A non-cooperative approach to the ordinal Shapley rule

Vidal-Puga, Juan

Universidade de Vigo

14 January 2013

A non-cooperative approach to the ordinal Shapley rule*

Juan Vidal-Puga†

vidalpuga@uvigo.es

January 14, 2013

Abstract

In bargaining problems, a rule satisfies ordinal invariance if it does not depend on order-preserving transformations of the agents' utilities. In this paper, a simple non-cooperative game for three agents, based on bilateral offers, is presented. The ordinal Shapley rule arises in subgame perfect equilibrium as the agents have more time to reach an agreement.

Keywords: ordinal bargaining, ordinal Shapley rule

1 Introduction

In bargaining problems, a rule satisfies ordinal invariance if it does not depend on order-preserving transformations of the agents' utilities. For two agents, Shapley (1969) shows that no efficient rule, apart from the dictatorial one,

*Financial support by the Spanish Ministerio de Ciencia e Innovación through grant ECO2011-23460 and Xunta de Galicia through grant 10PXIB362299PR is gratefully acknowledged.

†Research Group of Economic Analysis and Departamento de Estatistica e IO. Universidade de Vigo. 36200 Vigo (Pontevedra), Spain.

satisfies ordinal invariance. However, this negative result does not hold any more for more than two agents. Shubik (1982) first documents an efficient, symmetric, and ordinal invariant rule for three agents. Even though there is no reference on the origin of this rule in Shubik (1982), Pérez-Castrillo and Wettstein (2006, p. 297) attribute it to Shapley (1969). Furthermore, Roth (1979, p. 72-73) mentions a three-agent ordinal bargaining rule proposed by Shapley and Shubik in a 1974 working paper. Kıbrıs (2004a, 2004b) refers to it as the Shapley-Shubik rule.

Kıbrıs (2004b) describes a class of three-agent problems which generate all bargaining problems. On this class, the ordinal Shapley rule coincides with the Egalitarian rule (Kalai, 1977) and the Kalai-Smorodinsky rule (Kalai and Smorodinsky, 1975), and moreover it is the only symmetric member of a class of ordinal monotone path rules. Kıbrıs (2004a) also shows that the ordinal Shapley rule is deeply related to a family of solutions defined by Bennett (1997) for the class of multilateral bargaining problems. Moreover, Kıbrıs (2012) characterizes the ordinal Shapley rule using a weaker version of Independence of Irrelevant Alternatives (Nash, 1950). On the other hand, Samet and Safra (2005) extend the ordinal Shapley rule for more than three agents using constructions similar to O'Neill, Samet, Wiener and Winter (2004). Safra and Samet (2004) provide yet another family of ordinal solutions.

Following a different approach, Pérez-Castrillo and Wettstein (2006) use the underlying physical environment generating the utility possibilities frontier. This allows them to define an ordinal extension for the Shapley value for an arbitrary number of agents. We will call this value the ordinal Shapley *value*.

Finally, a mixed approach is given by Calvo and Peters (2005), who study situations where there exist ordinal and cardinal agents.

The definitions of these values are normative. An alternative approach is to propose non-cooperative games whose equilibria yield the desired values. This is the basis of the so-called Nash program, first suggested by Nash (1953), and also related to the theory of implementation. Implementation in general environments is addressed by Moore and Repullo (1988) and Maniquet (2003). See also Serrano (2005, 2008) for two recent surveys on the Nash program.

A non-cooperative game yielding the ordinal Shapley value in subgame perfect equilibria for three agents is presented in Pérez-Castrillo and Wettstein (2005). This non-cooperative game is based on a bidding mechanism by Pérez-Castrillo and Wettstein (2001, 2002).

In this paper, we present a non-cooperative game whose unique subgame perfect equilibrium payoff allocation approaches the ordinal Shapley rule as the agents have more time to reach an agreement. As far as we know, no other similar result has been found for a purely ordinal rule.

Informally, the idea of the non-cooperative game is as follows: First, two of the agents decide a payoff allocation *a la Rubinstein*, i.e. by an alternating-offer procedure, with no discount and with one round passing by each time an offer is rejected. However, each time an offer is rejected, the third agent has the choice to replace the agent that made the rejection. Once an offer (if any) is agreed upon, it constitutes a pre-agreement between the two agents who reached it. The other agent can then choose one of them and make her a counter-proposal, which in case of being accepted would cancel the pre-agreement. However, if the counter-proposal is rejected, the unchosen agent makes a last offer with the pre-agreement remaining as a status quo in case of rejection. Moreover, before either the counter-proposal or the pre-agreement is implemented, the agent that did not participate in it has an option of renegotiation, which makes the process to be repeated in the next round. If no agreement is reached after a pre-specified number of rounds, the process finishes with the status quo as the final payoff allocation.

As the number of rounds increases, there exists a subgame perfect equilibrium whose payoff allocation approaches the ordinal Shapley rule. Under reasonable assumptions on the behavior of the agents when they are indifferent (tie-breaking rules), this equilibrium is unique.

The paper is organized as follows: In Section 2, we present the basic notation and definitions, as well as some preliminary results. In Section 3 we formally describe the non-cooperative game and provide the main results, as well as an overview of the proofs. We present the formal proofs in Section 4 (Appendix).

2 Preliminaries

Let $N = \{1, 2, 3\}$ be the set of agents. Given $x, y \in \mathbb{R}^N$, $x \leq y$ means $x_i \leq y_i$ for all $i \in N$, $x \ll y$ means $x_i < y_i$ for all $i \in N$, and $x < y$ means $x \leq y$ and $x \neq y$. Let Π be the set of all permutations of N, with generic element π.

A pair $(S, d) \in 2^N \times \mathbb{R}^N$ is a *bargaining problem* if $\{x \in S : y \leq x\}$ is compact for all $y \in \mathbb{R}^N$ and d belongs to the interior of S. A point $x \in S$ is *Pareto optimal* in S if there is no $y \in S$ such that $x < y$. Let $P(S)$ denote the set of Pareto optimal points in S. A point $x \in S$ is *weakly Pareto optimal* in S if there is no $y \in S$ such that $x \ll y$. Let $WP(S)$ denote the set of weakly Pareto optimal points in S.

A bargaining problem (S, d) is *strictly comprehensive* if $WP(S) = P(S)$ and for each $x \in S$, $y \leq x$ implies $y \in S$. Let $\mathcal{B}$ denote the set of all strictly comprehensive bargaining problems.

For each $(S, d) \in \mathcal{B}$, $x, y \in \mathbb{R}^N$ and $N = \{i, j, k\}$, agent i's *aspiration payoff* restricted to x_j and y_k is

$$a_i(S, \langle x_j, y_k \rangle) \equiv \max\{s_i : (s_i, x_j, y_k) \in S\}$$

and her *aspiration payoff allocation* restricted to x_j and y_k is

$$a(S, \langle x_j, y_k \rangle) \equiv (a_i(S, \langle x_j, y_k \rangle), x_j, y_k).$$

Let $(S, d) \in \mathcal{B}$. Define $p^0(S, d) \equiv d$ and $p^{0,ij}(S, d) \equiv a(S, \langle d_i, d_j \rangle)$ for all $i, j \in N$. For each $t = 1, 2, ...$, there exists a unique $p^t(S, d) \in \mathbb{R}^N$ such that

$$\begin{aligned}
p^{t,12}(S, d) &\equiv \left(p_1^t(S, d), p_2^t(S, d), p_3^{t-1}(S, d)\right) \in P(S) \\
p^{t,13}(S, d) &\equiv \left(p_1^t(S, d), p_2^{t-1}(S, d), p_3^t(S, d)\right) \in P(S) \\
p^{t,23}(S, d) &\equiv \left(p_1^{t-1}(S, d), p_2^t(S, d), p_3^t(S, d)\right) \in P(S).
\end{aligned}$$

For simplicity, we write $p^{t,ij}$ instead of $p^{t,ij}(S,d)$. For notational convenience, $p^{t,12} = p^{t,21}$ and so on.

It is easily checked (see Figure 1) that, given $\{i,j,k\} = \{1,2,3\}$ and $t > 0$,

$$p_i^{t-1,ij} = p_i^{t-1,ik} = p_i^{t,jk}. \tag{1}$$

Let $p_i^{t-1,\cdot i} \equiv p_i^{t-1,ij} = p_i^{t-1,ik}$. We also have

$$\begin{array}{ll} p_i^{t-1,\cdot i} < p_i^{t-1,jk} & \text{if } t \text{ is odd} \\ p_i^{t-1,\cdot i} > p_i^{t-1,jk} & \text{if } t \text{ is even} \end{array} \tag{2}$$

and

$$\begin{array}{ll} p_i^{t,\cdot i} < p_i^{t-1,jk} & \text{if } t \text{ is odd} \\ p_i^{t,\cdot i} > p_i^{t-1,jk} & \text{if } t \text{ is even} \end{array} \tag{3}$$

and

$$\begin{array}{ll} p_i^{t-1,\cdot i} < p_i^{t,\cdot i} & \text{if } t \text{ is odd} \\ p_i^{t-1,\cdot i} > p_i^{t,\cdot i} & \text{if } t \text{ is even.} \end{array} \tag{4}$$

Moreover, notice that,

$$p^{t-1,ij} = a\left(S, \left\langle p_i^{t-1,\cdot i}, p_j^{t-1,\cdot j}\right\rangle\right). \tag{5}$$

The sequence $\{p^t\}_{t=0}^{\infty}$ is uniquely defined and it is convergent. Also, for each $i,j \in N$,

$$\lim_{t\to\infty} p^t = \lim_{t\to\infty} p^{t,ij}.$$

A bargaining rule $F : \mathcal{B} \to \mathbb{R}^N$ assigns to each bargaining problem $(S,d) \in \mathcal{B}$ a feasible point $F(S,d) \in S$. For each $(S,d) \in \mathcal{B}$, the *ordinal Shapley rule*, $Sh : \mathcal{B} \to \mathbb{R}^N$ assigns the limit of the sequence $\{p^t\}_{t=0}^{\infty}$, namely:

$$Sh(S,d) \equiv \lim_{t\to\infty} p^t.$$

This bargaining rule is Pareto optimal, symmetric and ordinal invariant.

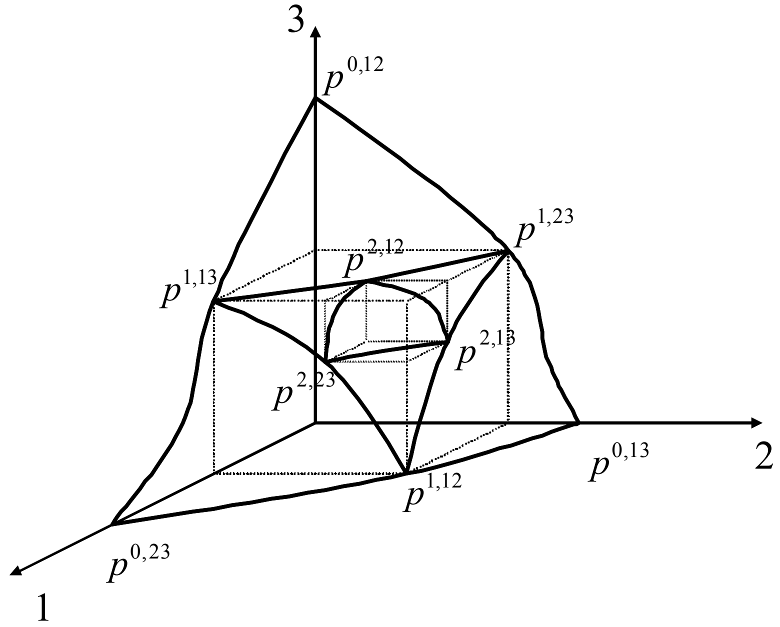

Figure 1: Position of the first points $p^{t,ij}$. The points p^t are at the vertices (of the polyhedrons) that do not lie on the frontier of S.

3 The non-cooperative game

We describe in detail the non-cooperative game depicted in the Introduction.

There are at most T negotiation rounds. If no agreement is reached after round T, the disagreement payoff allocation d is implemented. At each round, the agents play the roles of *first proposer*, *first responder*, and *pivot.* Say, w.l.o.g., that, in the first round, agent 1 is the first proposer, agent 2 is the first responder, and agent 3 is the pivot. Agent 1 proposes a payoff allocation $x \in S$. Agent 2 can then accept or reject this proposal.

A round passes by if agent 2 rejects this proposal. In this case, agent 3 can choose to replace agent 2, so that in the next round agent 1 keeps playing the role of first proposer, whereas agent 3 becomes the first responder and agent 2 becomes the pivot. In case agent 3 does not replace agent 2, then agent 2 plays the role of first proposer and agent 1 plays the role of first responder.

In case agent 2 accepts the proposal x, then agent 3 makes a counter-proposal $y \in S$ to either agent 1 or agent 2 (whoever agent 3 chooses). Let i be this agent and let j be the other one. Agent i should choose between the counter-proposal y and the pre-agreement x. Two cases are possible:

1. If agent i chooses y, then agent j can still ask for a renegotiation. If agent j does not ask for a renegotiation, y is implemented and the game finishes. If agent j ask for a renegotiation, a round passes by.

2. If agent i chooses x, then agent j makes a last proposal $z \in S$. Agent i should choose between z and x. In case z is chosen and agent 3 does not veto, this payoff allocation is implemented and the game finishes. If agent 3 vetoes, the final payoff is d. In case x is chosen, agent 3 can still ask for a renegotiation. If agent 3 does not ask for a renegotiation, x is implemented and the game finishes. If agent 3 ask for a renegotiation, a round passes by.

In case of renegotiation, in the next round agent 3 plays the role of first proposer, agent i plays the role of first responder, and agent j plays the role of pivot.

At round $T+1$, the game finishes and the final payoff allocation is d.

In order to fully formalize the non-cooperative game, a formal description is presented as follows (see also Figure 2). We denote the game as $B^t(\pi)$, where t is the number of rounds left (hence, we begin with $B^T(\pi)$) and $\pi \in \Pi$ is the order that specifies the roles: π_1 is the first proposer, π_2 is the first responder, and π_3 is the pivot. For simplicity, we write $B^t[\pi_1\pi_2\pi_3]$ instead of the more cumbersome $B^t([\pi_1\pi_2\pi_3])$.

The non-cooperative game is defined inductively on t. $B^0(\pi)$ is the trivial game with d as final payoff allocation.

Assume $B^s(\sigma)$ is defined for all $s < t$ and all $\sigma \in \Pi$. Assume w.l.o.g. that $\pi = [123]$, i.e. $\pi_i = i$ for all $i \in N$. We define $B^t[123]$ as follows:

> Agent 1 proposes $x \in S$. Agent 2 can accept or reject.
>
> If agent 2 rejects, agent 3 chooses between playing game $B^{t-1}[132]$ or $B^{t-1}[213]$. If agent 2 accepts, agent 3 chooses $i \in \{1,2\}$ and proposes $y \in S$. Let $j \in \{1,2\} \setminus \{i\}$ be the other agent.
>
> Agent i can choose x or y.
>
> If agent i chooses y, then agent j can ask for a renegotiation. If agent j ask for a renegotiation, then $B^{t-1}[3ij]$ is played. If agent j does not ask for a renegotiation, y is implemented and the game finishes.
>
> If agent i chooses x, then agent j proposes $z \in S$. Agent i can choose z or x. If agent i chooses z, then agent 3 can veto or not veto. If agent 3 does not veto, z is implemented and the game finishes. If agent 3 vetoes, d is implemented and the game finishes. If agent i chooses x, then agent 3 can ask for a renegotiation. If agent 3 does not ask for a renegotiation, x is implemented and the game finishes. If agent 3 ask for a renegotiation, then $B^{t-1}[3ij]$ is played.

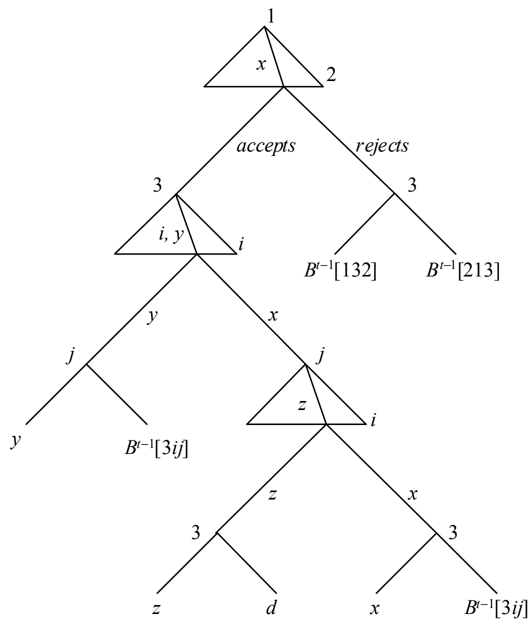

Figure 2: The non-cooperative game $B^t\,[123]$ with $t > 0$.

Notice that, in each round, any agreement (x, y or z) should be first achieved by two agents (x by agents 1 and 2, y by agents 3 and i, and z by agents 1 and 2 again). Once two agents have agreed upon a proposal, the third one has the choice to ask for a renegotiation and move to the next round. The only exception is given by agent 3 when z is chosen. In this case, agent 3 cannot ask for a renegotiation, but she can veto z and force the implementation of d.

Theorem 3.1 *For any $T > 0$ and $\pi \in \Pi$, there exists a subgame perfect equilibrium for the non-cooperative game $B^T(\pi)$ whose equilibrium payoff allocation is $p^{T,\pi_1\pi_3}$.*

An immediate corollary is the following:

Corollary 3.1 *As T increases, there exists a subgame perfect equilibrium in the non-cooperative game whose final payoff allocation approaches the payoff allocation given by the ordinal Shapley rule.*

In general, there can be more than one subgame perfect equilibrium. However, the above subgame perfect equilibrium is unique under the following Assumptions:

Assumption 1 The agents strictly prefer to finish in the earliest round.

Assumption 2 If the pivot (say k) is indifferent when choosing i, and x is such that k is bound to ask for renegotiating it, then k would choose the most harmful choice for the first proposer.

Assumption 1 follows from the fact that either a rejection or a renegotiation implies a delay. Hence, it seems natural that an agent would prefer to reach an agreement as soon as possible. Following this idea, Assumption 1 implies that, when an agent is due to receive the same final payoff when accepting or rejecting x, or whether she ask or not for a renegotiation, then she accepts and does not ask for a renegotiation. An equivalent assumption is to assume that all the agents have a lexicographic preference for the game ending in the first round. See Bag and Winter (1999, p. 79) for the intuition behind this assumption, which is also used in Mutuswami and Winter (2002). Other similar assumption is used in Moldovanu and Winter (1994), who assume that an agent prefers agreements which involve larger rather than smaller coalitions (provided her final payoff is the same in both agreements). Hart and Mas-Colell (1996) also assumed that agents "break ties in favor of quick termination of the game"[1].

Assumption 2 is needed for $T > 1$ and it has the following justification: since x comes from the first proposer, and it is not satisfactory for the pivot (in the sense that she would prefer to ask for a renegotiation), the threat of harming the first proposer is justifiable and hence credible. Moreover, the payoff allocation would be strictly smaller for the pivot in subgame perfect equilibria not satisfying Assumption 2. This latter idea is related with the justification the proper equilibria (Myerson, 1978) as a refinement of the

[1]However, tie-breaking rules are not needed in Hart and Mas-Colell's model.

"trembling-hand" perfect equilibria (Selten, 1975). The proper equilibria assumes that more costly mistakes are much less likely to happen.

Theorem 3.2 *Under Assumptions 1 and 2, for any $T > 0$ and $\pi \in \Pi$, $p^{T,\pi_1\pi_3}$ is the unique subgame perfect equilibrium payoff allocation for the non-cooperative game $B^T[\pi]$.*

An immediate corollary is the following:

Corollary 3.2 *Under Assumptions 1 and 2, as T increases the only equilibrium payoff allocation in the non-cooperative game $B^T[\pi]$ approaches the payoff allocation given by the ordinal Shapley rule.*

The formal proofs of Theorem 3.1 and Theorem 3.2 are located in the Appendix. We provide here the idea. Assume $\pi = [123]$. Basically, there are two cases to consider: $x_3 \geq p_3^{T,12}$ and $x_3 < p_3^{T,12}$.

In the first case ($x_3 \geq p_3^{T,12} \overset{(1)}{=} p_3^{T-1,\cdot 3}$), and under an induction hypothesis, agent 3 is bound to accept x in case agent i chooses x twice. From this, the optimal z for agent j is $a(S, \langle x_i, d_3\rangle)$ and that would be the final payoff allocation in the subgame that begins when agent j proposes z. Knowing that, the optimal y for agent 3 would be $a\left(S, \left\langle x_i, p_j^{T-1,\cdot j}\right\rangle\right)$ and that would be the final payoff allocation in the subgame that begins when agent 3 proposes y. This implies that an optimal x for agent 1 satisfies $x \in P(S)$, $x_3 = p_3^{T-1,\cdot 3}$ and $a_3\left(S, \left\langle x_1, p_2^{T-1,\cdot 2}\right\rangle\right) = a_3\left(S, \left\langle x_2, p_1^{T-1,\cdot 1}\right\rangle\right)$. These conditions imply $x = p^{T,12}$, which will make agent 2 indifferent between accepting or rejecting. The final payoff allocation will then be

$$a\left(S, \left\langle p_1^{T,12}, p_2^{T-1,\cdot 2}\right\rangle\right) \overset{(1)}{=} a\left(S, \left\langle p_1^{T,\cdot 1}, p_2^{T,13}\right\rangle\right) \overset{(5)}{=} p^{T,13}.$$

Assumption 1 is needed in order of agent 2 to accept $x = p^{T,12}$, since there is no possibility for agent 1 to find another x around $p^{T,12}$ which breaks this indifference. Inducing agent 2 to reject would be also harmful for agent 1, since agent 3 would choose the best (for her) between $B^{t-1}[132]$ and $B^{t-1}[213]$, which will coincide with the worst for agent 1.

In the second case ($x_3 < p_3^{T,12} \overset{(1)}{=} p_3^{T-1,\cdot 3}$), agent 3 is bound to ask for a renegotiation in case agent i chooses x twice. Under the induction hypothesis, the final payoff would be $p^{T-1,j3}$ when $T > 1$, and d when $T = 1$. From this, the optimal z for agent j is $a\left(S, \left\langle p_i^{T-1,j3}, d_3\right\rangle\right)$ when $T > 1$, and $a(S, \langle d_i, d_3\rangle) = p^{0,i3}$ when $T = 1$. That would be the final payoff in the subgame that begins when agent j proposes z. Knowing that, the optimal x for agent 3 would be $a\left(S, \left\langle p_i^{T-1,j3}, p_j^{T-1,\cdot j}\right\rangle\right) \overset{(5)}{=} p^{T-1,j3}$ when $T > 1$, and $a\left(S, \langle p_i^{0,\cdot i}, p_j^{0,\cdot j}\rangle\right) \overset{(5)}{=} p^{0,ij}$ when $T > 1$. That would be the final payoff in the game that begins when agent 3 proposes x.

At this point, Assumption 2 is needed for $T > 1$ so that agent 3 makes the most harmful choice of i for agent 1, so that it is not profitable for agent 1 to choose x with $x_3 < p_3^{T,12}$. If this happens, the final payoff for agent 3 would be strictly lower than $p_3^{T,\cdot 3}$, which gives an additional justification for Assumption 2 to hold.

Notice that Assumption 2 is not needed for $T = 1$ because the final payoff for player 1 would be d_1 (irrespectively of the choice of i) and $d_1 < p_1^{1,\cdot 1}$.

As a result, the equilibrium path in these results is the following: Agent 1 proposes $x = p^{T,12}$, agent 2 accepts, agent 3 chooses to negotiate with agent 1 and proposes $y = p^{T,13}$, agent 1 chooses y, and agent 2 does not ask for a renegotiation.

4 Appendix

In this section we formally prove Theorem 3.1 and Theorem 3.2. We will use the following lemmas:

Lemma 4.1 *Let $x \in S$ such that $x_3 \geq p_3^{t,12}$ for some $t \geq 0$. Then,*

$$\max\left\{a_3\left(S, \langle x_1, p_2^{t,13}\rangle\right), a_3\left(S, \langle x_2, p_1^{t,23}\rangle\right)\right\} \geq p_3^{t,12}$$

and, moreover, equality holds iff $x = p^{t,12}$.

Proof. Define three functions $f^1, f^2, f : \left\{y \in S : y_3 \geq p_3^{t,12}\right\} \to \mathbb{R}$ as follows:

$$\begin{aligned} f^1(y) &= a_3\left(S, \left\langle y_1, p_2^{t,13}\right\rangle\right) \\ f^2(y) &= a_3\left(S, \left\langle y_2, p_1^{t,23}\right\rangle\right) \\ f(y) &= \max\left\{f^1(y), f^2(y)\right\}. \end{aligned}$$

We will compute the minimum of f. Under our hypothesis on S, $f^1(y)$ is strictly decreasing in y_1, whereas $f_2(y)$ is strictly decreasing in y_2. Hence, f reaches a minimum when $y \in P(S)$, $y_3 = p_3^{t,12}$ and $f^1(y) = f^2(y)$, i.e.

$$\begin{aligned} y &\in P(S) \\ y_3 &= p_3^{t,12} \\ a_3\left(S, \left\langle y_1, p_2^{t,13}\right\rangle\right) &= a_3\left(S, \left\langle y_2, p_1^{t,23}\right\rangle\right). \end{aligned}$$

which has a unique solution: $y = p^{t,12}$, so that $f\left(p^{t,12}\right) = p_3^{t,12}$. ∎

Lemma 4.2 *Given $t > 0$,*

$$\max\left\{x_1 : x \in S, a_3\left(S, \left\langle x_1, p_2^{t-1,\cdot 2}\right\rangle\right) \geq a_3\left(S, \left\langle x_2, p_1^{t-1,\cdot 1}\right\rangle\right), x_3 \geq p_3^{t-1,\cdot 3}\right\} = p_1^{t,\cdot 1}$$

and, moreover, this maximum is only achieved when $x = p^{t,12}$.

Proof. Let $x \in S$ such that $a_3\left(S, \left\langle x_1, p_2^{t-1,\cdot 2}\right\rangle\right) \geq a_3\left(S, \left\langle x_2, p_1^{t-1,\cdot 1}\right\rangle\right)$ and $x_3 \geq p_3^{t-1,\cdot 3}$.

Assume first $x \notin P(S)$. Under the strict comprehensiveness of S, we can find some $\epsilon_1, \epsilon_2 > 0$ such that $x^\epsilon := (x_1 + \epsilon_1, x_2 + \epsilon_2, x_3) \in S$ and $a_3\left(S, \left\langle x_1^\epsilon, p_2^{t-1,\cdot 2}\right\rangle\right) \geq a_3\left(S, \left\langle x_2^\epsilon, p_1^{t-1,\cdot 1}\right\rangle\right)$. Hence, $x \in P(S)$ is necessary to achieve the supremum.

Assume now $x \in P(S)$ and $x_3 > p_3^{t-1,\cdot 3}$. Under the strict comprehensiveness of S, we can find some $\varepsilon_1 > 0$ such that $x^\varepsilon := \left(x_1 + \varepsilon_1, x_2, p_3^{t-1,\cdot 3}\right)$ satisfies $x \in S$, $x_3^\varepsilon \geq p_3^{t-1,\cdot 3}$ and $a_3\left(S, \left\langle x_1^\varepsilon, p_2^{t-1,\cdot 2}\right\rangle\right) \geq a_3\left(S, \left\langle x_2^\varepsilon, p_1^{t-1,\cdot 1}\right\rangle\right)$. Hence, $x_3 = p_3^{t-1,\cdot 3}$ is necessary to achieve the supremum.

Assume now $x \in P(S)$, $x_3 = p_3^{t-1,\cdot 3}$ and $a_3\left(S, \left\langle x_1, p_2^{t-1,\cdot 2}\right\rangle\right) > a_3\left(S, \left\langle x_2, p_1^{t-1,\cdot 1}\right\rangle\right)$. Under the strict comprehensiveness of S, we can find some $\delta_1, \delta_2 > 0$ such that $x^{\delta} := (x_1 + \delta_1, x_2 - \delta_2, x_3)$ satisfies $x^{\delta} \in S$ and $a_3\left(S, \left\langle x_1^{\delta}, p_2^{t-1,\cdot 2}\right\rangle\right) \geq a_3\left(S, \left\langle x_2^{\delta}, p_1^{t-1,\cdot 1}\right\rangle\right)$. Hence, $a_3\left(S, \left\langle x_1, p_2^{t-1,\cdot 2}\right\rangle\right) = a_3\left(S, \left\langle x_2, p_1^{t-1,\cdot 1}\right\rangle\right)$ is necessary to achieve the supremum.

Assume $x \in P(S)$, $x_3 = p_3^{t-1,\cdot 3}$ and $a_3\left(S, \left\langle x_1, p_2^{t-1,\cdot 2}\right\rangle\right) = a_3\left(S, \left\langle x_2, p_1^{t-1,\cdot 1}\right\rangle\right)$. We have to prove that $x = p^{t,12}$. Under (1), these conditions are equivalent to

$$\begin{aligned} x &\in P(S) \\ x_3 &= p_3^{t,12} \\ a_3\left(S, \left\langle x_1, p_2^{t,13}\right\rangle\right) &= a_3\left(S, \left\langle x_2, p_1^{t,23}\right\rangle\right). \end{aligned}$$

which has a unique solution: $x = p^{t,12}$. ■

We now prove Theorem 3.1 and Theorem 3.2. For any $t \geq 0$ and $i, j \in N$, $i \neq j$, let $q^{t,ij} \in \mathbb{R}^N$ be defined as $q^{0,ij} \equiv d$ and $q^{t,ij} \equiv p^{t,ij}$ otherwise. We will prove the following (stronger) result:

> For any $T \geq 0$ and $\pi \in \Pi$, there exists a subgame perfect equilibrium for the non-cooperative game $B^T(\pi)$ whose payoff allocation is $q^{T,\pi_1\pi_3}$. Moreover, this subgame perfect equilibrium payoff allocation is unique under Assumption 1 and Assumption 2.

Notice that, by definition, $q_i^{t,kj} = p_i^{t,jk}$ unless $t = 0$ and $i \notin \{j,k\}$. Hence, we write $p_i^{t,jk}$ instead of $q_i^{t,jk}$ unless case $q_i^{0,jk}$ with $i \notin \{j,k\}$ is possible.

Assume w.l.o.g. that, in the first round, agent 1 is the first proposer, agent 2 is the first responder, and agent 3 is the pivot. The proof is by induction on T. For $T = 0$, $B^0([123])$ is a trivial game and the unique final payoff allocation is $d = q^{0,13}$.

Assume the result is true for less than T rounds, $T > 0$. The subgame that arises in the second round of the game with T rounds is strategically equivalent to the game with $T - 1$ rounds. Hence, the continuation payoff in the second round is known by the agents. Under the induction hypothesis,

$q^{T-1,\pi_1\pi_3}$ is the continuation payoff in the second round when the order is given by $\pi \in \Pi$.

We first prove that there exists a subgame perfect equilibrium whose final payoff allocation is $p^{T,13}$.

Consider the following strategic profile in $B^T[123]$:

- At the beginning of the round, agent 1 proposes $x = p^{T,12}$.
- Agent 2 rejects x iff $x_3 \geq p_3^{T-1,\cdot 3}$, $a_3\left(S, \left\langle x_1, p_2^{T-1,\cdot 2}\right\rangle\right) < a_3\left(S, \left\langle x_2, p_1^{T-1,\cdot 1}\right\rangle\right)$ and $x_2 < p_2^{T-1,\cdot 2}$.
- If agent 2 rejects x , agent 3 chooses $B^{T-1}[132]$ if T is odd and $B^{T-1}[213]$ if T is even.
- If agent 2 accepts x, agent 3 chooses i and proposes y following the next rule:
  - If $x_3 < p_3^{T-1,\cdot 3}$, she chooses $i = 1$ if T is even and $i = 2$ if T is odd, and proposes $y = p^{T-1,j3}$.
  - If $x_3 \geq p_3^{T-1,\cdot 3}$, she chooses $i = 1$ if $a_3\left(S, \left\langle x_1, p_2^{T-1,\cdot 2}\right\rangle\right) \geq a_3\left(S, \left\langle x_2, p_1^{T-1,\cdot 1}\right\rangle\right)$ and $i = 2$ if $a_3\left(S, \left\langle x_1, p_2^{T-1,\cdot 2}\right\rangle\right) < a_3\left(S, \left\langle x_2, p_1^{T-1,\cdot 1}\right\rangle\right)$, and proposes $y = a\left(S, \left\langle x_i, p_j^{T-1,\cdot j}\right\rangle\right)$.
- After agent 3 chooses i and proposes y, agent i chooses either x or y following the next rule:
  - If $x_3 < p_3^{T-1,\cdot 3}$ and $y_j < p_j^{T-1,\cdot j}$, she chooses x.
  - If $x_3 < p_3^{T-1,\cdot 3}$ and $y_j \geq p_j^{T-1,\cdot j}$, she chooses x iff $q_i^{T-1,j3} > y_i$.
  - If $x_3 \geq p_3^{T-1,\cdot 3}$ and $y_j < p_j^{T-1,\cdot j}$, she chooses x iff $x_i > q_i^{T-1,j3}$.
  - If $x_3 \geq p_3^{T-1,\cdot 3}$ and $y_j \geq p_j^{T-1,\cdot j}$, she chooses x iff $x_i > y_i$.
- If agent i chooses y, then agent j asks for renegotiation iff $y_j < p_j^{T-1,\cdot j}$.
- If agent i chooses x, then agent j proposes z following the next rule:

- If $x_3 < p_3^{T-1,\cdot 3}$, she proposes $z = a\left(S, \left\langle q_i^{T-1,j3}, d_3\right\rangle\right)$.
- If $x_3 \geq p_3^{T-1,\cdot 3}$, she proposes $z = a\left(S, \langle x_i, d_3\rangle\right)$.

- After agent j proposes z, agent i chooses either z or x following the next rule:
  - If $z_3 < d_3$ and $x_3 < p_3^{T-1,\cdot 3}$, she chooses z iff $d_i \geq q_i^{T-1,j3}$.
  - If $z_3 < d_3$ and $x_3 \geq p_3^{T-1,\cdot 3}$, she chooses z iff $d_i \geq x_i$.
  - If $z_3 \geq d_3$ and $x_3 < p_3^{T-1,\cdot 3}$, she chooses z iff $z_i \geq q_i^{T-1,j3}$.
  - If $z_3 \geq d_3$ and $x_3 \geq p_3^{T-1,\cdot 3}$, she chooses z iff $z_i \geq x_i$.
- After agent i chooses z, agent 3 vetoes iff $z_3 < d_3$.
- After agent i chooses x, agent 3 asks for a renegotiation iff $x_3 < p_3^{T-1,\cdot 3}$.

In each $B^{T-1}(\sigma)$, we apply the induction hypothesis and assume the agents play a subgame perfect equilibrium profile that gives as final payoff $q^{T-1,\sigma_1\sigma_3}$.

Under (1) and the induction hypothesis, a backward reasoning shows that the above strategies constitute a subgame perfect equilibrium after agent i chooses y and after agent j proposes z.

To see that the proposed choice of z is optimal for agent j, we distinguish two cases:

1. If $x_3 < p_3^{T-1,\cdot 3}$, then agent i can assure herself $q_i^{T-1,j3}$ by choosing x. Thus, the maximum that agent j can get by making an acceptable offer is $a_j\left(S, \left\langle q_i^{T-1,j3}, d_3\right\rangle\right)$. This is what she would get by choosing $z = a\left(S, \left\langle q_i^{T-1,j3}, d_3\right\rangle\right)$, because it would induce agent i to choose z and agent 3 not to veto.

2. If $x_3 \geq p_3^{T-1,\cdot 3}$, then agent i can assure herself x_i by choosing x. Thus, the maximum that agent j can get is $a_j\left(S, \langle x_i, d_3\rangle\right)$. This is what she gets by choosing $z = a\left(S, \langle x_i, d_3\rangle\right)$, because it would induce agent i to choose z and agent 3 not to veto.

Hence, the final payoff allocation in this subgame (when agent j proposes z) is given by $a\left(S,\left\langle q_i^{T-1,j3},d_3\right\rangle\right)$ when $x_3<p_3^{T-1,\cdot 3}$ and by $a\left(S,\langle x_i,d_3\rangle\right)$ when $x_3\geq p_3^{T-1,\cdot 3}$.

Moreover, it is straightforward to check that, given y, the strategy of agent i is optimal for her.

We now check that the proposed choice of i and y is optimal for agent 3. Notice that, in case agent i chooses x, the strategies imply that the final payoff for agent 3 will be d_3.

We have two cases:

1. If $x_3<p_3^{T-1,\cdot 3}$, then $y=p^{T-1,j3}$ and the strategies determine that agent i chooses y and agent j does not ask for a renegotiation, so that the final payoff for agent 3 is $y_3=p_3^{T-1,\cdot 3}$. Hence, for such an x, the choice of i is indifferent. Moreover, this choice of y is optimal among those that induce agent i to choose y and agent j not to ask for a renegotiation.

2. If $x_3\geq p_3^{T-1,\cdot 3}$, then $y=a\left(S,\left\langle x_i,p_j^{T-1,\cdot j}\right\rangle\right)$ and the strategies determine that agent i chooses y and agent j does not ask for a renegotiation, so that the final payoff for agent 3 is $y_3=a_3\left(S,\left\langle x_i,p_j^{T-1,\cdot j}\right\rangle\right)$. Hence, for this y, the choice of i is optimal, so that the final payoff for agent 3 is $\max\left\{a_3\left(S,\left\langle x_1,p_2^{T-1,\cdot 2}\right\rangle\right),a_3\left(S,\left\langle x_2,p_1^{T-1,\cdot 1}\right\rangle\right)\right\}$. Moreover, this choice of y is optimal among those that induce agent i to choose y and agent j not to ask for a renegotiation. Under Lemma 4.1 and (1), $\max\left\{a_3\left(S,\left\langle x_1,p_2^{T-1,\cdot 2}\right\rangle\right),a_3\left(S,\left\langle x_2,p_1^{T-1,\cdot 1}\right\rangle\right)\right\}\geq p_3^{T-1,\cdot 3}$.

Hence, in both cases, agent 3 gets at least $p_3^{T-1,\cdot 3}$. There are two possible deviations: To induce agent i to choose y and agent j to ask for a renegotiation, and to induce agent i to choose x. In the first case, the induction hypothesis implies that the final payoff for agent 3 is $p_3^{T-1,\cdot 3}$, so she does not improve. In the second case, the strategies imply that the final payoff for agent 3 is d_3, which is again not higher than $p_3^{T-1,\cdot 3}$.

We now check that the proposed strategy for agent 3, after agent 2 rejects x, is optimal. Under the induction hypothesis, the final payoff for agent 3 is $q_3^{T-1,12}$ if she chooses $B^{T-1}[132]$, and $p_3^{T-1,\cdot 3}$ if she chooses $B^{T-1}[213]$. If $T=1$, agent 3 is indifferent (she gets d_3 in either case). If $T>1$, under (2), it is optimal to choose $B^{T-1}[132]$ if T is odd and $B^{T-1}[213]$ is T is even.

We now check that the proposed rule followed by agent 2 to reject x is optimal. Under the induction hypothesis, the final payoff for agent 2 in case of rejection is $p_2^{T-1,\cdot 2}$, irrespectively of T being odd or even. For the payoff in case of acceptance, we have two cases:

1. If $x_3 < p_3^{T-1,\cdot 3}$, the final payoff for agent 2 in case of acceptance is $p_2^{T-1,\cdot 2}$ if T is even, $p_2^{T-1,13}$ if T is odd and $T>1$, and d_2 if $T=1$. Hence, agent 2 is indifferent when T is even or (since $p_2^{0,\cdot 2}=d_2$) when $T=1$. Under (2), she is strictly better by accepting when T is odd and $T>1$.

2. If $x_3 \geq p_3^{T-1,\cdot 3}$, we have two subcases:

   (a) If $a_3\left(S,\left\langle x_1,p_2^{T-1,\cdot 2}\right\rangle\right) \geq a_3\left(S,\left\langle x_2,p_1^{T-1,\cdot 1}\right\rangle\right)$, the final payoff for agent 2 is $p_2^{T-1,\cdot 2}$. Hence, agent 2 is indifferent between accepting or rejecting. In particular, accepting is optimal.

   (b) If $a_3\left(S,\left\langle x_1,p_2^{T-1,\cdot 2}\right\rangle\right) < a_3\left(S,\left\langle x_2,p_1^{T-1,\cdot 1}\right\rangle\right)$, the final payoff for agent 2 is x_2. Hence, it is optimal to reject iff $x_2 < p_2^{T-1,\cdot 2}$.

We now check that $x=p^{T,12}$ is an optimal proposal for agent 1. If she does not deviate, then agent 2 will accept, agent 3 will choose $i=1$ and $y=a\left(S,\left\langle x_1,p_2^{T-1,\cdot 2}\right\rangle\right)$, agent $i=1$ will choose y and agent $j=2$ will not ask for a renegotiation, so that the final payoff for agent 1 will be $y_1=x_1=p_1^{T,\cdot 1}$.

Assume agent 1 deviates by proposing x with $x_3 < p_3^{T,12} \overset{(1)}{=} p_3^{T-1,\cdot 3}$. Then her final payoff becomes $p_1^{T-1,23}$ if T is even or $p_1^{T-1,\cdot 1}$ if T is odd. Under (3), agent 1 is strictly worse off when T is even. Under (4), agent 1 is strictly worse off when T is odd.

Assume now agent 1 deviates by proposing x with $x_3 \geq p_3^{T,12} \overset{(1)}{=} p_3^{T-1,\cdot 3}$ and $a_3\left(S,\left\langle x_1,p_2^{T-1,\cdot 2}\right\rangle\right) < a_3\left(S,\left\langle x_2,p_1^{T-1,\cdot 1}\right\rangle\right)$. Under (1) and Lemma 4.1, $a_3\left(S,\left\langle x_2,p_1^{T-1,\cdot 1}\right\rangle\right) > p_3^{T-1,\cdot 3}$ which implies $x_2 < p_2^{T-1,13}$.

We have two cases:

1. If $x_2 < p_2^{T-1,\cdot 2}$, then agent 2 rejects x. If T is odd, agent 3 chooses $B^{T-1}[132]$ and, under the induction hypothesis, the final payoff for agent 1 will be $p_1^{T-1,\cdot 1}$. Under (4), $p_1^{T-1,\cdot 1} < p_1^{T,\cdot 1}$ and agent 1 does not improve. If T is even, agent 3 chooses $B^{T-1}[213]$ and, under the induction hypothesis, the final payoff for agent 1 will be $p_1^{T-1,23}$. Under (3), $p_1^{T-1,23} < p_1^{T,\cdot 1}$ and agent 1 does not improve.

2. If $x_2 \geq p_2^{T-1,\cdot 2}$, then the final payoff allocation is $a\left(S,\left\langle x_2,p_1^{T-1,\cdot 1}\right\rangle\right)$. If T is odd, under (4), $p_1^{T-1,\cdot 1} < p_1^{T,\cdot 1}$ and agent 1 does not improve. If T is even, under (2), $p_2^{T-1,13} < p_2^{T-1,\cdot 2}$ but this not possible because $x_2 < p_2^{T-1,13}$ and $x_2 \geq p_2^{T-1,\cdot 2}$.

Assume now agent 1 deviates by proposing $x \neq p^{T,12}$ with $x_3 \geq p_3^{T-1,\cdot 3}$ and $a_3\left(S,\left\langle x_1,p_2^{T-1,\cdot 2}\right\rangle\right) \geq a_3\left(S,\left\langle x_2,p_1^{T-1,\cdot 1}\right\rangle\right)$. Then agent 2 accepts x, agent 3 chooses $i=1$ and proposes $y = a\left(S,\left\langle x_i,p_j^{T-1,\cdot j}\right\rangle\right)$, agent 1 finds it optimal to choose y so that agent 2 does not ask for a renegotiation and her final payoff is x_1. Under Lemma 4.2, $x_1 < p_1^{T,\cdot 1}$ and hence agent 1 does not improve.

We know prove that $p^{T,13}$ is the only subgame perfect equilibrium payoff allocation when Assumption 1 and Assumption 2 hold.

Assume we are in a subgame perfect equilibrium that satisfies Assumption 1 and Assumption 2. We proceed by a series of Claims:

Claim 4.1 *Assume $x_3 < p_3^{T,12}$. In the subgame that begins when j chooses $z \in S$, the final payoff allocation is $a\left(S,\left\langle q_i^{T-1,j3},d_3\right\rangle\right)$.*

Proof. Since $x_3 < p_3^{T,12}$, agent i can induce $q^{T-1,j3}$ by choosing x, knowing that agent 3 is bound to ask for a renegotiation and hence force $B^{T-1}[3ij]$.

Notice that, under the induction hypothesis, agent 3 would get $p_3^{T-1,\cdot 3} \overset{(1)}{=} p_3^{T,12}$ by asking for a renegotiation and x_3 by not doing so. In particular, agent i can assure herself $q_i^{T-1,j3}$.

On the other hand, agent 3 can assure herself a payoff of d_3 by vetoing any proposal (notice also that $d_3 \leq p_3^{T-1,\cdot 3}$).

Hence, agent j can assure herself $a_j \left(S, \left\langle q_i^{T-1,j3}, d_3 \right\rangle\right) - \epsilon_j$ for all $\epsilon_j > 0$ by proposing $z = a\left(S, \left\langle q_i^{T-1,j3}, d_3 \right\rangle\right) + (\epsilon_i, -\epsilon_j, \epsilon_3)$ for appropriate values of $\epsilon_i > 0$ and $\epsilon_3 > 0$. Thus, in subgame perfect equilibrium, agent j gets at least $a_j \left(S, \left\langle q_i^{T-1,j3}, d_3 \right\rangle\right)$. Since each agent can assure $a\left(S, \left\langle q_i^{T-1,j3}, d_3 \right\rangle\right)$, this is the only possible payoff allocation in subgame perfect equilibrium. ■

Claim 4.2 *Assume $x_3 \geq p_3^{T,12}$. In the subgame that begins when j chooses $z \in S$, the final payoff allocation is $a\left(S, \langle x_i, d_3 \rangle\right)$.*

Proof. Since $x_3 \geq p_3^{T,12}$, agent i can assure herself x_i by choosing x, knowing that agent 3 would not ask for a renegotiation (under Assumption 1 when $x_3 = p_3^{T,12}$). Notice that, under the induction hypothesis, agent 3 would get $p_3^{T-1,\cdot 3} \overset{(1)}{=} p_3^{T,12}$ by asking for a renegotiation and x_3 by not doing so. On the other hand, agent 3 can assure herself d_3 by vetoing any z. Hence, the subgame perfect equilibrium payoff for agent j is at most $a_j\left(S, \langle x_i, d_3 \rangle\right)$.

Hence, agent j can assure herself $a_j\left(S, \langle x_i, d_3 \rangle\right) - \epsilon_j$ for all $\epsilon_j > 0$ by proposing $z = a\left(S, \langle x_i, d_3 \rangle\right) + (\epsilon_i, -\epsilon_j, \epsilon_3)$ for appropriate values of $\epsilon_i > 0$ and $\epsilon_3 > 0$. Thus, in subgame perfect equilibrium, agent j gets at least $a_j\left(S, \langle x_i, d_3 \rangle\right)$. Since each agent can assure $a\left(S, \langle x_i, d_3 \rangle\right)$, this is the only possible payoff allocation in subgame perfect equilibrium. ■

Claim 4.3 *Assume $x_3 < p_3^{T,12}$. In the subgame that begins when agent 3 chooses $y \in S$, the final payoff allocation is $p^{T-1,j3}$ when $T > 1$, and $p^{0,12}$ when $T = 1$.*

Proof. Since $x_3 < p_3^{T,12}$, under Claim 4.1, agent i can induce $a\left(S, \left\langle q_i^{T-1,j3}, d_3 \right\rangle\right)$ by choosing x. If agent i chooses y, then, under the induction hypothesis, agent j can induce $q^{T-1,j3}$ by asking for a renegotiation.

Hence, agent 3 can assure herself $a_3\left(S,\left\langle q_i^{T-1,j3},p_j^{T-1,\cdot j}\right\rangle\right)-\epsilon_3$ for all $\epsilon_3>0$ by proposing $z=a\left(S,\left\langle q_i^{T-1,j3},p_j^{T-1,\cdot j}\right\rangle\right)+(\epsilon_i,\epsilon_j,-\epsilon_3)$ for appropriate values of $\epsilon_i>0$ and $\epsilon_j>0$. Thus, in subgame perfect equilibrium, agent 3 gets at least $a_3\left(S,\left\langle q_i^{T-1,j3},p_j^{T-1,\cdot j}\right\rangle\right)$.

When $T>1$,

$$a_3\left(S,\left\langle q_i^{T-1,j3},p_j^{T-1,\cdot j}\right\rangle\right)=a_3\left(S,\left\langle p_i^{T-1,j3},p_j^{T-1,\cdot j}\right\rangle\right)\overset{(5)}{=}p_3^{T-1,j3}$$

and , since each agent k can assure $p_k^{T-1,j3}$, this $p^{T-1,j3}$ is the only possible payoff allocation in subgame perfect equilibrium.

When $T=1$,

$$a_3\left(S,\left\langle q_i^{T-1,j3},p_j^{T-1,\cdot j}\right\rangle\right)=a_3\left(S,\langle d_i,d_j\rangle\right)=p_3^{0,ij}=p_3^{0,12}$$

and, since each agent k can assure $p_k^{0,12}$, this $p^{0,12}$ is the only possible payoff allocation in subgame perfect equilibrium. ■

Claim 4.4 *Assume $x_3\geq p_3^{T,12}$. In the subgame that begins when agent 3 chooses $i\in\arg\max_{k\in\{1,2\}}a_3\left(S,\left\langle x_k,p_{k^c}^{T-1,\cdot k^c}\right\rangle\right)$, and the final payoff allocation is $a\left(S,\left\langle x_i,p_j^{T-1,\cdot j}\right\rangle\right)$.*

Proof. Since $x_3\geq p_3^{T,12}$, under Claim 4.2, agent i can induce $a\left(S,\langle x_i,d_3\rangle\right)$ by choosing x. If agent i chooses y, then, under the induction hypothesis, agent j can induce $q^{T-1,j3}$ by asking for a renegotiation.

Hence, agent 3 can assure herself $a_3\left(S,\left\langle x_i,p_j^{T-1,\cdot j}\right\rangle\right)-\epsilon_3$ for all $\epsilon_3>0$ by proposing $z=a\left(S,\left\langle x_i,p_j^{T-1,\cdot j}\right\rangle\right)+(\epsilon_i,\epsilon_j,-\epsilon_3)$ for appropriate values of $\epsilon_i>0$ and $\epsilon_j>0$. Thus, in subgame perfect equilibrium, agent 3 gets at least $\max_{k\in\{1,2\}}a_3\left(S,\left\langle x_k,p_{k^c}^{T-1,\cdot k^c}\right\rangle\right)$. Under Lemma 4.1, this payoff is not lower than $p_3^{T,12}$. Hence it is neither lower than d_3, which would be the final payoff for agent 3 in case she induces agent i to choose x. Since each agent can assure $a\left(S,\left\langle x_i,p_j^{T-1,\cdot j}\right\rangle\right)$, this is the only possible payoff allocation in subgame perfect equilibrium. ■

Claim 4.5 *In the subgame that begins when agent 2 rejects x, the final payoff for agent 1 is strictly lower than $p_1^{T,\cdot 1}$.*

Proof. Under the induction hypothesis, the final payoff allocation is $q^{T-1,12}$ if agent 3 chooses $B^{T-1}[132]$, and $q^{T-1,23}$ if agent 3 chooses $B^{T-1}[213]$.

If $T = 1$, the final payoff for agent 1 would be $d_1 < p_1^{T,\cdot 1}$ irrespectively of agent 3's choice.

If T is odd and $T > 1$, then $p_3^{T-1,23} \overset{(2)}{<} p_3^{T-1,12}$ and hence agent 3 would strictly prefer to play $B^{T-1}[132]$. The final payoff for agent 1 will be $p_1^{T-1,12} \overset{(4)}{<} p_1^{T,\cdot 1}$.

If T is even, $p_3^{T-1,23} \overset{(2)}{>} p_3^{T-1,12}$ and hence agent 3 would strictly prefer to play $B^{T-1}[312]$. The final payoff for agent 1 will be $p_1^{T-1,23} \overset{(3)}{<} p_1^{T,\cdot 1}$. ■

Claim 4.6 *If agent 1 proposes some x with $x_3 < p_3^{T,12}$, her final payoff will be strictly lower than $p_1^{T,\cdot 1}$.*

Proof. If agent 2 rejects, under Claim 4.5 the final payoff for agent 1 is strictly lower than $p_1^{T,\cdot 1}$. If agent 2 accepts, under Claim 4.3, the final payoff will be $p^{T-1,j3}$ when $T > 1$ and $p^{0,12}$ when $T = 1$.

Assume first $T = 1$. Then, the final payoff for agent 1 is $d_1 < p_1^{T,\cdot 1}$.

Assume now $T > 1$. Under (1), agent 3 is indifferent on the choice of i and j. Under Assumption 2, agent 3 would choose i and j so that $p_1^{T-1,j3} < p_1^{T-1,i3}$. Under (2), this implies that the final payoff is $p^{T-1,13}$ if T is odd, and $p^{T-1,23}$ if T is even. If T is odd, $p_1^{T-1,13} \overset{(4)}{<} p_1^{T,\cdot 1}$. If T is even, $p_1^{T-1,23} \overset{(3)}{<} p_1^{T,\cdot 1}$. In either case, agent 1 will get less than $p_1^{T,\cdot 1}$. ■

Claim 4.7 *If agent 1 proposes some x with $x_3 \geq p_3^{T,12}$ and $a_3\left(S, \left\langle x_1, p_2^{T-1,\cdot 2}\right\rangle\right) < a_3\left(S, \left\langle x_2, p_1^{T-1,\cdot 1}\right\rangle\right)$, the final payoff for agent 1 will be strictly lower than $p_1^{T,\cdot 1}$.*

Proof. If agent 2 rejects, under Claim 4.5 the final payoff for agent 1 is strictly less than $p_1^{T,\cdot 1}$. Moreover, the induction hypothesis implies that agent

2 gets $p_2^{T-1,\cdot 2}$, irrespectively of agent 3's choice. Assume then agent 2 accepts, under Claim 4.4, the final payoff allocation is $a\left(S,\left\langle x_2,p_1^{T-1,\cdot 1}\right\rangle\right)$. Moreover, $x_2 \geq p_2^{T-1,\cdot 2}$ (otherwise, agent 2 would not accept).

If T is odd, $p_1^{T-1,\cdot 1} \overset{(4)}{<} p_1^{T,\cdot 1}$ and hence the result holds.

Assume T is even. We have $x_2 \geq p_2^{T-1,\cdot 2}$ and $x_3 \geq p^{T,12} \overset{(1)}{=} p_3^{T-1,\cdot 3}$, which implies $x_1 \leq p_1^{T-1,23}$ and hence

$$\begin{aligned} a_3\left(S,\left\langle x_1,p_2^{T-1,\cdot 2}\right\rangle\right) &\geq a_3\left(S,\left\langle p_1^{T-1,23},p_2^{T-1,\cdot 2}\right\rangle\right) \overset{(5)}{=} p_3^{T-1,23} \overset{(2)}{>} p_3^{T-1,12} \\ &\overset{(5)}{=} a_3\left(S,\left\langle p_2^{T-1,\cdot 2},p_1^{T-1,\cdot 1}\right\rangle\right) \\ &\overset{\left(x_2 \geq p_2^{T-1,\cdot 2}\right)}{\geq} a_3\left(S,x_2,p_1^{T-1,\cdot 1}\right). \end{aligned}$$

which is impossible. ■

Claim 4.8 *The final payoff allocation is* $p^{T,13}$.

Proof. It is enough to prove that each agent k can get at least $p_k^{T,13}$.

Agent 2: Under the induction hypothesis, agent 2 can assure herself $p_2^{T-1,\cdot 2} \overset{(1)}{=} p_2^{T,13}$ by rejecting any x.

Agent 1: Given any $\epsilon_1 > 0$, there exists some $\epsilon_2 > 0$ such that $x = p^{T,12} + (-\epsilon_1, \epsilon_2, 0) \in S$. Assume agent 1 proposes this x. Clearly, $x_3 \geq p_3^{T,12}$ and, moreover,

$$\begin{aligned} a_3\left(S,\left\langle x_1,p_2^{T-1,\cdot 2}\right\rangle\right) &= a_3\left(S,\left\langle p_1^{T,\cdot 1}-\epsilon_1,p_2^{T-1,\cdot 2}\right\rangle\right) > a_3\left(S,\left\langle p_1^{T,\cdot 1},p_2^{T-1,\cdot 2}\right\rangle\right) \\ &= a_3\left(S,\left\langle p_1^{T,\cdot 1},p_2^{T,13}\right\rangle\right) = p_3^{T,\cdot 3} = a_3\left(S,\left\langle p_1^{T,23},p_2^{T,\cdot 2}\right\rangle\right) \\ &> a_3\left(S,\left\langle p_1^{T,23},p_2^{T,\cdot 2}+\epsilon_2\right\rangle\right) = a_3\left(S,\left\langle p_1^{T,23},x_2\right\rangle\right). \end{aligned}$$

If agent 2 rejects x, under the induction hypothesis her final payoff will be $p_2^{T-1,\cdot 2}$. If agent 2 accepts x, under Claim 4.4 the final payoff allocation will be $a\left(S,\left\langle x_1,p_2^{T-1,\cdot 2}\right\rangle\right)$. Thus, agent 2 is indifferent between accepting and rejecting. Under Assumption 1, agent 2 accepts and the final payoff for agent 1 is x_1.

Hence, agent 1 can assure herself $p_1^{T,\cdot 1} - \epsilon_1$ for all $\epsilon_1 > 0$. Thus, in subgame perfect equilibrium, agent 1 gets at least $p_1^{T,\cdot 1}$.

Agent 3: Under Claim 4.6 and Claim 4.7, agent 1 will propose some x with $x_3 \geq p_3^{T,12}$ and $a_3\left(S, \left\langle x_1, p_2^{T-1,\cdot 2}\right\rangle\right) \geq a_3\left(S, \left\langle x_2, p_1^{T-1,\cdot 1}\right\rangle\right)$. Since $p_3^{T,12} \overset{(1)}{=} p_3^{T-1,\cdot 3}$, under Lemma 4.2, this implies $x_1 \leq p_1^{T,\cdot 1}$. Moreover, under Claim 4.5, agent 2 will accept it. Under Claim 4.4, the final payoff for agent 3 will be $a_3\left(S, \left\langle x_1, p_2^{T-1,\cdot 2}\right\rangle\right)$. Since $x_1 \leq p_1^{T,\cdot 1}$,

$$a_3\left(S, \left\langle x_1, p_2^{T-1,\cdot 2}\right\rangle\right) \geq a_3\left(S, \left\langle p_1^{T,\cdot 1}, p_2^{T-1,\cdot 2}\right\rangle\right) \overset{(1)}{=} a_3\left(S, \left\langle p_1^{T,\cdot 1}, p_2^{T,13}\right\rangle\right) \overset{(5)}{=} p_3^{T,13}.$$

Hence, agent 3 gets at least $p_3^{T,13}$. ■

References

[1] Bag P.K. and Winter E. (1999) *Simple subscription mechanisms for excludable public goods.* Journal of Economic Theory **87**, 72-94.

[2] Bennett E. (1997) *Multilateral bargaining problems.* Games and Economic Behavior **19**, 151-179.

[3] Calvo E. and Peters H. (2005) *Bargaining with ordinal and cardinal agents.* Games and Economic Behavior **52**(1), 20-33.

[4] Hart S. and Mas-Colell A. (1996) *Bargaining and value.* Econometrica **64**, 357-380.

[5] Kalai E. (1977) *Proportional solutions to bargaining situations: Interpersonal utility comparisons.* Econometrica **45**(7), 1623-1630.

[6] Kalai E. and Smorodinsky M. (1975) *Other solutions to Nash's bargaining problem.* Econometrica **43**, 513-518.

[7] Kıbrıs Ö. (2004a) *Ordinal invariance in multicoalitional bargaining.* Games and Economic Behavior **46**, 76-87.

[8] Kıbrıs Ö. (2004b) *Egalitarianism in ordinal bargaining: the Shapley-Shubik rule.* Games and Economic Behavior **49**, 157-170.

[9] Kıbrıs Ö. (2012) *Nash Bargaining in Ordinal Environments.* Review of Economic Design **16**(4), 269-282.

[10] Maniquet F. (2003) *Implementation under perfect information in economic environments.* Social Choice and Welfare **21**, 323-346.

[11] Moore J. and Repullo R. (1988) *Subgame perfect implementation.* Econometrica **46**, 1191-1220.

[12] Mutuswami S. and Winter E. (2002) *Subscription mechanisms for network formation.* Journal of Economic Theory **106**, 242-264.

[13] Myerson R. (1978) *Refinements of the Nash equilibrium concept.* International Journal of Game Theory **7**, 73-80.

[14] Nash J.F. (1950) *The bargaining problem.* Econometrica **18**, 155-162.

[15] Nash J.F. (1953) *Two-person cooperative games.* Econometrica **21**, 128-140.

[16] Pérez-Castrillo D. and Wettstein D. (2001) *Bidding for the surplus: A non-cooperative approach to the Shapley value.* Journal of Economic Theory **100**, 274-294.

[17] Pérez-Castrillo D. and Wettstein D. (2002) *Choosing Wisely: A Multi-bidding Approach.* American Economic Review **92**, 1577-1587.

[18] Pérez-Castrillo D. and Wettstein D. (2005) *Implementation of the Ordinal Shapley value for a three-agent economy.* Economics Bulletin **3**(48), 1-8.

[19] Pérez-Castrillo D. and Wettstein D. (2006) *An ordinal Shapley value for economic environments.* Journal of Economic Theory **127** (1), 296-308. DOI: 10.1016/j.jet.2004.11.007

[20] Roth A.E. (1979) *Axiomatic models of bargaining*. Springer-Verlag.

[21] Safra Z. and Samet D. (2004) *An ordinal solution to bargaining problems with many agents*. Games and Economic Behavior **46**(1),129-142.

[22] Samet D. and Safra Z. (2005) *A family of ordinal solutions to bargaining problems with many agents*. Games and Economic Behavior **50**(1), 89-106.

[23] Selten R. (1975) *Reexamination of the perfectness concept for equilibrium points in extensive games*. International Journal of Game Theory **4**, 25-55.

[24] Serrano R. (2005) *Fifty Years of the Nash Program, 1953-2003*. Investigaciones Economicas **29**, 219-258.

[25] Serrano R. (2008) *Nash Program*. The New Palgrave Dictionary of Economics (2nd edition). Ed. by S. Durlauf and L. Blume, McMillan, London.

[26] Shapley L.S. (1969) *Utility comparison and the theory of games*. In: La Décision, Agrégation et Dynamique des Ordres de Préférence, Editions du Centre de la Recherche Scientifique. Paris: 251-263.

[27] Shubik (1982) *Game Theory in the Social Sciences*. MIT Press. Cambridge, MA.