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EIGENVALUE DISTRIBUTION AND THE PRODUCTION PRICE-PROFIT RATE RELATIONSHIP IN LINEAR SINGLE-PRODUCT SYSTEMS: THEORY AND EMPIRICAL EVIDENCE

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ABSTRACT

A typical finding in many empirical studies is that the production price-profit rate relationship is, by and large, monotonic. This paper derives, in terms of the usual single-product model, the spectral conditions that make possible the appearance of such monotonicity. Furthermore, using data from input-output tables for a number of countries and years, it examines the extent to which actual economies fulfil those spectral conditions.

Key words: Eigenvalue distribution, production prices, spectral analysis, Standard systems

JEL classifications: B51, C67, D46, D57, E11

1. Introduction

In a world of fixed input-output coefficients and at least three commodities, produced by means of themselves and homogeneous labour, long-period relative prices can change in a complicated way as income distribution changes, a fact that has critical implications for the traditional theories of capital, value, distribution and international trade. In accordance with Classical, Marxian, Austrian and neoclassical theory, Sraffa (1960) noted that '[t]he key to the movement of relative prices consequent upon a change in the wage lies in the inequality of the proportions in which labour and means of production are employed in the various industries.' (*ibid.*, §15). Nevertheless, taking into account that 'the means of production of an industry are themselves the product of one or more industries which may in their turn employ a still lower proportion of labour to means of production' (*ibid.*, §19), he was able to show that 'as the wages fall the price of the product of a low-proportion […] industry may rise or may it may fall, or it may even alternate in rising and falling, relative to its means of production' (*ibid*.). Thus, he finally detected the fundamental consequence of the existence of complicated patterns of price-movement in the internal logic of the traditional (Austrian and neoclassical) theories of capital as follows: 'The reversals in the direction of the movement of relative prices, in the face of unchanged methods of production, cannot be reconciled with *any* notion of capital as measurable quantity independent of distribution and prices.' (*ibid*., p. 38).[1]

However, typical findings in many *empirical* studies of single-product systems are that[2] (i) the production price-profit rate curves are, more often than not, *monotonic* (in the economically significant interval of the profit rate); (ii) non-monotonic production price-profit rate curves are not only rare but also have no more than *one* extremum point; (iii) cases of reversal in the direction of deviation between production prices and labour values are *more* rare;[3] therefore, (iv) the approximation of the production prices through Bienenfeld's (1988) linear and, *a fortiori*, quadratic formulae works pretty well; and (v) the so-called 'wage-profit relationships' are *almost linear* irrespective of the numeraire chosen (*i.e.*, the correlation coefficients between the wage and profit rates tend to be above 99%), which implies, in its turn, that there is empirical basis for searching for an '*approximate* surrogate production function' (Schefold, 2008a, b). For example, our study on ten 19 x 19 input-output tables of the Greek economy, spanning the period 1988-1997 (Tsoulfidis and Mariolis, 2007), in which all capital is (by assumption) circulating capital and the vector of production prices is normalized with the use of Sraffa's (1960, ch. 4) 'Standard commodity', shows that the movement of prices is, by and large, governed by the relevant 'vertically integrated' (Pasinetti, 1973) capital-labour ratios, and detects 36 cases of non-monotonic movement (*i.e.*, $36/190 \simeq 19\%$) and 29 cases of

[1] For a compact exposition of the Sraffa-based critique of the traditional theories, see Kurz and Salvadori (1995, chs 4, 5 and 14). Sraffa's (1960, chs 3 and 6) analysis of the movement of relative prices has been extended by Schefold (1976), Pasinetti (1977, Section 5.7), Caravale and Tosato (1980, pp. 85-87), Parys (1982) and Bidard (1991, pp. 56-58). Moreover, Mainwaring (1978, pp. 16-17) has constructed and analyzed a very interesting numerical example for the three-commodity case, which indicates that non-monotonic movements of relative prices need not imply 'factor-intensity reversal'. Finally, it should also be noted that, more recently, C. Bidard, H. G. Ehrbar, U. Krause and I. Steedman have detected some 'monotonicity (theoretical) laws' for the relative prices (see Bidard and Ehrbar, 2007, and the references provided there).

[2] See Sekerka *et al*. (1970; Czechoslovakia), Krelle (1977; Germany), Ochoa (1984, ch. 7; USA), Leontief (1985; USA), Petrović (1987, 1991; Yugoslavia), Cekota (1988; Canada), Da Silva and Rosinger (1992; Brazil), Marzi (1994; Italy), Shaikh (1998; USA), Han and Schefold (2006; OECD), *inter alia*.

[3] Since prices are proportional to labour values at a zero profit rate, non-monotonicity is a necessary, but not sufficient, condition for price-labour value reversal.

price-labour value reversals (*i.e.*, 15%). Furthermore, as it has recently been argued, the said typical findings, which do not, of course, invalidate the Sraffa-based critique, could be connected to the distribution of the eigenvalues of the vertically integrated technical coefficients matrices of actual economies.[4]

The claim that this paper raises is that we can further investigate, both theoretically and empirically, the monotonicity issue. More specifically, first, we derive, in terms of the usual linear single-product model, the spectral conditions that make possible the appearance of such monotonicity and, second, using input-output data of many diverse economies, *i.e.*, China, Greece, Japan, Korea and USA, for which it is *already* known that the production price-profit rate and/or the wage-profit relationships have the aforementioned typical forms,[5] we examine the extent to which actual economies fulfil those conditions.

The remainder of the paper is structured as follows. Section 2 presents a spectral decomposition of the price system and derives conditions for the monotonicity of the price-profit rate relationship. Section 3 brings in the empirical evidence by examining actual input-output data. Section 4 concludes.

2. Theory

Consider a closed, linear system, involving only single products, basic commodities (in the sense of Sraffa, 1960, §6) and circulating capital. Furthermore, assume that (i) the input-output coefficients are fixed; (ii) the system is 'viable', *i.e.*, the Perron-Frobenius (P-F hereafter) eigenvalue of the irreducible $n \times n$ matrix of input-output coefficients, $\mathbf{A}$, is less than 1,[6] 'diagonalizable', *i.e.*, $\mathbf{A}$ has a complete set of n linearly independent eigenvectors, and 'regular' (in the sense of Schefold, 1971, pp.

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[4] See Schefold (2008b, c) and Mariolis and Tsoulfidis (2009). Nevertheless, Bienenfeld (1988, p. 255) has already shown that, in the extreme case in which the non-dominant eigenvalues of the said matrix equal zero, the production prices are strictly linear functions of the profit rate, and Shaikh (1998, p. 244) has noted that '[a] large disparity between first and second eigenvalues is another possible source of linearity.' (see also *ibid.*, p. 250, note 9).

[5] For the economy of China, 1997, see Mariolis and Tsoulfidis (2009). For Greece, 1970 and 1988-1997, see Tsoulfidis and Maniatis (2002) and Tsoulfidis and Mariolis (2007), respectively. For Japan, 1970, 1975, 1980, 1985 and 1990, see Tsoulfidis (2008) and Mariolis and Tsoulfidis (2010). For Korea, 1995 and 2000, see Tsoulfidis and Rieu (2006). Finally, for USA, 1947, 1958, 1963, 1967, 1972 and 1977, see Ochoa (1984), Bienenfeld (1988), Chilcote (1997) and Shaikh (1998).

[6] Matrices (and vectors) are denoted by boldface letters. The transpose of an $n \times 1$ vector $\mathbf{x}$ is denoted by $\mathbf{x}^{\mathrm{T}}$. $\lambda_{\mathbf{A}1}$ denotes the P-F eigenvalue of a semi-positive $n \times n$ matrix $\mathbf{A}$ and $(\mathbf{x}_{\mathbf{A}1}, \mathbf{y}_{\mathbf{A}1}^{\mathrm{T}})$ the corresponding eigenvectors, whilst $\lambda_{\mathbf{A}k}$, $k = 2, ..., n$ and $|\lambda_{\mathbf{A}2}| \geq |\lambda_{\mathbf{A}3}| \geq ... \geq |\lambda_{\mathbf{A}n}|$, denotes the non-dominant eigenvalues of $\mathbf{A}$ and $(\mathbf{x}_{\mathbf{A}k}, \mathbf{y}_{\mathbf{A}k}^{\mathrm{T}})$ the corresponding eigenvectors.

11-23, 1976; see also Bidard and Salvadori, 1995, p. 389), *i.e.*, no (real or complex) right eigenvector of $\mathbf{A}$ is orthogonal to the vector of direct labour coefficients, $\mathbf{l}^{\mathrm{T}}\ (>\mathbf{0}^{\mathrm{T}})$;[7] (iii) the rate of profits, r, is uniform; (iv) labour is not an input to the household sector and may be treated as homogeneous because relative wage rates are invariant (see Sraffa, 1960, §10; Kurz and Salvadori, 1995, pp. 322-325); and (v) wages are paid at the end of the common production period.[8]

On the basis of these assumptions we can write

$$\mathbf{p}^{\mathrm{T}} = w\mathbf{l}^{\mathrm{T}} + (1+r)\mathbf{p}^{\mathrm{T}}\mathbf{A} \tag{1}$$

where $\mathbf{p}$ denotes a vector of prices of production and w the money wage rate. Equation (1) after rearrangement gives:

$$\mathbf{p}^{\mathrm{T}} = w\mathbf{v}^{\mathrm{T}} + r\mathbf{p}^{\mathrm{T}}\mathbf{H}$$

or

$$\mathbf{p}^{\mathrm{T}} = w\mathbf{v}^{\mathrm{T}} + \rho\mathbf{p}^{\mathrm{T}}\mathbf{J} \tag{2}$$

or, if $\rho < 1$,

$$\mathbf{p}^{\mathrm{T}} = w\mathbf{v}^{\mathrm{T}}[\mathbf{I}-\rho\mathbf{J}]^{-1} = w\mathbf{v}^{\mathrm{T}}\sum_{t=0}^{\infty}\rho^{t}\mathbf{J}^{t} \tag{3}$$

where $\mathbf{H} \equiv \mathbf{A}[\mathbf{I}-\mathbf{A}]^{-1}\ (>\mathbf{0})$ denotes the 'vertically integrated technical coefficients matrix', $\mathbf{I}$ the identity matrix, $\mathbf{v}^{\mathrm{T}} \equiv \mathbf{l}^{\mathrm{T}}[\mathbf{I}-\mathbf{A}]^{-1}\ (>\mathbf{0}^{\mathrm{T}})$ the vector of vertically integrated labour coefficients or 'labour values', $R \equiv (\lambda_{\mathbf{A}1})^{-1} - 1\ (= (\lambda_{\mathbf{H}1})^{-1})$ the maximum rate of profits, *i.e.*, the rate of profits corresponding to $w = 0$ and $\mathbf{p} > \mathbf{0}$, $\rho \equiv rR^{-1}$, $0 \le \rho \le 1$, the 'relative rate of profits', and $\mathbf{J} \equiv R\mathbf{H}$, with $\lambda_{\mathbf{J}1} = R\lambda_{\mathbf{H}1} = 1$, $\lambda_{\mathbf{J}k} = R\lambda_{\mathbf{H}k} = R\lambda_{\mathbf{A}k}(1-\lambda_{\mathbf{A}k})^{-1}$ and $|\lambda_{\mathbf{J}k}| < 1$.[9]

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[7] Schefold argues that 'non-diagonalizable' and 'irregular' systems are of measure zero in the set of all systems and thus not generic (*ibid.*; see also Schefold, 1978, pp. 268-269, whilst for a similar argument, see Goodwin, 1976, p. 130, footnote 1). As is well known, given any $\mathbf{A}$ and an arbitrary $\varepsilon \neq 0$, it is possible to perturb the entries of $\mathbf{A}$ by an amount less than $|\varepsilon|$ so that the resulting matrix is diagonalizable (see, *e.g.*, Aruka, 1991, pp. 74-76). Finally, it may also be noted that the concepts of 'regularity' and 'controllability' (in the sense of Kalman, 1961) are algebraically equivalent (see Mariolis, 2003).

[8] It would make no relevant difference to our analysis the assumption of *ex ante* payment of wages (for the general case, see, *e.g.*, Steedman, 1977, pp. 103-105).

[9] If $\lambda_{\mathbf{A}k}$ is positive, then $\lambda_{\mathbf{A}k} < \lambda_{\mathbf{A}1}$. If it is negative or complex, then $|\lambda_{\mathbf{A}k}| \le \lambda_{\mathbf{A}1}$ (the equality holds iff $\mathbf{A}$ is imprimitive) and $|1-\lambda_{\mathbf{A}k}| > 1-|\lambda_{\mathbf{A}k}|$. Hence, $|\lambda_{\mathbf{J}k}| < 1$ holds for all k.

If commodity $\mathbf{z} \geq \mathbf{0}$, with $\mathbf{v}^{\mathrm{T}}\mathbf{z} = 1$, is chosen as the standard of value or numeraire, *i.e.*, $\mathbf{p}^{\mathrm{T}}\mathbf{z} = 1$, then (3) implies that

$$w = (\mathbf{v}^{\mathrm{T}}[\mathbf{I} - \rho\mathbf{J}]^{-1}\mathbf{z})^{-1} \tag{4}$$

which gives a trade-off between w measured in terms of $\mathbf{z}$ and ρ, known as the $w - \rho$ relationship. Finally, substituting (4) in (3) gives

$$\mathbf{p}^{\mathrm{T}} = (\mathbf{v}^{\mathrm{T}}[\mathbf{I} - \rho\mathbf{J}]^{-1}\mathbf{z})^{-1}\mathbf{v}^{\mathrm{T}}[\mathbf{I} - \rho\mathbf{J}]^{-1} \tag{5}$$

Since $\mathbf{A}$ is assumed to be diagonalizable, $\mathbf{v}^{\mathrm{T}}$ can be expressed as a linear combination of the basis vectors $\mathbf{y}_{\mathbf{A}m}^{\mathrm{T}}$, *i.e.*,

$$\mathbf{v}^{\mathrm{T}} = \sum_{m=1}^{n} c_m \mathbf{y}_{\mathbf{A}m}^{\mathrm{T}} \tag{6}$$

and $\mathbf{z}$ can be expressed as a linear combination of the basis vectors $\mathbf{z}_{\mathbf{A}m} \equiv [\mathbf{I} - \mathbf{A}]\mathbf{x}_{\mathbf{A}m}$, *i.e.*,

$$\mathbf{z} = \sum_{m=1}^{n} d_m \mathbf{z}_{\mathbf{A}m} \tag{7}$$

Post-multiplying (6) by $\mathbf{z}_{\mathbf{A}m}$ gives

$$\mathbf{v}^{\mathrm{T}}\mathbf{z}_{\mathbf{A}m} = c_m \mathbf{y}_{\mathbf{A}m}^{\mathrm{T}}\mathbf{z}_{\mathbf{A}m} \tag{8}$$

since, for any two distinct eigenvalues of a matrix, the left eigenvector of one eigenvalue is orthogonal to the right eigenvector of the other. Pre-multiplying (7) by $\mathbf{v}^{\mathrm{T}}$ gives

$$\mathbf{v}^{\mathrm{T}}\mathbf{z} = \sum_{m=1}^{n} d_m \mathbf{v}^{\mathrm{T}}\mathbf{z}_{\mathbf{A}m} \tag{9}$$

Hence, if $\mathbf{y}_{\mathbf{A}m}^{\mathrm{T}}$, $\mathbf{z}_{\mathbf{A}m}$ are normalized by setting

$$\mathbf{y}_{\mathbf{A}m}^{\mathrm{T}}\mathbf{z}_{\mathbf{A}m} = 1 \text{ and } \mathbf{v}^{\mathrm{T}}\mathbf{z}_{\mathbf{A}m} = 1 \tag{10}$$

then (8), (9) and $\mathbf{v}^{\mathrm{T}}\mathbf{z} = 1$ imply that

$$c_m = 1 \text{ and } \sum_{m=1}^{n} d_m = 1 \tag{11}$$

Moreover, pre-multiplying (7) by $\mathbf{y}_{\mathbf{A}1}^{\mathrm{T}}$ gives

$$\mathbf{y}_{\mathbf{A}1}^{\mathrm{T}}\mathbf{z} = d_1 \mathbf{y}_{\mathbf{A}1}^{\mathrm{T}}\mathbf{z}_{\mathbf{A}1} = d_1 \tag{11a}$$

and, therefore,

$$d_1 > 0 \tag{11b}$$

since $\mathbf{y}_{\mathbf{A}1} > \mathbf{0}$. Thus, the substitution of (6), (7) and (11) in (4) and (5) yields

$$w = [(1-\rho)^{-1} d_1 + \sum_{k=2}^{n} (1-\rho\lambda_{\mathbf{J}k})^{-1} d_k]^{-1} \tag{12}$$

or

$$w = \Pi_0 (\sum_{m=1}^{n} \Pi_m d_m)^{-1} \tag{12a}$$

and

$$\mathbf{p}^{\mathrm{T}} = [(1-\rho)^{-1} d_1 + \sum_{k=2}^{n} (1-\rho\lambda_{\mathbf{J}k})^{-1} d_k]^{-1} [(1-\rho)^{-1} \mathbf{y}_{\mathbf{A}1}^{\mathrm{T}} + \sum_{k=2}^{n} (1-\rho\lambda_{\mathbf{J}k})^{-1} \mathbf{y}_{\mathbf{A}k}^{\mathrm{T}}] \tag{13}$$

or

$$\mathbf{p}^{\mathrm{T}} = (\sum_{m=1}^{n} \Pi_m d_m)^{-1} (\sum_{m=1}^{n} \Pi_m \mathbf{y}_{\mathbf{A}m}^{\mathrm{T}}) \tag{13a}$$

where

$$\Pi_0 \equiv (1-\rho)(1-\rho\lambda_{\mathbf{J}2})...(1-\rho\lambda_{\mathbf{J}n}) = \det[\mathbf{I} - \rho\mathbf{J}]$$

and

$$\Pi_m \equiv \prod_{\substack{j=1 \\ j\neq m}}^{n} (1-\rho\lambda_{\mathbf{J}j})$$

Moreover, since $w' \equiv dw/d\rho < 0$ and $(\mathbf{p}/w)' > \mathbf{0}$ (see Sraffa, 1960, §49), differentiation of (12) and (13) with respect to ρ implies that

$$(1-\rho)^{-2} d_1 + \sum_{k=2}^{n} (1-\rho\lambda_{\mathbf{J}k})^{-2} \lambda_{\mathbf{J}k} d_k > 0$$

and

$$(1-\rho)^{-2} \mathbf{y}_{\mathbf{A}1}^{\mathrm{T}} + \sum_{k=2}^{n} (1-\rho\lambda_{\mathbf{J}k})^{-2} \lambda_{\mathbf{J}k} \mathbf{y}_{\mathbf{A}k}^{\mathrm{T}} > \mathbf{0}^{\mathrm{T}}$$

respectively.[10]

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[10] It should be noted that Steedman's (1999a) numeraire, which is *not* necessarily semi-positive, entails that $(\sum_{m=1}^{n} \Pi_m d_m)^{-1} = 1$ and, therefore, $w = \Pi_0$, $w' < 0$, and $\mathbf{p}^{\mathrm{T}} = \sum_{m=1}^{n} \Pi_m \mathbf{y}_{\mathbf{A}m}^{\mathrm{T}}$ (see (12a) and (13a)). Thus, the $w-\rho$ and $\mathbf{p}-\rho$ relationships take on simpler forms in the sense that the former is expressed solely in terms of the eigenvalues of $\mathbf{J}$, whilst the latter is expressed in terms of powers of ρ up to ρ^{n-1}. For example, for $n = 2$, we get $\mathbf{p}^{\mathrm{T}} = \mathbf{y}_{\mathbf{A}1}^{\mathrm{T}} + \mathbf{y}_{\mathbf{A}2}^{\mathrm{T}} - \rho(\lambda_{\mathbf{J}2}\mathbf{y}_{\mathbf{A}1}^{\mathrm{T}} + \mathbf{y}_{\mathbf{A}2}^{\mathrm{T}})$ or, since $\mathbf{v}^{\mathrm{T}} = \mathbf{y}_{\mathbf{A}1}^{\mathrm{T}} + \mathbf{y}_{\mathbf{A}2}^{\mathrm{T}}$ (see (6) and (11)), $\mathbf{p}^{\mathrm{T}} = [1, \rho]\mathbf{B}$, where $\mathbf{B} \equiv [\mathbf{v}^{\mathrm{T}}, (1-\lambda_{\mathbf{J}2})\mathbf{y}_{\mathbf{A}1}^{\mathrm{T}} - \mathbf{v}^{\mathrm{T}}]^{\mathrm{T}}$, and $[1, \rho]$ are the coordinates of the price vector in terms of the basis $\mathbf{B}$ (see *ibid.*, pp. 7-8 and 12).

From equations (12) and (13), which constitute the *spectral forms* of the $w-\rho$ and $\mathbf{p}-\rho$ relationships, respectively, we derive the following:

(i). If Sraffa's Standard commodity is chosen as numeraire, *i.e.*, $\mathbf{z}=[\mathbf{I}-\mathbf{A}]\mathbf{x}_{\mathbf{A}1}$, then $d_1=1$ and $d_k=0$. Thus, (12) becomes

$$w=1-\rho \tag{14}$$

i.e., the $w-\rho$ relationship is a straight line,[11] and (13) becomes

$$\mathbf{p}^{\mathrm{T}}=\mathbf{y}_{\mathbf{A}1}^{\mathrm{T}}+(1-\rho)\sum_{k=2}^{n}(1-\rho\lambda_{\mathbf{J}k})^{-1}\mathbf{y}_{\mathbf{A}k}^{\mathrm{T}} \tag{15}$$

or

$$\mathbf{p}^{\mathrm{T}}=[1,(1-\rho)(1-\rho\lambda_{\mathbf{J}2})^{-1},...,(1-\rho)(1-\rho\lambda_{\mathbf{J}n})^{-1}]\mathbf{B}_{\mathrm{E}} \tag{15a}$$

where $\mathbf{B}_{\mathrm{E}}\equiv[\mathbf{y}_{\mathbf{A}1}^{\mathrm{T}},\mathbf{y}_{\mathbf{A}2}^{\mathrm{T}},...,\mathbf{y}_{\mathbf{A}n}^{\mathrm{T}}]^{\mathrm{T}}$ is a left eigenbasis and

$$[1,(1-\rho)(1-\rho\lambda_{\mathbf{J}2})^{-1},...,(1-\rho)(1-\rho\lambda_{\mathbf{J}n})^{-1}]$$

are the coordinates of the price vector in terms of $\mathbf{B}_{\mathrm{E}}$. Differentiation of (15) with respect to ρ gives

$$(\mathbf{p}^{\mathrm{T}})'=-\sum_{k=2}^{n}(1-\lambda_{\mathbf{J}k})(1-\rho\lambda_{\mathbf{J}k})^{-2}\mathbf{y}_{\mathbf{A}k}^{\mathrm{T}}$$

which implies that the individual components of $\mathbf{p}$ can change in a complicated way as ρ changes. Nevertheless, it can be shown that there are commodity bundles whose prices decrease monotonically as ρ increases. Post-multiplying (15) by $\mathbf{z}_{\mathbf{A}\mu}$, $\mu=2,...,n$ and $\mu\neq k$, gives

$$\mathbf{p}^{\mathrm{T}}\mathbf{z}_{\mathbf{A}\mu}=f_{\mu}(\rho) \tag{16}$$

where $f_{\mu}(\rho)\equiv(1-\rho)(1-\rho\lambda_{\mathbf{J}\mu})^{-1}$. Now, it is necessary to distinguish between the following two cases:

Case 1: If $\mathbf{z}_{\mathbf{A}\mu}$ is a real eigenvector, then $f_{\mu}(\rho)\geq 0$ is a strictly *decreasing* function of ρ, which is strictly concave (convex) to the origin for $\lambda_{\mathbf{J}\mu}>(<)\,0$,[12] whilst it

[11] The system consisting of equations (3) and (14) has been investigated intensively by Bienenfeld (1988), Steedman (1999b), Mariolis and Tsoulfidis (2009, pp. 4-10) and Mariolis (2010).

[12] It is easily checked that

$$f_{\mu}'(\rho)=-(1-\lambda_{\mathbf{J}k})(1-\rho\lambda_{\mathbf{J}k})^{-2}<0$$

since $|\lambda_{\mathbf{J}k}|<1$, and

$$f_{\mu}''(\rho)=-2(1-\lambda_{\mathbf{J}\mu})\lambda_{\mathbf{J}\mu}(1-\rho\lambda_{\mathbf{J}\mu})^{-3}$$

coincides with $1-\rho$ for $\lambda_{\mathbf{J}\mu}=0$ and tends to 1 (to $(1-\rho)(1+\rho)^{-1}$) as $\lambda_{\mathbf{J}m}\to 1$ ($\lambda_{\mathbf{J}m}\to -1$) (see Figure 1). Finally, multiplying both sides of (16) by $\lambda_{\mathbf{J}\mu}$ gives

$$k_{\mu}(k_{\mathrm{S}})^{-1}=(1-\rho)(R_{\mu}R^{-1}-\rho)^{-1} \tag{17}$$

where $R_{\mu}\equiv(\lambda_{\mathbf{A}\mu})^{-1}-1$, $k_{\mu}\equiv\mathbf{p}^{\mathrm{T}}\mathbf{z}_{\mathbf{A}\mu}R_{\mu}^{-1}$ equal the ratio of the net product to the means of production (or 'Standard ratio') and the capital-intensity of the vertically integrated sector producing $\mathbf{z}_{\mathbf{A}\mu}$ (or, alternatively, of an economically insignificant, non-Sraffian real (non-complex) Standard system),[13] respectively, $k_{\mathrm{S}}\equiv R^{-1}$ equals the capital-intensity of the Sraffian Standard system, and $|k_{\mu}|<k_{\mathrm{S}}$, since $R<|R_{\mu}|$ (see also Figure 2, which represents equation (14): because of equation (17) $\tan a_{\mu}$ gives $|k_{\mu}|k_{\mathrm{S}}^{-1}$ at $\rho=\rho^{\mathrm{I}}$, where $R_2>0$ and $R_3<0$).

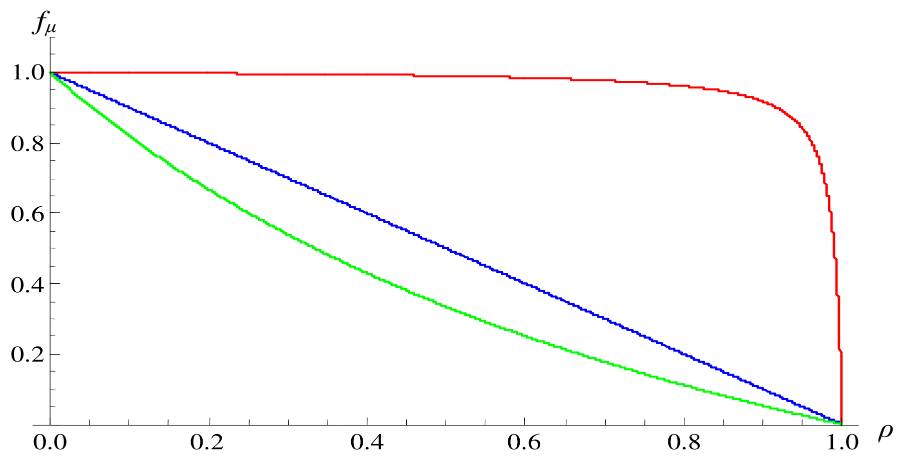

Figure 1. *The prices of non-Sraffian real Standard commodities in terms of the Sraffian Standard commodity as functions of the relative rate of profits*

[13] See Sraffa (1960, §42, footnote 2, and §§56, 64). For the non-Sraffian, real and/or complex, Standard commodities-systems, see also Goodwin (1976, 1977), Velupillai (1990, Part III), Aruka (1991) and Steenge (1995).

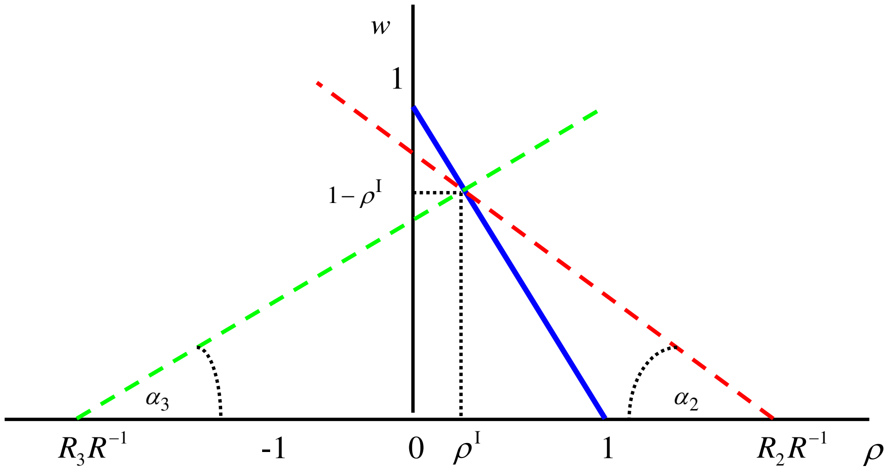

Figure 2. *The $w-\rho$ relationship and the capital-intensities of non-Sraffian real Standard Systems in terms of the Sraffian Standard commodity*

Case 2: If $\mathbf{z}_{\mathbf{A}\mu}$ is a complex eigenvector associated with $\lambda_{\mathbf{J}\mu} = \alpha + i\beta$, $i \equiv \sqrt{-1}$, $\left|\lambda_{\mathbf{J}\mu}\right| \equiv \sqrt{\alpha^2 + \beta^2} < 1$, $\beta \neq 0$, then from (15) we get

$$\mathbf{p}^{\mathrm{T}}(\mathbf{z}_{\mathbf{A}\mu} + \overline{\mathbf{z}}_{\mathbf{A}\mu}) = F_\mu(\rho) \tag{18}$$

where ‘ $^{-}$ ’ signifies the complex conjugate, and

$$F_\mu(\rho) \equiv f_\mu(\rho) + \overline{f}_\mu(\rho) = 2(1-\rho)(1-\rho\alpha)[(1-\rho\alpha)^2 + \rho^2\beta^2]^{-1} \geq 0 \tag{19}$$

or

$$F_\mu(\rho) = 2(1-\rho)(1-\rho\left|\lambda_{\mathbf{J}\mu}\right|\cos\theta)(1-2\rho\left|\lambda_{\mathbf{J}\mu}\right|\cos\theta + \rho^2\left|\lambda_{\mathbf{J}\mu}\right|^2)^{-1} \tag{19a}$$

where $\theta \equiv \arccos(\alpha\left|\lambda_{\mathbf{J}\mu}\right|^{-1})$. Given that (19) can be written as

$$2^{-1}F_\mu(\rho) = (g(\rho) + h(\rho))^{-1}$$

where

$$g(\rho) \equiv [(1-\rho)(1-\rho\alpha)]^{-1}(1-\rho\alpha)^2 \text{ and } h(\rho) \equiv [(1-\rho)(1-\rho\alpha)]^{-1}(\rho^2\beta^2)$$

are strictly increasing functions of ρ,[14] it follows that $F_\mu(\rho)$ is a strictly *decreasing* function of ρ. Moreover, equation (19a) implies that $2^{-1}F_\mu(\rho)$ tends to $(1-\rho)$ as

[14] It is easily checked that

$$g'(\rho) = (1-\rho)^{-2}(1-\alpha)$$

and

$$h'(\rho) = [(1-\rho)(1-\rho\alpha)]^{-2}\rho\beta^2[2-\rho(1+\alpha)]$$

Hence, $g'(\rho) > 0$ and $h'(\rho) > 0$, since $\left|\alpha\right| < 1$ and $\rho(1+\alpha) < 2$.

$\left|\lambda_{\mathbf{J}\mu}\right| \to 0$, to $(1-\rho)(1 \pm \rho\left|\lambda_{\mathbf{J}\mu}\right|)^{-1}$ (a function that is strictly concave (convex) to the origin) as $\cos\theta \to \pm 1$, and to $(1-\rho)(1+\rho^2\left|\lambda_{\mathbf{J}\mu}\right|^2)^{-1}$ (a function that has an inflection point in the interval $2-\sqrt{3}$ $(\simeq 0.270) < \rho < 1/3$) as $\cos\theta \to 0$ (see also Figures 3a-b, which represent $2^{-1}F_{\mu}(\rho)$ and its second derivative with respect to ρ, respectively, for $\left|\lambda_{\mathbf{J}\mu}\right| = 0.6$ and $\cos\theta = \pm 59/60,\ \pm 1/6$; the dashed line, in Figure 3a, represents $1-\rho$). Furthermore, the ratio of the capital-intensity, $k_{\mu} + \bar{k}_{\mu}$, of the vertically integrated sector producing $\mathbf{z}_{\mathbf{A}\mu} + \bar{\mathbf{z}}_{\mathbf{A}\mu}$ to the capital-intensity of the Sraffian Standard system is given by

$$(k_{\mu} + \bar{k}_{\mu})k_{\mathrm{S}}^{-1} = f_{\mu}(\rho)\lambda_{\mathbf{J}\mu} + \bar{f}_{\mu}(\rho)\bar{\lambda}_{\mathbf{J}\mu} \tag{20}$$

from which it follows that

$$\left|k_{\mu} + \bar{k}_{\mu}\right| k_{\mathrm{S}}^{-1} < 2\left|f_{\mu}(\rho)\lambda_{\mathbf{J}\mu}\right| = 2(1-\rho)\left|\lambda_{\mathbf{J}\mu}\right|\left|1-\rho\lambda_{\mathbf{J}\mu}\right|^{-1}$$

or

$$\left|k_{\mu} + \bar{k}_{\mu}\right| k_{\mathrm{S}}^{-1} < 2(1-\rho)\left|\lambda_{\mathbf{J}\mu}\right|(1-\rho\left|\lambda_{\mathbf{J}\mu}\right|)^{-1} < 2$$

Finally,

$$(\left|k_{\mu}\right|^{-1} k_{\mathrm{S}})^2 = \left|f_{\mu}(\rho)\lambda_{\mathbf{J}\mu}\right|^{-2} = [(1-\rho\alpha)^2 + \rho^2\beta^2][(1-\rho)\sqrt{\alpha^2+\beta^2}\,]^{-2} \tag{21}$$

is a strictly increasing function of ρ, since $\rho\alpha < 1$,[15] and, therefore, $\left|k_{\mu}\right|$ is a strictly decreasing function of ρ (however, $\left|k_{\mu} + \bar{k}_{\mu}\right|$ does not necessarily decreases with ρ; see, *e.g.*, Figure 4, which is associated with Figure 3a and represents $\left|k_{\mu} + \bar{k}_{\mu}\right| k_{\mathrm{S}}^{-1}$ as functions of ρ, respectively).

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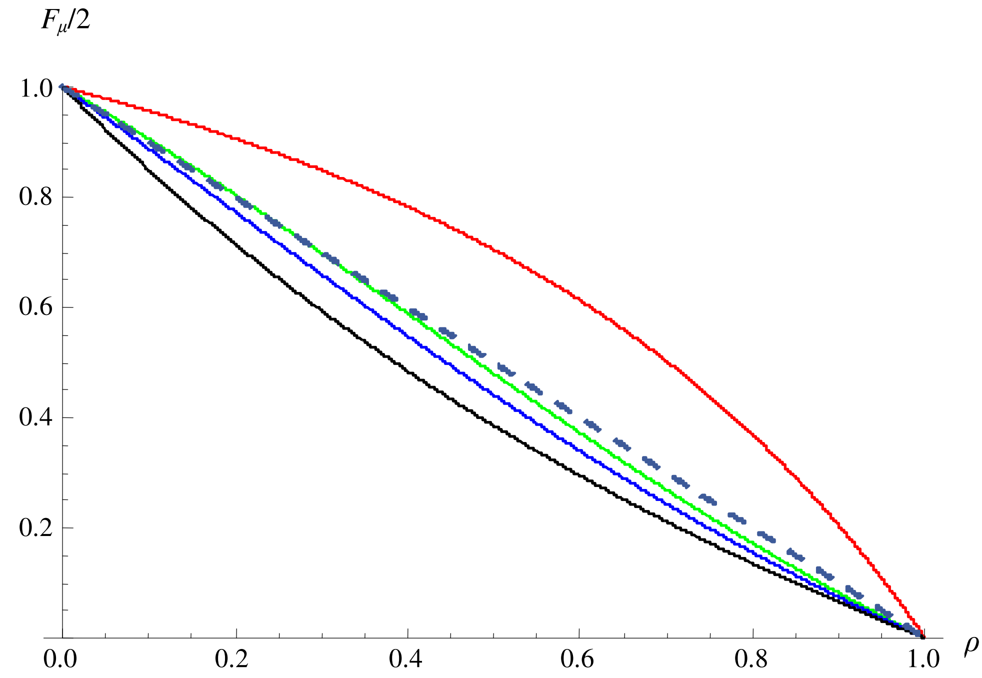
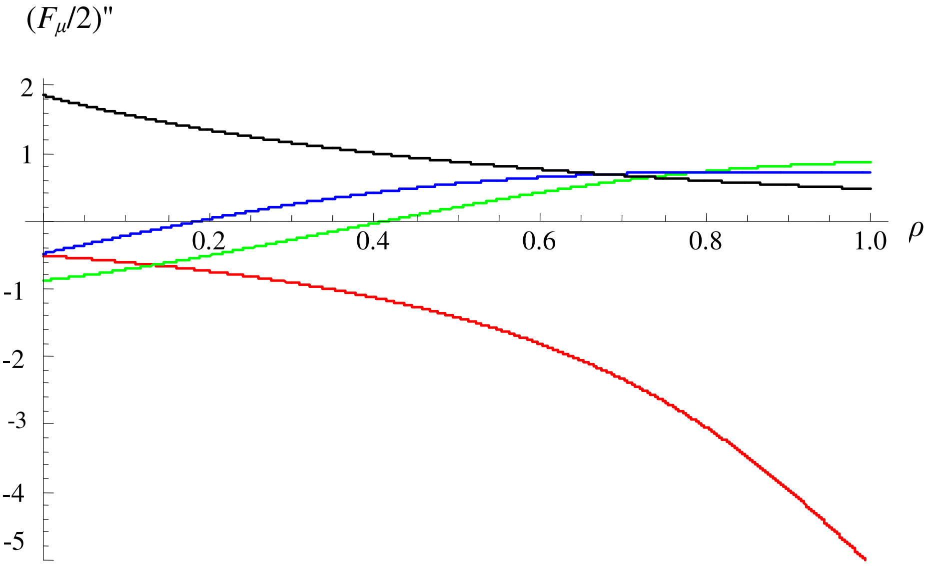
[15] It is easily checked that the first derivative of $(\left|k_{\mu}\right|^{-1} k_{\mathrm{S}})^2$ with respect to ρ equals

$$2[(1-\alpha\rho)(1-\alpha) + \rho\beta^2][(1-\rho)^3\sqrt{\alpha^2+\beta^2}\,]^{-1}$$



(a)



(b)

Figure 3. *The prices of the sum of complex conjugate non-Sraffian Standard commodities in terms of the Sraffian Standard commodity as functions of the relative rate of profits*

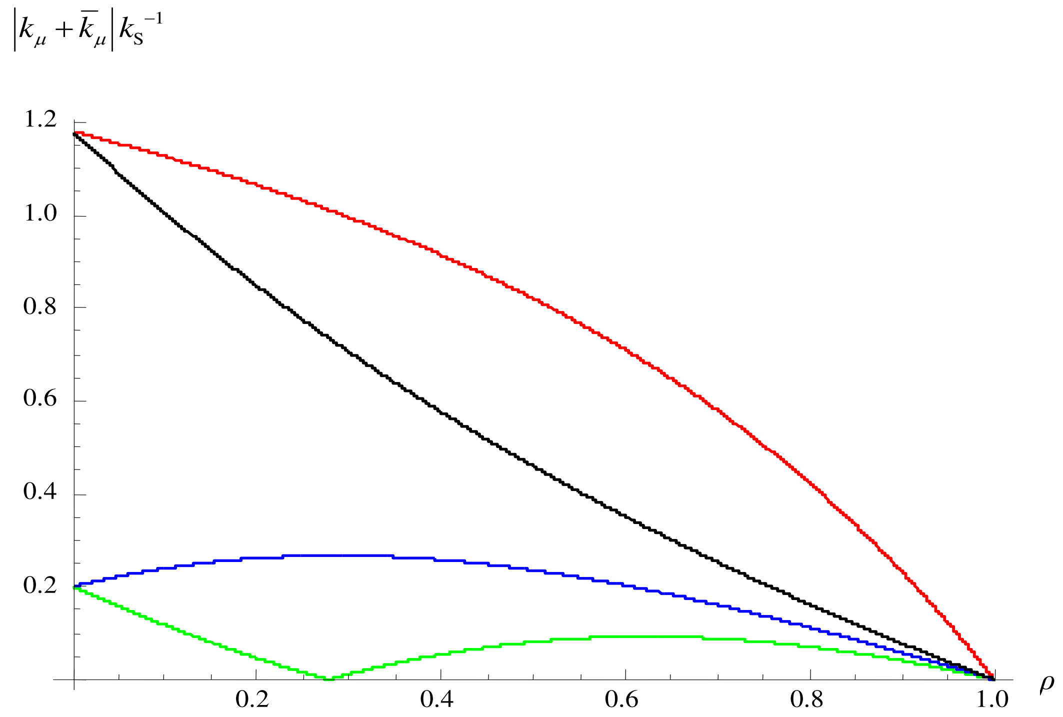


Figure 4. *The absolute value of the capital-intensities of vertically integrated sectors producing the sum of complex conjugate non-Sraffian Standard commodities in terms of the Sraffian Standard commodity as functions of the relative rate of profits*

Thus, we may conclude that, when Sraffa's Standard commodity is chosen as numeraire, the well-known Ricardo's (1951, p. 46) statement regarding the relationship between production prices and changes in income distribution holds true with respect to the (real) commodity bundles $\mathbf{z}_{\mathbf{A}\mu}$ and $\mathbf{z}_{\mathbf{A}\mu} + \bar{\mathbf{z}}_{\mathbf{A}\mu}$: they are labour-intensive relative to the numeraire, in the sense that $\left|k_\mu\right| < k_S$ and $\left|k_\mu + \bar{k}_\mu\right| < 2k_S$, respectively, and their prices decrease with increasing ρ.[16] However, this conclusion is not generally independent of the arbitrary choice of numeraire, since $\left|k_\mu\right|(k_S)^{-1}$ and, therefore, $\mathbf{p}^{\mathrm{T}}\mathbf{z}_{\mathbf{A}\mu}$ are not necessarily monotonic functions of ρ when $\mathbf{z} \neq [\mathbf{I} - \mathbf{A}]\mathbf{x}_{\mathbf{A}1}$

[16] It may be said that this is not unanticipated on the basis of Goodwin's (1976, 1977) method of 'general co-ordinates'. By following an approach which is closer to our, Bidard and Ehrbar (2007, pp. 203-204) show that $\left|k_\mu\right|$ decrease with ρ, and if k_μ is complex, then the derivative of its argument does not change sign, *i.e.*, k_μ moves monotonically either clockwise or counterclockwise across the complex plane. Since there are statements in the theory of international trade (*e.g.*, Stolper-Samuelson effect, 'factor price' equalization theorem) that depend crucially on the existence of monotonic price-profit rate relationships, our conclusion would seem to be of some importance for that theory (see also Metcalfe and Steedman, 1979; Mariolis, 2004).

(see, *e.g.*, Figure 5, where k_2 attains equal values at different values of ρ, and compare with Figure 2).

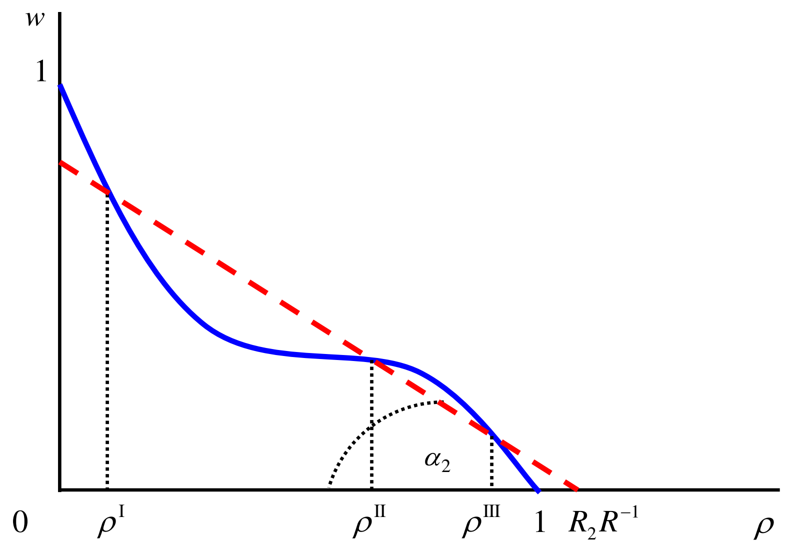


Figure 5. *The* $w-\rho$ *relationship and the capital-intensity of a non-Sraffian positive Standard system in terms of an arbitrary numeraire*

(ii). If the non-dominant eigenvalues of $\mathbf{J}$ are real and very close to each other, *i.e.*,

$$(\lambda_{\mathbf{J}2}, \lambda_{\mathbf{J}3}, \ldots, \lambda_{\mathbf{J}n}) \approx \lambda$$

or, in economic terms, the non-Sraffian Standard systems are real (non-complex) and their Standard ratios are very close to each other, then (12) reduces to

$$w \approx [(1-\rho)^{-1} d_1 + (1-\rho\lambda)^{-1} \sum_{k=2}^{n} d_k]^{-1}$$

or, recalling (11), *i.e.* $\sum_{k=2}^{n} d_k = 1 - d_1$, and ignoring the error,

$$w = (1-\rho)(1-\rho\lambda)\{1-\rho[1-d_1(1-\lambda)]\}^{-1} \tag{22}$$

Double differentiation of (22) with respect to ρ gives

$$w'' = 2d_1(d_1-1)(1-\lambda)^2\{1-\rho[1-d_1(1-\lambda)]\}^{-3} \tag{23}$$

which implies that the $w-\rho$ curve has *no* inflection points irrespective of the numeraire chosen. Moreover, (13) reduces to

$$\mathbf{p}^{\mathrm{T}} = \{1-\rho[1-d_1(1-\lambda)]\}^{-1}[(1-\rho\lambda)\mathbf{y}_{\mathbf{A}1}^{\mathrm{T}} + (1-\rho)\sum_{k=2}^{n} \mathbf{y}_{\mathbf{A}k}^{\mathrm{T}}]$$

or

$$\mathbf{p}^{\mathrm{T}} = \{1-\rho[1-d_1(1-\lambda)]\}^{-1}[\mathbf{y}_{\mathbf{A}1}^{\mathrm{T}} + \sum_{k=2}^{n}\mathbf{y}_{\mathbf{A}k}^{\mathrm{T}} - \rho(\lambda\mathbf{y}_{\mathbf{A}1}^{\mathrm{T}} + \sum_{k=2}^{n}\mathbf{y}_{\mathbf{A}k}^{\mathrm{T}})]$$

or, recalling (6) and (11),

$$\mathbf{p}^{\mathrm{T}} = \{1-\rho[1-d_1(1-\lambda)]\}^{-1}\{\mathbf{v}^{\mathrm{T}} + \rho[(1-\lambda)\mathbf{y}_{\mathbf{A}1}^{\mathrm{T}} - \mathbf{v}^{\mathrm{T}}]\}$$

or, taking into account the price vectors associated with the extreme values of ρ $(=0$ and $1)$, *i.e.*, $\mathbf{p}^{\mathrm{T}}(0) = \mathbf{v}^{\mathrm{T}}$ and $\mathbf{p}^{\mathrm{T}}(1) = d_1^{-1}\mathbf{y}_{\mathbf{A}1}^{\mathrm{T}}$ (see the price normalization equation and (11a)),

$$\mathbf{p}^{\mathrm{T}} = \{1-\rho[1-d_1(1-\lambda)]\}^{-1}\{\mathbf{p}^{\mathrm{T}}(0) + \rho[d_1(1-\lambda)\mathbf{p}^{\mathrm{T}}(1) - \mathbf{p}^{\mathrm{T}}(0)]\} \qquad (24)$$

Since (24) constitutes a rational function of degree 1, it follows that the $p_j - \rho$ curves are monotonic irrespective of the numeraire chosen.[17] Thus, the system retains all the essential properties of *two*-sector economies, in which, however, the 'neoclassical parable relations' do not necessarily hold (see Garegnani, 1970, pp. 408-410, and Kurz and Salvadori, 1995, chs 3 and 14).

Now, it seems to be appropriate to focus on the following three cases:

Case 1: If $\lambda \approx 1$, then (22) and (24) imply that

$$w = 1-\rho \qquad (22a)$$

and

$$\mathbf{p} = \mathbf{p}(0) \qquad (24a)$$

i.e., the 'pure labour theory of value' (Pasinetti, 1977, pp. 76-78) holds true (like in a *one*-sector economy).

Case 2: If $|\lambda| \approx 0$ (clearly, this case is also associated with complex eigenvalues), then (22) and (24) imply that

$$w = (1-\rho)[1-\rho(1-d_1)]^{-1} \qquad (22b)$$

and

$$\mathbf{p}^{\mathrm{T}} = [1-\rho(1-d_1)]^{-1}[\mathbf{p}^{\mathrm{T}}(0) + \rho(d_1\mathbf{p}^{\mathrm{T}}(1) - \mathbf{p}^{\mathrm{T}}(0))] \qquad (24b)$$

Thus, for $d_1 = 1$ we get

[17] For a similar exploration, which focuses on the curvature of the $w-\rho$ curve, see Schefold (2008b, c). Furthermore, it is easily checked that, when we adopt Steedman's numeraire (see footnote 10), (24) takes the form

$$\mathbf{p}^{\mathrm{T}} = (1-\rho\lambda)^{n-2}\{\mathbf{p}^{\mathrm{T}}(0) + \rho[(1-\lambda)^{-n+2}\mathbf{p}^{\mathrm{T}}(1) - \mathbf{p}^{\mathrm{T}}(0)]\}$$

where $\mathbf{p}^{\mathrm{T}}(1)$ is now equal to $(1-\lambda)^{n-1}\mathbf{y}_{\mathbf{A}1}^{\mathrm{T}}$. Hence, the $p_j - \rho$ curves are not necessarily monotonic.

$$\mathbf{p}^{\mathrm{T}} = \mathbf{p}^{\mathrm{T}}(0) + \rho(\mathbf{p}^{\mathrm{T}}(1) - \mathbf{p}^{\mathrm{T}}(0)) \tag{24c}$$

which coincides with Bienenfeld's (1988) *linear* (approximation) formula for the price vector.[18, 19]

Case 3: If $(\lambda_{\mathbf{J}2}, \lambda_{\mathbf{J}3}, ..., \lambda_{\mathbf{J}\kappa}) \approx 1$ and $(\lambda_{\mathbf{J}\kappa+1}, \lambda_{\mathbf{J}\kappa+2}, ..., \lambda_{\mathbf{J}n}) \approx \lambda$, then (22) and (24) still hold, provided only that d_1 is replaced by $d_1 + d_2 + ... + d_\kappa$. However, if $(\lambda_{\mathbf{J}2}, \lambda_{\mathbf{J}3}, ..., \lambda_{\mathbf{J}\kappa}) \approx \lambda^*$ and $(\lambda_{\mathbf{J}\kappa+1}, \lambda_{\mathbf{J}\kappa+2}, ..., \lambda_{\mathbf{J}n}) \approx \lambda$, $(\lambda^*, \lambda) \neq 1$, *or* if $(\lambda_{\mathbf{J}2}, \lambda_{\mathbf{J}3}, ..., \lambda_{\mathbf{J}n}) \approx \alpha \pm i\beta$, $\beta \neq 0$, *i.e.*, the non-dominant eigenvalues are complex and very close to each other, then the system tends to behave as a *three*-sector economy and, therefore, the $w - \rho$ curve may exhibit inflection points and the $p_j - \rho$ curves may be non-monotonic (see also the 3×3 numerical examples provided by Mainwaring, 1978, pp. 16-17, and, Shaikh, 1998, pp. 229-230; the latter presents a price-labour value reversal).[20]

(iii). In the same vein, let us assume that $d_1 = 1$ and $\rho|\lambda_{\mathbf{J}k}| << 1$, which implies that[21]

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[18] It should be noted that Bienenfeld (1988) derives $t-$th order polynomial approximations, $t = 1, 2, ...$, from (i) the so-called 'reduction of prices to dated quantities of *embodied* labour' (Kurz and Salvadori, 1995, p. 175), *i.e.*, $\mathbf{p}^{\mathrm{T}} = (1-\rho)\mathbf{p}(0)^{\mathrm{T}} \sum_{t=0}^{\infty} \rho^t \mathbf{J}^t$ (see (3)); and (ii) the fact that for *any* semi-positive row vector $\mathbf{y}^{\mathrm{T}}$, the vector $\mathbf{y}^{\mathrm{T}}\mathbf{J}^t$ tends to the left P-F eigenvector of $\mathbf{J}$ as t tends to infinity, from which it follows that, for a sufficiently large value of t, we can write $\mathbf{p}(0)^{\mathrm{T}}\mathbf{J}^t \approx \mathbf{p}(0)^{\mathrm{T}}\mathbf{J}^{t+1} \approx ... \approx \mathbf{p}^{\mathrm{T}}(1)$. The accuracy of Bienenfeld's approximations is directly related to the magnitudes of $|\lambda_{\mathbf{J}k}|^{-1}$, and in the (extreme) case in which $\mathbf{A}$ has rank 1, then $\lambda_{\mathbf{J}k} = 0$, $\mathbf{p}^{\mathrm{T}}(0)\mathbf{J} = \mathbf{p}^{\mathrm{T}}(1)$ and, therefore, equation (24c) holds *exactly* (see also Mariolis and Tsoulfidis, 2009, pp. 7-9).

[19] Numerical examples presented in the Appendix to this paper illustrate the points made above.

[20] Garegnani (1970, p. 419, n. 2) notes that 'the wage-curve is a ratio between a polynomial of the n th degree and one of the $(n-1)$th degree in r. […] [S]uch rational functions admit up to $(3n-6)$ points of inflexion. […] Further inquiry would be needed to find whether that maximum number can be reached in the relevant interval $0 < r < R$.'

[21] Consider the $n \times n$ column stochastic matrix $\mathbf{M} \equiv \hat{\mathbf{y}}_{\mathbf{A}1}\mathbf{J}\hat{\mathbf{y}}_{\mathbf{A}1}^{-1}$, which is similar to $\mathbf{J}$, and the elements of which are independent of the choice of physical measurement units and the normalization of $\mathbf{y}_{\mathbf{A}1}$. Applying Hopf's upper bound for the modulus of the subdominant eigenvalue of a positive matrix we get: $\max\{|\lambda_{\mathbf{J}k}|\} \leq (L-s)(L+s)^{-1} < 1$, where L (s) represents the largest (smallest) element of $\mathbf{M}$, and, therefore, we may conclude that when (but not only when) the elements of $\mathbf{M}$ are 'similar', approximation (25) works pretty well (for Hopf's bound, as well as for other, more complicated representations of the upper bounds for the modulus of the subdominant eigenvalue of non-negative matrices, see, *e.g.*, Rothblum and Tan, 1985). Furthermore, from Bródy's (1997) conjecture it directly follows that, when $\mathbf{M}$ is a random matrix, with identically and independently

$$(1-\rho\lambda_{\mathbf{J}k})^{-1} = 1+\rho\lambda_{\mathbf{J}k} + (\rho\lambda_{\mathbf{J}k})^2 + \ldots \approx 1+\rho\lambda_{\mathbf{J}k} \tag{25}$$

Then, ignoring the error, (15) reduces to

$$\mathbf{p}^{\mathrm{T}} = \mathbf{y}_{\mathbf{A}1}^{\mathrm{T}} + (1-\rho)\sum_{k=2}^{n}(1+\rho\lambda_{\mathbf{J}k})\mathbf{y}_{\mathbf{A}k}^{\mathrm{T}}$$

or

$$\mathbf{p}^{\mathrm{T}} = \mathbf{y}_{\mathbf{A}1}^{\mathrm{T}} + \sum_{k=2}^{n}\mathbf{y}_{\mathbf{A}k}^{\mathrm{T}} - \rho\sum_{k=2}^{n}(1-\lambda_{\mathbf{J}k})\mathbf{y}_{\mathbf{A}k}^{\mathrm{T}} - \rho^2\sum_{k=2}^{n}\lambda_{\mathbf{J}k}\mathbf{y}_{\mathbf{A}k}^{\mathrm{T}} \tag{26}$$

Since (26) constitutes a polynomial function of degree 2, it follows that the $p_j - \rho$ curves have at most one extremum point. Moreover, post-multiplying (6) by $\mathbf{J}$, and recalling (11), we get

$$\mathbf{v}^{\mathrm{T}}\mathbf{J} = \mathbf{y}_{\mathbf{A}1}^{\mathrm{T}} + \sum_{k=2}^{n}\lambda_{\mathbf{J}k}\mathbf{y}_{\mathbf{A}k}^{\mathrm{T}} \tag{27}$$

or

$$\mathbf{v}^{\mathrm{T}}\mathbf{J} - \mathbf{v}^{\mathrm{T}} = -\sum_{k=2}^{n}(1-\lambda_{\mathbf{J}k})\mathbf{y}_{\mathbf{A}k}^{\mathrm{T}} \tag{27a}$$

Substituting (27) and (27a) in (26), and recalling $\mathbf{p}^{\mathrm{T}}(0) = \mathbf{v}^{\mathrm{T}}$ and $\mathbf{p}^{\mathrm{T}}(1) = \mathbf{y}_{\mathbf{A}1}^{\mathrm{T}}$ (since $d_1 = 1$), yields

$$\mathbf{p}^{\mathrm{T}} = \mathbf{p}^{\mathrm{T}}(0) + \rho(\mathbf{p}^{\mathrm{T}}(0)\mathbf{J} - \mathbf{p}^{\mathrm{T}}(0)) + \rho^2(\mathbf{p}^{\mathrm{T}}(1) - \mathbf{p}^{\mathrm{T}}(0)\mathbf{J}) \tag{28}$$

which coincides with Bienenfeld's (1988) *quadratic* formula.[22] An alternative, but rather *different* approximation formula, which is also exact at the extreme values of ρ, can be deduced as follows: writing $(1-\rho\lambda_{\mathbf{J}k})^{-1}$ as $1+\rho\lambda_{\mathbf{J}k}(1-\rho\lambda_{\mathbf{J}k})^{-1}$ and substituting in (15) yields

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distributed entries, $\max\{|\lambda_{\mathbf{J}k}|\}$ tends to zero, with speed $n^{-0.5}$, when n tends to infinity (as Sun, 2008, shows, Bródy's conjecture can be proved using theorems provided by Goldberg *et al.*, 2000).

[22] See footnote 18. Since the modulus of the relative error of the approximation (25) equals $(\rho|\lambda_{\mathbf{J}k}|)^2$, the accuracy of (28) increases with decreasing ρ. It should also be noted that, in terms of a sector j, (28) can be written as

$$p_j p_j(0)^{-1} = 1 + \rho(k_j(0)k_{\mathrm{S}}^{-1} - 1) + \rho^2(k_j(1)k_{\mathrm{S}}^{-1} - k_j(0)k_{\mathrm{S}}^{-1}) \tag{28a}$$

where $k_j(0) \equiv \mathbf{p}^{\mathrm{T}}(0)\mathbf{H}_j p_j(0)^{-1}$, $k_j(1) \equiv \mathbf{p}^{\mathrm{T}}(1)\mathbf{H}_j p_j(0)^{-1} = p_j(1)(Rp_j(0))^{-1}$ denote the capital-intensity of the vertically integrated sector producing commodity j at $\rho = 0$ and $\rho = 1$, respectively, and $\mathbf{H}_j$ denotes the j-th column of $\mathbf{H}$. From (28a) it follows that $\rho^{**} = 2\rho^{*}$, where $\rho^{*} \equiv 2^{-1}(k_{\mathrm{S}} - k_j(0))(k_j(1) - k_j(0))^{-1}$ denotes the value of ρ at which the approximate $p_j - \rho$ curve has an extremum point, and ρ^{**} the approximate value of ρ at which there is a price-labour value reversal, *i.e.*, $p_j p_j(0)^{-1} = 1$.

$$\mathbf{p}^{\mathrm{T}} = \mathbf{y}_{\mathbf{A}1}^{\mathrm{T}} + (1-\rho)\sum_{k=2}^{n}\mathbf{y}_{\mathbf{A}k}^{\mathrm{T}} + (1-\rho)\sum_{k=2}^{n}\rho\lambda_{\mathbf{J}k}(1-\rho\lambda_{\mathbf{J}k})^{-1}\mathbf{y}_{\mathbf{A}k}^{\mathrm{T}}$$

or

$$\mathbf{p}^{\mathrm{T}} = \mathbf{p}^{\mathrm{T}}(0) + \rho(\mathbf{p}^{\mathrm{T}}(1) - \mathbf{p}^{\mathrm{T}}(0)) + (1-\rho)\sum_{k=2}^{n}\rho\lambda_{\mathbf{J}k}(1-\rho\lambda_{\mathbf{J}k})^{-1}\mathbf{y}_{\mathbf{A}k}^{\mathrm{T}} \tag{29}$$

Thus, if the moduli of the last $n-\nu$, $2 \le \nu \le n-1$, eigenvalues are sufficiently small that can be considered as negligible, then (29) reduces to

$$\mathbf{p}^{\mathrm{T}} \approx \mathbf{p}^{\mathrm{T}}(0) + \rho(\mathbf{p}^{\mathrm{T}}(1) - \mathbf{p}^{\mathrm{T}}(0)) + (1-\rho)\sum_{k=2}^{\nu}\rho\lambda_{\mathbf{J}k}(1-\rho\lambda_{\mathbf{J}k})^{-1}\mathbf{y}_{\mathbf{A}k}^{\mathrm{T}} \tag{30}$$

where the sum of the first two terms coincides with Bienenfeld's linear approximation (see equation (24c)), and if $\lambda_{\mathbf{J}k}$ is positive (negative), then the non-linear term $f_{sk}(\rho) \equiv (1-\rho)\rho\lambda_{\mathbf{J}k}(1-\rho\lambda_{\mathbf{J}k})^{-1}$ is a semi-positive (semi-negative) and strictly concave (convex) function of ρ, which is maximized (minimized) at $\rho' \equiv (1-\sqrt{1-\lambda_{\mathbf{J}k}})\lambda_{\mathbf{J}k}^{-1}$, where $-1+\sqrt{2} \simeq 0.414 < \rho' < 1$ and $-3+2\sqrt{2} \simeq -0.172 < f_{sk}(\rho') < 1$, since $|\lambda_{\mathbf{J}k}| < 1$. Relation (30) could be called a '$\nu-$th order spectral approximation'.

On the basis of this analysis, it may be argued that the monotonicity of the $p_j - \rho$ curves depends to a great extent on the distribution of the eigenvalues of matrix $\mathbf{J}$ or, alternatively, the exploration of the relationship between production prices and the profit rate may be reduced to an exploration of the aforementioned eigenvalue distribution.

Finally, it need hardly be said that, in terms (at least) of the well-known 'Leontief-Bródy approach' (see Bródy, 1970, ch. 1.2; Mathur, 1977), our analysis remains valid for the (more realistic) case of fixed capital and/or differential profit rates. Equation (1) becomes

$$\mathbf{p}^{\mathrm{T}} = w\mathbf{l}^{\mathrm{T}} + \mathbf{p}^{\mathrm{T}}\mathbf{A}^{+} + \mathbf{p}^{\mathrm{T}}\mathbf{K}\hat{\mathbf{r}} \tag{1a}$$

where $\mathbf{A}^{+} \equiv \mathbf{A} + \mathbf{D}$, $\mathbf{D}$ denotes the matrix of depreciation coefficients, $\mathbf{K}$ the matrix of capital stock coefficients and $\hat{\mathbf{r}}$ the diagonal matrix of the sectoral rates of profits, r_i. Provided that r_i exhibit a stable structure in relative terms, which implies that $\hat{\mathbf{r}}$ can be written as $r\hat{\hat{\mathbf{r}}}$, where $\hat{\hat{\mathbf{r}}}$ represents the relative magnitudes of the rates of profits in different sectors and r now represents the 'overall level' of the rates of profits (or, alternatively, the 'reference' rate of profits),[23] (1a) can be written as

$$\mathbf{p}^{\mathrm{T}} = (w\mathbf{l}^{\mathrm{T}} + r\mathbf{p}^{\mathrm{T}}\mathbf{K}\hat{\mathbf{r}})[\mathbf{I} - \mathbf{A}^{+}]^{-1}$$

or

$$\mathbf{p}^{\mathrm{T}} = w[\mathbf{v}^{+}]^{\mathrm{T}} + \rho^{+}\mathbf{p}^{\mathrm{T}}\mathbf{J}^{+} \tag{2a}$$

where $[\mathbf{v}^{+}]^{\mathrm{T}} \equiv \mathbf{l}^{\mathrm{T}}[\mathbf{I} - \mathbf{A}^{+}]^{-1}$ denotes the vector of labour values, $\mathbf{J}^{+} \equiv R^{+}\mathbf{H}^{+}$, $\mathbf{H}^{+} \equiv \mathbf{K}\hat{\mathbf{r}}[\mathbf{I} - \mathbf{A}^{+}]^{-1}$, $R^{+} \equiv (\lambda_{\mathbf{H}^{+}1})^{-1}$ and $\rho^{+} \equiv r(R^{+})^{-1}$. It then follows that (2a) is formally equivalent to (2).

3. Empirical Evidence

The application of the previous analysis to the input-output tables of actual economies (*i.e.*, China, Greece, Japan, Korea, and USA) gives the results summarized in Tables 1 to 3.

The two-part Table 1 reports the moduli of the eigenvalues of $\mathbf{J}$ (in descending order)[24] and six measures of the distribution of the moduli of the non-dominant eigenvalues of $\mathbf{J}$, namely, (i) the arithmetic mean, *AM*, that gives equal weight to all moduli; (ii) the geometric mean, *GM*, which in our case can be written as $|\det\mathbf{J}|^{1/(n-1)}$ and assigns more weight to lower moduli, and, therefore, is more appropriate for detecting the central tendency of an exponential set of numbers; (iii) the so-called spectral flatness, *SF*, defined as the ratio of the geometric mean to the arithmetic mean; (iv) $\pi_2 \equiv \max\{\pi_k \equiv |\lambda_{\mathbf{J}k}| / \sum_{k=2}^{n}|\lambda_{\mathbf{J}k}|\}$, where π_k represents a set of relative frequencies; (v) the relative (or normalized) entropy, *RE*, defined as the ratio of the 'information content or Shannon entropy', *E*, to its maximum possible value, *i.e.*, $RE \equiv E / E_{\max}$, where $E \equiv -\sum_{k=2}^{n}\pi_k \log\pi_k$ and $E_{\max} \equiv \log(n-1)$ is the maximum value of *E* corresponding to $\pi_k = 1/(n-1)$ for all *k*; and (vi) the relative 'equivalent

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[23] For instance, this rate could be the average or the minimum rate of profits of the system. See, *e.g.*, Steedman (1977, pp. 180-181); Reati (1986, pp. 159-160).

[24] The dimensions of the symmetric input-output tables (SIOT) vary from 19 sectors (Greece, 1988-97) to 39 sectors (USA). The tables of China and Japan are available from the OECD STAN database. Those of Greece and Korea are provided by the National Statistical Service of Greece and the Bank of Korea, respectively. Finally, those of USA are from the Bureau of Economic Analysis (BEA) and have been compiled by Juillard (1986) (the data used in the studies by Ochoa, 1984, Bienenfeld, 1988, and Shaikh, 1998, are from the same source although at 71 x 71 sector detail).

number', *REN*, defined as $EN/(n-1)$, where *EN* denotes the so-called equivalent number, which is determined by the equation $\log EN = E$ and represents the number of eigenvalues with equal moduli that would result in the same amount of entropy. *SF* and *RE* are known to be alternative, but different, measures of similarity (or closeness) of the moduli and take on values from near zero to one: when all $|\lambda_{\mathbf{J}k}|$ are equal to each other, then $AM = GM$, $\pi_k = 1/(n-1)$ and, therefore, $SF = RE = REN = 1$. However, a low *SF* rather reflects the presence of a much lower than the average $\min\{\pi_k\}$, whereas a low *RE* rather reflects the presence of a much higher than the average π_2.[25]

[25] Finkelstein and Friedberg (1967) discuss *E* and *EN*, and apply them to studies of industrial competition and concentration, whilst Jasso (1982) and Bailey (1985) discuss *SF* and *RE*, respectively, and apply them to studies of income distribution. It may also be noted that there is a connection between *SF* and entropy: using π_k, the former can be expressed as

$$SF = (n-1)\prod_{k=2}^{n} \pi_k^{1/(n-1)}$$

or, taking the logarithm of both sides,

$$\log SF = E_{\max} - [-(n-1)^{-1}\sum_{k=2}^{n} \log\pi_k]$$

where $\log SF$ is known as the Wiener entropy and the term in brackets can be conceived as a 'cross-entropy' expression.

Table 1. *The distribution of the moduli of the non-dominant eigenvalues; China, Greece, Japan, Korea and USA*

	CHN 1997	**GRC 1970**	**GRC 1988**	**GRC 1989**	**GRC 1990**	**GRC 1991**	**GRC 1992**	**GRC 1993**	**GRC 1994**	**GRC 1995**	**GRC 1996**	**GRC 1997**
Rank												
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	0.376	0.726	0.643	0.683	0.675	0.657	0.624	0.667	0.678	0.655	0.664	0.641
3	0.304	0.539	0.416	0.436	0.418	0.397	0.443	0.433	0.420	0.382	0.382	0.350
4	0.282	0.470	0.409	0.377	0.376	0.382	0.443	0.353	0.357	0.382	0.382	0.307
5	0.231	0.453	0.362	0.377	0.376	0.382	0.406	0.320	0.327	0.281	0.313	0.279
6	0.224	0.319	0.259	0.308	0.311	0.326	0.308	0.268	0.261	0.246	0.233	0.249
7	0.224	0.319	0.187	0.207	0.218	0.226	0.242	0.234	0.207	0.202	0.214	0.249
8	0.167	0.243	0.187	0.207	0.218	0.226	0.242	0.234	0.207	0.202	0.214	0.210
9	0.167	0.243	0.083	0.104	0.110	0.101	0.108	0.110	0.109	0.098	0.098	0.103
10	0.165	0.218	0.083	0.082	0.089	0.094	0.105	0.105	0.097	0.092	0.088	0.098
11	0.142	0.201	0.079	0.082	0.089	0.094	0.105	0.105	0.097	0.092	0.088	0.098
12	0.126	0.201	0.079	0.080	0.080	0.078	0.081	0.083	0.082	0.085	0.086	0.087
13	0.122	0.166	0.071	0.080	0.080	0.078	0.081	0.068	0.082	0.085	0.086	0.042
14	0.114	0.106	0.071	0.031	0.039	0.034	0.053	0.068	0.059	0.023	0.072	0.035
15	0.114	0.106	0.027	0.031	0.028	0.034	0.029	0.026	0.026	0.023	0.029	0.035
16	0.102	0.103	0.027	0.024	0.022	0.023	0.027	0.026	0.026	0.015	0.029	0.017
17	0.102	0.100	0.020	0.024	0.022	0.023	0.027	0.017	0.023	0.015	0.019	0.017
18	0.062	0.092	0.009	0.007	0.009	0.008	0.005	0.006	0.007	0.005	0.002	0.013
19	0.058	0.088	0.006	0.006	0.006	0.005	0.003	0.002	0.006	0.004	0.001	0.001
20	0.058	0.074	….	….	….	….	….	….	….	….	….	….
21	0.052	0.060	….	….	….	….	….	….	….	….	….	….
22	0.044	0.060	….	….	….	….	….	….	….	….	….	….
23	0.041	0.043	….	….	….	….	….	….	….	….	….	….
24	0.041	0.043	….	….	….	….	….	….	….	….	….	….
25	0.034	0.037	….	….	….	….	….	….	….	….	….	….
26	0.034	0.037	….	….	….	….	….	….	….	….	….	….
27	0.033	0.030	….	….	….	….	….	….	….	….	….	….
28	0.025	0.029	….	….	….	….	….	….	….	….	….	….
29	0.025	0.023	….	….	….	….	….	….	….	….	….	….
30	0.021	0.015	….	….	….	….	….	….	….	….	….	….
31	0.021	0.008	….	….	….	….	….	….	….	….	….	….
32	0.018	0.008	….	….	….	….	….	….	….	….	….	….
33	0.006	0.003	….	….	….	….	….	….	….	….	….	….
34	0.006	….	….	….	….	….	….	….	….	….	….	….
35	0.005	….	….	….	….	….	….	….	….	….	….	….
36	0.005	….	….	….	….	….	….	….	….	….	….	….
37	0.002	….	….	….	….	….	….	….	….	….	….	….
38	0.001	….	….	….	….	….	….	….	….	….	….	….
AM	0.096	0.161	0.168	0.175	0.176	0.176	0.185	0.174	0.171	0.161	0.167	0.157
GM	0.048	0.083	0.086	0.086	0.088	0.087	0.089	0.081	0.088	0.074	0.074	0.073
SF	0.499	0.517	0.511	0.490	0.500	0.495	0.483	0.469	0.513	0.459	0.446	0.462
π_2	11%	14%	21%	22%	21%	21%	19%	21%	22%	23%	22%	23%
RE	0.873	0.856	0.829	0.824	0.829	0.831	0.837	0.835	0.836	0.822	0.834	0.832
REN	62%	59%	61%	61%	61%	61%	61%	61%	61%	61%	61%	61%

contd.

	JPN 1970	**JPN 1975**	**JPN 1980**	**JPN 1985**	**JPN 1990**	**KOR 1995**	**KOR 2000**	**USA 1947**	**USA 1958**	**USA 1963**	**USA 1967**	**USA 1972**	**USA 1977**
Rank													
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	0.652	0.711	0.762	0.735	0.737	0.638	0.683	0.620	0.571	0.638	0.639	0.648	0.527
3	0.434	0.445	0.474	0.653	0.604	0.421	0.517	0.462	0.571	0.582	0.552	0.512	0.386
4	0.388	0.381	0.474	0.572	0.604	0.373	0.422	0.436	0.451	0.479	0.421	0.400	0.378
5	0.346	0.381	0.362	0.538	0.424	0.314	0.321	0.390	0.451	0.461	0.421	0.400	0.378
6	0.303	0.332	0.321	0.396	0.351	0.271	0.303	0.334	0.376	0.461	0.399	0.306	0.330
7	0.303	0.340	0.318	0.396	0.351	0.266	0.303	0.325	0.358	0.323	0.277	0.306	0.330
8	0.263	0.261	0.318	0.336	0.320	0.266	0.286	0.282	0.327	0.264	0.268	0.286	0.263
9	0.244	0.261	0.292	0.328	0.320	0.185	0.198	0.257	0.261	0.264	0.265	0.242	0.226
10	0.244	0.258	0.270	0.219	0.303	0.111	0.141	0.205	0.255	0.257	0.265	0.236	0.226
11	0.218	0.200	0.260	0.219	0.236	0.111	0.128	0.205	0.236	0.237	0.255	0.236	0.220
12	0.177	0.169	0.165	0.157	0.191	0.107	0.128	0.197	0.230	0.237	0.243	0.212	0.220
13	0.152	0.169	0.153	0.152	0.178	0.079	0.127	0.197	0.230	0.216	0.228	0.212	0.198
14	0.152	0.067	0.153	0.137	0.166	0.068	0.127	0.185	0.212	0.203	0.228	0.196	0.180
15	0.116	0.067	0.144	0.132	0.152	0.062	0.093	0.161	0.212	0.203	0.182	0.182	0.147
16	0.107	0.149	0.120	0.132	0.146	0.048	0.076	0.139	0.174	0.181	0.182	0.150	0.147
17	0.094	0.109	0.120	0.132	0.143	0.048	0.076	0.131	0.174	0.171	0.160	0.150	0.137
18	0.094	0.109	0.088	0.132	0.143	0.047	0.073	0.131	0.163	0.171	0.150	0.142	0.137
19	0.082	0.116	0.085	0.123	0.124	0.033	0.036	0.102	0.161	0.138	0.150	0.126	0.116
20	0.056	0.058	0.082	0.099	0.105	0.033	0.036	0.102	0.120	0.138	0.138	0.126	0.102
21	0.046	0.058	0.067	0.070	0.100	0.027	0.028	0.096	0.120	0.133	0.129	0.107	0.102
22	0.046	0.098	0.055	0.070	0.085	0.015	0.024	0.091	0.116	0.133	0.129	0.107	0.086
23	0.037	0.041	0.048	0.058	0.051	0.015	0.022	0.083	0.101	0.090	0.088	0.096	0.086
24	0.036	0.090	0.048	0.051	0.051	0.004	0.018	0.080	0.101	0.090	0.088	0.078	0.082
25	0.036	0.051	0.040	0.051	0.039	0.001	0.005	0.080	0.097	0.089	0.085	0.078	0.082
26	0.034	0.051	0.037	0.050	0.039	….	….	0.071	0.060	0.089	0.085	0.066	0.059
27	0.034	0.036	0.037	0.036	0.027	….	….	0.066	0.060	0.076	0.075	0.051	0.059
28	0.028	0.020	0.030	0.026	0.027	….	….	0.066	0.057	0.053	0.075	0.047	0.046
29	0.011	0.020	0.019	0.020	0.026	….	….	0.051	0.057	0.041	0.046	0.036	0.035
30	0.011	0.004	0.019	0.020	0.026	….	….	0.031	0.030	0.041	0.046	0.036	0.031
31	0.008	0.004	0.014	0.014	0.024	….	….	0.029	0.030	0.036	0.037	0.031	0.031
32	0.008	0.003	0.009	0.012	0.024	….	….	0.029	0.026	0.036	0.037	0.031	0.030
33	0.001	0.005	0.000	0.008	0.003	….	….	0.025	0.024	0.027	0.033	0.026	0.030
34	….	….	….	….	….	….	….	0.008	0.024	0.027	0.033	0.026	0.024
35	….	….	….	….	….	….	….	0.008	0.019	0.024	0.020	0.019	0.019
36	….	….	….	….	….	….	….	0.006	0.014	0.018	0.015	0.016	0.014
37	….	….	….	….	….	….	….	0.006	0.012	0.012	0.015	0.009	0.008
38	….	….	….	….	….	….	….	0.004	0.002	0.012	0.015	0.009	0.008
39	….	….	….	….	….	….	….	0.004	0.002	0.009	0.002	0.007	0.007
AM	0.149	0.158	0.168	0.190	0.191	0.148	0.174	0.150	0.171	0.175	0.171	0.156	0.144
GM	0.074	0.079	0.074	0.103	0.108	0.068	0.098	0.078	0.090	0.104	0.101	0.091	0.086
SF	0.495	0.497	0.440	0.544	0.562	0.459	0.563	0.523	0.527	0.593	0.591	0.583	0.597
π_2	14%	14%	14%	12%	12%	18%	16%	11%	9%	10%	10%	11%	10%
RE	0.863	0.866	0.866	0.863	0.875	0.837	0.862	0.880	0.888	0.891	0.897	0.888	0.894
REN	61%	63%	63%	59%	63%	58%	63%	63%	66%	66%	68%	66%	66%

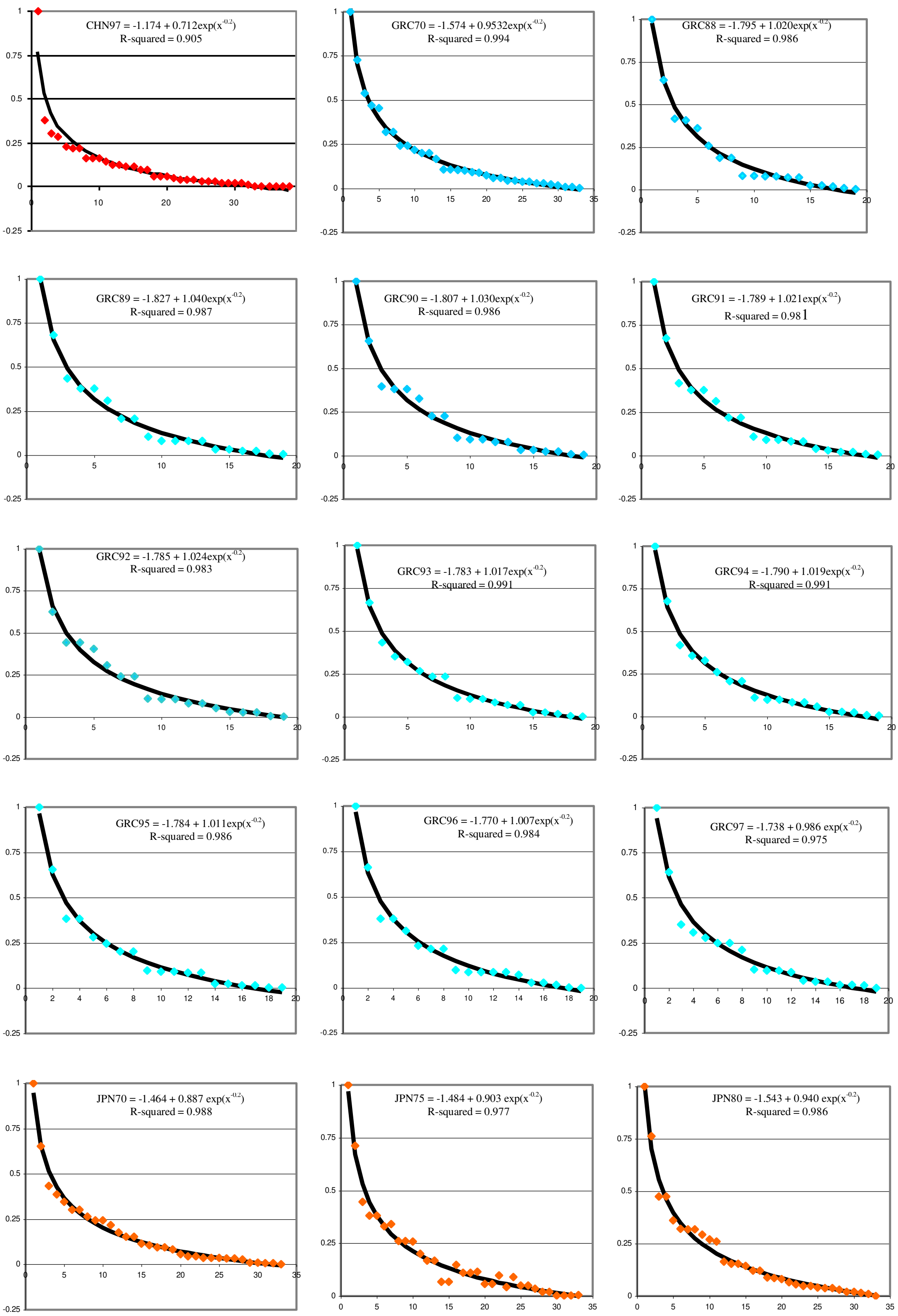
From the numerical results of Table 1 it becomes apparent that the moduli fall quite rapidly in the 'beginning' and then constellate in much lower values. In plotting these data for each of the countries and years, and after experimentation with various possible functional forms, we found that a single exponential functional form fits all the data pretty well, as this can be judged by the high R-square (*i.e.*, in the range of 90.5% (China)-99.4% (Greece, 1970)) as well as by the fact that all coefficients are statistically significant with zero probability value. This form is

$$y = c + b\exp(x^{-0.2})$$

where -1.827 (Greece, 1989) $\leq c \leq$ -1.174 (China) and 0.721 (China) $\leq b \leq 1.040$ (Greece, 1989) (see Figure 6).[26] It is expected, therefore, that the *SF* would be relatively low and that the opposite would hold true regarding *RE*. Indeed, it is found that the former is in the range of 0.440 (Japan, 1980)-0.597 (USA, 1977), whilst the latter is in the range of 0.822 (Greece, 1995)-0.897 (USA, 1967) and the relevant maxima relative frequencies, π_2, are 23% and 10%, respectively. Moreover, the *REN* is in the range of 58% (Korea, 1995)-68% (USA, 1967).[27] Thus, it could be concluded that these measures in combination give a quite good description of the central tendency and also the skewness of the distribution of the moduli.

[26] In fact, we tried an optimization procedure to find the best possible form, and from the many possibilities we opted for a simple but, at the same time, general enough to fit the moduli of the eigenvalues of all countries and years.

[27] It should be noted that we have also experimented with the input-output tables of Canada (1997, 34 x 34; source: OECD STAN database), Japan (1995-1997, 41 x 41; source: OECD STAN database), UK (1998, 40 x 40; source: OECD STAN database) and USA (1997, 40 x 40; source: BEA, compilation through the OECD STAN database), and the results were quite similar, *i.e.*, *SF*: 0.359 (USA)-0.500 (UK), π_2: 8% (UK)-18% (Canada), *RE*: 0.811 (Canada)-0.888 (UK), and *REN*: 52% (Canada)-67% (UK). The same holds true for the results reported by Mariolis *et al.* (2010) regarding the 59 x 59 input-output tables (source: Eurostat) of Denmark (2000, 2004), Finland (1995, 2004), France (1995, 2005), Germany (2000, 2002) and Sweden (1995, 2005): *SF*: 0.450 (France, 1995)-0.603 (Denmark, 2004), π_2: 6% (Germany, 2000 and 2002)-15% (Finland, 2004), *RE*: 0.821(Finland, 2004)-0.900 (Germany, 2000 and 2002), and *REN*: 50% (Finland, 1995, and Sweden, 1995)-66% (Germany, 2000 and 2002).


CHN97 = -1.174 + 0.712exp($x^{-0.2}$)
R-squared = 0.905
GRC70 = -1.574 + 0.9532exp($x^{-0.2}$)
R-squared = 0.994
GRC88 = -1.795 + 1.020exp($x^{-0.2}$)
R-squared = 0.986
GRC89 = -1.827 + 1.040exp($x^{-0.2}$)
R-squared = 0.987
GRC90 = -1.807 + 1.030exp($x^{-0.2}$)
R-squared = 0.986
GRC91 = -1.789 + 1.021exp($x^{-0.2}$)
R-squared = 0.981
GRC92 = -1.785 + 1.024exp($x^{-0.2}$)
R-squared = 0.983
GRC93 = -1.783 + 1.017exp($x^{-0.2}$)
R-squared = 0.991
GRC94 = -1.790 + 1.019exp($x^{-0.2}$)
R-squared = 0.991
GRC95 = -1.784 + 1.011exp($x^{-0.2}$)
R-squared = 0.986
GRC96 = -1.770 + 1.007exp($x^{-0.2}$)
R-squared = 0.984
GRC97 = -1.738 + 0.986 exp($x^{-0.2}$)
R-squared = 0.975
JPN70 = -1.464 + 0.887 exp($x^{-0.2}$)
R-squared = 0.988
JPN75 = -1.484 + 0.903 exp($x^{-0.2}$)
R-squared = 0.977
JPN80 = -1.543 + 0.940 exp($x^{-0.2}$)
R-squared = 0.986

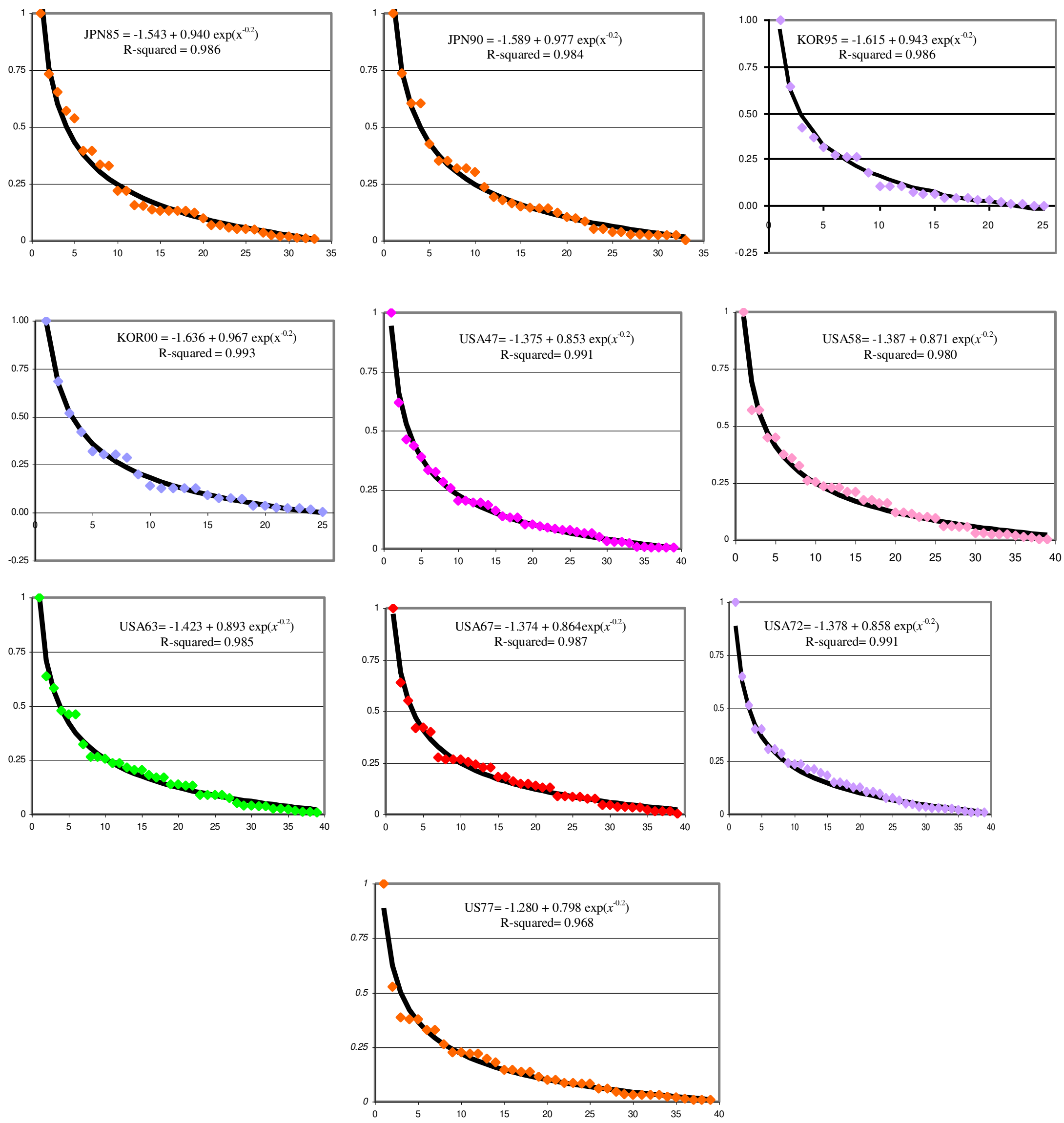


Figure 6. *Exponential fit of the distribution of the moduli of the eigenvalues; China, Greece, Japan, Korea and USA*

For reasons of clarity of presentation and economy of space, the numerical results displayed in Table 2 are only associated with the input-output tables of Japan and seek to detect the dependence of the distribution of the moduli on the level of aggregation, that is to say, n.[28] More specifically, we experimented with input-output tables for every 5 years starting from 1980 until 2005 for the 100 x 100 industry structure and we also repeated the experiment aggregating each of these input-output tables into 21 sectors.[29] In our aggregation, we put together similar industries and we

[28] See footnote 21.
[29] The original input-output data comprised 108 sectors comes from the Statistical Service of Japan. The problem with this data set is that 8 of the sectors have zero rows (*i.e.*, they do not deliver any

kept mainly the manufacturing as the most disaggregated from all the sectors. Finally, for reasons of economy in space, we present only the first 30 moduli and the last six rows display the statistical measures of the distribution. Clearly, the results suggest that *RE* decreases, whilst π_2 and *REN* increases, with decreasing n. On the other hand, they do *not* suggest that the modulus of the subdominant eigenvalues (as well as *SF*) tends to increase with decreasing n: it could be considered as rigid and the 'small' relative changes that we observe go to either direction (varying from -8.3% to 3.6%). Moreover, in Figure 7a below we display the histogram of the distribution of the moduli of the non-dominant eigenvalues associated with the 21 x 21 tables and in Figure 7b we display the histogram associated with the 100 x 100 tables, *i.e.*, 120 and 594 observations, respectively. On the top of each bar we report the number of observations in each of our 5 bins, the mean value of each bin and the bin edges. Clearly, the majority of the observations (*i.e.*, 62 (52%) or 411 (69.2%), respectively) constellate in the lowest bin, whereas 9 (7.5%) or 10 (1.7%), respectively, observations are on an average less than one-half of the dominant eigenvalue.

output to the other sectors and to themselves), which give rise to an input-output structure with 'non-basic' (in the sense of Sraffa) sectors, and, therefore, zero eigenvalues corresponding to each of these 8 sectors. To side step this problem we aggregated each of these 8 sectors to corresponding similar sectors so as the resulting input-output structure consists of dimensions 100 x 100 'basic' sectors. Finally, it should be noted that the results displayed in Table 2 are not comparable with these displayed in Table 1, since the 33 sectors input-output tables of Japan are constructed using different sources and also methodology.

Table 2. *The distribution of the moduli of the non-dominant eigenvalues and the level of aggregation; Japan, 1980-2005*

	1980	1980	1985	1985	1990	1990	1995	1995	2000	2000	2005	2005
n	**21**	**100**	**21**	**100**	**21**	**100**	**21**	**100**	**21**	**100**	**21**	**100**
Rank												
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	0.529	0.522	0.546	0.527	0.520	0.550	0.496	0.541	0.465	0.499	0.517	0.519
3	0.342	0.379	0.391	0.390	0.415	0.448	0.496	0.497	0.436	0.499	0.443	0.512
4	0.342	0.379	0.383	0.386	0.413	0.410	0.383	0.393	0.436	0.410	0.443	0.421
5	0.330	0.351	0.383	0.386	0.413	0.410	0.383	0.393	0.355	0.410	0.342	0.421
6	0.301	0.351	0.295	0.342	0.316	0.352	0.359	0.363	0.355	0.370	0.310	0.394
7	0.200	0.296	0.276	0.342	0.229	0.352	0.249	0.363	0.264	0.352	0.236	0.355
8	0.159	0.296	0.145	0.309	0.229	0.345	0.249	0.346	0.219	0.352	0.179	0.331
9	0.140	0.270	0.145	0.309	0.145	0.337	0.182	0.339	0.219	0.333	0.144	0.308
10	0.140	0.270	0.140	0.271	0.133	0.337	0.145	0.339	0.133	0.323	0.144	0.282
11	0.097	0.251	0.115	0.254	0.133	0.334	0.122	0.257	0.133	0.238	0.082	0.258
12	0.097	0.251	0.096	0.213	0.079	0.232	0.079	0.257	0.068	0.238	0.062	0.258
13	0.075	0.192	0.066	0.205	0.079	0.232	0.079	0.247	0.068	0.230	0.062	0.232
14	0.075	0.191	0.066	0.201	0.079	0.230	0.067	0.247	0.060	0.225	0.058	0.191
15	0.054	0.191	0.051	0.188	0.071	0.230	0.067	0.234	0.060	0.225	0.044	0.182
16	0.022	0.166	0.044	0.184	0.071	0.226	0.061	0.219	0.048	0.202	0.044	0.182
17	0.012	0.166	0.019	0.184	0.016	0.218	0.013	0.196	0.021	0.190	0.023	0.177
18	0.012	0.144	0.011	0.156	0.015	0.205	0.012	0.196	0.018	0.190	0.018	0.177
19	0.010	0.144	0.011	0.147	0.010	0.197	0.005	0.172	0.018	0.188	0.018	0.175
20	0.005	0.136	0.007	0.133	0.007	0.191	0.005	0.172	0.006	0.182	0.013	0.175
21	0.002	0.128	0.005	0.133	0.005	0.174	0.004	0.164	0.003	0.162	0.003	0.166
22	….	0.124	….	0.125	….	0.163	….	0.164	….	0.152	….	0.149
23	….	0.124	….	0.125	….	0.163	….	0.162	….	0.152	….	0.146
24	….	0.123	….	0.123	….	0.156	….	0.162	….	0.150	….	0.146
25	….	0.123	….	0.123	….	0.147	….	0.162	….	0.150	….	0.138
26	….	0.122	….	0.121	….	0.147	….	0.156	….	0.149	….	0.138
27	….	0.120	….	0.117	….	0.142	….	0.156	….	0.149	….	0.119
28	….	0.110	….	0.117	….	0.142	….	0.143	….	0.139	….	0.119
29	….	0.107	….	0.106	….	0.137	….	0.140	….	0.137	….	0.115
30	….	0.107	….	0.100	….	0.132	….	0.140	….	0.128	….	0.115
AM	0.147	0.090	0.160	0.091	0.169	0.106	0.173	0.110	0.169	0.105	0.159	0.099
GM	0.067	0.035	0.078	0.040	0.085	0.051	0.077	0.056	0.083	0.052	0.079	0.048
SF	0.452	0.389	0.487	0.440	0.501	0.482	0.448	0.511	0.491	0.498	0.499	0.487
π_2	18%	6%	17%	6%	15%	5%	14%	5%	14%	5%	16%	5%
RE	0.840	0.879	0.841	0.878	0.852	0.887	0.844	0.894	0.850	0.888	0.837	0.878
REN	62%	57%	62%	57%	64%	59%	62%	61%	63%	60%	61%	57%

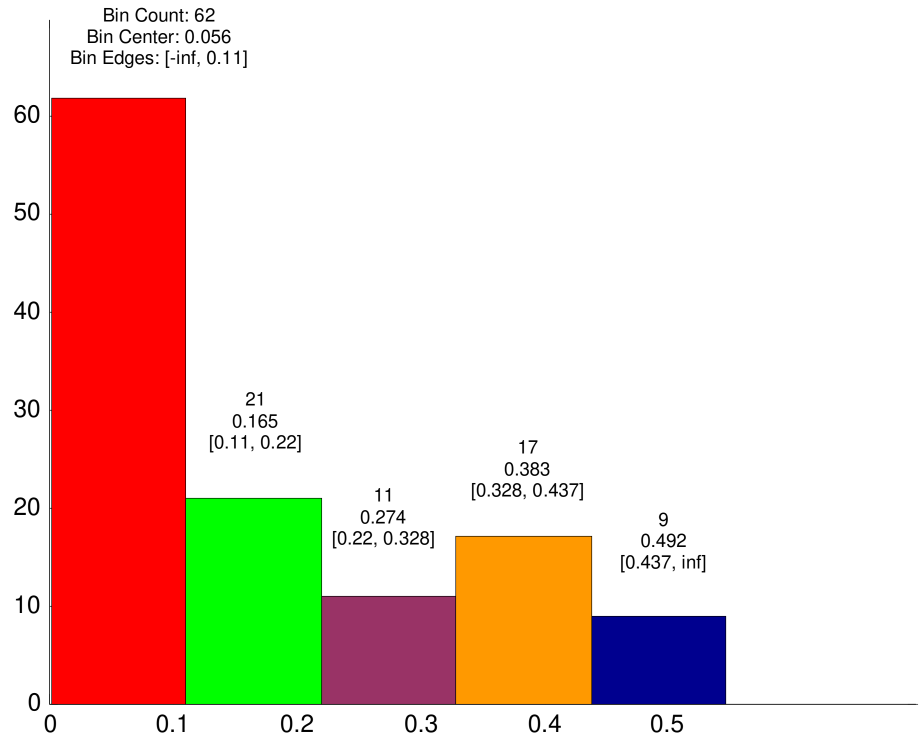


Figure 7a. *Histogram of the distribution of the moduli of the non-dominant eigenvalues; Japan, 1980-2005, 21 x 21 input-output tables*

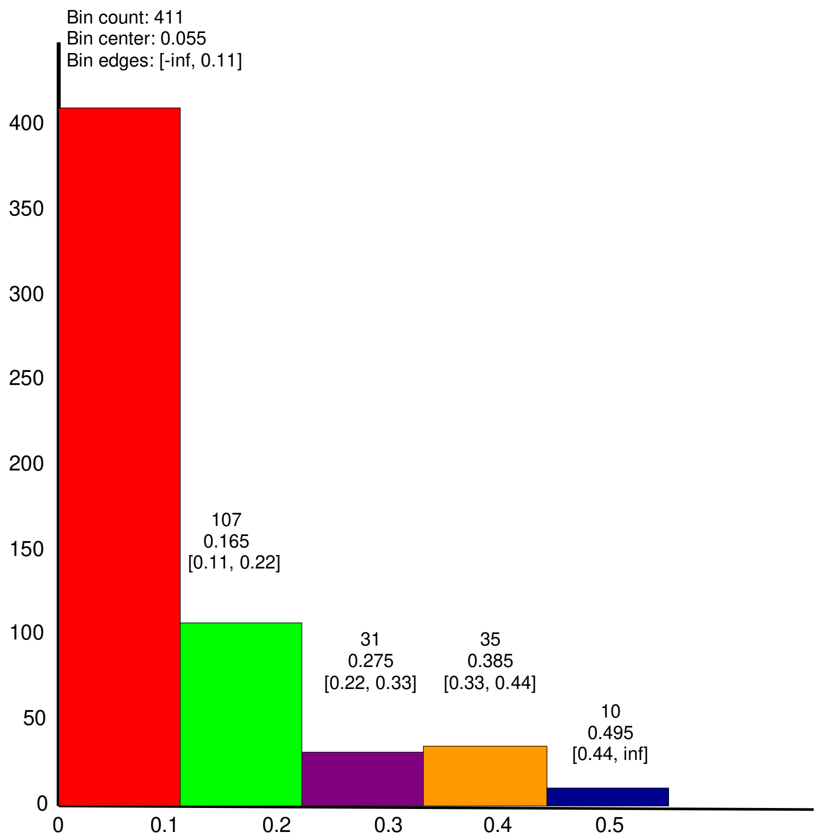


Figure 7b. *Histogram of the distribution of the moduli of the non-dominant eigenvalues; Japan, 1980-2005, 100 x 100 input-output tables*

Finally, Table 3 reports the moduli of the eigenvalues for the case of fixed capital stock (and a uniform profit rate; see equation (2a)) as well as the relevant statistical measures of distribution. The matrix of capital stock is rarely available in the official statistics and one should estimate it from the available data on the basis of some simplifying assumptions. More specifically, starting with the investment matrix of the same size and industry structure as of the input-output table we form weights which post-multiplied, element-by-element, by the vector of capital stock per unit of output gives the matrix of capital stock coefficients. The assumption here is that the matrix of capital stock is proportional to investment matrix. It is important to stress at this point that in the capital stock matrix, the consumer goods producing industries as they do not normally sell investment goods their respective rows will contain many zeros or near zero (higher than the fifth decimal) elements, and, therefore, we end up with many zero or near zero eigenvalues.

We could have side stepped the problem of zero eigenvalues by accounting as part of the matrix of capital stock the inventories as well as the matrix of workers necessary consumption (‘wage fund’). However, the data on turnover times are hard to come by with the exception of the US economy, where they can be approximated through the inventories to sales ratio. Thus in the interest of brevity and clarity of presentation we opted not to use inventories and in the same spirit, we did not use matrices of depreciation coefficients. Thus, in what follows we present estimates of the moduli of eigenvalues only for the economies that we had access to data on their capital stock and also we have an idea from past studies about the shape of the $w-\rho$ curves. Table 3 below displays the data for Greece (1970), Korea (1995 and 2000) and the USA (1947, 1958, 1963, 1967, 1972, and 1977).

Table 3. *The distribution of the moduli of the non-dominant eigenvalues for the case of fixed capital; Greece, Korea and USA*

	GRC 1970	KR 1995	KR 2000	USA 1947	USA 1958	USA 1963	USA 1967	USA 1972	USA 1977
Rank									
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	0.037	0.084	0.063	0.408	0.309	0.309	0.473	0.549	0.461
3	0.035	0.059	0.063	0.117	0.090	0.105	0.116	0.069	0.069
4	0.035	0.057	0.057	0.069	0.057	0.057	0.065	0.069	0.058
5	0.015	0.057	0.057	0.069	0.045	0.057	0.065	0.062	0.058
6	0.012	0.026	0.025	0.050	0.045	0.053	0.051	0.062	0.054
7	0.004	0.007	0.009	0.048	0.043	0.049	0.051	0.062	0.054
8	0.004	0.004	0.009	0.048	0.043	0.049	0.047	0.054	0.041
9	0.002	0.004	0.008	0.044	0.037	0.043	0.034	0.049	0.036
10	0.002	0.000	0.003	0.044	0.035	0.043	0.034	0.049	0.036
11	0.000	0.000	0.003	0.040	0.035	0.042	0.034	0.034	0.036
12	0.000	0.000	0.000	0.040	0.033	0.038	0.032	0.034	0.036
13	0.000	0.000	0.000	0.040	0.031	0.038	0.032	0.034	0.031
14	0.000	0.000	0.000	0.029	0.031	0.030	0.029	0.034	0.031
15	0.000	0.000	0.000	0.029	0.029	0.027	0.029	0.030	0.029
16	0.000	0.000	0.000	0.027	0.018	0.022	0.028	0.030	0.029
17	0.000	0.000	0.000	0.024	0.017	0.021	0.028	0.029	0.023
18	0.000	0.000	0.000	0.021	0.017	0.021	0.027	0.027	0.022
19	0.000	0.000	0.000	0.017	0.017	0.021	0.021	0.021	0.022
20	0.000	0.000	0.000	0.017	0.017	0.021	0.021	0.020	0.017
21	0.000	0.000	0.000	0.016	0.014	0.018	0.017	0.020	0.016
22	0.000	0.000	0.000	0.016	0.013	0.016	0.016	0.019	0.015
23	0.000	0.000	0.000	0.016	0.013	0.016	0.016	0.015	0.014
24	0.000	0.000	0.000	0.012	0.012	0.010	0.015	0.012	0.014
25	0.000	0.000	0.000	0.012	0.012	0.008	0.015	0.011	0.012
26	0.000	0.000	0.000	0.012	0.007	0.007	0.010	0.011	0.012
27	0.000	0.000	0.000	0.010	0.006	0.007	0.009	0.010	0.009
28	0.000	0.000	0.000	0.008	0.006	0.007	0.009	0.010	0.008
29	0.000	0.000	0.000	0.007	0.005	0.006	0.009	0.010	0.008
30	0.000	0.000	0.000	0.007	0.005	0.006	0.007	0.006	0.005
31	0.000	0.000	0.000	0.006	0.005	0.006	0.007	0.006	0.002
32	0.000	0.000	0.000	0.003	0.002	0.004	0.007	0.005	0.006
33	0.000	0.000	0.000	0.003	0.002	0.004	0.006	0.003	0.006
34	….	….	….	0.002	0.002	0.003	0.002	0.003	0.003
35	….	….	….	0.001	0.002	0.002	0.001	0.002	0.003
36	….	….	….	0.001	0.002	0.002	0.001	0.002	0.000
37	….	….	….	0.001	0.000	0.001	0.001	0.001	0.001
38	….	….	….	0.000	0.000	0.000	0.001	0.000	0.001
39	….	….	….	0.000	0.000	0.000	0.000	0.000	0.000
AG	0.009	0.027	0.027	0.035	0.028	0.031	0.036	0.039	0.034
GM	2.2E-06	0.006	0.010	0.012	0.009	0.012	0.014	0.015	0.013
SF	2.4E-04	0.223	0.387	0.352	0.338	0.406	0.394	0.377	0.394
π_2	25%	28%	21%	31%	29%	26%	35%	38%	36%
RE	0.668	0.734	0.791	0.767	0.782	0.800	0.753	0.736	0.748
REN	39%	53%	61%	43%	45%	48%	41%	38%	40%

An inspection of the results reveals that the presence of fixed capital stock leads to considerably lower moduli and to higher π_2 than the corresponding flow data. Thus, we observe reductions in *SF*, *RE* and *REN*.[30]

From all these tables, the associated numerical results and the hitherto analysis we arrive at the following conclusions:

(i). The moduli of the non-dominant eigenvalues fall quite rapidly in the 'beginning' and figuratively speaking their falling pattern can be described by an exponential curve that approaches asymptotically much lower values, where it is observed a concentration of moduli. Further analysis reveals that the distribution of the moduli tends to be remarkably *uniform* across countries and over time.

(ii). The complex (as well as the negative) eigenvalues tend to appear in the lower ranks, *i.e.*, their modulus is relatively small. However, even in the cases that they appear in the higher ranks, *i.e.*, second (USA, 1958, see Table 1, and Japan, 1995, 21 x 21, see Table 2) or third rank (Greece, 1992, 1995 and 1996, USA 1958, Japan 1980 and 1990, see Table 1; see also the cases displayed in Table 2), the real part has been found to be much larger than the imaginary part (*i.e.*, $\cos\theta \simeq 1$; see Section 2, point (i)), which is equivalent to saying that the imaginary part may even be ignored (*e.g.*, in the Greek economy the real part is from 19 to 50 times larger than the imaginary part). Moreover, in the fewer cases that the imaginary part of an eigenvalue exceeds the real one, not only their ratio is relatively small but also the modulus of the eigenvalue can be considered as a negligible quantity (*e.g.*, the imaginary part of the fifth (sixteenth) eigenvalue of the Greek economy, 1997, is 1.1 (1.65) times higher than the real part, nevertheless the modulus equals 0.098 (0.017)). Finally, by inspecting all of our eigenvalues we observe that, in general, the imaginary part *tends* to fall. Consequently, first, the already detected distribution of the moduli can be viewed as a fair representation of the distribution of the eigenvalues and, second, the majority of the prices of the non-Sraffian Standard commodities in terms of the Sraffian Standard commodity are almost linear functions of ρ and close to the $w-\rho$

[30] It may be noted that we also experimented with an aggregation in a 3 x 3 input-output table for the USA (1977): in the flow version, the modulus of the subdominant (complex) eigenvalue equals 0.146; in the stock version, the subdominant eigenvalue equals 0.031, whilst the third eigenvalue equals − 0.0001. The aggregation in a 3 x 3 input-output table for Greece (1970) did not give any different results: in the flow version, the modulus of the subdominant (complex) eigenvalue equals 0.087; in the stock version, the subdominant eigenvalue equals − 0.027, whilst the third eigenvalue equals zero (see Tsoulfidis, 2010, pp. 150-155). See also the evidence provided by Steenge and Thissen (2005).

curve, $1-\rho$. For example, we may consider the representative case of the Greek economy, 1994: Table 4 reports the non-dominant eigenvalues and the mean of the relative error, *MRE*, between $f_{\mu}(\rho)$ or $2^{-1}F_{\mu}(\rho)$ and $1-\rho$,[31] and Figure 8 represents $f_2(\rho)$ to $f_4(\rho)$ (dotted lines), $2^{-1}F_{\mu}(\rho)$ (solid lines) and $1-\rho$ (dashed line)).

Table 4. *Non-dominant eigenvalues and mean of the relative error between the prices of the non-Sraffian Standard commodities and the wage-profit curve in terms of the Sraffian Standard commodity; Greece 1994*

$\lambda_{\mathbf{J}k}$	*MRE*
0.678	67.1%
0.420	29.1%
0.357	23.7%
0.327	21.1%
0.261	15.9%
0.199 $\pm i$ 0.057	11.3%
0.109	5.9%
-0.071 $\pm i$ 0.066	3.5%
0.071 $\pm i$ 0.041	3.7%
0.059	3.1%
-0.013 $\pm i$ 0.023	0.7%
0.023	1.2%
-0.007	0.3%
0.006	0.3%
	AM=13.4%

[31] $MRE \equiv \int_0^1 \left| [1-f(\rho)(1-\rho)^{-1}] \right| d\rho$, where $f(\rho)$ denotes $f_{\mu}(\rho)$ or $2^{-1}F_{\mu}(\rho)$.

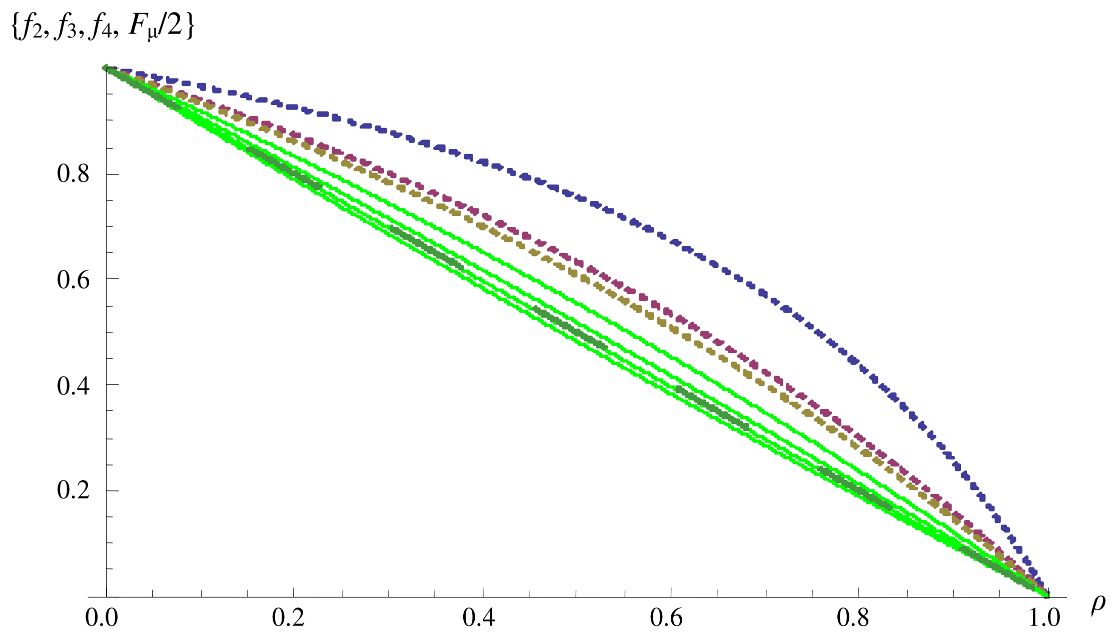


Figure 8. *The prices of the first three non-Sraffian Standard commodities, the prices of the sum of complex conjugate non-Sraffian Standard commodities and the money wage rate, in terms of the Sraffian Standard commodity, as functions of the relative rate of profits; Greece, 1994*

(iii). The actual economies do not fulfill the polar spectral conditions that guarantee a linear or even monotonic price-profit rate relationship (see Section 2, point (ii)). Nevertheless, those conditions constitute useful '*ideal types*', since the actual eigenvalue distribution is indeed polarized and, therefore, both Bienenfeld's quadratic formula (see equation (28)) and a spectral formula, which involves few non-dominant eigenvalues (see relation (30)), track down accurately enough the trajectories of the actual prices of production (see Section 2, point (iii)). For example, consider the graphs of Figure 9, which are associated with the Greek economy, 1994, and display trajectories of the actual prices (depicted by solid lines) and the relevant trajectories corresponding to (i) a fourth-order polynomial approximation (depicted by dashed lines that cross the $\rho-$axis at $\rho=1$) in terms of 'dated quantities of embodied labour' (see equation (3), Steedman, 1999b, and Tsoulfidis and Mariolis, 2007, p. 429), *i.e.*,

$$\mathbf{p}^{\mathrm{T}} = (1-\rho)\mathbf{p}^{\mathrm{T}}(0)[\mathbf{I}+\rho\mathbf{J}+(\rho\mathbf{J})^2+(\rho\mathbf{J})^3+(\rho\mathbf{J})^4]$$

or

$$\mathbf{p}^{\mathrm{T}} = (1-\rho)\mathbf{p}^{\mathrm{T}}(0)\mathbf{X}[\mathbf{I}+\rho\mathbf{\Lambda_J}+...+(\rho\mathbf{\Lambda_J})^4]\mathbf{X}^{-1}$$

where $\mathbf{X}$ and the diagonal matrix $\mathbf{\Lambda_J}$ denote matrices formed from the right eigenvectors and the eigenvalues of $\mathbf{J}$, respectively; (ii) Bienenfeld's quadratic approximation (dashed lines); and (iii) a third-order spectral approximation (dotted lines), *i.e.*,

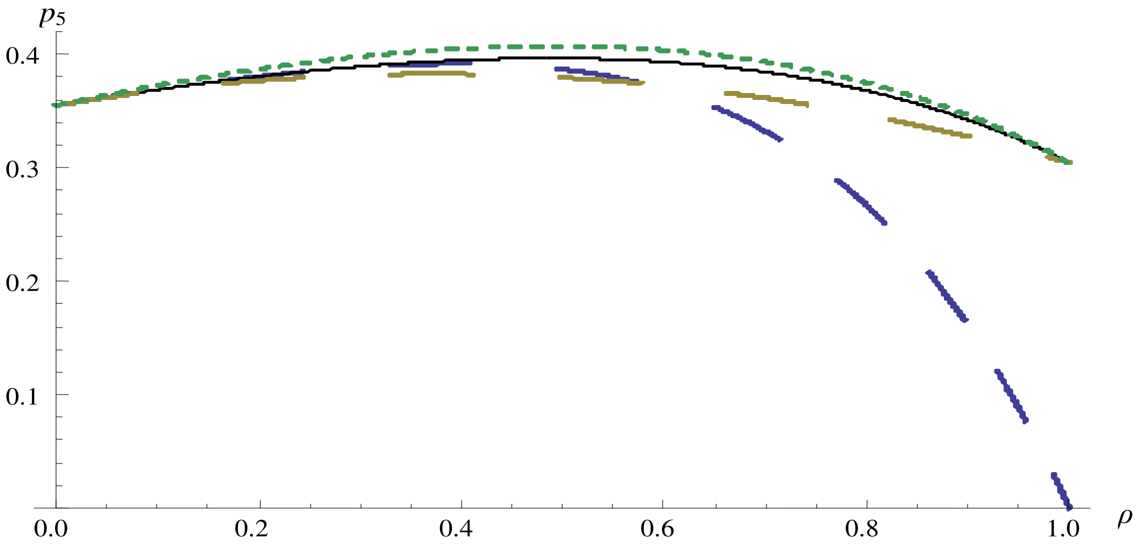
$$\mathbf{p}^{\mathrm{T}} \approx \text{B.L.A.} + \text{S.N-L.T.}$$

where B.L.A. denotes Bienenfeld's *linear* approximation and

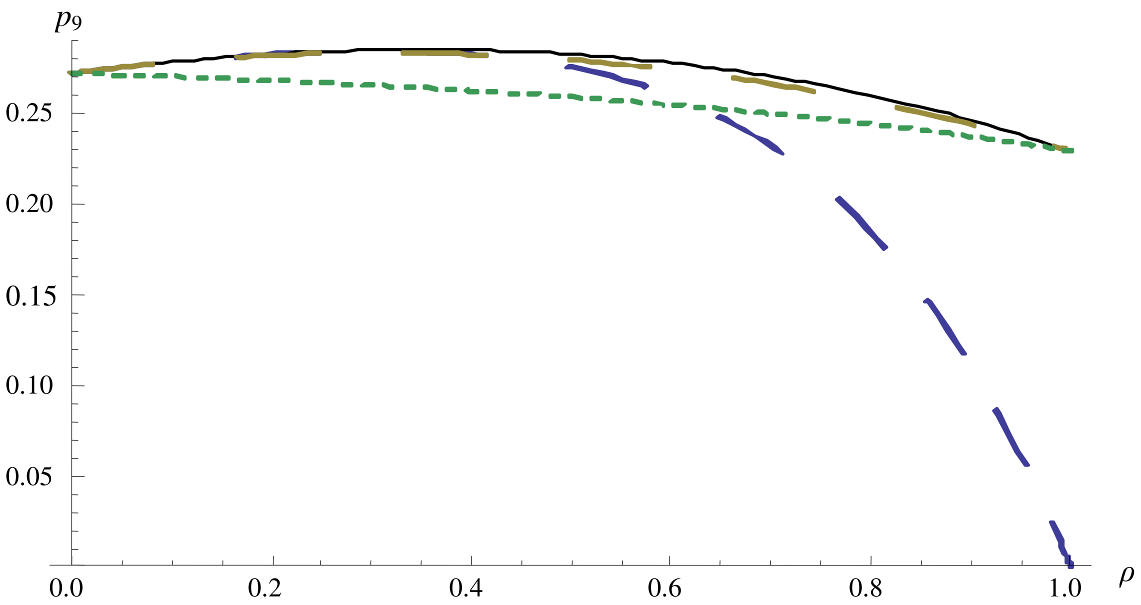
$$\text{S.N.-L.T} \equiv (1-\rho)\{[\rho 0.678(1-\rho 0.678)^{-1}]\mathbf{y}_{\mathbf{A}2}^{\mathrm{T}} + [\rho 0.420(1-\rho 0.420)^{-1}]\mathbf{y}_{\mathbf{A}3}^{\mathrm{T}}\}$$

the sum of the two non-linear terms (see Table 4). In fact, for reasons of economy of space, we focus on the four sectors displaying an extremum point (*i.e.*, sectors 5, 9, 14 and 18; setting aside sector 14, they also give a price-labour value reversal), on two sectors (10 and 12) that give strictly rising curves and on two sectors (1 and 7) that give strictly falling curves. Moreover, in each graph we report (i) the actual and the approximate values of ρ at which occur extrema points and reversals; and (ii) the mean of the relative error (*MRE*) between the approximate and the actual curves (the subscripts 'a, p, B and s' indicate 'actual, polynomial, Bienenfeld and spectral', respectively).[32]

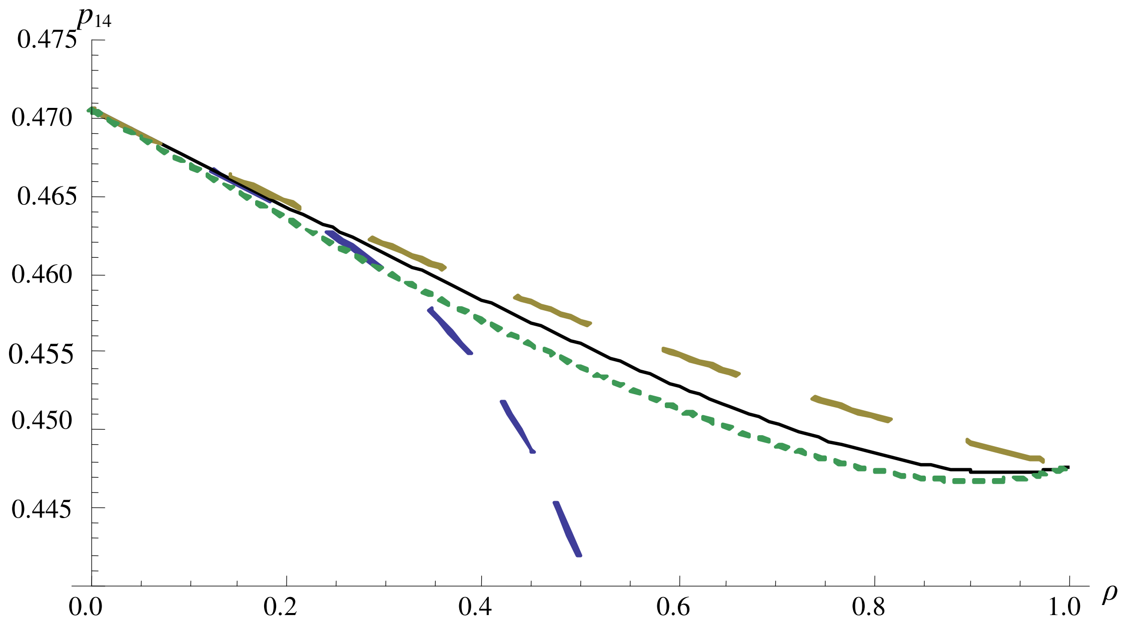
[32] It should be noted that in *all* sectors that present an extremum point, B.L.A. is found to be decreasing, whilst in the remaining sectors its monotonicity coincides with that of the actual curve. The S.N.-L.T. presents a minimum point in the sectors 12 and 14, whilst in the remaining sectors it presents a maximum.



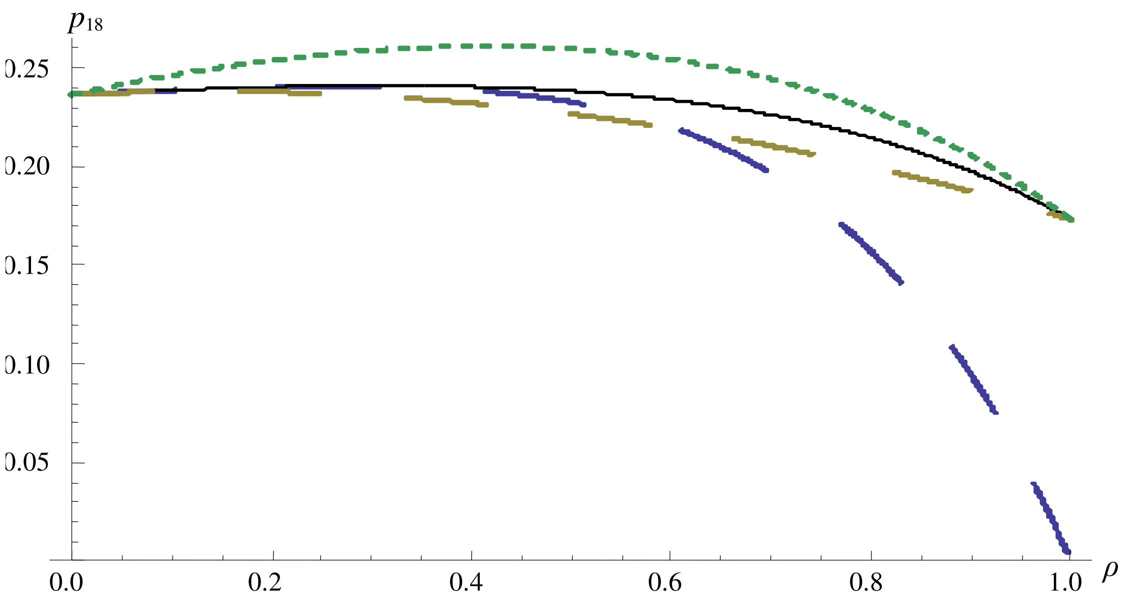
Sector 5: Max: { $\rho_a = 0.488$, $\rho_p = 0.397$, $\rho_B = 0.373$, $\rho_s = 0.491$ }, Reversal: { $\rho_a = 0.856$, $\rho_p = 0.646$, $\rho_B = 0.746$, $\rho_s = 0.874$ }, $MRE_p = 14.8\%$, $MRE_B = 3.3\%$, $MRE_s = 1.8\%$



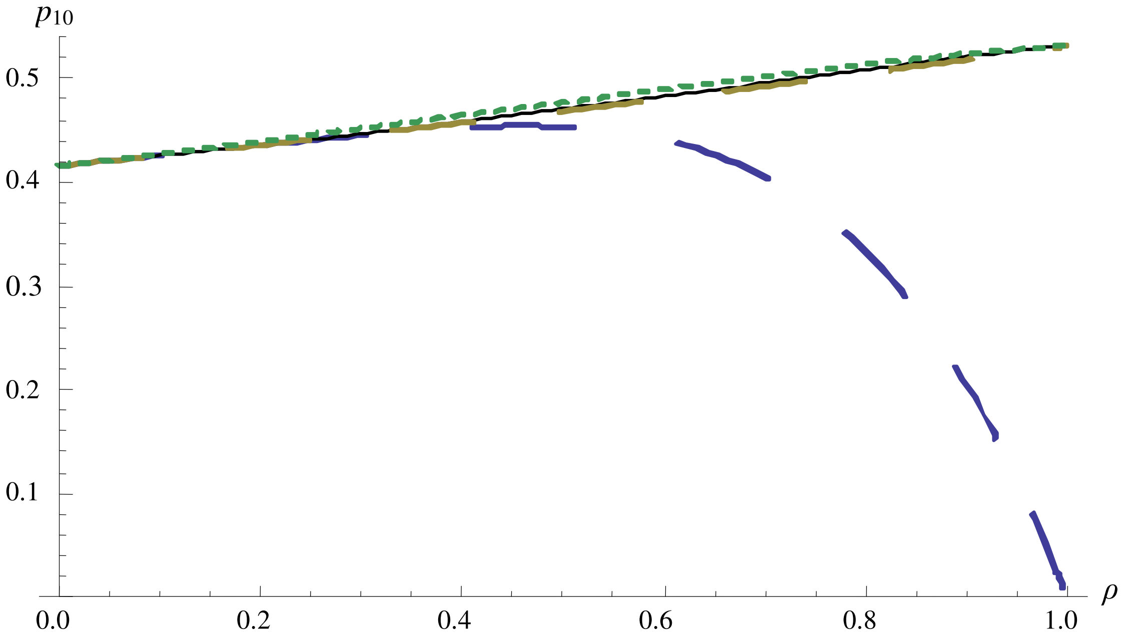
Sector 9: Max: { $\rho_a = 0.352$, $\rho_p = 0.307$, $\rho_B = 0.315$, $\rho_s > 1$ }, Reversal: { $\rho_a = 0.680$, $\rho_p = 0.529$, $\rho_B = 0.629$ $\rho_s > 1$ }, $MRE_p = 15.3\%$, $MRE_B = 0.8\%$, $MRE_s = 5.7\%$



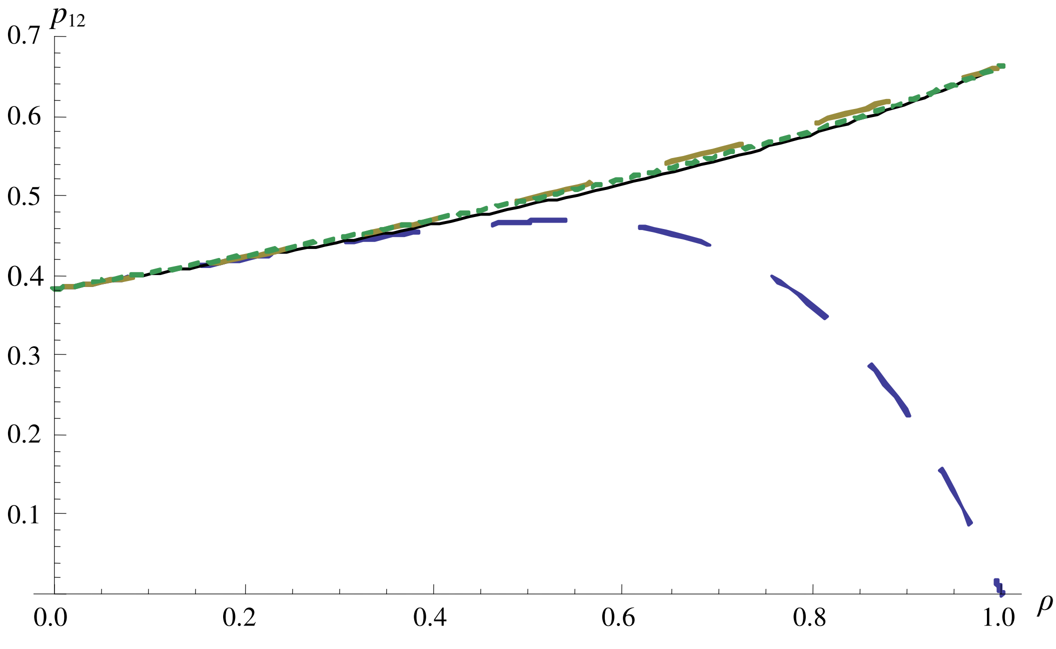
Sector 14: Min: { $\rho_a = 0.938$, $\rho_B > 1$, $\rho_s = 0.902$ }, Reversal: { $\rho_a > 1$, $\rho_B > 1$, $\rho_s > 1$ }, $MRE_p = 16.6\%$, $MRE_B = 0.3\%$, $MRE_s = 0.2\%$



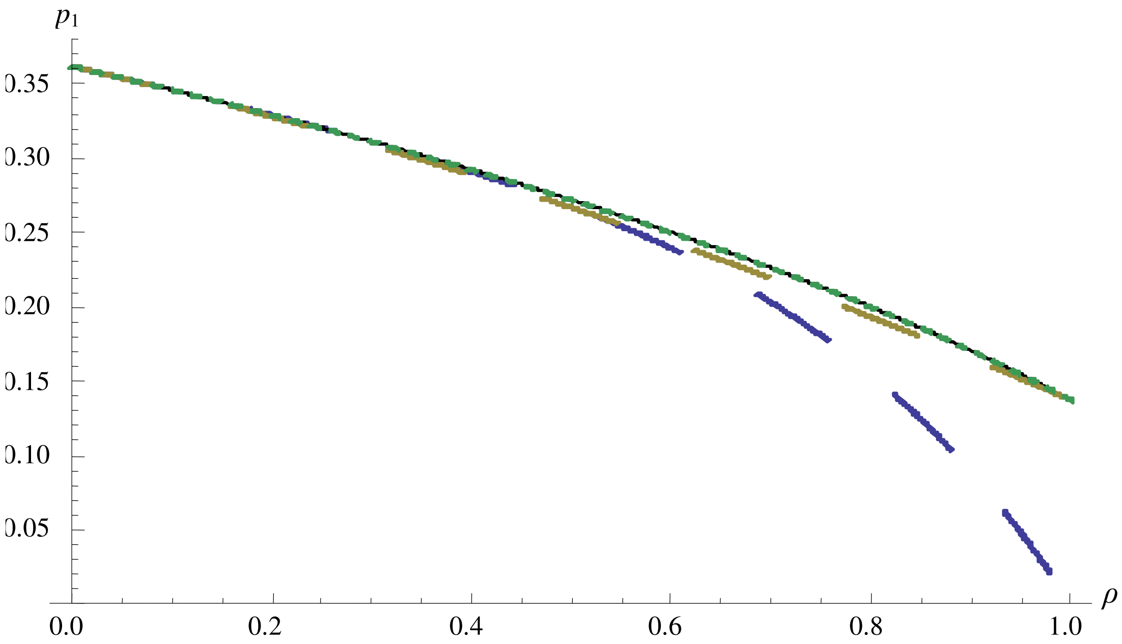
Sector 18: Max: { $\rho_a = 0.313$, $\rho_p = 0.272$, $\rho_B = 0.137$, $\rho_s = 0.415$ }, Reversal: { $\rho_a = 0.546$, $\rho_p = 0.447$, $\rho_B = 0.274$, $\rho_s = 0.754$ }, $MRE_p = 14.6\%$, $MRE_B = 3.8\%$, $MRE_s = 6.2\%$



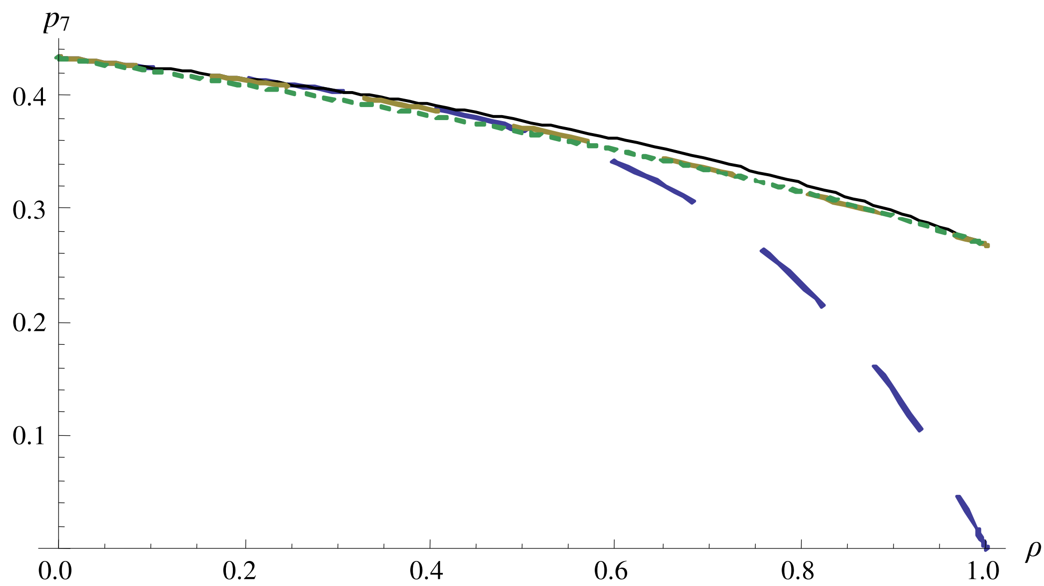
Sector 10: $MRE_{\mathrm{p}} = 17.2\%$, $MRE_{\mathrm{B}} = 0.3\%$, $MRE_{\mathrm{s}} = 0.9\%$



Sector 12: $MRE_{\mathrm{p}} = 18.3\%$, $MRE_{\mathrm{B}} = 1.4\%$, $MRE_{\mathrm{s}} = 0.9\%$



Sector 1: $MRE_p = 13.1\%$, $MRE_B = 1.7\%$, $MRE_s = 0.1\%$



Sector 7: $MRE_p = 14.8\%$, $MRE_B = 1.2\%$, $MRE_s = 1.9\%$

Figure 9. *Actual prices, fourth-order polynomial approximation, Bienenfeld's quadratic approximation and third-order spectral approximation; Greece, 1994*

Clearly, even the polynomial approximation works pretty well, *although* for 'low' or, more precisely, 'realistic' values of the relative rate of profits: in the considered case of the Greek economy, the 'actual' value of the relative rate of profits (*i.e.*, that associated with the 'actual' real wage rate, estimated on the basis of the available input-output data) is almost 0.272, provided that wages are paid *ex ante* (see Tsoulfidis and Mariolis, 2007, p. 428, Table 1), and, to our knowledge, there is no relevant empirical study where it is greater than 0.40 (and less than 0.17).[33] It could also be added that, regarding sector 9, for which the MRE_s is greater, the accuracy of the spectral approximation is improved considerably by including the sixth eigenvalue, in the sense that the relevant curve presents a maximum point (as well as a price-labour value reversal; see Figure 10, where the dotted curve below (above) the actual one represents the sixth-order (thirteenth-order) spectral approximation, and Figure 11, where the solid (dotted) curve represents the sum of *all* the real (complex) non-linear terms of the spectral approximation). Finally, if $w^{S}\ (=1-\rho)$, p_j^{S} denote the money wage rate and the price of commodity j, respectively, in terms of the Sraffian Standard commodity, and w^j the money wage rate corresponding to the normalization equation $p_j = v_j\ (= p_j(0))$, then $w^j = p_j(0) w^{S} (p_j^{S})^{-1}$ from which it follows that the w^j curves of the sectors 5, 9 and 18 cross the w^{S} curve at the points where occur price-labour value reversals, whilst the w^j curves of the remaining sectors are below or above the w^{S} curve, for $0 < \rho < 1$. Moreover, since the approximation of p_j^{S} through Bienenfeld's quadratic formula is pretty accurate, it is expected that the latter w^j curves will *tend* to be either strictly convex or strictly concave to the origin, whilst nothing guarantees that the means of the relative errors between the w^j and the w^{S} curves will be low. Indeed, Figure 12, which displays all the differences $w^j - w^{S}$ as functions of ρ, and Table 5, which reports the values of ρ at which the w^j curves change their shape from convex (cx) to concave (ce) or vice versa and the *MRE*, show that this statement holds true. More specifically, setting aside the sectors that display price-labour value reversals, turning points are detected in the sectors 6, 11 and 14, the sectroral *MRE* are in the range of 74.6% (sector 19)-3.1% (sector 14) and their arithmetic mean is almost 29.8%.

[33] See, for example, the empirical studies mentioned in footnotes 2 and 5.

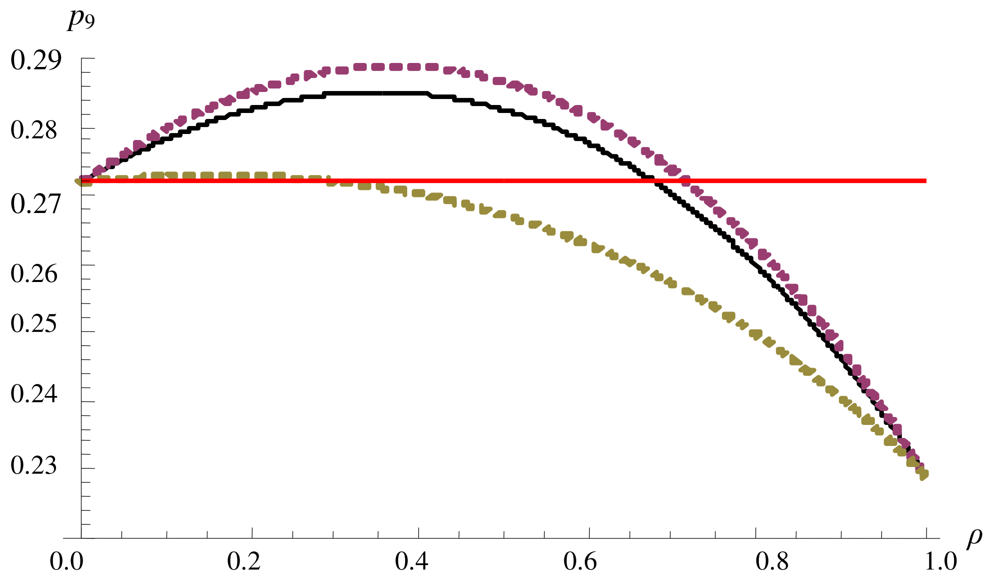


Figure 10. *Actual price, sixth-order and thirteenth-order spectral approximations ($MRE_s = 3.7\%$ and 1.0%, respectively); Greece, 1994, Sector 9*

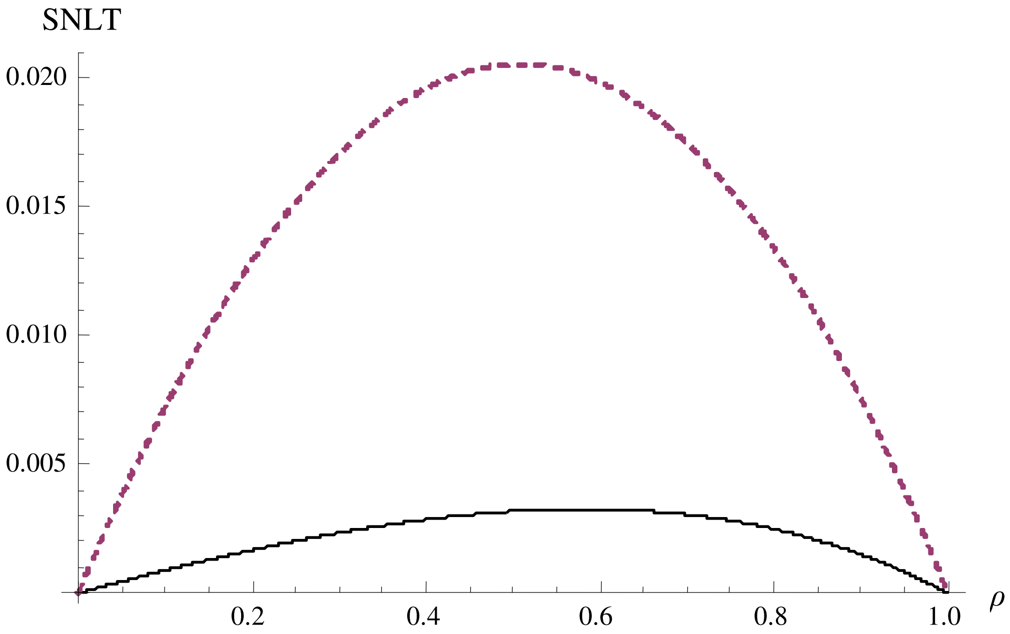


Figure 11. *The sums of all the real and complex non-linear terms of the spectral approximation as functions of the relative rate of profits; Greece 1994, Sector 9*

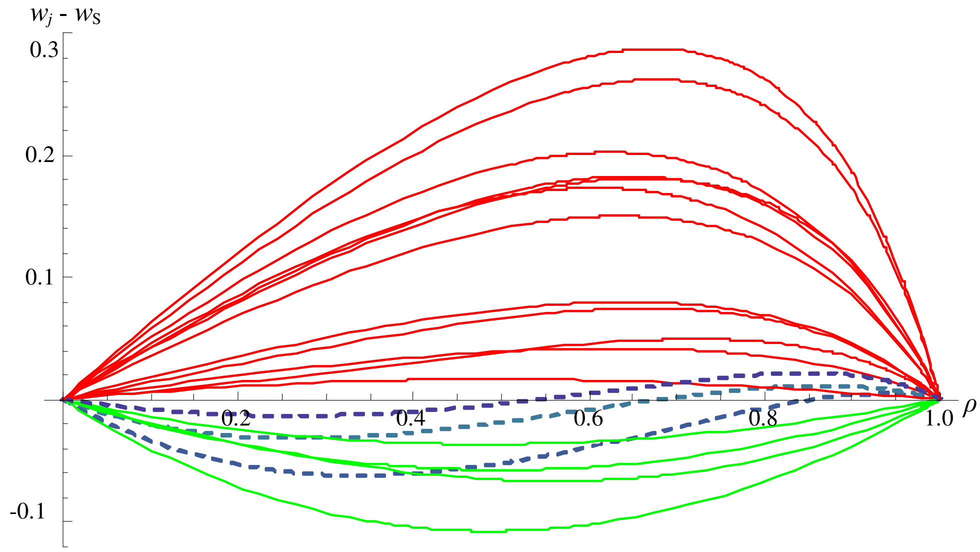


Figure 12: *The difference between the money wage rate in terms of the commodity j and the money wage rate in terms of the Sraffian Standard commodity as a function of the relative rate of profit; Greece, 1994*

Table 5. *Curvatures of the wage- profit curves in terms of the commodity j; Greece, 1994*

w^j	Curvature	*MRE*
1	ce	46.1%
2	ce	68.8%
3	ce	40.3%
4	ce	48.4%
5	cx-ce, $\rho \simeq 0.697$	5.1%
6	cx-ce, $\rho \simeq 0.351$	12.8%
7	ce	19.4%
8	ce	10.0%
9	cx-ce, $\rho \simeq 0.594$	0.3%
10	cx	11.4%
11	cx-ce, $\rho \simeq 0.127$	18.2%
12	cx	21.4%
13	cx	14.3%
14	ce-cx, $\rho \simeq 0.959$	3.1%
15	ce	45.0%
16	cx	7.2%
17	ce	36.4%
18	cx-ce, $\rho \simeq 0.630$	4.5%
19	ce	74.6%
		AM=25.6%

(iv). Although the level of aggregation affects both the central tendency and skewness of the eigenvalue distribution, it is expected that it does not drastically affect the monotonicity of the price-profit rate relationship, since the higher non-dominant eigenvalues exhibit small relative changes that go to either direction.

(v). Moving from the flow to (the more realistic) stock input-output data the above conclusions are strengthened; inasmuch as, we found that the subdominant eigenvalue falls even more abruptly, whereas the third or fourth eigenvalues become indistinguishable from the rest lending further support to the idea of approximating the trajectories of the actual prices of production linearly.[34]

4. Concluding Remarks

On the basis of a spectral decomposition of linear single-product systems, it has been shown that the monotonicity of the production price-profit rate relationship depends to a great extent on the distribution of the eigenvalues of the vertically integrated technical coefficients matrices. The examination of input-output data of many diverse economies suggested that the majority of the non-dominant eigenvalues concentrate at very low values and this means that the actual price-wage-profit systems can be adequately described by only a few non-Sraffian Standard systems. It follows therefore that the production price-profit rate relationship tends to be monotonic and its approximation through low-order formulae, like Bienenfeld's quadratic formula and a third or fourth-order spectral formula, works extremely well. In the more realistic case with capital stocks, we found that the non-dominant eigenvalues are much lower than that of the flow case and thus the linear or a second-order spectral formula approximate accurately enough the movement of production prices.

A salient feature of our analysis is the tendency towards uniformity in the eigenvalue distribution across countries and over time. Such a typical finding could be viewed as a manifestation of technological characteristics embedded in the structure of actual economies and these may become the focus of future research efforts.

[34] Thus, it comes as no surprise that both Ochoa (1984) and Shaikh (1998) find that their linear approximations are quite accurate and they further claim that there is no necessity for higher order terms. Bienenfeld (1988), on the other hand, using the same flow input-output data of the US economy, but not stock data, confirms that his quadratic approximation constitutes a marginal improvement over the linear one.

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Appendix: Numerical Examples for Cases of Polar Distributions of Eigenvalues

Consider a 4×4 system where all the diagonal (off-diagonal) elements of $\mathbf{A}$ equals 0.3 (equals a) and $\mathbf{l}^{\mathrm{T}}=[1,2,3,6]$ (by invoking perturbation theorems, see, *e.g.*, Horn and Johnson, 1990, pp. 371-373, the reader will be able to ascertain that, within certain limits, the following results are robust to differential parameterization of $\mathbf{A}$, say $\mathbf{A}(\varepsilon)=\mathbf{A}+\varepsilon\mathbf{E}$, where $\mathbf{E}$ denotes a fixed perturbation matrix). It is obtained that the moduli of the eigenvalues of $\mathbf{A}$ are strictly monotonic functions of a and $\lambda_{\mathbf{A}1}<1$ for $a<7/30\simeq0.233$ (see Figure A.1.1).

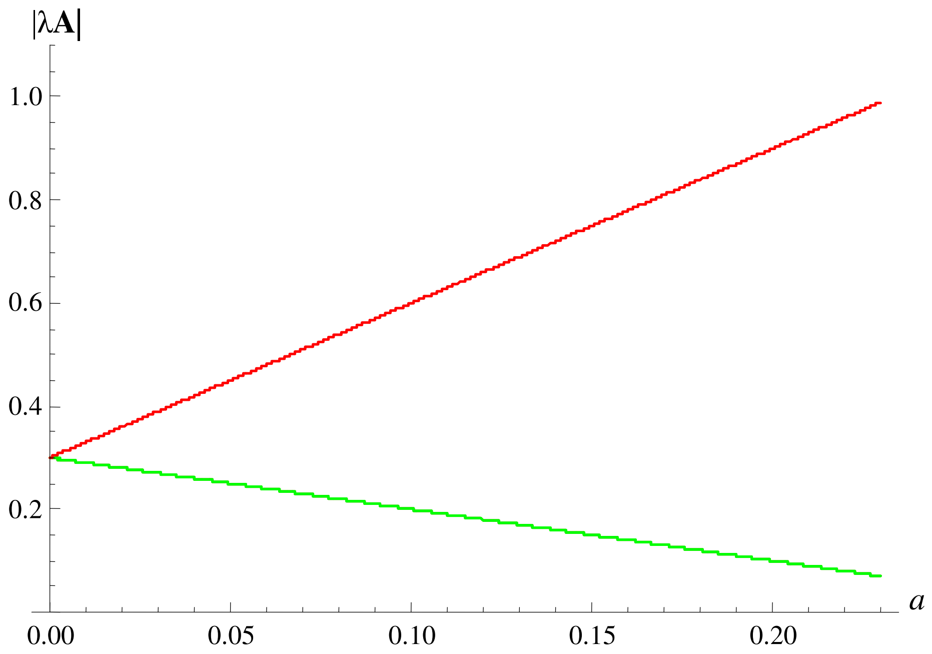


Figure A.1.1 *The moduli of the eigenvalues of the system as functions of the input-output coefficient* a

For $a=0.001$ it follows that $\lambda\simeq0.981$ and $\mathbf{v}^{\mathrm{T}}\simeq[1.451,2.878,4.304,8.584]$. Therefore, the $w-\rho$ curves tend to coincide with $1-\rho$, and the production prices tend to be insensitive to ρ. The Figures A.1.2 a-b represent the $w-\rho$ and the $p_j-\rho$ curves, respectively, in terms of $\mathbf{z}=90.289^{-1}[1,1,10,5]^{\mathrm{T}}$, *i.e.*, $\mathbf{v}^{\mathrm{T}}\mathbf{z}=1$ and $d_1\neq1$ (the dashed line, in Figure A.1.2 a, represents $1-\rho$). In fact, $w''(\rho)<0$ and, for example,

$$p_1=1.3913(\rho-1.0592)(\rho-1.0155)^{-1}$$

or, using the Taylor expansion about $\rho=0$,

$$p_1 \approx v_1 + 0.059\rho + 0.058\rho^2 + 0.057\rho^3$$

whilst the deviation of the vector of prices from the vector of labour values, measured by the '$d -$distance' (which is a numeraire-free measure; see Steedman and Tomkins, 1998), is less than, say, 10% for $\rho < 0.873$ (see Figure A.1.3)

On the other hand, for $a = 0.23$ it follows that $\lambda \simeq 0.008$ and $\mathbf{v}^{\mathrm{T}} \simeq [297.85, 298.93, 300.0, 303.22]$. Therefore, for $d_1 = 1$, *i.e.*, $\mathbf{z} = 1200^{-1}[1,1,1,1]^{\mathrm{T}}$, the $p_j - \rho$ curves tend to be linear (see Figure A.1.4), and using the Taylor expansion about $\rho = 0$, we get

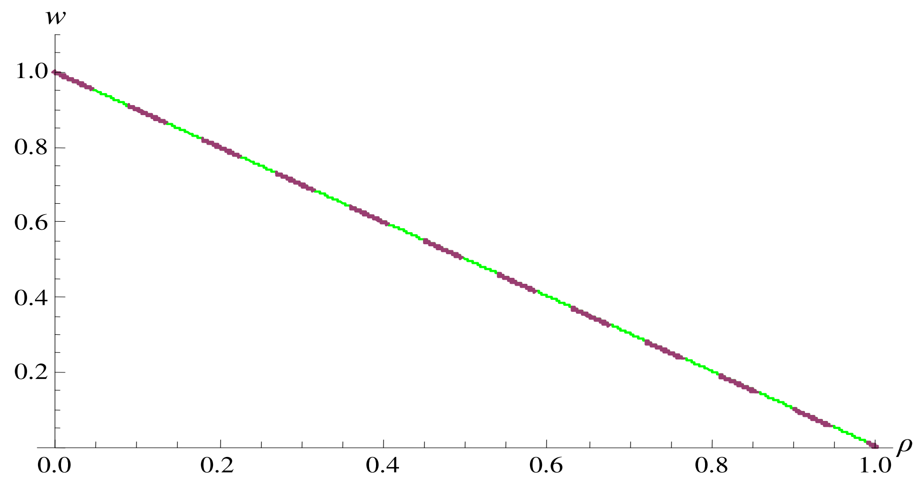
$$p_1 \approx v_1 + 2.149\rho + 0.002\rho^2 + (1.242\times 10^{-6})\rho^3$$

$$p_2 \approx v_2 + 1.074\rho + 0.001\rho^2 + (6.211\times 10^{-7})\rho^3$$

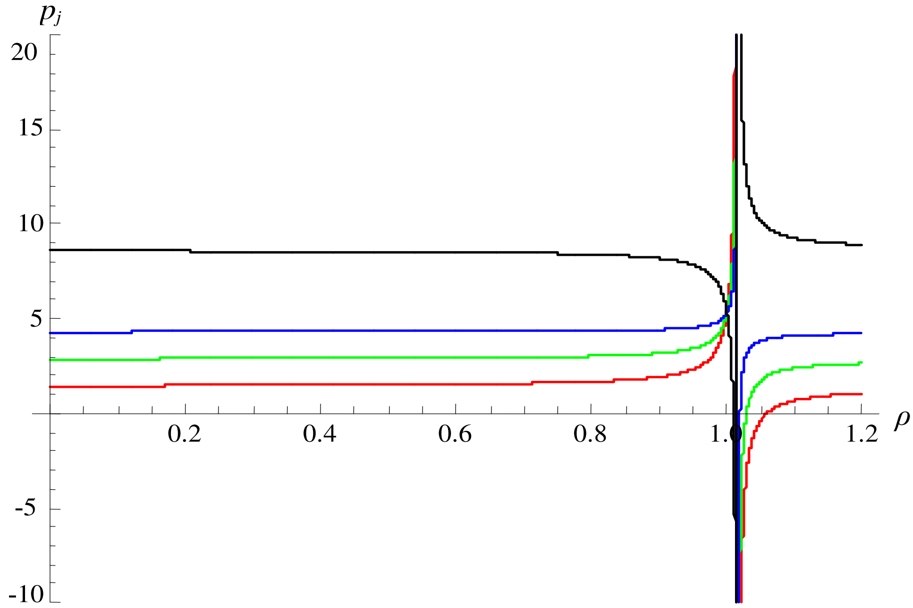
$$p_3 \approx v_3 - (9.888\times 10^{-14})\rho$$

$$p_4 \approx v_4 - 3.223\rho - 0.002\rho^2 - (1.863\times 10^{-6})\rho^3$$

which show that the first-order approximations work pretty well.



(a)



(b)

Figure A.1.2. *$w-\rho$ and $p_j-\rho$ curves;* $a = 0.001$, $\lambda \simeq 0.981$, $d_1 \neq 1$

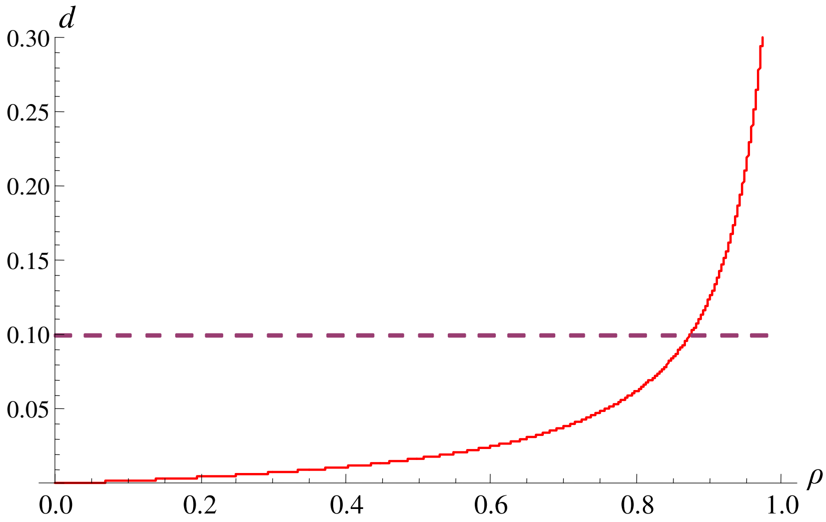


Figure A.1.3. *The '$d-$distance' as a function of the relative rate of profits;* $a = 0.001$, $\lambda \simeq 0.981$

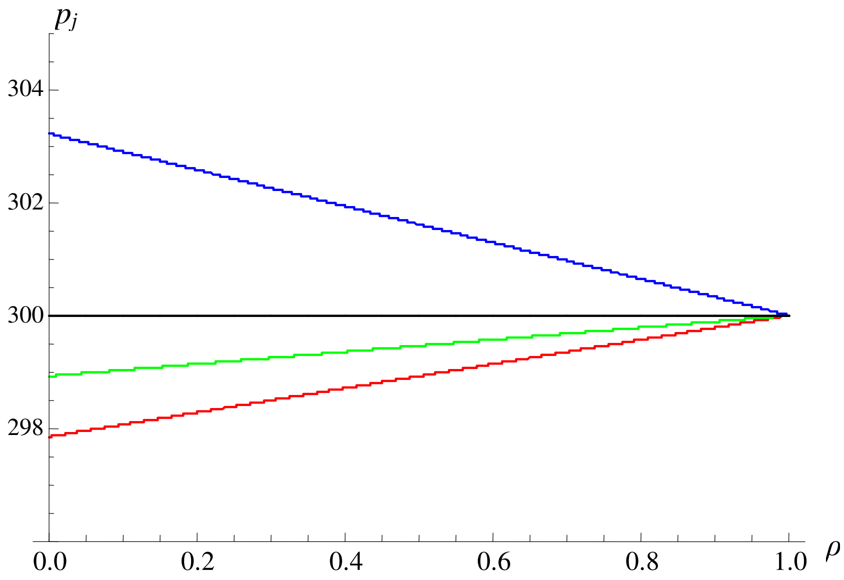


Figure A.1.4. $p_j - \rho$ *curves;* $a = 0.23$, $\lambda \simeq 0.008$, $d_1 = 1$