Collaborative Dominance: When Doing Unto Others As You Would Have Them Do Unto You Is Reasonable.

Souza, Filipe and Rêgo, Leandro

Federal University of Pernambuco

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COLLABORATIVE DOMINANCE: WHEN DOING UNTO OTHERS AS YOU WOULD HAVE THEM DO UNTO YOU IS REASONABLE

Filipe Costa de Souza[1]
Federal University of Pernambuco

Leandro Chaves Rêgo[2]
Federal University of Pernambuco

Abstract

In this article, we analyze how reasonable it is to play according to some Nash equilibria if players have a preference for one of their opponents' strategies. For this, we propose the concepts of collaborative dominance and collaborative equilibrium. First we prove that, when the collaborative equilibrium exists it is always efficient, what can be seen as a focal property. Further we argue that a reason for players choosing not to collaborate is if they are focusing in security instead of efficiency, in which case they would prefer to play maximin strategies. This argument allows us to reduce the hall of reasonable equilibria for games where a collaborative equilibrium exists. Finally, we point out that two-player zero-sum games do not have collaborative equilibrium and, moreover, if there exists a strategy profile formed only by collaboratively dominated actions it is a Nash equilibrium in such kind of game.

Keywords: Nash Equilibrium, Collaborative Dominance, Two-Players Zero-Sum Games.

[1] Accounting and Actuarial Science Department, Federal University of Pernambuco, Recife, PE, 50740-040, Brazil, filipe.costas@ufpe.br (corresponding author).
[2] Statistics Department, Federal University of Pernambuco, Recife, PE, 50740-040, Brazil, . leandro@de.ufpe.br, Fax: +55-81-2126-7438.

1. Introduction

Even though other authors had previously made isolated contributions to the Theory of Games, many researchers, including Kuhn & Tucker (1958) and Leonard (1995), considered von Neumann as the father of the Theory of Games because of his initial work in the late 1920 and his further collaboration with Oskar Morgenster that culminated in the publication in 1944 of the classical book *Theory of Games and Economic Behavior*. Even von Neumann holding this prestigious position in the game theory field, John Nash is the main star in this constellation, largely because of his equilibrium's definition (Nash, 1951). Nowadays, it is almost impossible to talk about game theory without saying Nash's name.

Due to the relevance of this equilibrium concept, many criticism and even more applications appeared. The first known criticism to the Nash equilibrium idea was made by Merrill Flood and Melvin Dresher in an experiment at RAND Corporation, where the researchers analyzed how players behaved in a finitely repeated iteration of the prisoners' dilemma (Poundstone, 1992 and Nasar, 1998). On the other hand, the applications started with economic and military subjects and rapidly spread to many other fields such as political science, evolutionary biology, etc.

In this paper, our aim is to provide a theoretical criticism on some Nash equilibrium in non-cooperative games where players have a strict preference for one of the pure strategies of the other players. There are many games in applied and theoretical literature as, for example, the Stag-Hunt game, in which players have a strict preference for one of the other player's pure strategies, and still some Nash equilibria do not recommend or predict that player should or will collaborate. Thus, to criticize some Nash equilibria, we will analyze when doing unto others as you would have them do unto you could be considered a reasonable behavior.

At this point, it is worth noting that different from the existent Nash equilibrium criticism, our argument against some Nash equilibria is neither based in a new refinement of the Nash equilibrium nor in a new concept of equilibrium selection. First, it is not a refinement because in some games, players do not have a preference for one of the other players' pure strategies regardless of their own actions; secondly, even when the game satisfies this property and our argumentation classifies some equilibria as unreasonable, we not advocate that any other specific equilibrium should be selected. The remaining sections of this paper are organized as follows: in Section 2, we make a brief review about Nash equilibrium and its main criticisms; in Section 3, we introduce the concept of collaborative dominance and collaborative equilibrium discussing its rationality principles and efficiency property; in Section 4, we present some results about collaborative dominance related with two-player zero-sum games; and in Section 5, we present the final remarks.

2. Background

Roughly speaking, Game Theory is a branch of mathematics that study conflict and cooperation between subjects when the action of each one of them influences their final outcome in the strategic situation. A strategic situation (or simply a game) is composed by three main parts: the players, the strategies available for each player and a function (utility function) that for every possible combination of actions of the players gives a real number that represents players' preferences (the higher such number is, the more preferred is the consequence induced by the given combination of actions).

Formally, a strategic or normal form game G is represented by $\mathrm{G} = (\mathrm{N}, (\mathrm{S_i})_{\mathrm{i \in N}}, (\mathrm{U_i})_{\mathrm{i \in N}})$ where N={1, …, n} is the finite set of players, $\mathrm{S_i}$ is the finite set of (pure) strategies available for player $\mathrm{i} \in \mathrm{N}$, and $U_i \times_{\mathrm{i \epsilon N}} \mathrm{S_i} \rightarrow \mathbb{R}$ is the utility function of player $\mathrm{i} \in \mathrm{N}$. A strategy profile $\mathrm{s} = \times_{\mathrm{i \epsilon N}} \mathrm{s_i}$ is a collection of pure strategies, one for each player in G, and $\mathrm{S} = \times_{\mathrm{i \epsilon N}} \mathrm{S_i}$ is the set of all pure strategy profiles. Furthermore, to simplify the notation, we assume that $\mathrm{S_{-k}} = \times_{\mathrm{i \in N \setminus \{k\}}} \mathrm{S_i}$ is the set of all strategies profiles of the other players in G, except for player k. A mixed strategy for player i is a probability distribution over his pure strategies. Let $\Delta(\mathrm{S_k})$ be the set of all mixed strategies of player k, where $\sigma_k(s_k)$ indicates the probability that k gives to his pure strategy s_k when implementing the mixed strategy $\sigma_k \in \Delta(S_k)$. So we say that a strategy profile $\sigma = (\times_{i \in N} \sigma_i)$ is a Nash equilibrium in G if, and only if: $U_i(\sigma) \geq U_i(\sigma_{-i}, \tau_i), \forall i \in N, \forall \tau_i \in \Delta(S_i)$, where $U_i(\sigma)$ is player *i*'s expected utility when the mixed strategy profile σ is implemented. Thus, $U_i(\sigma) = \sum_{s \in S}\left(\prod_{j \in N} \sigma_j(s_j)\right) U_i(s)$. On the other hand, (σ_{-k}, τ_k) is the mixed strategy profile where player k plays according to the mixed strategy τ_k and the other players play according to the strategies specified in σ. Thus, $U_i(\sigma_{-i}, \tau_i) = \sum_{s \in S}\left(\prod_{j \in N \setminus \{i\}} \sigma_j(s_j)\right) \tau_i(s_i) U_i(s)$. In words, a mixed strategy profile is a Nash equilibrium if no player, acting individually, is capable of improving his expected payoff by deviating from the strategy prescribed by the equilibrium.

At this point it is worth noting that the Nash equilibrium could be interpreted in many different ways. Nash originally provided two main interpretations of his equilibrium concept: the rational and the evolutionary. Besides Nash, other renowned authors such as Harsanyi (1973) and Aumann (1987) also provided their view on the subject. However, there is no consensus regarding which of the interpretations is the most appropriate. An interesting discussion about the interpretations of the Nash equilibrium can be seen in Osborne and Rubinstein (1994). In this book it is possible to see the authors' disagreement on some points. Here we adopt the traditional rational interpretation to develop our argument, i.e., we study one-shot games in which players have some probabilistic mechanism available that enables them to actually adopt a mixed strategy.

Almost as well known as the Nash equilibrium are the existing criticisms to it, which spread themselves in many different directions especially regarding to the mixed equilibrium. Now we will briefly present some of them. In the core of this discussions are the refinements of the Nash equilibrium concept that seeks (by new equilibrium concepts) to eliminate implausible equilibrium points from the original set of equilibria (as for example, equilibrium profiles composed by weakly dominated strategies). In this literature we shall highlight the works of Selten (1975), Myerson (1978) and Kalai & Samet (1984). Following a different direction of the refinements, Aumann (1974) creates the concept of correlated equilibrium that seeks to increase the set of equilibria adding efficient points to the original set of equilibria (assuming the existence of an impartial mediator).

The others criticisms to the Nash equilibrium are mainly focused in the mixed equilibrium (hereafter, mixed means mixed in the non-degenerate sense). In this context first we shall start with the instability argument from Harsanyi (1973) and Harsanyi & Selten (1988). In a mixed strategy Nash equilibrium, players are indifferent among their pure strategies that receive positive probability according to the mixed strategy, i.e., among the pure strategies in the mixed strategy's support. In fact, as the players are indifferent among these pure strategies, any probability distribution over them will give them the same expected utility. Therefore, because there are infinitely many strategies that act as best response to the opponent's mixed strategy, the mixed Nash equilibrium is classified as unstable.

The final criticism of this section is from Aumann & Maschler (1972). The authors inquired why players should play according to the mixed equilibrium in games where they can guarantee the same expected utility by playing their maximin strategies regardless of the others players' actions. Even though these authors presented an interesting argument against the mixed equilibrium, they failed to present a convincing solution to how players should behave in a game, since the maximin strategies of the players may not be a best response to each other. Further discussions about this subject can be found in Pruzhansky (2011).

3. Doing unto others as you would have them do unto you

Consider the games shown in Figure 1. First, note that in all of them both players would prefer that the other choose a specific strategy, regardless of which strategy they intend to play. So we may wonder what would lead players choosing not to collaborate in those games? But before analyze this problem, we should formalize this idea. Let $\Gamma = (N, (S_i)_{i \in K}, (U_i)_{i \in N})$ be a normal form game, we define:

Weak Collaborative Dominance (or non-strict): For any normal form game Γ, for any player $i \in N$ and for any pure strategy $s_i \in S_i$, s_i is said weakly collaboratively dominated for player $j \in N$, with $j \neq i$, if exist any mixed strategy $\sigma_i \in \Delta(S_i)$ such that $U_j(s_i, s_{-i}) \leq \sum_{s_i^{'} \in S_i} \sigma_i(s_i^{'}) U_j(s_i^{'}, s_{-i})$, $\forall\, s_{-i} \in S_{-i}$, with at least one strict inequality. In this case, we say that σ_i collaboratively dominates s_i for player *j*.

(i)

		Player 2	
		W	*Z*
Player 1	*X*	*(3, 3)*	*(0, 2)*
	Y	*(2, 0)*	*(1, 1)*

(ii)

		Player 2	
		W	*Z*
Player 1	*X*	*(0, 0)*	*(3, 1)*
	Y	*(1, 3)*	*(2, 2)*

(iii)

		Player 2	
		W	*Z*
Player 1	*X*	*(1, 1)*	*(0, 1)*
	Y	*(1, 0)*	*(0, 0)*

(iv)

		Player 2	
		W	*Z*
Player 1	*X*	*(3, 0)*	*(0, 1)*
	Y	*(2, 3)*	*(1, 2)*

Figure 1

For a strong version of the collaborative dominance definition made above, we should simply use a strict inequality signal (<). Furthermore, by simplicity, in a two-player game, when player *i* has a collaboratively dominant strategy for player *j*, we say that play *i* has a collaboratively dominant strategy. So, based on the concept of collaborative dominance, we can reformulate the problem presented in the beginning of this section in a more precise way: as long as there is a pure strategy s_i of player *i* that collaboratively dominates all of his other strategies for player *j* and *vice-versa*, why would players choose not to collaborate? A possible explanation for players' lack of collaboration is supported by stability purposes, as shown next.
By analyzing this problem we want to classify what are the Nash equilibria that should be seen as unreasonable in games with collaborative dominant strategies. For this purpose, let us analyze in more details the games shown in Figure 1. In all of them, both players have a collaboratively dominant strategy. So what happen if each player decides *to do unto others as he would have them do unto him*? (This idea is known as Luke's Golden Rule (Luke, 6:31)). Or in a game theory context, if they decide to play the strategy that is collaboratively dominant for the other. If both choose the collaboratively dominant strategy, then they will achieve a Nash equilibrium in Games *(i)* and *(iii)* which is efficient and, therefore, provides an expected utility higher than (or at least equal to) the expected utility of any other Nash equilibria in the respective game for both players. This happens because each player collaboratively dominant strategy is a best response to the other player's collaboratively dominant strategy. However, for Games *(ii)* and *(iv)* the pair of collaboratively dominant strategies leads the players to achieve points in the payoff matrix that do not correspond to a Nash equilibrium, i.e., points that are not stable, even if desired by both players. This happens because, for at least one player, the collaboratively dominant strategy is not a best response to the other player's collaboratively dominant strategy. When a collaborative strategy of player *i* is not a best response to player *j*'s collaborative strategy, but the inverse is true, then we say that the collaborative profile is unstable for player *i*. For this reason, we realize that the concept of collaborative dominance is not sufficient to ensure that collaboration will occur. In fact, in Game *(iv)*, the mixed equilibrium is the unique stable strategy profile. Thus, we argue that collaboration (or any other strategy) is a reasonable option if it is part of a Nash equilibrium. Therefore, we define:

Stable Collaborative Dominance (collaborative equilibrium): For a given normal form game $\Gamma = (N, (S_i)_{i\in N}, (U_i)_{i \in N})$, if $(\times\, s_{i \in N}^*)$ is such that, for all $i \in N$ and for all $s_i \in S_i$, s_i is weakly collaboratively dominated by the pure strategy s_i^*, for all players $j \neq i$, then, we say that the strategy profile $(\times\, s_{i \in N}^*)$ is *collaboratively stable* or a *collaborative equilibrium* if it is a pure

Nash equilibrium, i.e., a strategy profile such that any player that deviates unilaterally from it is not in a better situation.

The next theorem shows that the collaborative equilibrium is always efficient.

Theorem 1: Let Γ be a game in strategic form. If $(\times\, s^*_{i\in N})$ is a collaborative equilibrium, then, for all player i in N, $U_i(\times\, s^*_{i\in N})$ is the highest possible utility for player i in Γ.

Proof: Assume that $(\times\, s^*_{i\in N})$ is a collaborative equilibrium. Thus, by the idea of stability we have: $U_i(s^*_i, s^*_{-i}) \geq U_i(s_i, s^*_{-i})$, $\forall s_i \in S_i$ and $\forall i \in N$. In turn, applying the concept of collaborative dominance $\|N\| - 1$ times, we have: $U_i(s_i, s^*_{-i}) \geq U_i(s_i, s_{-i})$, $\forall s_i \in S_i$, $\forall\, s_{-i} \in S_{-i}$ and $\forall i \in N$. Therefore, $U_i(\times\, s^*_{i\in N})$ is higher than or equal to any other utility that player i can achieve in game Γ. This conclusion applies $\forall i \in N$, ending the proof. ■

Intuitively, Theorem 1 provides us with an important result, that if there is a collaborative equilibrium in a given strategic form game, then no player can get a better result than what he would get if the collaborative equilibrium is achieved. Note that this is stronger than the usual Pareto efficiency property since it is not only the case that players cannot improve the collaborative equilibrium payoffs without harming another player, but it is the case that every player is getting his most preferred outcome in the game. This happens because the concept of collaborative equilibrium (when applied) makes use of two principles of rationality: the selfish and the altruistic. The selfish rationality refers to the traditional own utility maximization idea; and the altruistic rationality, in the opposite sense, refers to the maximization of others' utility. Thus, by joining these two ideas, it is possible to ensure that the best outcomes can be achieved, fact that is not guaranteed by each one of these principles alone.
It is important to highlight that for a collaborative equilibrium to be achieved, players do not need to try to maximize their opponents' utility, it is only necessary that they are trying to maximize their own utility and that they believe that the collaborative profile will be played (the same assumptions necessaries to any Nash equilibrium to be achieved). So the fact that the utilities of others players are maximized in a collaborative equilibrium is an additional feature that may turn such equilibrium focal. However, this focal property is lost if players are focusing on security, for example. Hereafter, a secure equilibrium is an equilibrium composed by maximin strategies. Game (*i*) in Figure 1, known in the literature by the name Stag-Hunt game, is an example of this efficiency-risk dilemma. Even though, there is a collaborative equilibrium (*X, W*), which is efficient by Theorem 1, there is another equilibrium (*Y, Z*), which is secure. Having said that, note that the mixed equilibrium of this game does not have this security property so, besides the stability property of such equilibrium, players have no rational reason to play according to the mixed Nash equilibrium. So we argue that in games with a collaborative equilibrium (like Game (*i*)) the only reason for not playing according to it, is if players are seeking for security in which case they should play a secure equilibrium. So by this argument, we can classify the mixed equilibrium of Game (*i*) as unreasonable, because in such game, the mixed equilibrium is neither efficient nor secure, and if players are expecting that other will play according to the mixed equilibrium, then any strategy (including the collaborative or the secure) will act as a best response.
Now, we are able to classify which are the equilibria that should be considered unreasonable in a strategic analysis: in a finite normal form game, if a collaborative equilibrium exists, any equilibrium that is not the collaborative or a secure equilibrium should be considered unreasonable. Moreover, it is important do highlight that if the game does not have a collaborative equilibrium, then we do not have any new critique to any Nash equilibrium of the game.
Again, we do not advocate that players should choose the collaborative equilibrium, which is efficient; they can adopt another pure or mixed equilibrium with security properties, for example. To illustrate what are the Nash equilibria that we classify as unreasonable let us look at Figure 2.

		Player 2			
		A	*B*	*C*	*D*
Player 1	*A*	*(3, 3)*	*(0, 2)*	*(0, 2)*	*(1, 1)*
	B	*(2, 0)*	*(1, 1)*	*(0, 0)*	*(0, 0)*
	C	*(2, 0)*	*(0, 0)*	*(1, 1)*	*(0, 0)*
	D	*(1, 1)*	*(0, 0)*	*(0, 0)*	*(0, 0)*

Figure 2

In such game, strategy *A* for both players is collaboratively dominant, and (*A, A*) is a collaborative equilibrium. Therefore, the game satisfies the requirements for our critique. This game has seven equilibria, namely: (*A, A*), (*B, B*), (*C, C*), [(½, ½, 0, 0), (½, ½, 0, 0)], [(½, 0, ½, 0), (½, 0, ½, 0)], [(0, ½, ½, 0), (0, ½, ½, 0)] and [(1/3, 1/3, 1/3, 0), (1/3, 1/3, 1/3, 0)], in which the last one is a secure equilibrium (because it is composed by maximin strategies). So in our approach, the game in Figure 2 has two reasonable equilibria (*A, A*) and [(1/3, 1/3, 1/3, 0), (1/3, 1/3, 1/3, 0)], because the remaining equilibria do not provide any advantage for the players and therefore, playing any one of them could be considered as an unreasonable behavior. So our argumentation excludes these five equilibria from the hall of reasonable equilibria.

4. Collaborative dominance in two-player zero-sum games.

Based on the concept of collaborative equilibrium and on our early conclusion, we can still point out that two-player zero-sum games do not have collaborative equilibrium. This result is quite intuitive, since zero-sum games have a competitive nature and are characterized by the conflict of interests between the players.

Corollary 2: A two-player zero-sum game does not have collaborative equilibrium.

Proof: By Theorem 1, if a collaborative equilibrium exists, then all players get their highest available utilities in the game. Thus, in a two-player zero-sum this cannot happen, since if a player gets the highest utility the other one will get the lowest; and if all players' payoffs are equal to zero, then the game does not have a collaborative equilibrium. ■

Theorem 3: In a two-player zero-sum game, as illustrated in Figure 3, if α_i is strongly (resp. weakly) collaboratively dominant with respect to α_j for the column player, then α_j will be strongly (resp. weakly) dominant with respect to α_i.

$$\begin{array}{cc} & \begin{array}{ccc} \beta_1 & \cdots & \beta_m \end{array} \\ \begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_n \end{array} & \begin{bmatrix} (a_{1,1}, -a_{1,1}) & \cdots & (a_{1,m}, -a_{1,m}) \\ \vdots & \ddots & \vdots \\ (a_{n,1}, -a_{n,1}) & \cdots & (a_{n,m}, -a_{n,m}) \end{bmatrix} \end{array}$$

Figure 3

Proof: Admit Figure 3. Suppose, for simplicity, that the strategy α_1 of Player 1 is strongly (resp. weakly) collaboratively dominant with respect to strategy α_n for Player 2, implying that $-a_{1,j}>-a_{n,j}$, for *j=1,...,m* (resp. $-a_{1,j}\geq-a_{n,j}$, for *j=1,...,m*, with at least one strict inequality). But $-a_{1,j}>-a_{n,j}$ if, and only if, $a_{1,j} < a_{n,j}$ (resp. $-a_{1,j}\geq-a_{n,j}$ if, and only if, $a_{1,j} \leq a_{n,j}$), then this leads us to conclude that the strategy α_1 is strongly (resp. weakly) dominated by α_n. A similar conclusion is also valid for Player 2. ■

Corollary 4: If in a two-player zero-sum game, as illustrated in Figure 3, the strategy α_i of Player 1 is collaboratively dominated by all other strategies α_j for Player 2 and if the strategy β_j of Player 2 is collaboratively dominated by all other strategies β_k for Player 1, then the pair (α_i, β_j)

is a Nash equilibrium of the game. Furthermore, if the collaborative dominance is strong, then the equilibrium is unique.

Proof: By Theorem 3, we know that if a strategy is strongly (resp. weakly) collaboratively dominated by another strategy, then it is strongly (resp. weakly) dominant in relation to this strategy. Therefore, given the profile (α_i, β_j), no player has an incentive to deviate from this profile, so it is a Nash equilibrium. If dominance is strong, then we have the profile (α_i, β_j), is the only one that survives the process of elimination of strongly dominated strategies. Therefore, the equilibrium is unique. ■

5. Final Remarks

As it is the most important solution concept in game theory, Nash equilibrium is constantly kept under review. In this article, we analyze how reasonable it is to play according to some Nash equilibria if players have a preference for one of their opponents' strategies. For this, we propose the concepts of collaborative dominance and collaborative equilibrium (which is a strict pure Nash equilibrium). First we proved that, when the collaborative equilibrium exists it is always efficient, what could be seen as a focal property. Further we argue that a reason for players not to collaborate is if they are focusing in security instead of efficiency, and therefore they shall play a secure equilibrium. This argument allows us to reduce the hall of reasonable equilibria for games where a collaborative equilibrium exists. Finally, we point out that two-player zero-sum games do not have collaborative equilibrium and, on the other hand, if there exists a strategy profile formed only by collaboratively dominated actions it will be a Nash equilibrium.

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