# Sub-interval analysis and possibilities of its use

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**Sub-Interval Analysis**
**and Possibilities of its Use**

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This paper is a brief review and development of a part of the plenary report in the Moscow Institute of Physics and Technology. Two existing tools of sub-interval analysis are reviewed and elements of two new tools (the sub-interval layerwise analysis and the sub-interval smoothing) are presented. The sub-interval analysis may be used, e.g., in economics: micro- and macroeconomics, accounting, econometrics, utility theory; internet.

Contents

## Introduction

This paper is a brief review and development of a part of the plenary report (Harin 2012b) "Sub-interval analysis and possibilities of its applications" that took place 23 November 2012 in the Moscow Institute of Physics and Technology in the department of Innovations and High Technology.

The sub-interval analysis was founded in 2011 in (Harin 2011) as a new branch of the interval analysis (see, e.g., Moore 1966, Shary 2010, Nguyen et al. 2012). At present, there are about 20 reports and papers and one book (Harin 2012a) devoted to the sub-interval analysis.

## 1. What is it?

### 1.1. What is a sub-interval?

A sub-interval is simply a part of an interval. More accurately: An interval is a closed segment. A sub-interval is an interval which is a part of another interval. You may see an example of an interval $\boldsymbol{X}$ and its sub-interval $\boldsymbol{X_1}$ in Figure 1.

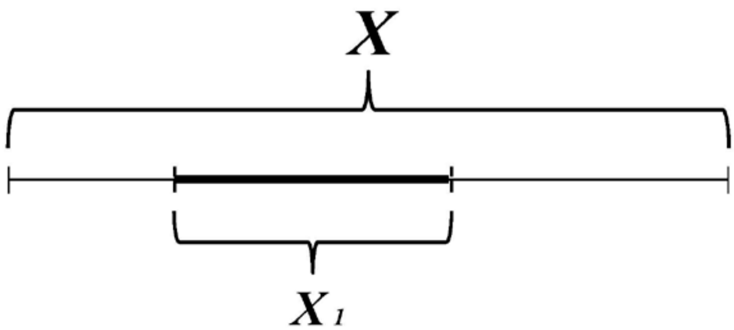


Figure 1. An example
of an interval $\boldsymbol{X}$ with a sub-interval $\boldsymbol{X_1}$

## 1.2. What is a sub-interval analysis?

A sub-interval analysis is a resource

1) to calculate the characteristics of a whole interval and
2) to represent a whole interval

with the help of its sub-intervals.

## 1.3. Prerequisites of use of sub-interval analysis

There are two basic prerequisites of use of sub-interval analysis.

1) The major prerequisite of use of sub-interval analysis is the fact that

**A sub-interval can be determined by only 3 (Three) quantities.**

They are 2 bounds co-ordinates and 1 weight of the sub-interval.

This fact may give the gain of orders of value for information storage and for some other operations in comparison with the continuous, pointwise description.

2) **The sub-interval analysis allows exact and reliable calculations and evaluations**, which are exact and reliable not less than initial data (in comparison with, e.g., probability theory).

## 1.4. An example. Box and sub-boxes

### 1.4.1. Box without sub-boxes

Suppose a box without sub-boxes. Such a box is drawn in Figure 2. The box contains two balls (apples, tennis-balls, cannon-balls or something else). The balls can be located at any points of the box. The weights of the balls are depicted by arrows. The center of gravity, the mean weight position of the box is denoted as $M$.

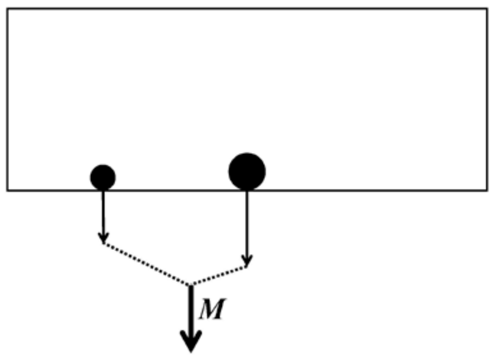


Figure 2. Box without sub-boxes

Suppose two balls are located in the utmost left point of the box (see Figure 3). At that, the mean weight position $M$ is also located in this point.

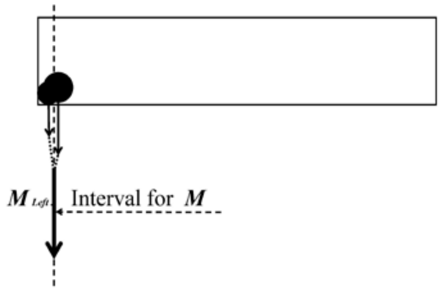


Figure 3. The balls in the utmost left point

Suppose two balls are located in the utmost right point of the box (see Figure 4). At that, $M$ is also located in this point.

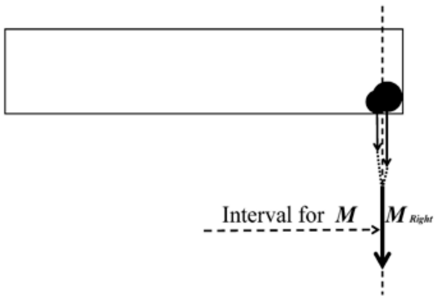


Figure 4. The balls in the utmost right point

Figure 5 represents the interval for the possible mean weight positions $M$.

We see, if both balls can be located at any point of the box, the mean weight position $M$ may be located at any point of the box as well. The precision of determining of $M$ is the whole box.

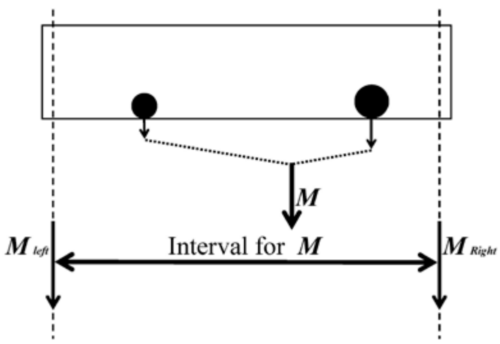


Figure 5. The interval for the possible mean weight positions $M$

So, in the case of no sub-boxes, the precision is the worst.

#### 1.4.2. Box with two sub-boxes

Let us suppose a box with two sub-boxes. Every sub-box contains one ball. The balls can be located at any points of the sub-boxes. The positions of balls are arbitrary. The center of gravity, the mean weight position of the box is as well denoted as $M$.

If the balls are in the utmost left positions, then the mean weight position $M$ is also in the utmost left position (see Figure 6)

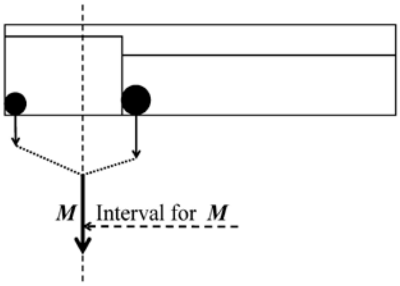


Figure 6. The mean weight position $M$ at the utmost left positions of the balls

If the balls are in the utmost right positions, then the mean weight position $M$ is also in the utmost right position (see Figure 7)

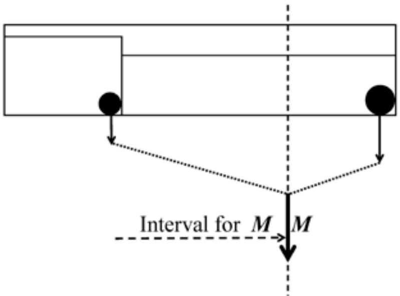


Figure 7. The mean weight position $M$ at the utmost right positions of the balls

Figure 8 represents the interval for the possible mean weight positions $M$.

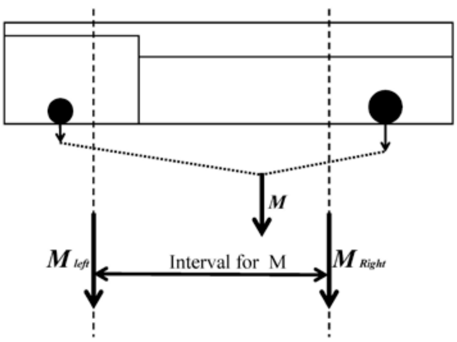


Figure 8. The interval for the possible mean weight positions $M$

We see, while the first ball can be located at any point of the first sub-box and the second ball can be located at any point of the second sub-box, the interval for the possible mean weight positions $M$ may be essentially less than the dimension of the box.

The precision of determining of $M$ is essentially better, than the size of whole box. So, in the case of sub-boxes, the precision may be better, than in the case without sub-boxes.

## Advantages of Sub-Boxes

Let us compare the interval of the mean weight position for a box without sub-boxes and that for a box with sub-boxes. It is evident the interval for a box with sub-boxes is less than the interval for a box without sub-boxes (see Figure 9)

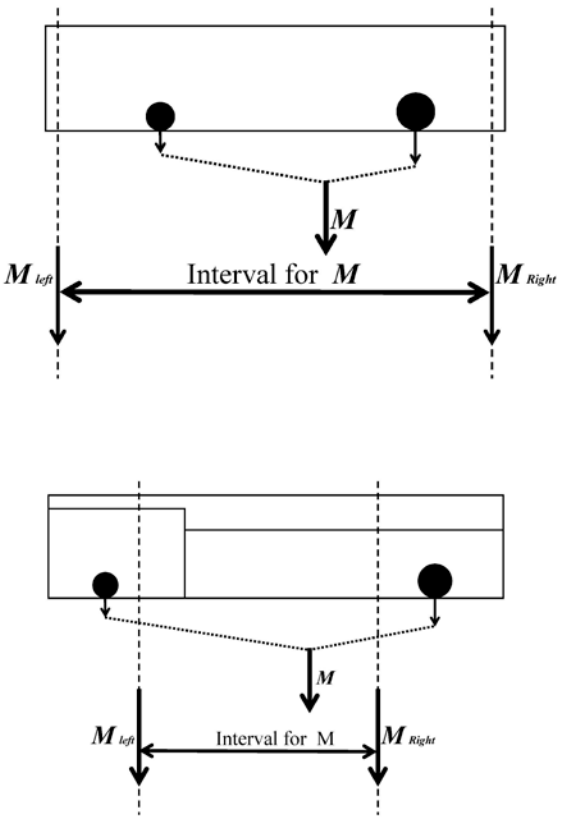


Figure 9. The intervals for the possible mean weight positions $M$

We see sub-boxes (as a particular case of sub-intervals) improve the precision about twice (for two sub-boxes). Note, a box without sub-boxes needs 3 quantities: 2 bounds co-ordinates and 1 weight of the box. A box with 2 sub-boxes needs 5 quantities: 3 bounds co-ordinates and 2 weights of the sub-boxes. So 2 (two) additional quantities may improve the precision about twice.

## 2. Basic tools of sub-interval analysis

### 2.1. Sub-interval arithmetic

A **sub-interval arithmetic** is a tool to calculate analogs of moments and other quantitative characteristics of a whole interval with the help of characteristics of its sub-intervals.

Let us consider a general case of a system of $S$ : $2 \le S < \infty$ adjacent sub-intervals.

Suppose an interval $X=[A, B]$ : $0<(B-A)<\infty$, and a quantity or a function $w(x_k)$, : $k=1, 2, \ldots K$ : $1 \le K \le \infty$ and $w(x_k) \ge 0$ for $A \le x_k \le B$ and

$$\sum_{k=1}^{K} w(x_k) = W < \infty \ .$$

For simplicity, let us suppose by default $W=1$.

Let us define a moment (an analog of a moment) of $n$-th : $1 \le n < \infty$, order of a quantity $w(x_k)$ relative to a point $x_0$ as

$$E(X - X_0)^n \equiv \sum_{k=1}^{K} (x_k - x_0)^n w(x_k)$$

and a mean of a quantity $w(x_k)$ as

$$M \equiv \sum_{k=1}^{K} x_k w(x_k) \ .$$

Suppose a quantity $w(x_k)$ is known within the accuracy of a system of adjacent sub-intervals $\{X_s\}$ : $s=1, 2, \ldots S$ : $2 \le S \le \infty$, and denote $X_{1..S} \equiv X$ so that

$$\begin{gathered} A \equiv \underline{X} \equiv \underline{X_{1..S}} = \underline{X_1} < \overline{X_1} = \underline{X_2} < \ldots \\ \ldots < \overline{X_S} = \overline{X_{1..S}} \equiv \overline{X} \equiv B \end{gathered} .$$

Evidently, at this condition, many characteristics of the interval $X$ are intervals.

The system of formulas of the sub-interval arithmetic is based mainly on the so-called "Ring of formulas"

$$\begin{aligned} & wid M_{1..S} = \\ & = \sum_{s=1}^{S} w_s \, wid X_s = \\ & = wid X_{1..S} - \sum_{s=1}^{S} w_s \sum_{m=1,..,N|m \ne s} wid X_m = \\ & = wid X_{1..S} - \sum_{s=1}^{S} wid X_s \sum_{m=1,..,N|m \ne s} w_m = \\ & = wid M_{1..S} \end{aligned}$$

where

*wid* - width;

$M_{1..S}$ - interval for the mean value;

$w_s$ - weight of $s$-th sub-interval.

The first formula of the Ring of formulas is named "formula of Novosyolov".

## 2.2. Sub-interval analysis of incomplete data

The sub-interval analysis allows exact calculations and exact evaluations at incomplete information (both exact and inexact). A **sub-interval analysis of incomplete data** (or incomplete data analysis) is a tool to calculate analogs of moments and other quantitative characteristics of a whole interval with the help of characteristics of its sub-intervals at the condition when the information on the characteristics of sub-intervals is incomplete.

When the information is incomplete, e.g. when only $wid\_X_{First}$ and $w_{First}$ of $\boldsymbol{X_{First}}$ are known, then the Ring of formulas may be rewritten as

$$
\begin{aligned}
& w_{First} wid\, X_{First} \leq \\
& \leq wid\, M_{1..S} \leq \\
& \leq wid\, X_{1..S} - w_{First}(wid\, X_{1..S} - wid\, X_{First}) = \\
& = w_{First} wid\, X_{First} + wid\, X_{1..S}\ (1 - w_{First})
\end{aligned}
$$

## 2.3. Sub-interval images

A **sub-interval image** of a two-dimensional picture (or of a $N$-dimensional object) is a sub-interval representation of the picture that requires memory by orders of magnitude smaller than the original picture ($N$-dimensional object).

### An example of a process of formation of a (standard minimal) sub-interval image

Let us choose any two-dimensional picture, e.g. a picture of a digit "4" or "5" (see Figure 10) for a postal envelope.

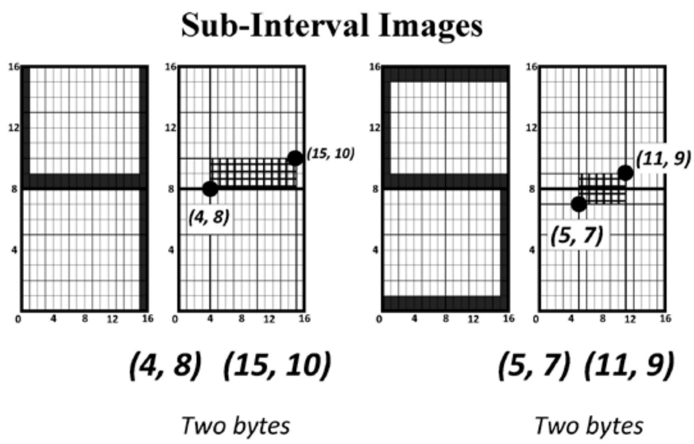


Figure 10. The digits "4" and "5" and their sub-interval images

This or any other two-dimensional picture is divided by the grid 16x16 and by 3x3 sub-intervals. Then sub-interval images are formed (see Figures 11-16). (In this particular case, due to the character of the postal envelope digits, the grid is elongated two times in the vertical direction)

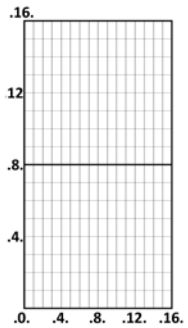


Figure 11. The grid 16x16

For the sake of simplicity let us demonstrate the process of formation of a sub-interval image on the uniform picture.

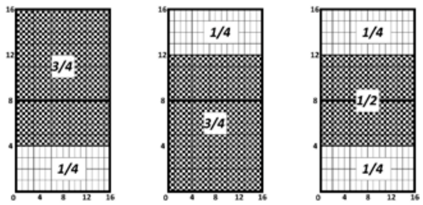


Figure 12. The formation of three vertical sub-intervals
1/4, 1/2, 1/4 on the uniform picture

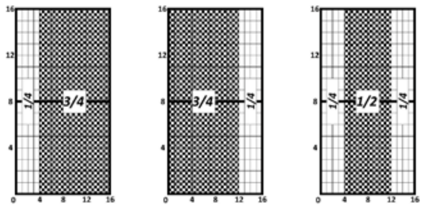


Figure 13. The formation of three horizontal sub-intervals
1/4, 1/2, 1/4 on the uniform picture

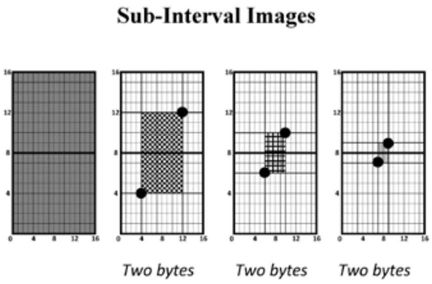


Figure 14. The sub-interval images of the uniform picture.
The central sub-interval weights equal to 1/2, 1/4, 1/8

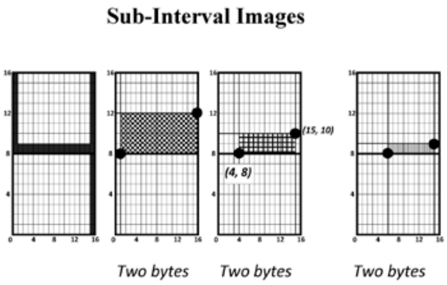


Figure 15. The sub-interval images of the digit "4".
The central sub-interval weights equal to 1/2, 1/4, 1/8

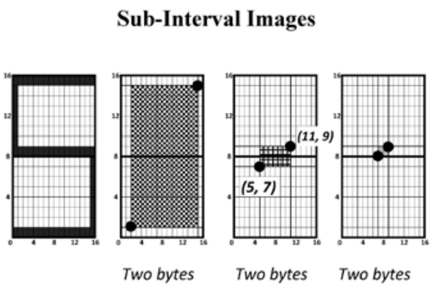


Figure 16. The sub-interval images of the digit "5".
The central sub-interval weights equal to 1/2, 1/4, 1/8

Sub-interval images allow to display the $N$-dimensional picture of image size, e.g., $1000^N$ pixels into the amount of $N$ bytes for preliminary analysis and recognition.

## 2.4. Sub-interval calculus

A **sub-interval calculus** (a sub-interval smoothing, a sub-interval averaging) is a resource for sub-interval representation of information for application of analytical methods (e.g., differential and integral calculus). The sub-interval calculus may be used for other tools and parts of the sub-interval analysis.

### Sub-interval function (sub-interval smoothing)

Let there is a quantity $f(x)$, defined on some points and/or sub-intervals of an interval $\boldsymbol{X}$ and there is a set of sub-intervals $\{\boldsymbol{X_s}\}$ : $s=1, 2, \ldots, S : S<\infty$.

Let us take an additional sub-interval $\boldsymbol{X_\delta}$ with its width $wid\_X_\delta=\delta\geq\delta_0>0$ and $\delta<<Min(wid\_X_s)$ and let us continuously move this sub-interval $\boldsymbol{X_\delta}$ from the left bound of the interval $\boldsymbol{X}$ to the right one. At that, the normalized weight $w_\delta=weight\_X_\delta/weight\_X$ of the sub-interval $\boldsymbol{X_\delta}$ will run some continuous sequence of values, defining a function in this way.

Let us define a **sub-interval function** $\boldsymbol{SI(f(x))}$ or $SI(f(x), \delta)$ (or a sub-interval smoothing, averaging) with a parameter $\delta$ as the function

$$SI(f(x)) \equiv SI(f(x),\delta) \equiv \frac{1}{\delta}\int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} f(x_1)dx_1 \ .$$

Let us define a **sub-interval function of *N*-th order** ($N<\infty$) with a parameter $\delta$ as a function $SI(f(x), N)$ or $SI(f(x), N, \delta)$ such as

$$SI(f(x),N) \equiv (\frac{N}{\delta})^N \int_{x-\frac{\delta}{2N}}^{x+\frac{\delta}{2N}} .. \int_{x_{N-1}-\frac{\delta}{2N}}^{x_{N-1}+\frac{\delta}{2N}} f(x_N)dx_1..dx_N \ .$$

Let us define the sub-interval function of zero order $SI(f(x), 0)$ as $SI(f(x), 0)\equiv f(x)$.

### Some properties of sub-interval functions

The interval of definition of the sub-interval function $SI(f(x), N)$ is less than the interval $\boldsymbol{X}$ by $\delta$.

If a sub-interval $X_s$ is more than $\boldsymbol{X_\delta}$ and the first derivative of a function $f(x)$ (or a function $f(x)$) are constant on $X_s$, then on the sub-interval $X_s$-$\boldsymbol{X_\delta}$ the sub-interval function is exactly equal to the function $f(x)$. That is

$$SI(f(x),N) = f(x) \,|\, x \in [\underline{X}_s + \frac{wid\ X_\delta}{2}, \overline{X}_s - \frac{wid\ X_\delta}{2}] \ .$$

A discrete quantity $f(x_a)$ is transformed by the smoothing (function) $SI(f(x))$ into the rectangular column of the width $\delta$ and the height $f(x_a)/\delta$.

A step is transformed by the smoothing (function) $SI(f(x))$ into the slope of the width $\delta$.

A rectangular column of the width $\varepsilon : \varepsilon<\delta$ and the height $H$ is transformed by the smoothing sub-interval function $SI(f(x))$ with the sub-interval $\boldsymbol{X_\delta}$ into the truncated pyramid with the width of its flat top $\delta\text{-}\varepsilon$, the height of its flat top $H\varepsilon/\delta$ and the width of its slope $\varepsilon$ (see Figure 17)

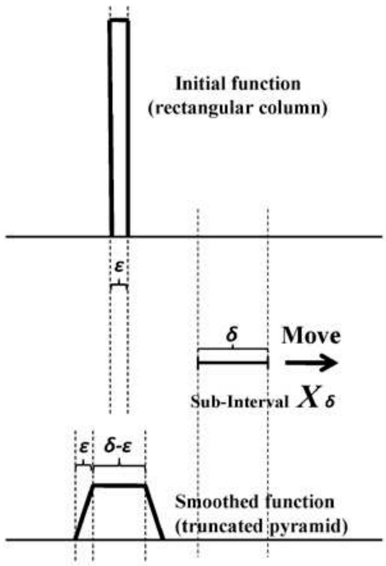


Figure 17. Sub-interval smoothing. The initial function (rectangular column) is smoothed to the smoothed function (truncated pyramid) by the sub-interval $\boldsymbol{X_\delta}$

The sub-interval function of the second order $SI(f(x), 2)$ is continuous (even for a discrete $f(x_k)$).

The $n$-th derivative of the $(n+2)$-th order sub-interval function is continuous (even for a discrete $f(x_k)$).

The fundamental property of sub-interval functions is that they may model and represent processes of extended (e.g., linear or volumetric) measurements.

## Notes

Note 1. The idea of the sub-interval layerwise analysis was created due to the advice of Astafyev (2012a) to be acquainted with Astana Economic Forum and ideas of G-Global of Nazarbayev (2012) that G-8 or G-20 may be insufficient to solve adequately some Global problems. The sub-interval calculus was created primarily as a support for the sub-interval layerwise analysis.

Note 1. Krivtsov (2012) proposed to tend the order of sub-interval functions to infinity and to use generalized functions.

Note 1. Astafyev (2012b) proposed to tend the number of sub-intervals to infinity to come to continuous calculus and to verify the principles of the sub-interval analysis.

## 2.5. Sub-interval layerwise analysis

A **sub-interval layerwise analysis** is a tool to determine and analyze levels or layers of description of an examined phenomenon.

### An example of levels of description

Global economic and policy phenomena may be described, e.g., in terms of G-2, G-3, G-8, G-20, …

### Parts of the sub-interval layerwise analysis

1) Determination and analysis of required precision
2) Determination of minimal number of required sub-intervals
3) Determination of minimal size of sub-interval
4) Determination and analysis of features for minimal representation
5) Determination and analysis of generalized characteristics

Let us consider some of these parts, namely the parts 1 and 4.

### 1) Determination and analysis of required precision
### Uniform sub interval grid

For the uniform sub interval grid the precision equals

$$wid\, M_{1..S} = \sum_{s=1}^{S} wid\, X_s\, w_s = wid\, X_k \sum_{s=1}^{S} w_s = $$
$$= wid\, X_k \times 1 = wid\, X_k = \frac{wid\, X_{1..S}}{S} .$$

### Approximation and interpolation

A) Let us approximate a function *f(x)* by the mean values of sub-intervals.

B) Let us perform the interpolation *Interp(x)* such as to connect the sub intervals' midpoints (medians) by straight lines.

If the function *f(x)* obeys the Lipshitz condition

$$| f(x_2) - f(x_1) | \le L \, | x_2 - x_1 | .$$

then, for the cases A and B the precision $\Delta_{Precis}$, that is the difference between the mean value and the function *f(x)* (accuracy of the mean) and the difference between the interpolation *Interp(x)* and the function *f(x)* for the *s*-th sub interval, is no worse than

$$\Delta_{Precis}(x \,|\, x \in X_s) \le L \frac{wid\, X_s}{2} .$$

4) Determination and analysis of
features for minimal representation

One can highlight some of the salient features that should be displayed, including extremums, fields of constancy and fields of changes. Let us consider some of these features.

Fields of constancy

If there is a field $\boldsymbol{X}$ such that, its width $wid\_X$ is more than some maximum width $wid_{Max}$

$$wid\ X > wid_{Max}$$

and the change of the sub-interval function of the second order $SI(f(x), 2)$ (the sub-interval change of the function) is less than a certain minimum value $\Delta_{Min}$

$$|\,\Delta(SI(f(x\,|\,x \in X),2))\,| < \Delta_{Min}\ ,$$

then such field should be singled out by means of not less than one sub-interval.

Fields of changes.
Fields of sharp changes

If there is a field $\boldsymbol{X}$ such that, its width $wid\_X$ is less than some minimum width $wid_{Min}$

$$wid\ X < wid_{Min}$$

and the change of the sub-interval function of the second order $SI(f(x), 2)$ (the sub-interval change of the function) is more than a certain minimum value $\Delta_{Max}$

$$|\,\Delta(SI(f(x\,|\,x \in X),2))\,| > \Delta_{Max}\ ,$$

then the boundary between sub-intervals should pass across this field.

Fields of changes.
Fields of monotonous changes

If there is a field $\boldsymbol{X}$ such that, its width $wid\_X$ is more than some maximum width $wid_{Max}$

$$wid\ X > wid_{Max}\ ,$$

the sub-interval function of the second order $SI(f(x), 2)$ is monotonous, e.g. if $x_1 \leq x_2$ then

$$SI(f(x_1),2) \leq SI(f(x_2),2)\ ,$$

and the change of $SI(f(x), 2)$ (the sub-interval change of the function) is more than a certain minimum value $\Delta_{Max}$

$$|\,\Delta(SI(f(x\,|\,x \in X),2))\,| > \Delta_{Max}\,,$$

then such field should be represented out by means of not less than one additional sub-interval.

## 3. Fields of possible use of sub-interval analysis

### 3.1. General prerequisites of use

A sub-interval needs only two coordinates of its boundaries and one number of its weight. This fact may give the gain of orders of value for information storage and for some other operations in comparison with the continuous, pointwise description. The speed of processing of sub-interval information may be by orders more high than that of point information.

The incomplete data analysis allows monitoring, planning and correcting of majority of durable processes both scientific and practical.

Sub-interval analysis may be used to deal with widespread time sub-intervals such as year, month, day, etc.

Sub-interval analysis may be used to deal with widespread space sub-intervals such as island, continent, state, province, city, etc.

### 3.2. Economics: micro- and macroeconomics, accounting and audit, econometrics, utility theory, …

Economics is one of the most natural and wide field of applications of the sub-interval analysis.

Sub-intervals in microeconomics may be presented, for example, in forms of time sub-intervals, such as months, quarters and years, and sub-intervals, sub-parts of nomenclatures of corporations, storehouses, retail trade firms, etc.

Macroeconomics is a natural application of space sub-intervals as town, city, province, state, etc. It may be represented by means of time sub-intervals such as year-sub-interval GDP. It may be represented also by, e.g., sub-intervals in Leontiev's interindustry balances.

Accounting is a natural application of time sub-intervals as months, quarters for gain, profit, etc. Additional types of sub-intervals are presented in a form of sections and sub-sections, parts and sub-parts of accounting reports. Sub-intervals of a balance sheet such as assets, liabilities and capital are divided among more fine sub-intervals such as current assets, long-term assets and so on.

An audit examination may check all operations of audited firm. Though, when it is not the overall examination, it deals usually only with parts of total volume of information. Audit data may be processed by sub-interval analysis. Sub-interval incomplete knowledge methods may help to make reliable conclusions about the whole information using only parts of this information. The tasks are to develop criteria for such conclusions.

A result of a measurement may be interpreted as only a part of a series measurements. This hypothesis may be used in the econometrics and improve its results.

The theorems of existence of restrictions may help to explain basic decision and utility theory paradoxes, such as the underweighting of high and the overweighting of low probabilities, risk aversion, etc.

### 3.3. Internet

Internet is a prospective field for sub-interval analysis. It is easy to see that Web page, screen, site, search page are, in a sense, intervals and sub-intervals.

Internet databases, E-stores, E-shops, forums, blogs, Internet search etc. may be applications of sub-interval analysis.

### 3.4. Complex systems

The important examples of complex systems are really complex firms that are represented by year's sub-interval gains and losses sheets or really complex lands, states that are represented by year-sub-interval and quarter-sub-interval GDP, public accounts, ecological and population data etc.

## Conclusions

The brief review and development of a part of the plenary report (Harin 2012b) "Sub-interval analysis and possibilities of its applications" in the Moscow Institute of Physics and Technology in the department of Innovations and High Technology is presented.

The sub-interval arithmetic and sub-interval images as the existing tools of sub-interval analysis are reviewed. The elements of the sub-interval calculus and sub-interval layerwise analysis as the new tools of sub-interval analysis are described.

It should be noted, that 3 of 5 basic tools of the sub-interval analysis were first time proposed on the conferences of the Moscow Institute of Physics and Technology in the department of Innovations and High Technology (see Harin 2011b and 2012b).

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