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30 June 2012

Online at https://mpra.ub.uni-muenchen.de/43605/
MPRA Paper No. 43605, posted 07 Jan 2013 03:46 UTC

Capital Substitution in an Industrial Revolution

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January 6, 2013

Abstract

A unified growth model is presented in which productivity growth is driven by learning-by-doing. We show that the growth rate of productivity is an increasing function of the share of capital. It is assumed that the industrial sector has a higher capital share than the agricultural sector and that the ability to substitute one output for the other in the construction of capital goods slowly rises over time. Two distinct regimes of constant growth emerge, connected by a rapid transition in which the growth rates of population and income increase by an order of magnitude, indicative of simultaneous agricultural and industrial revolutions.

Keywords: Growth, Industrial Revolution, Capital, Substitution

JEL classifications: N10, O41

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1 Introduction

The preindustrial era was characterized by low rates of productivity growth, a heavy reliance on the agricultural sector both as a source of food and as a source of raw materials for industry, and low levels of fixed capital. Since the Industrial Revolution the growth rate of total factor productivity has increased by an order of magnitude, the relative size of the agricultural sector has shrunk considerably, and the economy has become reliant on vast quantities of fixed capital in the form of machines and infrastructure. This paper presents a unified growth model that ties together these observations based on Wrigley's thesis that a key enabler of the Industrial Revolution was a shift in the source of raw materials and energy from agriculture to industry (Wrigley, 1988, 2010). In the present model, this shift is sufficient to trigger many of the changes we have come to associate with the transition to a modern economy.

The starting point for the present effort is the "Malthus to Solow" model of Hansen & Prescott (2002). In the Hansen & Prescott (HP) model there are two sectors, a land-intensive "Malthus" sector that grows slowly, and a capital-intensive "Solow" sector that grows quickly. The outputs of the two sectors are assumed to be perfect substitutes. Both sectors grow at all times but the Solow sector is not utilized until its productivity reaches a critical value, after which it gradually replaces the Malthus sector.

The HP model succeeds in describing the broad outlines of the Industrial Revolution, but has the following drawbacks:

1. The productivity takeoff associated with the Industrial Revolution is not explained but is simply assumed by assigning a higher rate of growth to the Solow sector than to the Malthus sector.

2. The productivity of the Solow sector grows *before* industrialization, even though that sector is not used until *after* industrialization.

3. The demographic transition is captured in an ad-hoc manner instead of being tied to an underlying model of preferences.

We have developed a new model of the Industrial Revolution that is inspired by the

HP model and addresses the concerns listed above. To address the first concern we have assumed that productivity growth is driven by learning-by-doing due to capital investment (Arrow, 1962). In a one-sector version of the model to be described in the next section, the growth rate of total factor productivity is an increasing function of the share of capital. The reason that the Solow sector has the potential to grow faster than the Malthus sector is that the Solow sector has a larger capital share than the Malthus sector.

To address the second concern we have assumed that the two sectors are used at all times. The outputs of the two sectors cannot be substituted for the purposes of consumption but they *can* be substituted for the purposes of capital formation. The Malthus sector produces food and some components of capital goods (e.g. horses), while the Solow sector produces manufactured goods for consumption and some other components of capital goods. Following Wrigley we assume that the output of the Malthus sector is a required component of capital prior to the Industrial Revolution. This constrains the learning-by-doing mechanism and leads to a low rate of growth in the Solow sector. We further assume that after the Industrial Revolution the outputs of the Malthus and Solow sectors become substitutes for the purposes of capital formation.[1] This closes the loop in the Solow sector, allowing Solow output to be reinvested into capital goods for the Solow sector, i.e. for the manufacturing sector, and allowing the learning-by-doing mechanism to live up to its full potential. Solow output is also used in the formation of agricultural capital, leading to an agricultural revolution.

To address the third concern, we have included a simple Cobb-Douglas model of utility. Individual preferences for consumption, investment, and childbearing are assumed constant over time. Each child requires a fixed amount of Malthus output,

[1] In the preindustrial era farm plows used iron components from the preindustrial Solow sector (horseshoes, buckles, plowshares), but relied on horses from the Malthus sector, and the productivity of horses did not grow at modern rates. The preindustrial Solow sector contained things like iron foundries that were built using iron components, but that also required working capital derived from the Malthus sector such as wood for heat, and water or horses for mechanical energy. The supply of wood represented a very serious constraint since it depended on an exploitive activity that was almost pre-agricultural in nature, with very little invested capital. Some of the most famous inventions of the Industrial Revolution, such as Watt's separate condenser for the steam engine and Cort's puddling process for iron production, had the effect of lessening the dependence of industry on land-based production. (Wrigley, 1988, 2010, Sieferle, 2001).

e.g. food. After the Industrial Revolution the Solow sector grows faster than the Malthus sector so the price of Malthus output increases, which increases the cost of childrearing. The higher cost of childrearing reduces the demand for children so the growth rate of population settles to a steady value. However, the final population growth rate is higher than it was before the Industrial Revolution.

One of the challenges in constructing a unified growth model is to capture the discontinuity associated with the Industrial Revolution without the use of an exogenous shock (Galor, 2011). One way to make the Industrial Revolution endogenous is to assume that population and income grow slowly until one of the variables of the model reaches a critical value, triggering the transition to a modern economy. In the HP model, the trigger is pulled when the productivity of the Solow sector reaches a critical value.

In the present model, a different variable is used to trigger the Industrial Revolution. We assume that a given quantity of capital (e.g. a certain number of machines) can be produced using various mixtures of Malthus output and Solow output, and the isoquant is convex. To simplify the modeling we assume a Constant Elasticity of Substitution (CES) production function for capital. The elasticity of substitution (EoS) between Malthus output and Solow output starts out small (EoS < 1), but slowly increases over time as people learn to substitute Solow output for Malthus output in the production of capital goods. The rationale for assuming a constantly increasing EoS is that it is *always* profitable for a firm to introduce a new technique that expands the possibilities for substitution. The mathematical proof of this assertion is based on the Envelope Theorem, and is described in Section 3. A property of the CES production function is that when EoS is less than 1, then for any prescribed level of output there is a minimum required level of each input. But when EoS is greater than 1 that constraint is released. An industrial revolution is triggered when EoS reaches the value of 1. The economy then undergoes a structural transformation whereby the output of the land-based Malthus sector diminishes and is replaced by the output of the capital-intensive Solow sector.

The model predicts a sharp increase in the rate of growth income *and* population when EoS surpasses the value of one. This prediction is consistent with the accepted view of historians that there were simultaneous industrial and agricultural revolu-

tions in England in the late eighteenth and early nineteenth centuries. The model predicts an Industrial Revolution because the industrial sector is released from the constraints of agriculture. But it also predicts an *agricultural revolution* because capital goods on the farm are increasingly constructed using industrial outputs. This link between industry and agriculture is opposite in direction to the usual view that agricultural improvement was a prerequisite of the Industrial Revolution.[2] Is there any evidence that farm output increased because of industrialization?

Adam Smith observed that agriculture tended to flourish in regions located near large towns and cities, and offered three possible explanations for that link.[3] First, he believed that towns spurred agricultural improvement because the demand for food was higher near towns than elsewhere. Second, he believed that towns and cities introduced good governance into the surrounding country. Third, he believed that urban merchants invested their wealth into the country because they saw opportunities that were not apparent to the country gentlemen. Smith did not detail the forms that these investment might have taken, but one possibility is that urban capitalists were introducing capital goods into the countryside.

One example where manufacturing had a positive impact on agriculture during the eighteenth century was in the provisioning of plows (Brunt, 2003). Plowshares made of iron had about twice the efficiency of the older wooden plowshares (Sieferle, 2001), and were supplied by blacksmiths working in the neighboring towns. Brunt (1997) has conducted a cross-sectional study of wheat yields in the late eighteenth century, looking to uncover significant drivers of productivity. He has found that the most significant improvement factors, all of which were forms of capital investment, were the planting of turnips, the use of seed drills, the use of fertilizers such as marl and lime, and the use of drainage pipes. Turnips assisted in the provisioning of humus to the soil, and can be thought of as a purely agricultural innovation. The other three factors were linked to industry. Seed drills were a manufactured product that

[2] A common view is that labour was released from the countryside and there was a growing surplus of food to support the growing urban population. The best-known proponent of this line of reasoning is Rostow (1960).

[3] Chapter IV, "How the Commerce of the Towns Contributed to the Improvement of the Country", Smith (1776). Jacobs (1970) postulated that cities came before agriculture and that cities have always been strong drivers of agricultural growth.

allowed for more efficient weeding. Marl and lime had to be transported from distant locations and so benefited from the development of the canal and railway networks (Mathew, 1993). Regarding drainage pipes, Brunt notes that they were made of clay in the eighteenth century, but were replaced with much cheaper ceramic pipes in the nineteenth century, leading to a large increase in use. Taking a longer point of view, agriculture experienced huge increases in productivity in the twentieth century after the introduction of tractors, pesticides and industrial fertilizers.

The model presented in this paper is broadly consistent with the view of Crafts (1995), who has analyzed the applicability of recent growth models to the Industrial Revolution. In Crafts view, the most promising approach is to combine exogenous shocks (inventions) with learning-by-doing effects (innovation). We have modeled increases in the elasticity of substitution between land-based output and industrial output as an exogenous process, and have then allowed the mechanism of learning-by-doing to follow through with its far-reaching effects on growth. The model is also consistent with the theme of Allen's study of the blast furnace industry, which showed that capital investment was a key driver of productivity improvements through learning effects (Allen, 1983).

As Lucas (2008) has pointed out, it is widely accepted amongst economists that the process of industrialization is an ongoing intellectual achievement. One of the defining characteristics of the Industrial Revolution is that it introduced a sharp discontinuity in the rate of growth of human knowledge. If we accept the premise that people were just as mentally capable 2,000 years ago as they are today, this discontinuity represents a deep mystery. The goal of unified growth theory is to develop a model that captures the dynamics of both the preindustrial economy and the modern economy without invoking any special exogenous shocks to explain the Industrial Revolution. The present model fits into that mold because the dynamical equations and their parameters are unchanging over time, yet a discontinuity emerges. The implication of this paper is that people have been striving to substitute the products of their hands for the products of nature for millenia, and this process finally reached a threshold of sustained capitalist learning about two hundred years ago.

The plan of the paper is as follows. In the next section, we briefly introduce a sim-

ple one-sector model designed to motivate the development of the model. Section 3 contains a description of the unified growth model. In section 4 we write down analytic expressions for the growth rates of population and income in the preindustrial and modern regimes. These expressions are used to derive predictions that are compared with historical data. We also calibrate the parameters of the model to selected historical data, run a simulation of the model, and then make further predictions that are tested against historical data. Finally, section 5 discusses the limitations of the model and some possible extensions, and concludes. An Appendix is included that describes the asymptotic behavior of the model in the two historical regimes.

2 Malthus to Solow via Arrow

Amongst the many changes to occur during the Industrial Revolution, the two most relevant to the present discussion were the increase in the share of capital and the large increase in the rate of growth of Total Factor Productivity (TFP). The purpose of this section is to show that these two phenomena can be linked using a variant of Arrow's learning-by-doing model (Arrow, 1962). A simple toy model will demonstrate that even a modest increase in the share of capital can lead to a productivity takeoff.

The production function is

$$Y = AK^{\alpha}N^{1-\alpha}, \quad 0 < \alpha < 1, \tag{1}$$

where Y is output, A is TFP, K is capital, and N is labour. For the purposes of this exercise we will assume that labour is held fixed. The focus of our attention will be on the relationship between A and K. Let us also assume that savings are a fixed fraction s of output:

$$\dot{K} = sY - \delta K, \tag{2}$$

where δ is the rate of depreciation of capital. Productivity grows exogenously at a small rate g_0 and also grows due to gross investment in capital as in Arrow's model.

Since savings are a fixed fraction of output, we have

$$\frac{d}{dt}\ln A = g_0 + \xi \frac{d}{dt}\ln Y, \tag{3}$$

where ξ is a parameter that captures the spillover effects of learning due to capital investment.

We wish to study the steady-state growth path associated with this model. We assume that all quantities grow exponentially at constant rates which we label $\{g_x\}$, where $\{x\}$ stands for whatever quantities are under consideration. Dividing both sides of equation (2) by K we have $g_K = sY/K - \delta$. Since g_K is assumed constant the capital/output ratio must be constant. Therefore, from (1) we find

$$g_A = (1-\alpha)g_K, \tag{4}$$

and from equation (3) we see that

$$g_A = g_0 + \xi g_K. \tag{5}$$

Combining (4) and (5), we obtain the following expression for g_A

$$g_A = \frac{1-\alpha}{1-\alpha-\xi}g_0, \quad \xi < 1-\alpha. \tag{6}$$

Finally, the growth rate of wages is

$$g_w = \frac{1}{1-\alpha-\xi}g_0, \quad \xi < 1-\alpha. \tag{7}$$

Notice that the growth rate of TFP is proportional to g_0, with the constant of proportionality being greater than 1 when $\xi > 0$. Arrow's learning-by-doing mechanism is acting as a lever on the exogenous growth rate.[4] Notice also that the growth rates of TFP and wages are monotonically increasing in α (thereby linking growth rates with the share of capital) and are essentially unbounded.

[4]This model avoids the kind of "knife-edge" condition normally required in a purely endogenous model.

Consider the following numerical example: $g_0 = 0.1\%$ per annum, $\xi = 0.55$ and $\alpha = 0.1$, the latter corresponding to the share of capital in the Malthus sector of Hansen & Prescott's (2002) model. The resulting growth rate of wages is 0.29% per annum. If we increase α to 0.4 (the capital share in the Solow sector of Hansen & Prescott's model) the growth rate of wages increases to 2% per annum.

We now proceed to describe the unified growth model, incorporating the above learning mechanism in a two-sector framework.

3 The Model

3.1 Production

Following Hansen & Prescott (2002), we define two sectors, a "Malthus" sector, which produces Y_{Mt} at time t, and a "Solow" sector, which produces Y_{St} at time t. The production functions are:

$$Y_{Mt} = K_{Mt}^{\phi} N_{Mt}^{\mu}, \tag{8}$$

$$Y_{St} = K_{St}^{\theta} N_{St}^{1-\theta}, \tag{9}$$

where K_{Mt} and K_{St} are the quantities of capital employed in the Malthus sector and Solow sector, respectively, and N_{Mt} and N_{St} are the quantities of labour employed in the Malthus sector and Solow sector, respectively. It is assumed that $\phi < \theta$, reflecting that the capital share in the Malthus sector is lower than the capital share in the Solow sector. It is also assumed that $\phi + \mu < 1$, reflecting the hidden presence of land in the Malthus sector, which is normalized to 1. There are no productivity factors in the above production functions. Instead, productivity is embodied in capital, as will be explained in the next subsection.

The production side of the economy solves the following optimization problem:

$$\max \left\{ p_t Y_{Mt} - w_{Mt} N_{Mt} - r_{Kt} K_{Mt} - r_{Lt} L_t \right\}, \quad L_t = 1, \tag{10}$$

$$\max \left\{ Y_{St} - w_{St} N_{St} - r_{Kt} K_{St} \right\}, \tag{11}$$

where w_{Mt} is the wage in the Malthus sector, w_{St} is the wage in the Solow sector, r_{Kt} is the rent on capital, r_{Lt} is the rent on land, and p_t is the price of goods produced by the Malthus sector. The output of the Solow sector acts as numeraire.

We assume that capital is perfectly mobile so there is a unique rent on capital. However, we distinguish two different wages w_{Mt} and w_{St}. The reason has to do with our treatment of land. Our focus is on capital and labor, so consistent with common practice in this field we assume that laborers in the Malthus sector own all of the land that they use for production. Hence, their total earnings consist of wages plus rents. We also assume that land has no value, so labour will freely migrate until the total income earned in the Malthus sector (wages plus land rents) is equal to the wage earned in the Solow sector.[5] Let us define total earnings y_t to be

$$y_t = w_{Mt} + \frac{r_{Lt}}{N_{Mt}} = w_{St}. \tag{12}$$

Competitive firms will then hire quantities of labour and capital such that

$$r_{Kt} = \frac{\partial Y_{St}}{\partial K_{St}} = \frac{\partial Y_{Mt}}{\partial K_{Mt}} p_t, \tag{13}$$

$$y_t = \frac{\partial Y_{St}}{\partial N_{St}} = \frac{Y_{Mt} - r_{Kt} K_{Mt}}{N_{Mt}} p_t, \tag{14}$$

or, in terms of labour and capital quantities:

$$r_{Kt} = \phi \frac{N_{Mt}^{\mu}}{K_{Mt}^{1-\phi}} p_t = \theta \frac{N_{St}^{1-\theta}}{K_{St}^{1-\theta}}, \tag{15}$$

$$y_t = (1-\phi) \frac{K_{Mt}^{\phi}}{N_{Mt}^{1-\mu}} p_t = (1-\theta) \frac{K_{St}^{\theta}}{N_{St}^{\theta}}. \tag{16}$$

3.2 Capital

Capital goods are constructed using two types of material: X_{Mt}, which is directly produced by the Malthus sector, and X_{St}, which is directly produced by the Solow

[5] This modeling choice avoids the need to iteratively solve for the price of land, as was necessary in Hansen & Prescott (2002).

sector.[6] That is, X_{Mt} is some portion of past output as produced according to equation (8), and X_{St} is some portion of past output as produced according to equation (9). The Malthus and Solow materials are combined to form total capital $K_t = K_{Mt} + K_{St}$ via a Constant Elasticity of Substitution (CES) production function:

$$K_t = \left[(1-\gamma)\left(A_{Mt}X_{Mt}\right)^{\rho_t} + \gamma\left(A_{St}X_{St}\right)^{\rho_t}\right]^{\frac{1}{\rho_t}}, \tag{17}$$

where A_{Mt} is the Malthus capital-augmenting productivity, A_{St} is the Solow capital-augmenting productivity, γ is the share of the Solow sector in capital goods production, and ρ_t is a monotonic increasing function of the Elasticity of Substitution (EoS) σ_t:

$$\rho_t = 1 - \frac{1}{\sigma_t}. \tag{18}$$

The parameter ρ_t lies between $-\infty$ and 1, while σ_t lies between 0 and ∞. Note that there is no overall productivity factor outside the square brackets in equation (17). This means that if σ_t is low, the growth-rate of productivity is constrained by the slowest-growing productivity factor. We can offer an energy interpretation of equation (17) inspired by Wrigley (1988, 2010). Capital goods are low-entropy systems that require energy in the form of work for their construction, ongoing maintenance and operation. It is assumed that energy is embodied in $A_{Mt}X_{Mt}$ and/or $A_{St}X_{St}$. Traditionally, capital goods may have been produced using Solow output ($A_{St}X_{St}$) but the energy was embodied in $A_{Mt}X_{Mt}$. During industrialization it became embodied in $A_{St}X_{St}$ as well, allowing substitution to take place. Historical examples include coal replacing wood as a source of heat for the iron industry, and steam engines replacing water wheels and horses for mechanical energy.

We assume that the suppliers of capital minimize the cost of each unit of capital by solving

$$\min\left\{p_t\, x_{Mt} + x_{St}\right\}, \tag{19}$$

subject to the constraint

$$0 = 1 - K_t\left(X_{Mt} = x_{Mt},\, X_{St} = x_{St}\right), \tag{20}$$

[6] Working capital can be thought of as a capital good having a high rate of depreciation.

where K_t is given by (17). In a competitive economy, the price of capital q_t is equal to its marginal cost. Since the production function for capital is first-order homogeneous, marginal cost equals average cost. Defining C_t^* to be the minimum cost, x_{Mt}^* to be the optimal quantity of Malthus material and x_{St}^* to be the optimal quantity of Solow material, we have

$$q_t = C_t^* = p_t\, x_{Mt}^* + x_{St}^*. \tag{21}$$

The Lagrangian for this optimization problem is

$$\mathcal{L} = p_t x_{Mt} + x_{St} + \lambda(1 - K_t), \tag{22}$$

and the first-order conditions for cost minimization are

$$\frac{\partial \mathcal{L}}{\partial X_{Mt}} = 0, \quad \frac{\partial \mathcal{L}}{\partial X_{St}} = 0, \quad 1 - K_t = 0. \tag{23}$$

The solution for q_t, x_{Mt}^* and x_{St}^* is

$$q_t = \left\{ (1-\gamma)^{\frac{1}{1-\rho}} \left(\frac{p_t}{A_{Mt}}\right)^{-\frac{\rho}{1-\rho}} + \gamma^{\frac{1}{1-\rho}} \left(\frac{1}{A_{St}}\right)^{-\frac{\rho}{1-\rho}} \right\}^{-\frac{1-\rho}{\rho}}, \tag{24}$$

$$x_{Mt}^* = q_t^{\frac{1}{1-\rho}} (1-\gamma)^{\frac{1}{1-\rho}} \left(\frac{p_t}{A_{Mt}}\right)^{-\frac{\rho}{1-\rho}} \frac{1}{p_t}, \tag{25}$$

$$x_{St}^* = q_t^{\frac{1}{1-\rho}} \gamma^{\frac{1}{1-\rho}} \left(\frac{1}{A_{St}}\right)^{-\frac{\rho}{1-\rho}}. \tag{26}$$

We will assume that capital completely depreciates at the end of each time step and that a time step equals 35 years, consistent with Hansen & Prescott (2002). The annualized interest rate is then

$$i = \left(\frac{r_{Kt}}{q_t} - 1\right)^{\frac{1}{35}} - 1, \tag{27}$$

where r_{Kt} is given by equation (15).

3.3 Preferences

Building on Hansen & Prescott (2002), we assume an overlapping generations model. Each person lives for two periods of time, and the population includes a mix of young people and old people. Young people work to earn an income, which they use to purchase three things: manufactured goods from the Solow sector for their own consumption, capital goods for investment purposes, and Malthus output (e.g. food and child care) to support the raising of children. It is assumed that old people receive enough Malthus output in childhood to sustain them for life. Hence, old people consume Solow goods only, which are paid for by the returns on their investments. Land has no value, so each generation simply confiscates it upon their entry into the Malthusian labour force.

We assume the following utility function

$$u_t = \alpha \log c_{1,t} + \beta \log c_{2,t+1} + (1 - \alpha - \beta) \log n_t, \tag{28}$$

where $c_{1,t}$ is consumption in the first period (the working period), $c_{2,t+1}$ is consumption in the second period, and n_t is the number of children born to each worker at the end of the time step (this is the new element not included in Hansen & Prescott's utility function). Children become part of the labor force in the next time step $t+1$. Assuming that children require one unit of output from the Malthus sector, which is sufficient to sustain them throughout life, the budget constraints are

$$y_t = c_{1,t} + q_t k_{t+1} + p_t n_t, \tag{29}$$

$$c_{2,t+1} = r_{t+1} k_{t+1}, \tag{30}$$

where y_t is the total income earned by young people in period 1 (wages and rent on land), k_{t+1} is the invested capital (savings) for period 2, and q_t and p_t are the prices of capital and Malthus output, respectively.

The solution to the above optimization problem is

$$c_{1,t} = \alpha y_t, \tag{31}$$

$$k_{t+1} = \beta \frac{y_t}{q_t}, \tag{32}$$

$$n_t = (1 - \alpha - \beta) \frac{y_t}{p_t}. \tag{33}$$

The first result says that young workers spend a fixed percentage of their earnings on output from the Solow sector. The second result states that savings are a fixed fraction of income. The third result describes Malthusian population dynamics: the higher the earnings and the lower the price of output from the Malthus sector, the more children.

The above results can be used to define real earnings. Consistent with our assumption of Cobb-Douglas utility, we define the cost-of-living index as the geometric average of the factor prices, with expenditure shares used as weights.[7] Real earnings y_{Rt} are then defined as nominal earnings y_t divided by the cost-of-living index:

$$y_{Rt} \equiv \frac{y_t}{q_t^{\beta} p_t^{1-\alpha-\beta}}. \tag{34}$$

We can also define earnings in terms of food y_{Ft} as:

$$y_{Ft} \equiv \frac{y_t}{p_t}. \tag{35}$$

3.4 Productivity Growth

We now specify a dynamical process for the capital-augmenting productivity parameters A_{Mt} and A_{St}, and for the elasticity of substitution σ_t. Two processes are assumed for A_{Mt} and A_{St}. First, productivity grows exogenously at some small rate g_0. Second, we assume that productivity also increases as a side effect of gross capital investment (Arrow, 1962). The relative changes in $A_{M\,t}$ and $A_{S\,t}$ are thus

[7] Clark (2005) uses a weighted geometric average in his construction of a cost-of-living index covering the years 1209-1869. Allen (2001) uses both geometric and arithmetic averages to compute inflation indices for several cities in Europe covering the years 1350 to 1750 and finds little difference in their values.

driven by two factors, the first related to exogenous growth and the second related to total gross investment:

$$\frac{A_{M\,t+1}}{A_{M\,t}} = e^{g_0}\left(\frac{\tilde{X}_{M\,t+1}}{\tilde{X}_{M\,t}}\right)^{\xi}, \tag{36}$$

$$\frac{A_{S\,t+1}}{A_{S\,t}} = e^{g_0}\left(\frac{\tilde{X}_{S\,t+1}}{\tilde{X}_{S\,t}}\right)^{\xi}, \tag{37}$$

where ξ is a new parameter of $\mathcal{O}(1)$ that captures the spillover effects of learning, and

$$\tilde{X}_{M\,t} = \sum_{t_i \leq t} X_{M\,t_i}, \tag{38}$$

$$\tilde{X}_{S\,t} = \sum_{t_i \leq t} X_{S\,t_i}. \tag{39}$$

Here $X_{M\,t_i}$ and $X_{S\,t_i}$ are, respectively, the quantities of Malthus output and Solow output used in the construction of capital at time t_i:

$$X_{M\,t_i} = \frac{\beta y_{t_i} N_{t_i} x^*_{m\,t_i}}{q_{t_i}}, \tag{40}$$

$$X_{S\,t_i} = \frac{\beta y_{t_i} N_{t_i} x^*_{s\,t_i}}{q_{t_i}}. \tag{41}$$

Here, we have made use of equation (32) and multiplied by $x^*_{m\,t_i}$ and $x^*_{s\,t_i}$, which are the Malthus and Solow components, respectively, of each unit of capital, and we have also multiplied by N_{t_i} to obtain total quantities. The above model of learning assumes that the efficiency of each component of capital grows in direct response to its use. Note that the parameters driving productivity growth (g_0, ξ) are identical in both sectors.

The final key parameter to consider is σ (or ρ). Since $0 < \sigma < \infty$, it is natural to assume an exponential growth process:

$$\sigma_t = \sigma_0 e^{g_\sigma t}, \tag{42}$$

where $\sigma_0 < 1$ and g_σ is a new growth parameter.

The rationale for assuming a constantly increasing σ_t is that it is always (temporarily) profitable for a capital-producing firm to introduce a new technique that expands the opportunities for substitution. To show this, it suffices to see that the cost of capital is a declining function of ρ_t (recall that ρ_t is monotonic increasing in σ_t). Application of the Envelope Theorem to the optimization problem in section 3.2 results in

$$\frac{\partial C^*}{\partial \rho} = \frac{\partial \mathcal{L}}{\partial \rho} = -\lambda \frac{\partial K}{\partial \rho}.$$

K is of the form of a generalized mean, which has the property that it is a monotonic increasing function of ρ.[8] The Lagrange multiplier λ is equal to C_t^*, which is positive.[9] Hence

$$\frac{\partial C^*}{\partial \rho} < 0.$$

Therefore, it is always profitable for a firm to introduce a new capital-goods production technique that embeds a higher elasticity of substitution than is currently prevailing.

3.5 The Equilibrium Path

The purpose of this section is to complete the set of equations required to find the equilibrium prices and allocations at each time step, and to describe the changes in the key variables over time. The initial conditions for the model are the quantities of labour and capital at time zero: N_0 and K_0. During each time step t, the economy optimally allocates labour and capital to the Malthus and Solow sectors. One complication is that in solving for the equilibrium allocations, one must take into account the optimal mix of materials required in the construction of capital goods to be used in the next time step. The resulting equilibrium income earned by young workers is the key determinant of the quantities of labour and capital in the next time step (N_{t+1} and K_{t+1}) via the preference equations (31) - (33). The end

[8] The proof is contained in Hardy *et al.*, 1934, p. 26. See also La Grandville, 2009 (Appendix of Chapter 4).

[9] One can use the first-order conditions listed in equation (23), to show that $\lambda = q_t$ and by Euler's theorem for homogeneous functions, we have $q_t = C_t^*$.

result is that we can take the total quantities of labour and capital as 'given' at the beginning of each time step.

The market-clearing conditions are

$$N_t = N_{Mt} + N_{St}, \tag{43}$$

$$K_t = K_{Mt} + K_{St}, \tag{44}$$

$$Y_{Mt} = (1 - \alpha - \beta)\frac{y_t}{p_t}N_t + \beta\frac{y_t}{q_t}N_t x^*_{Mt}, \tag{45}$$

$$Y_{St} = \alpha y_t N_t + r_{Kt}K_t + \beta\frac{y_t}{q_t}N_t x^*_{St}. \tag{46}$$

The first two conditions listed above simply equate total labour and capital with the supply of those factors at the beginning of each time step. The last two conditions equate the supply and demand of final outputs. Equation (45) says that total Malthus output is equal to the amount required to support children plus the amount required in the construction of capital goods. Equation (46) says that total Solow output is equal to total consumption by young workers plus total consumption by old people (investment returns) plus the amount required for capital goods. We now have sufficient conditions to determine the equilibrium solution at each time step.[10]

Finally, from equations (32) and (33) the quantities of labour and capital at time $t + 1$ are

$$N_{t+1} = (1 - \alpha - \beta)\frac{y_t}{p_t}N_t, \tag{47}$$

$$K_{t+1} = \beta\frac{y_t}{q_t}N_t. \tag{48}$$

The model is now fully specified.

4 Model Predictions and Tests

Figures 1 to 3 summarize the behavior of the model, showing how the economy evolves in response to increases in the elasticity of substitution (EoS) starting from

[10] In fact, we have one too many equations! However one of (45) or (46) is redundant.

the Malthusian epoch ($\sigma < 1$), through an Industrial Revolution ($\sigma = 1$), to the modern era ($\sigma > 1$). The figures are based on a simulation of the model using parameters listed in Table 1. We see that there are two distinct regimes where the growth rates becomes asymptotically constant, and these two regimes are connected by a rapid transition in which the growth rates of population and income increase by an order of magnitude. This overall pattern is a general characteristic of the model, and is not an artifact of the particular set of parameters listed in Table 1.

In the following we provide some analytic solutions that can be used to test the model against historical data. We then calibrate the parameters of the model in order to run a numerical simulation and explore the quantitative and qualitative aspects of the model's behavior.

4.1 Analytic Solutions

It is possible to derive analytic expressions for the asymptotic growth rates in the two regimes (preindustrial and modern). This may be accomplished by assuming constant growth rates of y_t, p_t, K_t, N_t, q_t, A_{Mt} and A_{St}, etc., represented for example by $y_t \sim e^{g_y t}$, thereby introducing new symbols g_y, g_p, g_K, g_N, g_q, g_{A_M}, g_{A_S} etc.

The Appendix contains the derivation of steady-state growth rates, which are summarized here. The growth rates of population before and after the Industrial Revolution are, respectively:

$$g_N^{before} = \frac{g_0}{\frac{1-(1+\xi)\theta}{\phi}(1-\mu-\phi) + (1-\xi)(1-\theta-\mu) + (1+\xi)(1-\mu)\frac{\theta-\phi}{\phi} - \xi}, \quad (49)$$

$$g_N^{after} = \frac{g_0}{\frac{1-\mu-\phi}{\phi}[1-\theta(1+\xi)] - \xi}. \quad (50)$$

In order that the growth rates be positive, the parameters must be constrained such that the denominators in the above expressions are positive. All other growth rates can be expressed in terms of these population growth rates. The following expres-

sions hold in both regimes so we can drop the "before" and "after" superscripts:

$$g_y = g_{GDPc} = \frac{\theta\left(1-\mu-\phi\right)}{\phi} g_N, \tag{51}$$

$$g_p = g_y, \tag{52}$$

$$g_q = -\frac{1-\theta}{\theta} g_y, \tag{53}$$

$$g_{yR} = \left(\alpha + \frac{\beta}{\theta}\right) g_y. \tag{54}$$

Here g_y is the growth rate of nominal earnings, g_{GDPc} is the growth rate of GDP per capita, g_p and g_q are the growth rates of the prices of Malthus output and capital, respectively, and g_{yR} is the growth rate of real earnings, which is the same as the growth of of real GDP per capita. Note that a unit of time is thirty-five years. However, the above four expressions are also valid when the time unit is one year.

As mentioned previously, labour earnings in units of food are constant in both regimes. From equation (47) the wage expressed in units of food is

$$y_F = \frac{1}{1-\alpha-\beta} e^{g_N}, \tag{55}$$

from which we have

$$\frac{y_F^{after}}{y_F^{before}} = \exp\left(g_N^{after} - g_N^{before}\right). \tag{56}$$

4.2 Tests of the Analytic Solutions

The most important prediction of the model is that the growth rate of earnings and GDP per capita is positive at all times and is a fixed multiple of the population growth rate. In other words, the standard of living improves at all times. Since the population growth rate was much lower in preindustrial times than it is today, this prediction implies a low rate of growth of the standard of living in preindustrial times.

At first this prediction of a growing standard of living in preindustrial times would seem to contradict the logic of a Malthusian economy, which has much empirical

support (Ashraf & Galor, 2011, Clark, 2007). Recall that our model is Malthusian in the sense that population growth is a linear function of $y_{Ft} = y_t/p_t$ (see equation (33)). The Malthusian mechanism leads to stable levels of income in units of food (y_{Ft}) as seen in figure 2. However income in units of manufacturing output (y_t) grows, and in fact grows at the same rate as p_t. The presence of a Solow sector in the preindustrial epoch allows income to grow in the context of a Malthusian economy, which accords with our intuition that there must have been individual material progress even before the Industrial Revolution. An English citizen circa 1700 was surely richer in material terms than a hunter gatherer living in the same region 5,000 years earlier.

According to equations (51) and (54), the growth rate of real GDP per capita is a fixed multiple of the population growth rate. Given that we have a fairly good handle on population growth rates in both preindustrial and modern times, and given that we can measure the growth rate of income in the modern era, we can infer the growth rate of income in the preindustrial epoch. The latter can be checked against historical data shown in figure 4.

We assumed the following per-annum population growth rates

$$g_N^{before} = 0.05\%,$$
$$g_N^{after} = 0.5\%.$$

The first number is close to the rate of growth of world population between 0 and 1700 based on data from Maddison (2007).[11] The figure of 0.5% for the modern era is in the middle of the range of values for industrialized countries during the twentieth century.[12]

[11] A simple computation using Maddison data gives a growth rate of 0.058% per annum. A similar exercise using world population data from Kremer (1993) over the period -10,000 BCE (after the start of the Holocene) to 1700 gives a growth rate of 0.043% per annum. The population data from Broadberry *et al.* (2010) shown in figure 4 implies a growth rate 0.034% per annum. The challenge in using data from this period is that the English economy was not on a steady-state growth path during the late middle ages. The population suffered a heavy blow in the second half of the fourteenth century due to plague and famine, and did not fully recover until the mid-seventeenth century. Nevertheless the *overall* population growth rate between 1270 and 1700 of 0.034% per annum is close to the value of 0.05% assumed above.

[12] Based on Maddison data. At the high end of the range we have the "Western Offshoots"

We assumed a growth rate of income of 1.9% per annum for the modern era.[13] The corresponding ratio of income growth rate to population growth for the modern era is 3.8. This implies an income growth rate of 0.19% per annum for the preindustrial era. This figure is close to the estimate of 0.17% per annum over the years 1270 - 1700 based on Broadberry *et al.* data shown in figure 4. Hence the predicted growth rate of income in the preindustrial era is consistent with historical data.

According to equation (52), the model predicts that p_t should grow at a steady rate in both the preindustrial era and the modern era. This prediction is verified for the preindustrial era in figure 4, which shows a price series of food expressed in non-food terms (a proxy for p_t) between 1270 and 1830.[14] The growth rate of food prices between 1270 and 1700 is 0.13% per annum, which is close to the empirical growth rate of income of 0.17%. According to equation (52) the growth rate of p_t should be the same as the growth rate of *nominal* income, which is not the same as the growth rate of real income. So our model is only consistent with historical data if we assume the following (see equation 54):

$$\left(\alpha + \frac{\beta}{\theta}\right) \approx 1. \tag{57}$$

Taking $\beta = 0.2$, which is the approximate rate of savings in the U.S. (Jones, 2002) and assuming that $\theta = 0.4$ as in HP (2002), this implies $\alpha = 0.5$. This means that half of all earnings are spent on Solow goods, i.e. manufactured goods. This prediction also implies that 30% of earnings are spent on childcare, which includes expenditures on food. All we can say at this point is that our model is consistent with historical data if 30% of earnings are spent on childcare, so that is the prediction of the model that remains to be verified. In the remainder of this paper we will assume that equation (57) holds, i.e. that nominal earnings are equal to real earnings.

(U.S., Canada, Australia and New Zealand) which had growth rates above 1% per annum during the twentieth century. However those rates were heavily influenced by immigration. At the low end of the range we have Germany and the U.K., each with growth rates of about 0.36% per annum.

[13] According to Maddison data, the growth rate of real per-capita income during the twentieth century was 1.9% for the U.S. and 1.88% for Western Europe.

[14] The food prices represent the terms of trade between agriculture and industry, and is courtesy of R. Allen, Oxford University. A price series published by O'Brien (1985) shows similar trends between 1500 and 1830, but does not cover the period prior to 1500.

Equation (52) also predicts that p_t should grow at the same rate as income in the modern era. Recall that the Malthus sector is responsible for the production of children (see equation 33). In the modern era the main cost of children-rearing is the cost of labour (or the opportunity cost of not working), which also rises with income. So this prediction appears to be in accordance with observation, provided that we interpret the Malthus sector as including a child-care component.

Assuming $\theta = 0.4$, equation (53) predicts that the rate of deflation of the price of capital goods should be 50% higher than the rate of growth of earnings (in both regimes). This prediction remains to be verified against historical data.

Finally, we can test the prediction that wages in terms of food rose through the Industrial Revolution, as implied by equation (56). Using the population growth rates listed above (and remembering to scale them up by the factor of 35), we have the prediction

$$\frac{y_F^{after}}{y_F^{before}} = 1.17.$$

According to Broadberry (1997, Table 2) the average grain-wage in England between 1550 and 1700 was 5.23 kg of wheat per day. That rose to an average of 8.6 kg of wheat per day between 1800 and 1849, which implies a ratio of $8.6/5.23 = 1.64$. The empirical ratio is greater than 1 as predicted by the model, however the magnitude of the empirical ratio is larger than the predicted ratio.

4.3 Simulating the Model

In order to extract further predictions from the model we must resort to numerical simulation. The first task is to choose parameter values. There are nine parameters, which are listed in Table 1. As mentioned in the previous section, we choose $\beta = 0.2$ based on Jones (2002) and $\theta = 0.4$ based on HP (2002). The latter implies a labour share of 0.6 in the Solow sector. We also set the labour share in the Malthus sector to be equal to 0.6 ($\mu = 0.6$), consistent with HP. We assume $\alpha = 0.5$ as per the discussion in the previous section.

HP assume that the capital share in the Malthus sector (ϕ) is 0.1. We have been

unable to reconcile that value with a 10-fold increase in the population growth rate over the Industrial Revolution. Instead we have landed on $\phi = 0.04$, based on the following reasoning. First we set $\xi = 1$, which means that a percentage increase in a component of gross capital, either Malthus or Solow, leads to the same percentage increase in productivity of that component. We then set the ratio of equation (49) to (50) equal to 10, as implied by the population growth figures listed in the previous section. Solving, we obtain $\phi = 0.04$. To satisfy the requirement that the denominators of (49) and (50) are positive we must have $\xi < 1.174$, hence $\xi = 1$ is acceptable. The resulting value of ϕ (0.04) is lower than that used by HP (0.1), consistent with our idea that there is a missing labour-intensive child-care sector in the present model.

Next, we calibrate g_0 to the growth rate of population in the modern era using equation (50), resulting in $g_0 = 0.004$ per annum. We also set $g_\sigma = 0.004$ per annum. This value has no impact on the behavior of the model other than dictating the speed of the Industrial Revolution. Finally, we arbitrarily set $\gamma = 0.5$, which has no consequence for model behavior. We are now ready to simulate the model.

In order to run the simulation, we must solve a set of recursive-algebraic equations at each time step, consisting of two recursive equations, (47) and (48), coupled to the algebraic constraints (15), (16) and (43)-(45). In addition, the productivity parameters evolve according to equations (36) and (37), while the elasticity of substitution increases as in (42), being the ultimate driver for the changes in the system dynamics. Note that we choose the starting time of our simulation so that $\sigma = 1$ at $t = 1780$, coinciding with the Industrial Revolution but this choice is arbitrary and simply shifts the results in time.

We now introduce two new variables so as to derive a solution algorithm. These variables represent the fraction of population in the Solow sector

$$\eta_t = \frac{N_{st}}{N_t} \tag{58}$$

and the fraction of capital in the Solow sector

$$\kappa_t = \frac{K_{st}}{K_t}. \tag{59}$$

At the beginning of each time step, we know the values of $A_{M\,t}$, $A_{S\,t}$, N_t and K_t. In order to determine those quantities at time $t+1$ using (47) and (48), we need to know y_t, p_t and q_t. We note that y_t can be obtained from η_t and κ_t using equation (16). Hence, we need to solve the model equations for η_t, κ_t, p_t and q_t. The price of capital q_t can be expressed in terms of p_t using (24). The price of Malthus output p_t can in turn be obtained from η_t and κ_t using (15) in the form

$$p_t(\kappa_t, \eta_t) = \frac{\theta N_t^{1-\theta-\mu}}{\phi K_t^{\phi-\theta}} \frac{\eta_t^{1-\theta}(1-\kappa_t)^{1-\phi}}{(1-\eta_t)^{\mu}\kappa_t^{1-\theta}}. \tag{60}$$

We can also write κ_t in terms of η_t by dividing (15) by (16) to find

$$\kappa_t(\eta_t) = \frac{E(\eta_t)}{1+E(\eta_t)} \tag{61}$$

with $E = \frac{(1-\phi)\theta\eta_t}{\phi(1-\theta)(1-\eta_t)}$. Hence, we have just one remaining unknown: η_t, which can be computed by solving the following equation, derived from (8), (16) and (45):

$$\begin{aligned} G(\eta_t, p_t, q_t(p_t), x^*_{Mt}(p_t, q_t(p_t))) &= \frac{1}{p_t} - \frac{(1-\alpha-\beta)(1-\phi)}{p_t(1-\eta_t)} - \frac{\beta(1-\phi)x^*_{Mt}}{q_t(1-\eta_t)} \\ &= 0. \end{aligned} \tag{62}$$

A nonlinear root search technique must be used to find η_t. In the above equation, the quantity x^*_{Mt} can be expressed in terms of the other quantities, using (25).

The initial values K_0, N_0, A_{M0} and A_{S0} are arbitrary. The initial value of EoS is 0.01. We then let the system evolve according to the scheme described above.

The resulting growth patterns are shown in figures 1-3. Figure 1 shows the assumed evolution of σ_t, starting at the value of 0.01 and growing at the rate of 0.4% per year. As can be seen in figure 2, after an initial period of adjustment, population and real earnings grow slowly and steadily until σ reaches the value of one. For $\sigma > 1$, population and real earnings grow much faster but still roughly at a constant rate. This transition occurs within a few time steps, equivalent to about 150 years. The growth rates of population and real earnings are shown in figure 3. The main

observation is that the growth rates appear to settle to constant values for $\sigma < 1$ and for $\sigma > 1$. There is an order-of-magnitude increase in growth rates as the elasticity of substitution surpasses one, which can be interpreted as an Industrial Revolution.

As a check, we note that the growth rate of real income peaks at 1.9% per annum and then settles out to 1.8% per annum, close to the empirical value of 1.9% as discussed in the previous section. As explained in the appendix, the growth rates of earnings are the same as the growth rates of GDP per capita. It should be noted that in the simulation exercise the growth rates had not yet reached their asymptotic limits before the Industrial Revolution. If we allow the growth rates to converge to their theoretical values before the Industrial Revolution (e.g. by setting σ_0 to a smaller value), the growth rates overshoot quite a bit after the Industrial Revolution (see figure 5).

This is only one specific example. A crucial result of our numerical study, however, is that for any set of realistic parameter values, we find an order of magnitude jump in growth rates as σ exceeds one. Moreover, the growth rates are nearly independent of σ for $\sigma < 1$ and $\sigma > 1$. These results are consistent with the view that the Industrial Revolution was a sudden event when placed against the backdrop of recorded history. Even though innovation occurred over many centuries, the growth rates remained small (and nearly constant) until a crucial piece of innovation pushed σ passed one. In this sense, one can think of the Industrial Revolution as a phase transition.

4.4 Further Tests of the Model

According to the simulation results (using parameters as calibrated above), the model predicts a structural transformation away from the Malthus sector to the Solow sector. The predicted fractions of population employed in the Solow sector before and after the Industrial Revolution are

$$\eta^{before} = 55\%, \quad , \quad \eta^{after} = 71\%,$$

which implies that 45% of the population is employed in the Malthus sector prior to Industrial Revolution, and that fraction drops to 29% after the Industrial Revo-

lution.

Given the difficulty in determining the composition of a Malthus sector in modern times, we concentrate on comparing to empirical data up the the 19th century. According to Allen (2000, pgs. 8, 9), the average percentage of labour employed in the agricultural sector between 1300 and 1700 was 69%. Hence our prediction of 45% seems to be off the mark.[15]

According to Crafts (2001, Table 2), the share of agricultural employment in England in 1820 was 35%, dropping to 22.7% in 1870 and 11.8% in 1913. These figures can be compared to our prediction of 29%. The implication is that our model best describes the early part of the Industrial Revolution and not the latter part.[16]

As a further test of the structural transformation, we compute the share of GDP attached to the Malthus sector using the following formula:

$$s_{Mt} = \frac{p_t Y_{Mt}(1 - \phi(1 - s_t)) + Y_{St}\theta s_t}{p_t Y_{Mt} + Y_{St}}, \tag{63}$$

where

$$s_t = \frac{p_t x^*_{Mt}}{p_t x^*_{Mt} + x^*_{St}}.$$

Here s_{Mt} is the share of Malthus output in total GDP excluding the component of Malthus capital rent supporting Solow input to capital, and including the component of Solow capital rent supporting Malthus input to capital. The quantity s_t is the percentage of capital rent that is flowing to the Malthus sector. The simulation exercise predicts (figure 6) that s_{Mt} drops from 55% to 20% over the course of

[15] However as Weisdorf (2006) has pointed out, during the preindustrial era agricultural workers spent a large fraction of their time producing non-agricultural goods such as clothing. Hence the empirical estimate of 26% may not completely reflect the extent of labor allocated to industry. Prior to industrialization, farming communities were largely self-sufficient. It was only after industrialization that farmers devoted their entire working hours to agricultural production and traded their produce for manufactured goods.

[16] Two caveats should be noted. First, our model treats the economy as self-contained and closed to external trade. However, this is far from representing the British economy in the later nineteenth century, being embedded in the globalized trading system of the British empire. For example, agricultural goods were imported, displacing some of the employment in the agricultural sector. Also, as suggested previously the Malthus sector should ideally include a child care sector as well as an agricultural sector. The observed large drop in the share of agriculture in the late nineteenth century does not necessarily imply a large drop in the share of the Malthusian sector.

the Industrial Revolution. According to Broadberry (2010, Tables 16 and 20), the share of GDP devoted to agriculture in England was 42.4% in 1381, dropping to 28% in 1700 and further dropping to 22.1% in 1841. The last figure is in line with our model results, but the figure of 1381 is low in comparison with our model prediction. Nevertheless our model predicts a structural transformation that is roughly consistent with historical data.

Using equation (27) the interest rate in the two regimes is

$$i^{before} = 1.4\% \quad , \quad i^{after} = 8.0\%,$$

exhibited in figure 7. The direction of this prediction is counter-factual (Clark, 2007) but is consistent with Hansen & Prescott (2002) and is really an artifact of the assumed model of preferences.[17]

Figure 8 shows the paths of prices p and q. Figure 9 shows the path of productivities. It is important to note that according to figure 9, the productivity of the Solow sector does not take off until *after* the Industrial Revolution. This is in contrast to the HP paper which posits a rapidly growing Solow sector at all times in history.

An interesting story can be obtained by studying figures 8 and 9. In the preindustrial epoch both Solow output and Malthus output are required components of capital. This is reflected by near identical (and small) growth rates for the Malthus and Solow productivities, g_{A_M} and g_{A_S}, before the Industrial Revolution. After the Industrial Revolution substitution became easier. While g_{A_M} remains between half a percent and one percent during the entire simulation, g_{A_S} increases to near 3%, almost an order of magnitude larger than g_{A_M}. The reason for this divergence is as follows. As it becomes easier to use Solow material as a component of capital to replace Malthus material (e.g. steam replacing horses), the productivity of the Solow sector starts to accelerate due to learning-by-doing effects. This causes the

[17] A small modification of the model might suffice to reverse this result: the parameter β could be made an increasing function of time (increasing savings rate). This modification would have no effect on the growth rates of N_t or y_t but *would* affect the path of real earnings. One might also wish to move away from the assumption that land is free. Rural inhabitants could earn land rents in retirement whereas urban inhabitants could not, spurring a higher rate of the saving in the Solow sector.

price of Solow material to drop rapidly (p_t accelerates upwards), implying more and more of a shift from the Malthus sector to the Solow sector, spurring even greater learning-by-doing effects in the Solow sector, etc. This positive reinforcement cycle leads to the Solow sector taking over from the Malthus sector in the construction of capital goods. The end results is a lower price of capital, which also boosts the agricultural sector.

5 Summary and Discussion

The model presented above is consistent with the historical fact that between 1800 and 1900 the rates of growth of population and real earnings increased markedly in the western hemisphere (Maddison, 2007). The following predictions of the model are roughly confirmed by historical data:

1. The growth-rate of income in the preindustrial era is about 0.19% per annum.
2. The price of food expressed in non-food terms is increasing in the preindustrial era at the about the same rate as income.
3. The level of earnings expressed in food terms increases over the course of the Industrial Revolution.
4. There is a structural transformation during the Industrial Revolution away from agriculture towards industry, as measured by employment share and GDP share.

The one true failure of the model is the predicted rise in interest rates over the course of the Industrial Revolution, which is also a failure of the HP model. Several other predictions of the model require empirical verification. These are:

1. The budgetary share for childrearing in both the preindustrial and modern eras is 30%. The cost of childrearing grows in tandem with earnings at all times.

2. The rate of deflation of the price of capital goods is 50% greater than the rate of growth of income in both the preindustrial and modern eras.

We can now identify the model's shortcomings and suggest possible improvements.

First, it is worth pointing out a theoretical limitation of the model. It has been assumed that individual preferences are not influenced by the kind of factors that have been used, for example, in the unified growth models of Galor & Weil (2000) and Galor & Moav (2002). Human capital does not enter into the model, and there is no trade-off between quantity of children and quality of children. Hence, our model cannot account for a decline in fertility after the Industrial Revolution. Our fertility model is "Malthusian" at all times. Despite this restriction, the fertility model *does* provide a mechanism by which real earnings could rise after industrialization without a corresponding explosion in population. The presence of two sectors allows for the taming of population growth because the consumption of manufactured goods coincides with a rise in the cost of childrearing. One of the contributions of this study, therefore, is to suggest that growth-theorists might simplify their models of the demographic transition by introducing a second sector. For example, one could add a human capital or educational element tied to childcare to the present two-sector model.

Adding a labour-intensive child-care component to the present model might also help to better reconcile its predictions of employment weights with historical data and to reconcile the small capital share in the Malthus sector (4%) with the share of agriculture often used in these types of studies (e.g. 10% in HP). It would also assist in the interpretation of the price of Malthus output p_t in the modern era.

The model predicts that real interest rates rose during the Industrial Revolution, which is counter-factual. This failure is also present in the HP model. Possible remedies include having a more sophisticated model of preferences and assigning a non-zero value to land.

We should note that the model presented in this paper includes some restrictive assumptions that could be loosened. For example, the exogenous growth rates are assumed to be the same in the Malthus and Solow sectors. However, there may be good historical reasons to believe that innovation occurred faster for manufactured

goods than for agricultural goods. The workings of manufactured items might well have been understood by our ancestors, but the inner workings of agriculture goods such as horses were undoubtedly a complete mystery (physics came before biology). The learning-by-doing spillover parameter could also be made higher in the Solow sector than in the Malthus sector to reflect the tendency of industry to be located in densely populated towns and cities. It would also be worth investigating whether the growth of σ could be made endogenous.

Finally, given the fact that the most important examples of substitution during the Industrial Revolution were related to energy use and conversion, future empirical work should include an explicit energy component.

In conclusion, this paper has described a unified growth model based on Wrigley's thesis that a key enabler of the Industrial Revolution was a shift in the source of raw materials from agriculture to industry (Wrigley, 1988, 2010). The main contributions of the paper are as follows. First, we have demonstrated the benefit of working with two sectors in developing a unified demographic model. Second, we have shown that Arrow's learning-by-doing model allows one to link an increase in the growth rate of productivity to an increase in the share of capital. Finally, we have proposed that the elasticity of substitution between industrial output and agricultural output was slowly rising over history, and that once it reached the critical value of one, there were simultaneous agricultural and industrial revolutions.

Acknowledgment

We would like to thank R. Allen for the use of his price data series for England. P. Berg acknowledges support by an NSERC Discovery Grant.

A Appendix

This appendix contains a mathematically rigorous derivation of the analytical expressions for the growth rates.

A.1 Analytical Derivation of Growth Limits

Substituting the exponential time dependencies, e.g. $y_t \sim e^{g_y t}$, into the equations of our model, we can obtain several relations that are independent of σ_t, and hence valid in both regimes $\sigma < 1$ and $\sigma > 1$. Balancing terms in (47), we see that

$$g_y = g_p. \tag{64}$$

The above result is consistent with a constant population growth rate g_N (see equation (33)). From (16), this result implies that

$$\phi g_K = (1 - \mu) g_N. \tag{65}$$

The same equation also yields

$$g_y = \theta(g_K - g_N). \tag{66}$$

Substituting (65) into (66) leads to

$$g_y = \frac{\theta}{\phi}(1 - \mu - \phi) g_N, \tag{67}$$

implying that the ratio of the growth rate of nominal earnings to the growth rate of population is the same before and after the Industrial Revolution.

We can now derive an expression relating the growth rate of *real* earnings to the growth rate of population as follows. From (48) we have

$$g_K = g_y + g_N - g_q. \tag{68}$$

This, when combined with (66), implies that

$$g_q = -\frac{1 - \theta}{\theta} g_y. \tag{69}$$

Substituting this result along with (64) into (34), we obtain

$$g_{yR} = \left(\alpha + \frac{\beta}{\theta}\right) g_y. \tag{70}$$

Combining the above result with (67), we finally find

$$g_{yR} = \frac{(1-\mu-\phi)(\alpha\theta+\beta)}{\phi} g_N. \tag{71}$$

Hence, the ratio of the growth rate of real earnings to the growth rate of population is the same before and after the Industrial Revolution.

Given the relationship between the real earnings growth and population growth, we can now concentrate our attention on deriving expressions for g_N before and after the Industrial Revolution. Combining (69) and (67), we have

$$g_N = -\frac{\phi}{(1-\mu-\phi)(1-\theta)} g_q. \tag{72}$$

So the task now is to determine g_q (which must be negative).

Pulling A_{St} out of the bracket in (24), we obtain

$$q_t = \frac{1}{A_{St}} \left\{ (1-\gamma)^{\frac{1}{1-\rho}} \left(\frac{p_t A_{St}}{A_{Mt}}\right)^{-\frac{\rho}{1-\rho}} + \gamma^{\frac{1}{1-\rho}} \right\}^{-\frac{1-\rho}{\rho}}. \tag{73}$$

Let us define $psm := \frac{p_t A_{St}}{A_{Mt}}$. The simulation exercise revealed that in the case $\sigma < 1$ ($\rho < 0$), where the Malthusian sector dominates, psm is larger than one but small. When σ is small, ρ is large and negative, in which case the first term containing psm in the curly brackets dominates over the second term and we have

$$q_t \sim \frac{p_t}{A_{Mt}}. \tag{74}$$

Hence to determine the asymptotic growth rates before the Industrial Revolution, we need to determine the difference in growth rates between p_t and A_{Mt}.

After the Industrial Revolution when $\sigma > 1$ ($0 < \rho < 1$), where the Solow sector

dominates, psm approaches infinity. In that case, the second term in the curly brackets dominates over the first and we have

$$q_t \sim \frac{1}{A_{St}}. \tag{75}$$

So to determine the growth rates after the Industrial Revolution, we need only determine g_{A_S}.

Figure 9 supports our assumptions that lead to equations (74) and (75), namely that the growth of psm switches dramatically near $t = 1780$, meaning near $\sigma = 1$. The growth in efficiency of the Solow sector, A_S, begins to dominate that of the Malthusian sector, A_M before the Industrial Revolution. However, it is only with the Industrial Revolution that A_S, and likewise psm, begins to grow dramatically. Therefore, our asymptotic derivations for q_t above are consistent.

It should be noted that the growth rate of GDP per capita is the same as the growth rate of earnings. There are two components of income: labour earnings and investment income earned in retirement. Since we have an overlapping generations model, the population of old retirees is a fixed multiple of the population of young labourers. Hence to show that the growth rate of GDP per capita is equal to the growth rate of earnings we need only show that the growth rate of price-deflated capital per person is equal to the growth rate of earnings (recall that interest rates are constant). This last statement follows from equation (68).

We will now analyze the two regimes, starting with the second.

A.2 The Industrial Revolution: $\sigma > 1$

We have the setting described by (75), namely

$$g_q = -g_{A_S}. \tag{76}$$

We now need to utilize (37), rewritten as

$$35g_{A_S} = 35g_0 + \xi ln\left[1 + \frac{y_{t+1}N_{t+1}x^*_{St+1}/q_{t+1}}{\sum_{t_i \leq t} y_{t_i}N_{t_i}x^*_{St_i}/q_{t_i}}\right], \tag{77}$$

where we have introduced the factor of 35 since a time step corresponds to 35 years and we wish to express all growth rates in annualized terms. The trick now is to write the sum as a geometric series, which the sum approaches asymptotically

$$\sum_{t_i \leq t} y_{t_i}N_{t_i}x^*_{St_i}/q_{t_i} \rightarrow \sum_{t_i \leq t} e^{35(g_y+g_N+g_{x^*_S}-g_q)t_i} = \frac{1-e^{(\dots)(t+1)}}{1-e^{(\dots)}}, \tag{78}$$

where the dots represent $g_y + g_N + g_{x^*_S} - g_q$. Note that the initialization constants cancel in the above expression. The logarithm simplifies dramatically as $t \rightarrow \infty$ and (77) becomes

$$g_{A_S} = g_0 + \xi(g_y + g_N + g_{x^*_S} - g_q). \tag{79}$$

From (26) we find that

$$\frac{x^*_{St}}{q_t} \sim (A_{St}q_t)^{\frac{\rho}{1-\rho}} \rightarrow 1 \tag{80}$$

since the assumption was that $q_t \sim 1/A_{St}$. Hence, there is no ρ dependency and $g_q = g_{x^*_S}$. Accordingly, (79) simplifies to

$$g_{A_S} = g_0 + \xi(g_y + g_N). \tag{81}$$

Starting with (72) and (76), in (81) we can express g_{A_S} in terms of g_N and substitute (67) for g_y, yielding

$$g_N^{after} = \frac{g_0}{\frac{1-\mu-\phi}{\phi}[1-\theta(1+\xi)] - \xi}. \tag{82}$$

The above formula gives a population growth rate that matches numerical results very well, given sufficient simulation time. Analytical expressions for all other growth rates, e.g. g_{yR}, follow immediately by successive substitution into the previous equations.

A.3 Before the Industrial Revolution: $\sigma < 1$

The same type of analysis can now be applied to the Malthusian case

$$q_t \sim \frac{p_t}{A_{Mt}}. \tag{83}$$

Similarly, we end up with

$$g_{A_M} = g_0 + \xi(g_y + g_N - g_p), \tag{84}$$

leading to

$$g_N^{before} = \frac{g_0}{\frac{1-(1+\xi)\theta}{\phi}(1-\mu-\phi) + (1-\xi)(1-\theta-\mu) + (1+\xi)(1-\mu)\frac{\theta-\phi}{\phi} - \xi}. \tag{85}$$

Again, it matches numerical results very well when enough time is allowed for the system to evolve for constant, small σ.

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Parameter	Value
g_0	0.004
g_σ	0.004
α	0.5
β	0.2 (Jones, 2002)
γ	0.5
μ	0.6 (Hansen & Prescott, 2002)
ϕ	0.04
σ_0	0.01
θ	0.4 (Hansen & Prescott, 2002)
ξ	1.0

Table 1: *Parameters used in the simulation exercise.*

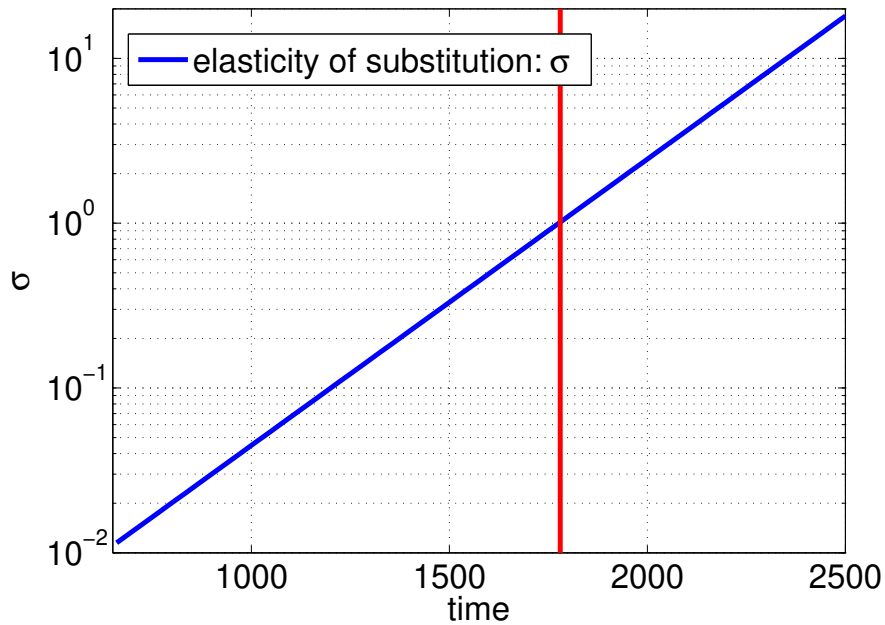

Figure 1: *The elasticity of substitution, σ, grows exponentially with time, starting at $\sigma_0 = 0.01$ and ending at $\sigma \gg 1$. We have shifted the values of the time axis in all figures so that $\sigma = 1$ at $t = 1780$ years, marked by a vertical line. The dividing line roughly corresponds to the start of the Industrial Revolution in England.*

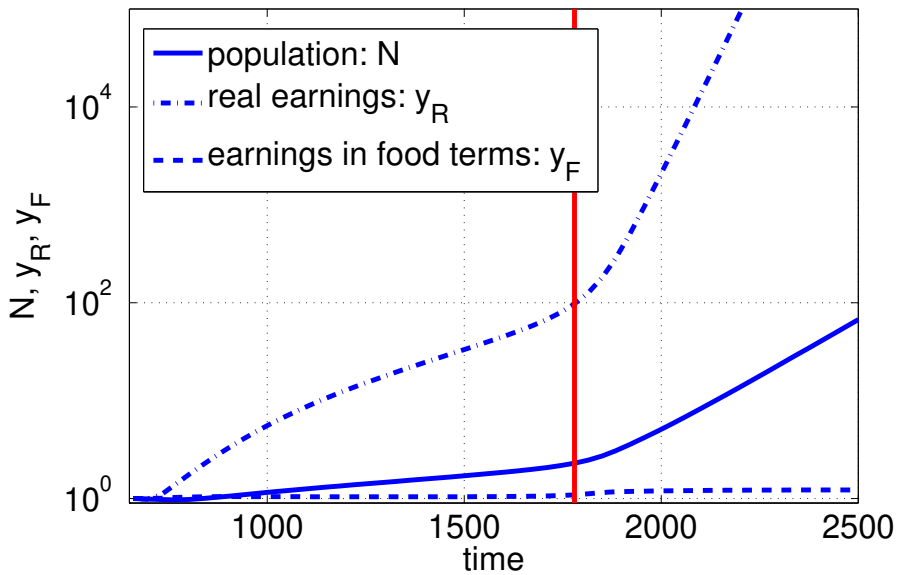

Figure 2: *Population, real earnings and earnings in terms of food as a function of time, normalized by their respective values at* $t_0 = 620$. *At* $t = 1780$, *equal to* 33 *time steps,* σ *passes one. Around this point in time we can clearly observe a transition in dynamics between two growth regimes.*

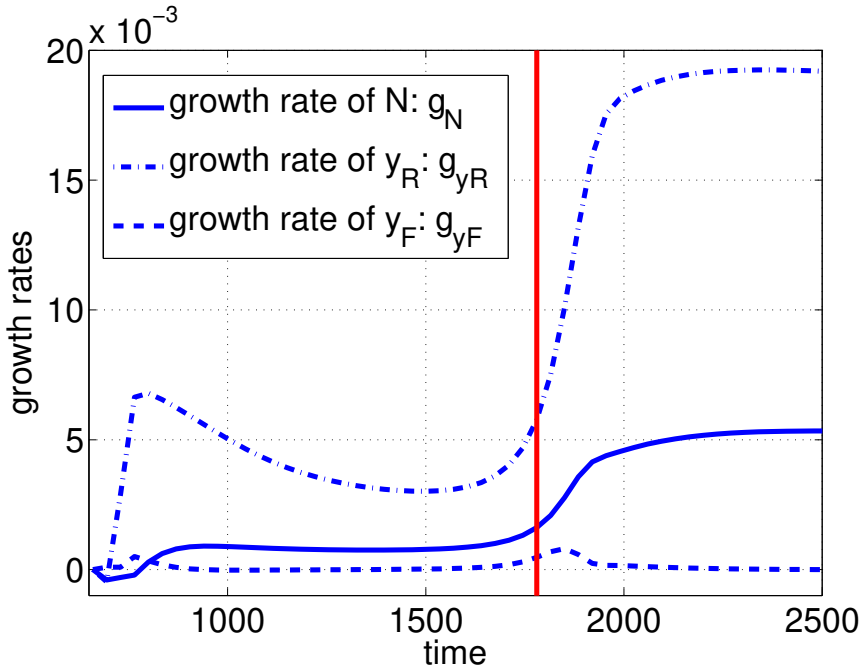

Figure 3: *Growth rates of population, real earnings and earnings in terms of food versus time. After a rapid initial adjustment ($t < 800$) when the full growth model is switched on at $t_0 = 620$, the population growth rate is nearly constant while the growth rate of real earnings converges from above to a constant value. However, as σ approaches one and eventually passes through one, these small growth rates change by an order of magnitude. In contrast, the growth rate of earnings in food terms increases from zero only temporarily during the transition between the two regimes. The transition occurs within about four to five generations, equivalent to 150 years.*

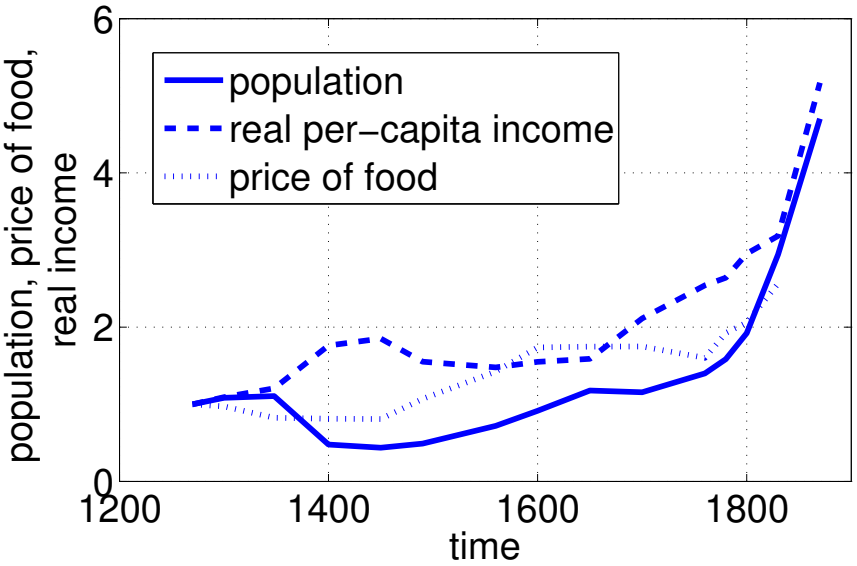

Figure 4: *Population, real per-capita income and the relative price of food (a proxy for p_t) for England, 1270-1879. Population and income data is from Broadberry* et al. *(2010). Food prices are in terms of non-food items (e.g. manufactured goods) and are courtesy of Robert Allen (Oxford University; personal communication). All series are normalized to the value of one in the year 1270.*

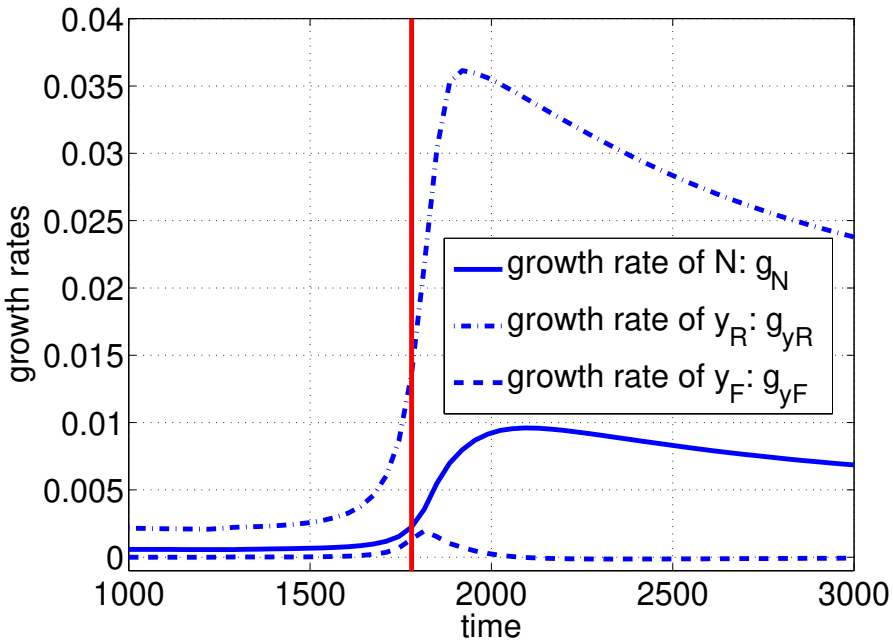

Figure 5: *Growth rates of population and earnings versus time when the rates are allowed to reach their asymptotic values before the Industrial Revolution.*

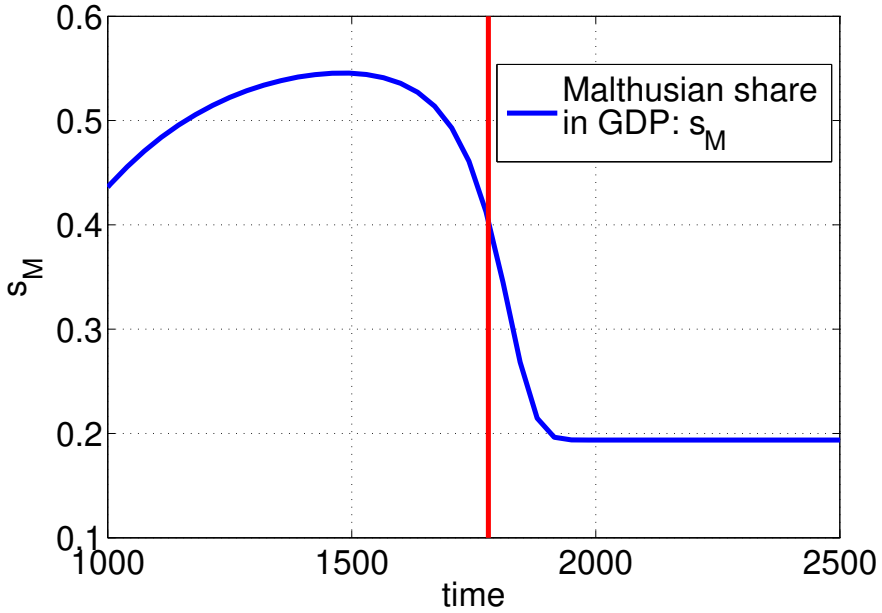

Figure 6: *Share of the Malthus sector in GDP, equation (63), versus time.*

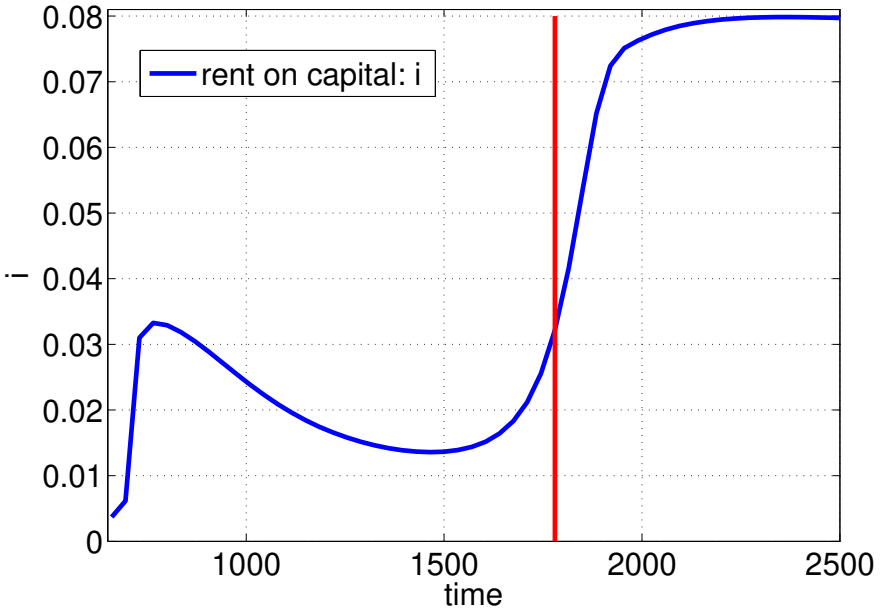

Figure 7: *Annualized interest rate, equation (27), versus time.*

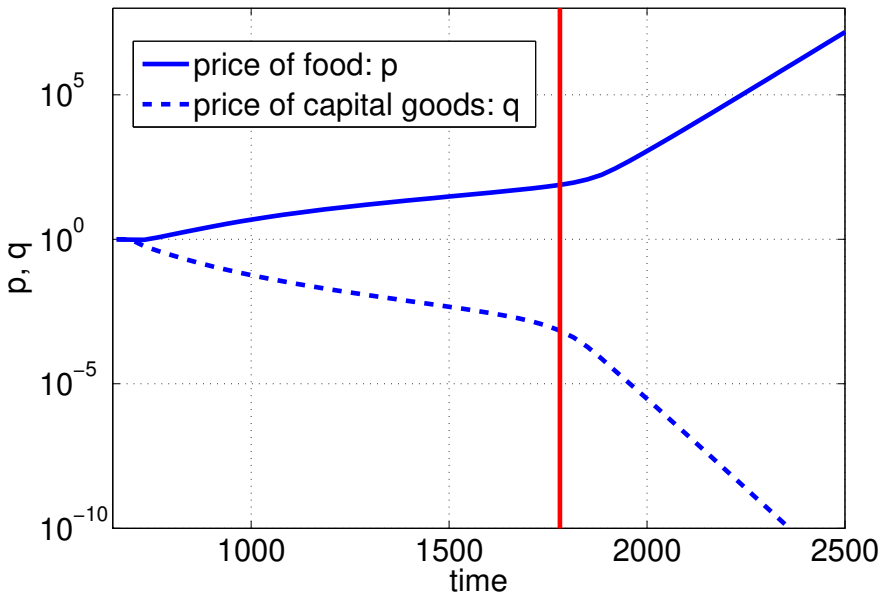

Figure 8: *Food prices grow at two distinct rates before and after the Industrial Revolution, spurred by a jump in the population growth rate. The price of capital makes a sudden turn downwards after the Industrial Revolution, spurred by learning-by-doing in the Solow sector.*

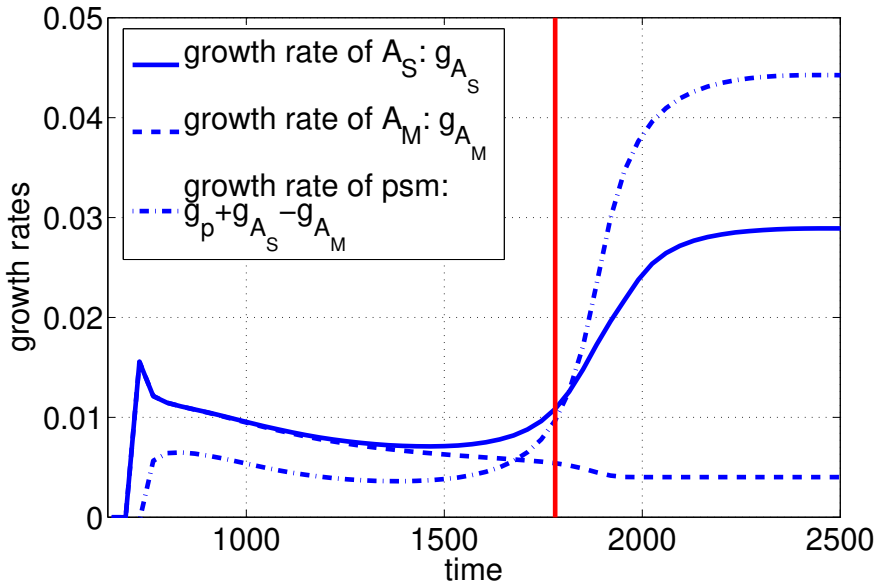

Figure 9: *Growth rates of Solow and Malthusian efficiencies, g_{A_S} and g_{A_M}, and growth of psm in equation (73). Remarkably, the efficiency of the Solow sector, A_S, already begins to dominate that of the Malthusian sector, A_M, ahead of $\sigma = 1$. However, this does not contradict our asymptotic analysis of q_t, reflected by psm.*