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ON BRÓDY'S CONJECTURE: FACTS AND FIGURES FROM THE US ECONOMY

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Bródy's conjecture is submitted to an empirical test using input-output flow data of varying size for the US economy for the benchmark years 1997 and 2002, as well as for the period 1998-2010. The results suggest that the ratio of the modulus of the subdominant eigenvalue to the dominant one increases both with the size of the matrix and, for the same matrix size, over the years lending support to the view of increasing instability (in the sense of Bródy) for the US economy over the period 1997-2010.

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JEL classifications: C62, C67, D57

1. INTRODUCTION

On the basis of the so-called 'power method', Bródy (1997) noticed that starting from an arbitrary $n-$ vector, which when is repeatedly post-multiplied by the $n \times n$ matrix of input-output coefficients, $\mathbf{A}$, converges to the equilibrium eigenvector.[1] He argued that the speed of convergence depends on the ratio of the modulus of the subdominant eigenvalue to the dominant one, $\rho_{\mathbf{A}2}$; the closer to zero $\rho_{\mathbf{A}2}$ is, the faster is the convergence to the equilibrium eigenvector.[2] Bródy, then experimenting with large random Leontief type matrices, namely with identically and independently distributed entries, and derived that in fact $\rho_{\mathbf{A}2}$ tends to zero, with speed $n^{-0.5}$, when n tends to

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[1] Matrices (and vectors) are delineated in boldface letters. The transpose of an $n \times 1$ vector $\mathbf{x}$ is denoted by $\mathbf{x}^{\mathrm{T}}$, and the diagonal matrix formed from the elements of $\mathbf{x}$ is denoted by $\hat{\mathbf{x}}$. $\lambda_{\mathbf{A}1}$ denotes the Perron-Frobenius eigenvalue of a semi-positive $n \times n$ matrix $\mathbf{A}$ and $(\mathbf{x}_{\mathbf{A}1}, \mathbf{y}_{\mathbf{A}1}^{\mathrm{T}})$ the corresponding eigenvectors, whereas $\lambda_{\mathbf{A}k}$, $k = 2,...,n$ and $|\lambda_{\mathbf{A}2}| \geq |\lambda_{\mathbf{A}3}| \geq ... \geq |\lambda_{\mathbf{A}n}|$, denotes the non-dominant eigenvalues of $\mathbf{A}$, and $\rho_{\mathbf{A}k} \equiv |\lambda_{\mathbf{A}k}| \lambda_{\mathbf{A}1}^{-1} \leq 1$ (the equality holds iff $\mathbf{A}$ is imprimitive). Finally, the symbols '>0', '≥ 0' denote strict positivity and semi-positivity, respectively.

[2] As is well-known, the convergence is asymptotically exponential, at a rate at least as fast as $\log \rho_{\mathbf{A}2}^{-1}$. The number $\rho_{\mathbf{A}2}^{-1}$ is called the 'damping ratio', in population dynamics theory, and can be considered as a measure of the intrinsic resilience of the state vector to disturbance (see, e.g. Keyfitz and Caswell, 2005, pp. 165-166, and Stott et al., 2011, pp. 960-961).

infinity.[3] Also the estimated eigenvalues fluctuate around their theoretical distribution, but the amplitude of these deviations progressively dwindles. Hence, the larger the system is the faster is the convergence, which is equivalent to saying that in a very large system convergence to equilibrium may be attained in just a few iterations.

2. FACTS AND FIGURES FROM THE US ECONOMY

Bródy's conjecture is not necessarily an investigation in pure mathematics and right from the introduction of his paper it becomes evident that the focus is not on mathematics per se, but rather on the behavior of the actual economies, as these are described by their flow and stock input-output structures. Bródy seems to assume that input-output data of dimensions of about one hundred industries would be adequate enough for what is true for the large random matrices, and therefore the same theorem might be applicable to actual economies.

In this paper we present results for the US economy for the Symmetric Input-Output Tables of the years 1997 and 2002; that is, two benchmark years for which we have input-output data of dimensions varying from 12 up to 488 industries.[4] More specifically, for the year 1997, the dimensions are of 12, 129 and 488 industries, while for the year 2002 the dimensions are of 15, 133 and 426 industries. These input-output tables provide an ideal terrain to test Bródy's conjecture in the context of actual data and not just on randomly generated matrices of various dimensions. The reason is that Bródy's conjecture holds for randomly generated matrices with size tending to infinity, however there are no published input-output tables larger than 488 industries for the US economy at least. Certainly given these dimensions, especially the extra large one, we can submit the conjecture to what we think fair empirical test. It should be stressed, however, that our investigation is carried out on the basis of circulating capital, as there are no available data for the construction of the corresponding matrices of fixed capital stocks.

In what follows, we present the results for the matrix $\mathbf{J} \equiv R_1\mathbf{H}$, where $R_1 \equiv \lambda_{\mathbf{A}1}^{-1} - 1 = \lambda_{\mathbf{H}1}^{-1}$ denotes the maximum uniform profit (growth) rate of the economy,

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[3] Bidard and Schatteman (2001) argue that Bródy's conjecture is of statistical nature based on the law of large numbers. Sun (2008) shows that it can be proved using theorems provided by Goldberg et al. (2000) (see also Goldberg and Neumann, 2003).

[4] This is the largest input-output table ever used and that larger dimensions must be rare if they exist in other countries.

$\mathbf{H} \equiv \mathbf{A}[\mathbf{I}-\mathbf{A}]^{-1}$ the 'vertically integrated technical coefficients matrix' (Pasinetti, 1973), and $\lambda_{\mathbf{J}1} = R_1 \lambda_{\mathbf{H}1} = 1$.[5] It should be noted that:

(i). If $\lambda_{\mathbf{A}k}$ is positive, then $\lambda_{\mathbf{A}k} < \lambda_{\mathbf{A}1}$. If it is negative or complex, then $|\lambda_{\mathbf{A}k}| \leq \lambda_{\mathbf{A}1}$ and $|1-\lambda_{\mathbf{A}k}| > 1-|\lambda_{\mathbf{A}k}|$. Hence,

$$\rho_{\mathbf{J}k} \equiv |\lambda_{\mathbf{J}k}|\lambda_{\mathbf{J}1}^{-1} = |\lambda_{\mathbf{J}k}| = R_1|\lambda_{\mathbf{A}k}||1-\lambda_{\mathbf{A}k}|^{-1} < \rho_{\mathbf{A}k}(1-\lambda_{\mathbf{A}1})(1-|\lambda_{\mathbf{A}k}|)^{-1} \leq \rho_{\mathbf{A}k}$$

or

$$\rho_{\mathbf{J}k} = |\lambda_{\mathbf{J}k}| < \rho_{\mathbf{A}k} \tag{1}$$

holds for all k. In other words, the eigenvalue ratios of $\mathbf{J}$ will be smaller than those of $\mathbf{A}$, which is equivalent to saying that we give more credence to Brody's conjecture.

(ii). $R_k \equiv \lambda_{\mathbf{A}k}^{-1} - 1 = \lambda_{\mathbf{H}k}^{-1}$ represents the maximum uniform profit (growth) rate of the $k-$th 'eigensector' (or non-Sraffian 'Standard system'; see Goodwin, 1976, 1977, and Sraffa, 1960, §42, footnote 2, and §§56, 64, respectively).

(iii). If $\mathbf{y}_{\mathbf{J}1} > \mathbf{0}$, then $\mathbf{J}$ is similar to the column stochastic matrix $\mathbf{M} \equiv \hat{\mathbf{y}}_{\mathbf{J}1}\mathbf{J}\hat{\mathbf{y}}_{\mathbf{J}1}^{-1}$ (the elements of which are independent of the choice of physical measurement units and the normalization of $\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}}$).

(iv). If $\mathbf{A}$ is irreducible, then the Hopf-Ostrowski and Deutsch upper bounds (or 'coefficients of ergodicity'; Seneta, 2006, pp. 63-64) imply that

$$\rho_{\mathbf{J}2} \leq 2^{-1}\max\{\sum_{f=1}^{n}|m_{fi}-m_{fj}|\} \leq (\tau-1)(\tau+1)^{-1} \leq (L-s)(L+s)^{-1} < 1 \tag{2}$$

where $\tau^2 \equiv \max\{(m_{ij}m_{gl}(m_{il}m_{gj})^{-1})\}$ and $L\,(s)$ represents the largest (smallest) element of $\mathbf{M} \equiv [m_{ij}]$ (see Ostrowski, 1963, and Maitre, 1970).[6] Thus, we may conclude that the

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[5] We focus on matrix $\mathbf{J}$ (instead of $\mathbf{A}$), since its eigenvalue distribution also regulates, to a great extent, the relationships between (i) with-profit prices and zero-profit prices (or 'labour values'); and (ii) wage rate and profit rate (see Bienenfeld, 1988, Steedman, 1999, Mariolis and Tsoulfidis, 2011, and Shaikh 2012): for example, when $\rho_{\mathbf{J}2} \approx 0$, the production price-profit rate relationships tend to be rational functions of degree 1. Furthermore, for an iterative procedure related to a price–wage–profit system and $\mathbf{J}$, see Mariolis and Tsoulfidis (2009, pp. 7-8) (and take into account Mariolis, 2010, pp. 563-564, for the case of fixed capital, joint production and/or differential profit rates). For a different iterative procedure, which is related to Marx's 'transformation problem', but also involves a matrix (the product of the profit factor with the 'augmented' input-output coefficients matrix) with Perron-Frobenius eigenvalue that is equal to 1, see Bródy (1970, pp. 88-91) and Morishima and Catephores (1978, pp. 160-166).

[6] It may be noted that $|m_{ij}m_{gl}(m_{il}m_{gj})^{-1} - 1|m_{il}m_{gj}$ equals 0 or the absolute value of the determinant of a 2×2 sub-matrix of $\mathbf{M}$.

closer to each other the columns of $\mathbf{M}$ are, the closer to zero is $\rho_{\mathbf{J}2}$ (if $\mathbf{A}$ has rank 1, then all the columns of $\mathbf{M}$ are equal to each other and $\rho_{\mathbf{J}2} = 0$).

The location of the eigenvalues in the complex plane for the year 1997 are displayed in the left-hand side graphs of Figure 1 that starting from the lower dimensions going to the intermediate and finally to the highest dimensions, while in the right hand side graphs, we display the location of all the eigenvalues for the year 2002. Moreover, Table 1 reports $\lambda_{\mathbf{H}1}$, $\rho_{\mathbf{J}2}$, $\rho_{\mathbf{J}3}$ and the arithmetic mean, *AM*, of $\rho_{\mathbf{J}k}$.

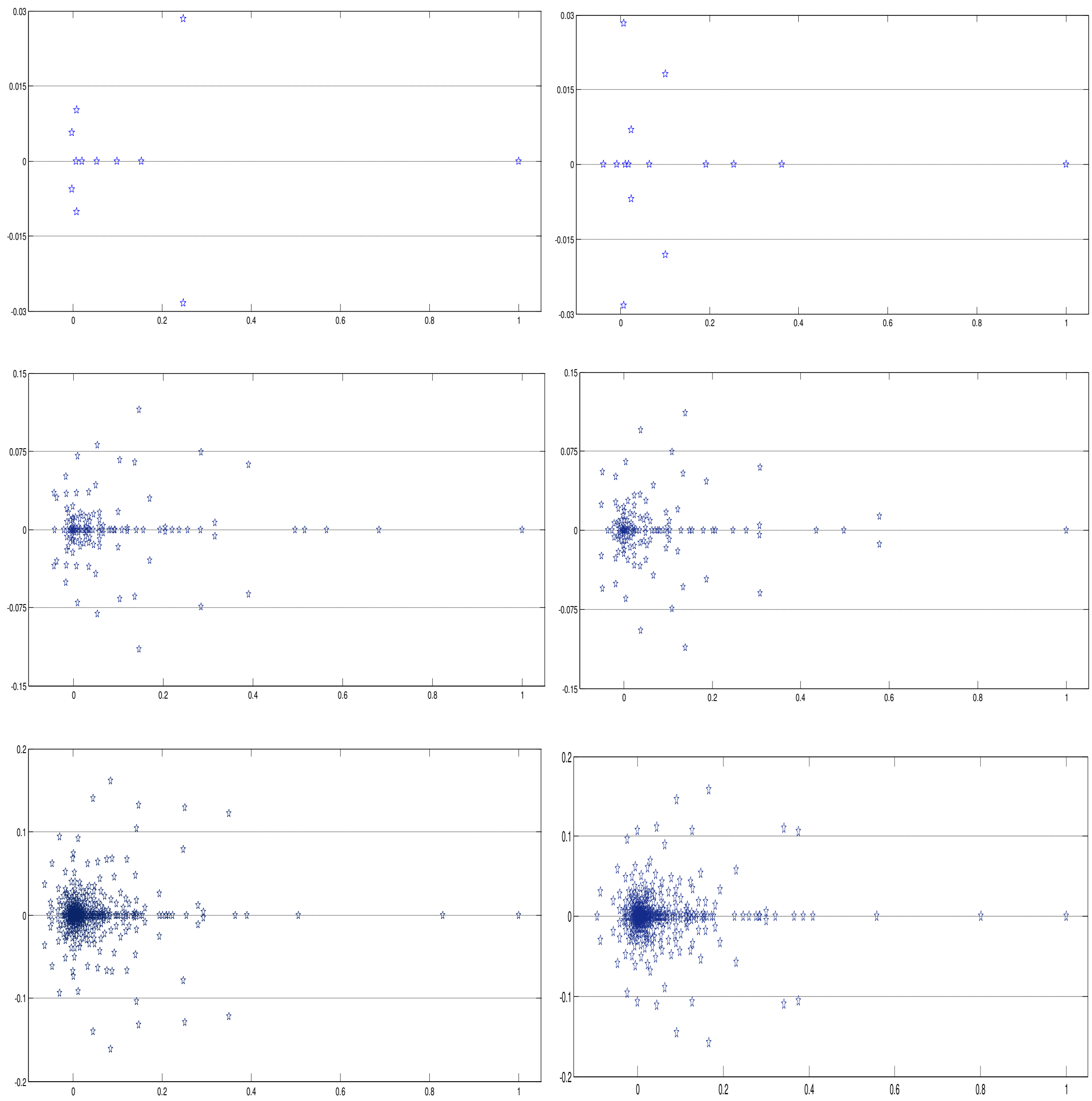


Figure 1. The location of the eigenvalues in the complex plane; years 1997 ($n = 12$, 129, 488) and 2002 ($n = 15$, 133, 426)

Table 1. The moduli of the second and third eigenvalues, and the average mean of the moduli of the non-dominant eigenvalues

	1997			2002		
n	12	129	488	15	133	426
$\lambda_{\mathbf{H}1}$	0.97	0.96	1.06	0.92	0.92	0.92
$\rho_{\mathbf{J}2}$	0.25	0.68	0.83	0.36	0.58	0.80
$\rho_{\mathbf{J}3}$	0.25	0.56	0.51	0.25	0.58	0.56
AM	0.08	0.08	0.05	0.08	0.08	0.05

From these results and the hitherto analysis we arrive at the following conclusions:

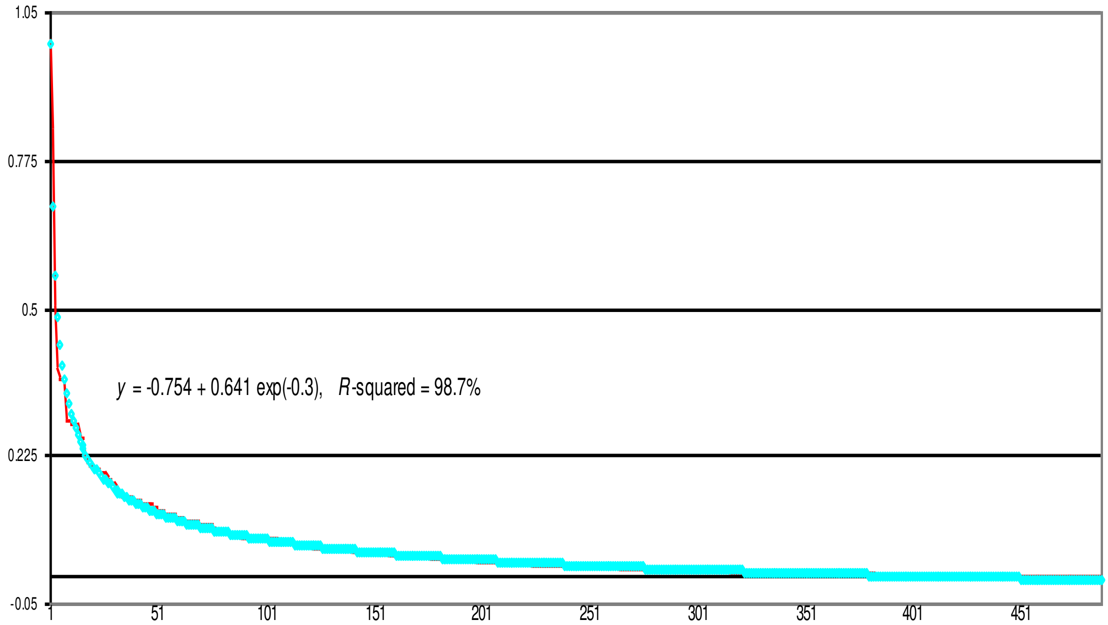
(i). Although $\lambda_{\mathbf{H}1}$ for all aggregations are near to each other (in fact, for the year 2002 they differ in the third decimal), $\rho_{\mathbf{J}2}$ increases with the size of the matrices casting doubt on Bródy's conjecture (see also relation (1)).[7] $\rho_{\mathbf{J}3}$ increases from the small size matrices to the large ones and decreases slightly for the extra large matrices.

(ii). The moduli of the first non-dominant eigenvalues fall quite rapidly and the rest constellate in much lower values forming a 'long tail' (see also the values of AM in Table 1): as Figure 2 indicates, the moduli follows an exponential pattern of the form $y = c + b\exp(x^a)$, $c < 0$, $b > 0$ and $a < 0$, similar to the findings of previous studies (Mariolis and Tsoulfidis, 2011, pp. 104-105, and Iliadi et al., 2012, pp. 10-12). This implies that the effective rank of $\mathbf{M}$ is relatively low (see relation (2)) and, therefore, the system can be adequately described by only a few eigensectors, which regulate its adjustment to equilibrium.[8]

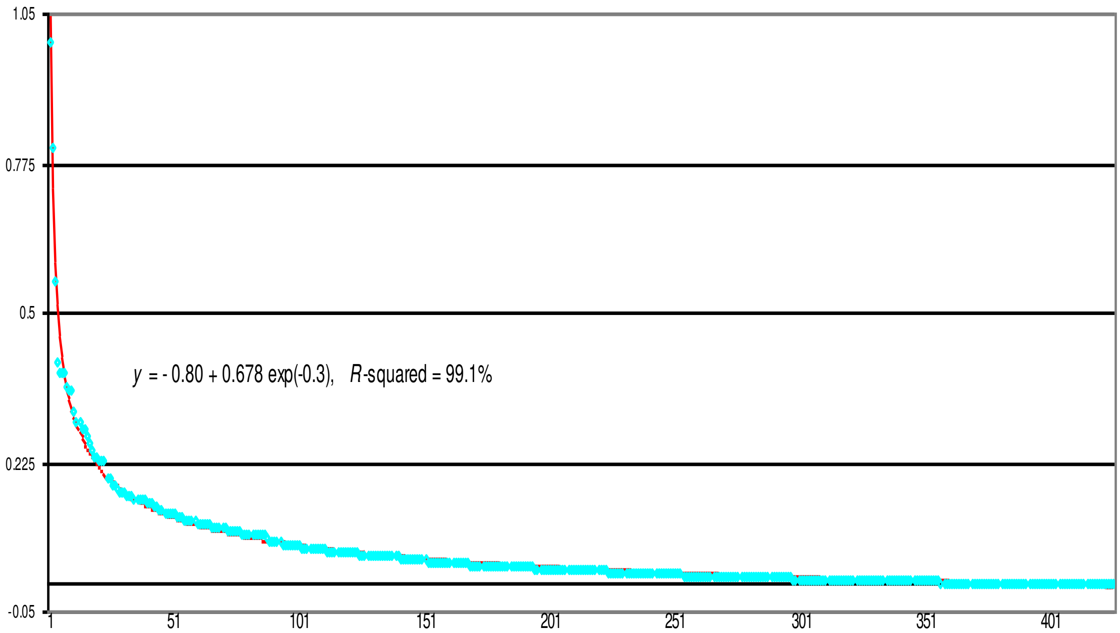
[7] Nevertheless, Bródy (1997, p. 255) notes: "[T]he coefficients computed from input-output tabulations are not evenly distributed and do not seem to follow a clear-cut distribution. Their pattern is skew, with a few large and many small and zero elements. […] A special distribution and/or a special structure of the matrix may still permit exceptions [to the conjecture]." (see also Molnár and Simonovits, 1998, who examine deterministic matrices, and Białas and Gurgul, 1998, whose focus is on column stochastic matrices).

[8] Bidard and Schatteman (2001, p. 297) note: "For economists, the crucial hypothesis in Bródy's conjecture is that the entries of I-O tables can be considered as i.i.d. random variables. The hypothesis is but the expression of our a priori ignorance. The difficulty does not come from the presence of many zeros in I-O tables but from specific linkages between some industries. It would be interesting to check whether the existence of patches of intense relationships between some sectors resists disaggregation and is sufficient to reverse the result established when the entries are chosen at random. […] Since the randomness hypothesis is economically unrealistic, an application of the theorem to actual I-O tables remains subject to practical tests.".

Figure 2. Exponential fit ($a = -0.3$) of the moduli of the eigenvalues; (a) 1997, $n = 488$; and (b) 2002, $n = 426$



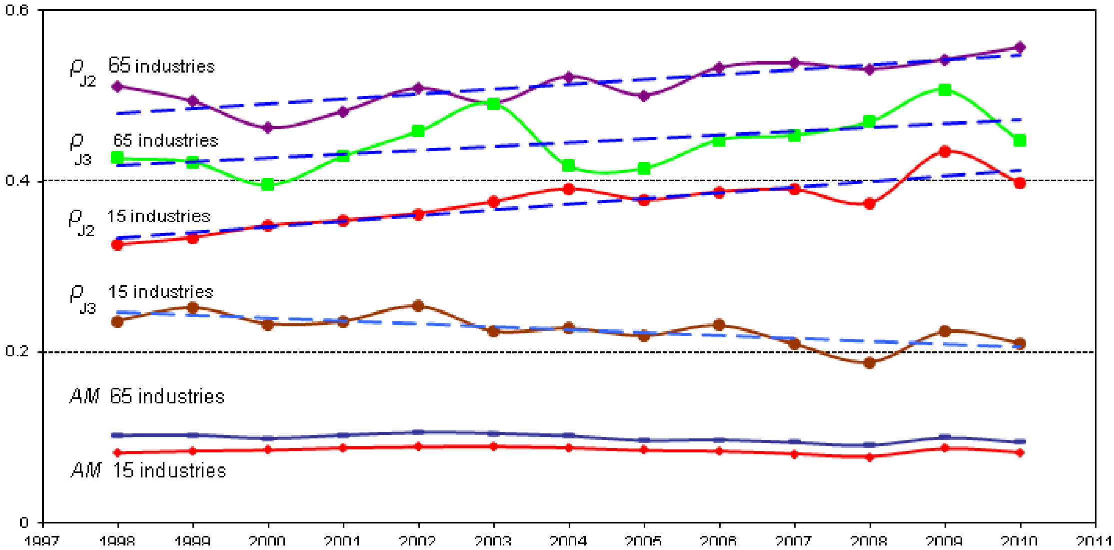
(a)



(b)

Apart from the benchmark years 1997 and 2002 the BEA provides input-output data spanning the period 1998-2010 and the dimensions of these tables are of 12 and 65 industries. The industry structure and the methods of assembling the data is the same and so we put together, in Figure 3, ρ_{J2}, ρ_{J3} and the AM of ρ_{Jk}, in order to observe their evolution during a period of 13 years (we also display the linear regression trend lines).

Figure 3. The evolution of the moduli of the second and third eigenvalues, and of the mean of the moduli of the non-dominant eigenvalues; period 1998-2010, $n = 15,\ 65$



We observe that the results are consistent with the findings for the years 1997 and 2002. More specifically, for both the 65 and the 15 industry detail, ρ_{J2} follow upward trends, and $(\rho_{J2})_{15} < (\rho_{J2})_{65}$. Moreover, $(\rho_{J3})_{15} < (\rho_{J3})_{65}$, where the former (the latter) follows a downward (upward) trend. Finally, for both 65 and 15 industry detail, the arithmetic mean move pretty much parallel to the horizontal axis and, as expected, the $(AM)_{65}$ ($\cong$ 0.100 over the whole period) is somewhat higher than $(AM)_{15}$ ($\cong$ 0.086 over the whole period) because the second and the third eigenvalues in the 15 industry structure are lower than those of the 65 industry structure.

Our results are in absolute accordance, both qualitatively and quantitatively, with those of many diverse economies (i.e. Canada, China, Denmark, Finland, France, Germany, Greece, Japan, Korea, Sweden and UK, where $19 \le n \le 100$; see Schefold, 2008, pp. 34-36, Mariolis and Tsoulfidis, 2011, pp. 101-109, Iliadi et al., 2012, pp. 8-12). Thus, it is reasonable to expect that there is a strong tendency towards uniformity in the eigenvalue distribution across countries and over time. However, moving from the flow to the more realistic stock input-output data, the available evidence suggests that ρ_{J2} falls even more abruptly, whereas ρ_{J3} or ρ_{J4} become 'indistinguishable' from the rest (see Steenge and Thissen, 2005; Mariolis and Tsoulfidis, 2011, pp. 109-111), indicating that Bródy's conjecture should be further investigated.

3. CONCLUDING REMARKS

In his conclusions Bródy (1997, p. 257) remarks that while one major undertaking in theoretical economies of the 20th century focuses on the existence of equilibrium, "[t]he question for the next [twenty-first] century seems to relate to whether the market can be rendered convergent. If not, why; if yes, how?". The results of our analysis for the US economy suggest that the ratio of the modulus of the subdominant eigenvalue to the dominant one increases both with the size of the input-output matrix and, for the same matrix size, over the years lending support to the view of increasing instability. Thus, it can be concluded that the actual input-output matrices do not share all the properties of random matrices. However, the fact that the majority of the non-dominant eigenvalues of the former matrices concentrates at low values indicates that there are considerable quasi-linear dependencies between the technical conditions of production in many industries.

Future research should use more disaggregated input-output data, also with fixed capital and joint production, and further investigate the detected tendency towards uniformity in the eigenvalue distribution.

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