq U_i(s_i', x_i) \tag{13}$$

for all $i \in N$, $x_i', x_i \in X_i$, and $s_i', s_i \in S_i$, ensures the monotonicity of R_i in a rather weak sense:

$$[s_i' > s_i \ \&\ x_i' \in R_i(s_i') \ \&\ x_i \in R_i(s_i)] \Rightarrow [\min\{x_i', x_i\} \in R_i(s_i) \ \text{ or } \ \max\{x_i', x_i\} \in R_i(s_i')] \tag{14}$$

for all $i \in N$ and $s_i', s_i \in S_i$. Since every $R_i(s_i)$ is compact, (14) implies, by Theorem 3.2 from Veinott (1989), the existence of an increasing selection r_i from R_i. Defining R_i^* by the closure of the graph of r_i, we immediately see that R_i^* is upper hemicontinuous and satisfies both (4) and (10). In other words, if the best responses are upper hemicontinuous and increasing in the sense of (14), while aggregation rules σ_i belong to the class described in Jensen (2010), i.e., allow the Huang-Dubey et al. trick to work, then the game admits a partial restricted Cournot potential.

8 "Counterexamples"

This section consists of examples showing the impossibility of easy generalizations.

Example 8.1. Let us consider a game where $N := \{1, 2\}$, $X_i := [0, 1] \cup \{2\}$, and the preferences are defined by these utility functions:

$$u_i(x_N) := \begin{cases} \min\{2x_i - x_{-i}, -2x_i + x_{-i} + 2\}, & x_N \in [0,1] \times [0,1]; \\ 1, & x_i = 2,\ x_{-i} \in [0, 1[; \\ 2, & x_i = 2,\ x_{-i} = 1; \\ x_i, & x_{-i} = 2. \end{cases}$$

Each utility function u_i is upper semicontinuous in x_N and continuous in x_i; the only discontinuity in x_{-i} happens when $x_{-i} = 1$ and $x_i = 2$. The best responses are easy to compute:

$$\mathcal{R}_i(x_{-i}) = \begin{cases} \{2\}, & x_{-i} \in \{1, 2\}; \\ \{2, x_{-i}/2 + 1/2\}, & x_{-i} \in [0, 1[. \end{cases}$$

There is a unique Nash equilibrium, $(2, 2)$.

To define a Cournot potential, we introduce an auxiliary function on $\mathbb{R}^2$: $\psi(x, y) := \min\{x, -x + y + 1\}$. Then we define a continuous function on X_N:

$$P(x_N) := \begin{cases} \max\{\psi(x_i, x_{-i}), \psi(x_{-i}, x_i)\}, & x_N \in [0, 1] \times [0, 1]; \\ 2 + \min_i x_i, & \text{otherwise.} \end{cases}$$

Claim 8.1.1. *If $y_N \rhd^{\text{BR}} x_N$, then $P(y_N) > P(x_N)$, i.e., P represents a Cournot potential.*

Proof of Claim 8.1.1. Let $y_N \rhd_i^{\text{BR}} x_N$; if $x_{-i} = 2$, we are home immediately. Let $x_{-i} \in [0, 1]$, hence $x_i \in [0, 1]$ too, hence $P(x_N) \leq 1$. If $y_i = 2$, then $P(y_N) \geq 2 > P(x_N)$.

Let $x_i \in [0, 1]$ and $x_{-i} \in [0, 1[$; then $y_i = x_{-i}/2 + 1/2 > x_{-i}$. We have $\psi(y_i, x_{-i}) = y_i > x_{-i} \geq \psi(x_{-i}, y_i)$, hence $P(y_N) = y_i$. Meanwhile, $\psi(x_{-i}, x_i) \leq x_{-i} < y_i$ and $\psi(x_i, x_{-i}) < y_i$; therefore, $P(y_N) > P(x_N)$ again. □

Since $\mathcal{R}_i$ are not upper hemicontinuous, neither Theorem 3.2, nor Theorem 3.5 is applicable here. Indeed, a Cournot path converging to $(1, 1)$, which is not an equilibrium, can be started from every strategy profile in $[0, 1[\times [0, 1[$. On the other hand, Theorem 3.7 *is* applicable; actually, the unique Nash equilibrium can be reached from every strategy profile after, at most, two best response improvements.

Example 8.2. Let us consider a game where $N := \{1, 2, 3\}$, $X_1 := X_2 := [0, 1]$, $X_3 := \{0, 1\}$, and the preferences are defined by these utility functions: $u_3(x_N) := 1$ if $x_N = (1, 1, 1)$, $u_3(x_N) := 0$ otherwise, whereas for $i \in \{1, 2\}$, $u_i(x_N) := \min\{2x_i - x_{3-i}, -2x_i + x_{3-i} + 2\}$. Both functions u_1, u_2 are continuous in x_N; u_3 is upper semicontinuous in x_N and continuous in x_3. The best responses are easy to compute: $\mathcal{R}_i(x_{-i}) = \{x_{3-i}/2 + 1/2\}$ for $i = 1, 2$, $\mathcal{R}_3(x_{-3}) = \{1\}$ if $x_{-3} = (1, 1)$, and $\mathcal{R}_3(x_{-3}) = X_3$ otherwise. There is a unique Nash equilibrium, $(1, 1, 1)$.

To define a Cournot potential, we use the same auxiliary function on $\mathbb{R}^2$: $\psi(x, y) := \min\{x, -x + y + 1\}$ and define a continuous function on X_N by $P(x_N) := \max\{\psi(x_1, x_2), \psi(x_2, x_1)\} + x_3$.

Claim 8.2.1. *If $y_N \rhd^{\text{BR}} x_N$, then $P(y_N) > P(x_N)$, i.e., P represents a Cournot potential.*

Proof of Claim 8.2.1. For players 1 or 2, the argument is the same as in the proof of Claim 8.1.1, one only has to consider fewer cases. The situation $y_N \rhd_3^{\text{BR}} x_N$ is only possible when $y_{-3} = x_{-3} = (1, 1)$, $x_3 = 0$ and $y_3 = 1$. □

Every Cournot path started from $[0,1[\times[0,1[\times\{0\}$ converges to $(1,1,0)$, which is not an equilibrium. Thus, the assumption $\#N=2$ in Theorem 3.2 is essential.

Example 8.3. In a plane with polar coordinates (ρ,φ) ($\rho\geq 0$, $0\leq\varphi<2\pi$), we define a compact subset

$$X := \bigl\{(\rho,\varphi) \mid 1\leq\rho\leq 2\bigr\}$$

and a mapping $f\colon X\to X$ by

$$f(\rho,\varphi) := \begin{cases} \bigl(1, \min\{3\varphi/2, \pi+\varphi/2\}\bigr), & \rho=1; \\ \bigl((\rho+1)/2, \min\{3\varphi/2, \pi+\varphi/2\}\oplus\pi/[1-\log_2(\rho-1)]\bigr), & \rho>1; \end{cases}$$

where $\oplus$ denotes addition modulo 2π. Clearly, f is continuous and $(1,0)$ is its unique fixed point. Defining $X^0 := \{(\rho,\varphi)\in X \mid \rho=1\}$ and $X^* := X\setminus X^0$, we immediately see that $f^k(x)$ converges to $(1,0)$ whenever $x\in X^0$ and to X^0 whenever $x\in X^*$.

Now we define a strategic game: $N:=\{1,2\}$, $X_1 := X_2 := X$, $u_i(x_N) := -d(x_i, f(x_{-i}))$, where d denotes distance in the plane. Both utilities are continuous; the best responses are unique, $\mathcal{R}_i(x_{-i}) = \{f(x_{-i})\}$. The strategy profile $((1,0),(1,0))$ is a unique Nash equilibrium.

Then we define a function $P\colon X\times X\to\mathbb{R}$ in this way:

$$P(x_1,x_2) := \begin{cases} 0, & \rho_1=\rho_2=1 \;\&\; \varphi_1=\varphi_2=0; \\ \min_i\varphi_i + \max_i u_i(x_N) - 2\pi, & \rho_1=\rho_2=1 \;\&\; \max_i\varphi_i>0; \\ \min_i(1-\rho_i) + \max_i u_i(x_N) - 2\pi, & \text{otherwise.} \end{cases}$$

The function is upper semicontinuous, but not continuous.

Claim 8.3.1. *If $x'_N \rhd^{\mathrm{BR}} x_N$, then $P(x'_N) > P(x_N)$, i.e., P represents a Cournot potential.*

Proof of Claim 8.3.1. Let $x'_{-i} = x_{-i}$ and $x'_i = f(x_{-i}) \neq x_i$, hence $u_i(x'_N) = 0 \geq u_{-i}(x'_N)$. If $x_{-i} = (1,0)$, then $P(x_N) < 0 = P(x'_N)$ and we are home.

Let $\rho_{-i}=1$ and $\varphi_{-i}>0$. Then $\rho'_i = 1$ and $\varphi'_i > \varphi_{-i}$, hence $P(x'_N) = \varphi_{-i} - 2\pi$. If $\rho_i>1$, then $P(x_N) < -2\pi < P(x'_N)$. If $\rho_i = 1$, then we consider two alternatives. If $\varphi_i\geq\varphi_{-i}$, then $\max_i u_i(x_N) < 0$, hence $P(x_N) < \varphi_{-i} - 2\pi = P(x'_N)$; if $\varphi_i < \varphi_{-i}$, then $P(x_N) \leq \varphi_i - 2\pi < P(x'_N)$.

Finally, let $\rho_{-i}>1$. Then $P(x'_N) = 1-\rho_{-i}-2\pi$. If $\rho_i\leq\rho_{-i}$, then $x_{-i}\neq f(x_i)$, hence $P(x_N) < 1-\rho_{-i}-2\pi = P(x'_N)$. If $\rho_i>\rho_{-i}$, then $P(x_N)\leq 1-\rho_i-2\pi < P(x'_N)$. $\square$

We see that the assumptions of Theorem 3.2 are satisfied. Moreover, the potential is upper semicontinuous and the best responses are single-valued. Meanwhile, every Cournot path started from $X^*\times X^*$ has an infinite number of cluster points besides the unique equilibrium, i.e., does not converge to the set of equilibria. Thus, the continuity of the potential in Theorem 3.5 is essential.

Example 8.4. We consider a modification of Example 8.3 with the same subset X

$$X := \{(\rho, \varphi) \mid 1 \leq \rho \leq 2\}$$

of the plane with polar coordinates and a different continuous mapping $f: X \to X$,

$$f(\rho, \varphi) := \big((\rho + 1)/2, \varphi \oplus \pi/[1 - \log_2(\rho - 1)]\big), \tag{15}$$

where $\oplus$ again denotes addition modulo 2π. Defining $X^0 := \{(\rho, \varphi) \in X \mid \rho = 1\}$ and $X^* := X \setminus X^0$, we immediately see that $f(x) = x$ whenever $x \in X^0$, and $f^k(x)$ converges to X^0 whenever $x \in X^*$.

Now we define a strategic game in exactly the same way as in Example 8.3: $N := \{1, 2\}$, $X_1 := X_2 := X$, $u_i(x_N) := -d(x_i, f(x_{-i}))$, where d denotes distance in the plane. Again, both utilities are continuous; the best responses are unique, $\mathcal{R}_i(x_{-i}) = \{f(x_{-i})\}$. The set of Nash equilibria of the game is $\{x_N \in X^0 \times X^0 \mid x_1 = x_2\}$.

Then we define a continuous function $P: X \times X \to \mathbb{R}$ by

$$P(x_N) := \min_i (1 - \rho_i) + \max_i u_i(x_N).$$

An argument similar to the proof of Claim 8.3.1, but even simpler, shows that P represents a Cournot potential. Meanwhile, the set of cluster points of any Cournot path started from $X^* \times X^*$ is the whole set of Nash equilibria of the game. We see that the assumptions of Theorem 3.5, even Theorem 4.3, do not ensure the convergence of every Cournot path to *a* Nash equilibrium.

The following example is essentially due to Powell (1973).

Example 8.5. Let us consider a game where $N := \{1, 2, 3\}$, $X_i := [-2, 2]$, and the preferences of each player are defined by the same continuous utility function:

$$u(x_N) := \sum_{i,j \in N,\, i \neq j} x_i \cdot x_j/2 - \sum_{i \in N} \big[\max\{x_i - 1, 0, -1 - x_i\}\big]^2.$$

Clearly, u is an exact potential of the game, hence a continuous Cournot potential as well. Note that the game belongs to the class considered in Section 7 with $\sigma_i(x_{-i}) := \sum_{j \neq i} x_j$; the strict single crossing condition (11) is easy to check. Note also that u is concave in each x_i.

The best responses are easy to compute; given $i \in N$ and $x_{-i} \in X_{-i}$, we denote $s_i := \sum_{j \neq i} x_j$.

$$\mathcal{R}_i(x_{-i}) = \begin{cases} \{2\}, & s_i \geq 2; \\ \{1 + s_i/2\}, & 0 < s_i \leq 2; \\ [-1, 1], & s_i = 0; \\ \{-1 + s_i/2\}, & -2 \leq s_i < 0; \\ \{-2\}, & s_i \leq -2. \end{cases}$$

There are two Nash equilibria maximizing the utility/potential: $(2, 2, 2)$ and $(-2, -2, -2)$. $(0, 0, 0)$ is also a Nash equilibrium.

Fixing an arbitrary $\delta \in]0, 1/4[$, we consider a sequential Cournot path starting at $x_N^0 := (1 + 4\delta, -1 - 2\delta, 1 + \delta)$: $x_N^1 = (-1 - \delta/2, -1 - 2\delta, 1 + \delta)$; $x_N^2 = (-1 - \delta/2, 1 + \delta/4, 1 + \delta)$; $x_N^3 = (-1 - \delta/2, 1 + \delta/4, -1 - \delta/8)$; $x_N^4 = (1 + \delta/16, 1 + \delta/4, -1 - \delta/8)$; $x_N^5 = (1 + \delta/16, -1 - \delta/32, -1 - \delta/8)$; $x_N^6 = (1 + \delta/16, -1 - \delta/32, 1 + \delta/64) \ldots$ Comparing x_N^0 and x_N^6, we see how the path will continue *ad infinitum*. Thus, it has six cluster points: $(1, -1, 1)$, $(-1, -1, 1)$, $(-1, 1, 1)$, $(-1, 1, -1)$, $(1, 1, -1)$, and $(1, -1, -1)$, none of which is an equilibrium.

We see that Theorem 3.5 cannot be extended to $n > 2$, while Theorem 4.3 is wrong without the uniqueness of the best responses.

Remark. Most likely, Theorem 4.2 does not hold for $\#N > 3$, but I have failed so far to come up with a fully developed example.

9 Concluding remarks

9.1. ω-transitivity of a Cournot potential alone ensures the "transfinite convergence" of every Cournot path to Nash equilibria; a formal exposition can be found in Kukushkin (2010). The concept might seem exotic, but there is something to it. If, e.g., we replace all $X_i = [-2, 2]$ in Example 8.5 with arbitrary finite subsets, then every Cournot path will reach an equilibrium in a finite number of steps. Therefore, one can argue that the problem illustrated by the example is just an artefact of the suboptimal way to introduce infinity: no such thing could happen with transfinite paths.

9.2. It is worth stressing once again: None of the results of this paper needs a *numeric* potential; moreover, in each of the "counterexamples" in Section 8, there is a numeric potential, which does not help. The upper semicontinuity of a Cournot potential also seems not to ensure any better properties of best response dynamics than just ω-transitivity. The same holds for the presence of a "best-response potential" (Voorneveld, 2000). The only difference is that the continuity of a best-response potential implies the upper hemicontinuity of all best response correspondences, hence Theorem 3.6 absorbs Theorem 3.7, and Examples 8.1 or 8.2 become impossible.

9.3. Theorem 2 of Jensen (2010) is neither weaker, nor stronger than any result of this paper. It establishes the convergence of sequential Cournot tâtonnement to Nash equilibria under an assumption concerning paths were the players consecutively replace one best response with another. The assumption is automatically satisfied if all best responses are single-valued, in which case our Theorem 4.3 is a bit stronger. It is worth noting that Example 8.2 shows Jensen's theorem to be, strictly speaking, wrong (upper semiconinuity of utility functions is not enough).

9.4. The Cournot path leading nowhere in Example 8.5 needs a carefully chosen initial point. It does not matter here since the only objective of the example is to demonstrate the invalidity of straightforward extensions of Theorems 3.5 and 4.3. Powell (1973) also provides a more complicated example where such paths can be started from every point in an open subset.

9.5. If we modify the constructions of Section 6, replacing the sum in (6) with the minimum, cf. Germeier and Vatel' (1974), then the leximin ordering on X_N will be a potential in the sense of (2) for coalition improvements, hence a Cournot potential as well. Since the ordering is not continuous, our main results are inapplicable even though no counterexample is known. Funnily, aggregative games of Section 7 with $\sigma_i(x_{-i}) = \min_{j \neq i} x_j$ for all $i \in N$ or $\sigma_i(x_{-i}) = -\min_{j \neq i} x_j$ for all $i \in N$ also admit ω-transitive Cournot potentials. And the existence of a continuous Cournot potential in every such game also remains neither proven, nor disproved so far.

9.6. Everything in this paper is about games with ordinal preferences. For applications of the idea of potential games to the best responses in the context of cardinal utilities, see, e.g., Morris and Ui (2004).

Acknowledgments

Financial support from the Russian Foundation for Basic Research (projects 11-07-00162 and 11-01-12136) and the Spanish Ministry of Education and Innovation (project ECO 2010-19596) is acknowledged. I thank Martin K. Jensen for helpful comment on a draft version.

References

Bernheim, B.D., 1984. Rationalizable strategic behavior. Econometrica 52, 1007–1028.

Dubey, P., O. Haimanko, and A. Zapechelnyuk, 2006. Strategic complements and substitutes, and potential games. Games and Economic Behavior 54, 77–94.

Facchinei, F., V. Piccialli, and M. Sciandrone, 2011. Decomposition algorithms for generalized potential games. Computational Optimization and Applications 50, 237–262.

Germeier, Yu.B., and I.A. Vatel', 1974. On games with a hierarchical vector of interests. Izvestiya Akademii Nauk SSSR, Tekhnicheskaya Kibernetika, 3, 54–69 [in Russian; English translation in Engineering Cybernetics 12(3), 25–40 (1974)].

Huang, Z., 2002. Fictitious play in games with a continuum of strategies. Ph.D. Thesis, State University of New York at Stony Brook, Department of Economics.

Jensen, M.K., 2010. Aggregative games and best-reply potentials. Economic Theory 43, 45–66.

Kandori, M., and R. Rob, 1995. Evolution of equilibria in the long run: A general theory and applications. Journal of Economic Theory 65, 383–414.

Kukushkin, N.S., 1994. A condition for the existence of a Nash equilibrium in games with public and private objectives. Games and Economic Behavior 7, 177–192.

Kukushkin, N.S., 2004. Best response dynamics in finite games with additive aggregation. Games and Economic Behavior 48, 94–110.

Kukushkin, N.S., 2005. "Strategic supplements" in games with polylinear interactions. EconWPA Paper #0411008

Kukushkin, N.S., 2007. Congestion games revisited. International Journal of Game Theory 36, 57–83.

Kukushkin, N.S., 2008. Maximizing an interval order on compact subsets of its domain. Mathematical Social Sciences 56, 195–206.

Kukushkin, N.S., 2010. Improvement Paths in Strategic Games: A Topological Approach. Russian Academy of Sciences, Dorodnicyn Computing Center, Moscow, 2010. (PDF)

Kukushkin, N.S., 2011. Nash equilibrium in compact-continuous games with a potential. International Journal of Game Theory 40, 387–392.

Milchtaich, I., 1996. Congestion games with player-specific payoff functions. Games and Economic Behavior 13, 111–124.

Milgrom, P., and J. Roberts, 1990. Rationalizability, learning, and equilibrium in games with strategic complementarities. Econometrica 58, 1255–1277.

Milgrom, P., and C. Shannon, 1994. Monotone comparative statics. Econometrica 62, 157–180.

Monderer, D., and L.S. Shapley, 1996. Potential games. Games and Economic Behavior 14, 124–143.

Morris, S., and T. Ui, 2004. Best response equivalence. Games and Economic Behavior 49, 260–287.

Moulin, H., 1984. Dominance solvability and Cournot stability. Mathematical Social Sciences 7, 83–102.

Novshek, W., 1985. On the existence of Cournot equilibrium. Review of Economic Studies 52, 85–98.

Powell, M.J.D., 1973. On search directions for minimization algorithms. Mathematical Programming 4, 193–201.

Rosenthal, R.W., 1973. A class of games possessing pure-strategy Nash equilibria. International Journal of Game Theory 2, 65–67.

Shannon, C., 1995. Weak and strong monotone comparative statics. Economic Theory 5, 209–227.

Topkis, D.M., 1979. Equilibrium points in nonzero-sum n-person submodular games. SIAM Journal on Control and Optimization 17, 773–787.

Veinott, A.F., Jr., 1989. Lattice Programming. Unpublished lectures.

Vives, X., 1990. Nash equilibrium with strategic complementarities. Journal of Mathematical Economics 19, 305–321.

Voorneveld, M., 2000. Best-response potential games. Economics Letters 66, 289–295.