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17 May 2013

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*The first author acknowledges Financial support of the University of Cape Town UCR fund. The second author acknowledges Financial Support by the Centre interuniversitaire de recherche en économie quantitative (CIREQ), the Centre interuniversitaire de recherche en analyse des organisations (CIRANO), and Suffolk University Summer Research Grant. A special thanks to Rene Garcia and Nikolaos Artavanis for their suggestions. We also thank participants at the 2011 Econometric Society North America Summer Meeting, the 2011 Canadian Economic Association Meeting, the 2011 Financial Management Association Meeting, the Université Laval, the University of New Orleans, and the Federal Reserve Board for their valuable comments. We are responsible for any remaining errors.
The views expressed in this paper are authors' views and do not necessarily reflect those of Ministère des Finances du Québec.
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The Implications of VaR and Short-Selling Restrictions on the Portfolio Manager Performance

ABSTRACT

The ability of a portfolio manager to deliver higher returns with relatively low risk is a fundamental issue in finance. We analyze here the performance of a portfolio manager under two different types of constraints. For a manager with private information, we compare the effect of value at risk (VaR) and short-selling constraints on the relation between the expected portfolio return and the market return. We find that in more volatile market, the VaR restriction will have a stronger effect on the manager performance compared to the short-selling restriction effect. The VaR constraint also strongly affects a manager with good quality of information while the short-selling restriction moderately affects manager with any level of information quality. For the manager attitude toward the risk, a too aggressive manager will find his overall performance more affected by the VaR constraint. Therefore, financial institutions such as large investment banks and hedge-funds with a strong ability to obtain superior information could be more affected by a very strong VaR restriction than by a short-selling restriction.

JEL classification: G11, G28, G32

Keywords: Performance valuation, Asymmetric information, Financial regulation, VaR restriction, Short-Selling restriction.

1 Introduction

Many economists have argued that the absence of strong regulation of investments banks as well as hedge-funds in US is a key element behind the spread of the subprime crisis from the stock market to almost all financial institutions (see, e.g., Davies and Green, 2008). It is then likely that sooner or later almost all financial institutions will be subject to some type of regulations in order to reduce the systemic problem in the market. Among others, two type of regulation which are likely to be strengthened are the short-selling constraint and the Value at Risk constraint.

In one hand, the short-selling (SS) regulation has been adopted by US regulators during the subprime crisis and the International Organization of Securities Commissions (IOSCO) has adopted a similar measure in a final regulation paper (see IOSCO, 2009). In another hand, the value at risk (VaR) has been widely adopted as a regulation which can mitigate the effect of exposure to extreme risk by the financial community.[1] The Basel II Accord regulatory framework sets the amount of capital that banks should hold in function of the VaR that they have reported. In order to reduce the level of capital, central managements usually set a level of VaR to different portfolio managers. These two regulatory constraints may have different type of effects on portfolio manager in different market conditions.

Gendron and Genest (1990) analyze the effect of short-selling constraint on the relation between expected portfolio return and the market return. But while a fair amount of work has already been done on the link between the VaR regulation and financial sector stability see (e.g., Basak and Shapiro (2001), and Alexander and Batista (2005) which show that VaR is not effective at reducing risk and Yiu (2004), and Cuoco, He and Isaenko (2008) who used differents methodology and showed that VaR can limit risk taking behavior), the impact of the VaR regulation on the performance of portfolio manager remains a challenging question which should be addressed. This paper has then two main objectives. The first is to address this issue and the second is to make a comparative analysis of these two regulatory constraints in respect to the performance of a portfolio manager with private information.

In this paper we analyze the effect of the VaR constraint on the relation between expected portfolio excess-return and the market excess-return and we make a comparative analysis with the relation obtained with the SS constraint. Following Gendron and Genest (1990), we adopt the mean-variance setup enriched with the manager private information model. We then add a VaR constraint. In this model, the portfolio manager has the objective to maximize the mean-variance of the excess-return of his portfolio under the condition that he meets the regulation constraint fixed by the regulator or the central management of the corporation. We assume that the central management sets this regulatory constraint in the form of a maximum loss that the portfolio manager should not exceed with a certain confidence level. In this model the portfolio manager has access to a private signal which gives him an indication on the future excess-return of the market. The information of the manager is characterized by his prediction of market excess-return which is assumed to be normally distributed around the actual market excess-return. Therefore,

[1] For an historical background of the adoption of VaR as the widely used measure of risk in financial risk management see Jorion (2000), and for an overview of the concept and calculating models, see Duffie and Pan (1997).

the quality of this information is given by the level of the standard deviation of this distribution. Small standard deviation means high information quality.

Gendron and Genest (1990) show that in the context of informed portfolio manager, unlike the quadratic relation obtained when no constraint exists, the short sell investment constraint leads to a type of relation conjectured by Treynor and Mazuy (1966). This relation is smooth and flat for lower market return and becomes increasing with stronger curvature while the market return becomes positive and large. Their result suggests that the portfolio manager is strongly affected by the short-selling constraint during the market downturn and typically gets a negative expected return in this situation.[2]

We obtain with VaR constraint a less asymmetric curvature than the one obtained with the short-selling constraint. When the average excess-return is null, the conditional expected portfolio excess-return is a symmetric function of the market excess return. It follows that the conditional expected portfolio excess-return of a manager under even a strong VaR restriction can be symmetric as in the case of unrestricted manager. This contrast with the asymmetric curvature of the conditional expected excess-return in the short-selling constraint. Besides, unlike in the short-selling regulated market, a VaR restriction allows the manager to enjoy positive expected conditional excess-return even during downturn market conditions since it allows the manager to short sell the market portfolio, while the short selling restriction does not. Moreover, as expected, the quality of private information improves the manager performance in any setup. The overall performance valuation instead shows that a strong VaR regulation is more detrimental to manager with better quality of information than short-selling regulation. Furthermore we find that in more volatile market, the conditional expected excess-return of a VaR manager is lower in extreme negative market situation, than in a less volatile market. This result is confirmed when using the overall portfolio valuation which shows that market volatility hampers strongly the manager performance under VaR. This shows implicitly that VaR may be effective at reducing risk taking behavior, a result close to the one of Yiu (2004).

Finally, under VaR regulation, an aggressive manager performs better than risk-averse manager. But compared with a manager under short-selling restriction, a very aggressive manager will find a strong VaR regulation more harmful. It follows that using the manager performance as the main criterion, institutions such as large investment banks and hedge-funds which are viewed as highly informed and very aggressive are better under a short-selling regulation than under a VaR regulation. Actually, the VaR restriction disallows highly leveraged long positions in the market portfolio, but the short selling restriction allows such positions. Since in normal times, the average excess return on the market is positive, highly leveraged long positions are beneficial.

The next section presents the portfolio manager optimization problem, and provides the analytical optimal decision of the portfolio manager followed by the expressions of the portfolio expected excess-return and performance valuation. The section 3 uses calibration exercises to analyze the similitudes and differences in the implications of VaR and short-selling constraints, and then to investigate the effect of the market volatility, the information quality as well as the risk aversion

[2]The short-selling constraint is often used for regulation purpose specially against speculation, while the VaR constraint is more common in practice for risk management and capital requirement.

on expected excess-return and performance valuation with and without restrictions. The section 4 concludes.

2 The Portfolio Manager Problem

Before introducing the problem of the portfolio manager in aVaR restricted setup, we present here conceptual distinction between the VaR and the short-selling constraints.

2.1 The Nature of the VaR and the Short-selling Constraints

It is important to stress the fact that although the VaR and the short-selling constraints are both used by regulators to prevent the market from systemic problems, there is a clear difference between their initial purpose. The VaR restriction is a risk constraint specially designed to control the level of the overall risk regardless the type of position taken by the manager, while the short-selling constraint is a trading restriction used to prevent negative speculation against some stocks, specially when the market is in a down move.[3] Gendron and Genest (1990) solve the problem of the portfolio manager under the latter constraint. Below we introduce and solve the problem of a portfolio manager with private information and under a VaR restriction.

2.2 Portfolio Manager Problem under VaR Restriction

We consider a portfolio manager who accesses to a private information on the future market excess-return and who has a mean-variance objective function. Following Gendron and Genest (1990) we assume that the market-excess (from risk free rate) return R_m is normally distributed with mean μ and variance σ_u^2.

$$R_m \sim N\left(\mu, \sigma_u^2\right) \tag{1}$$

The portfolio manager possesses a private information characterized by a signal (S) that provides some level of information about the future market excess-return. Conditional to the realization of the market excess-return this signal follows a normal distribution around the actual value of market excess-return, r_m and with variance, σ_s^2.

$$S\,|_{R_m=r_m} \sim N\left(r_m, \sigma_s^2\right) \tag{2}$$

The variance inverse $\left(1/\sigma_s^2\right)$ indicates the quality in terms of accuracy of manager private information. Smaller variance means a more accurate information. As stated in Gendron and Genest (1990), by the Bayes' theorem, we can easily obtain

$$R_m\,|S = s \sim N\left(m_s, \sigma^2\right) \tag{3}$$

[3] In February 2010, the US Securities and Exchange Commission approved some short-selling restrictions during periods of stress and volatility.

where $m_s = \sigma^2 \left(\mu/\sigma_u^2 + s/\sigma_s^2\right)$ and $\sigma^2 = \left(1/\sigma_u^2 + 1/\sigma_s^2\right)^{-1}$. So, with his private information, the manager observes a distribution of the market excess-return with less volatility (more accuracy).[4] The portfolio excess-return is

$$R_p = \theta R_m \tag{4}$$

where θ is the share invested in the market portfolio.[5]

The portfolio manager problem's is to maximize his utility subject to the VaR constraint set by the top management of the corporation or by the regulator. The investor Mean-Variance problem is therefore[6],

$$\begin{aligned} &\underset{\theta}{Max} \left\{ E\left(R_p \left| S = s\right.\right) - \tfrac{\rho}{2} Var\left(R_p \left| S = s\right.\right) \right\} \\ &s.t. \;\; VaR\left(R_p\right) \leq \overline{V} \end{aligned} \tag{5}$$

where ρ is the manager risk aversion coefficient and $\overline{V}$ is the maximum value of the VaR set by the top management.[7] This problem can be easily expressed in the explicit form as follows

$$\begin{aligned} &\underset{\theta}{Max} \left\{ \theta m_s - \tfrac{\rho}{2} \theta^2 \sigma^2 \right\} \\ &s.t. \quad \theta\mu + \Phi^{-1}\left(\alpha\right) \left|\theta\right| \sigma_u \geq -\overline{V} \end{aligned} \tag{6}$$

where α is the level of the VaR (the probability to violate the VaR limit), and Φ the *cdf* of the standard normal distribution.

The main difference with the traditional mean-variance problem is the information asymmetry, and the difference with the short-selling restriction solved by Gendron and Genest (1990) is the non-linearity in the constraint. The VaR restriction leads to a more complex solution.[8]

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[4] For a better understanding of this signaling model, let us consider three simple cases: i) The case $\left(\sigma_s^2 \to \infty\right)$, which corresponds to a manager with no private information. Therefore, his distribution of the market return is as public information $(R_m \left| S = s \sim N\left(\mu, \sigma_u^2\right)\right.$, although this case is interesting we will focus in the paper on he cases with private information); ii) When both public and private "information" have the same distribution. Therefore, the distribution of informed manager is $R_m \left| S = s \sim N\left(\mu, \frac{\sigma_u^2}{2}\right)\right.$, what means that the variance of the return distribution observed by the manager is cut by half; iii) The case where both "information" have the same variance but different means. The return distribution of informed manager is $R_m \left| S = s \sim N\left(\frac{\mu+s}{2}, \frac{\sigma_u^2}{2}\right)\right.$. What means that the private signal will not only reduce the uncertainty of informed manager, but also change his expectation of future return compared to uninformed manager.

[5] This expression of the manager portfolio return is consistent with the theorem of two portfolios stating that in the equilibrium, each agent ends up with a portfolio including a risk-free asset and a fraction of the market portfolio.

[6] Beyond the fact that the mean-variance utility function is a commonly used model with basis finance intuition of trade-off between risk and return, the use of this function here is essential to derive an analytical solution for the problem and subsequent performance measures.

[7] It is important here to notice that the distribution used in the VaR is not conditional on the manager private information since the assessment of the respect for the VaR limit is done at a different level where the private information may not be available.

[8] Alexander and Baptista (2004) in a traditional mean-variance model analyze and compare the effect of VaR and CVaR constraints on the portfolio selection, while Alexander et al. (2007) examine a similar problem with discrete distributions. The mean difference with the present framework is the private information.

Proposition 1 *The solution to the optimization problem in (6) is given by*

$$\widehat{\theta} = \frac{m_s}{\rho\sigma^2}\left[1_{A_1} + 1_{A_2}\right] - \frac{\overline{V}}{\mu + \Phi^{-1}(\alpha)\,\sigma_u}1_{A_3} - \frac{\overline{V}}{\mu - \Phi^{-1}(\alpha)\,\sigma_u}1_{A_4} \tag{7}$$

where

$$\begin{aligned}
A_1 &= \left\{m_s\mu + \Phi^{-1}(\alpha)\,m_s\sigma_u + \rho\sigma^2\overline{V} \geq 0, m_s \geq 0\right\} \\
A_2 &= \left\{m_s\mu - \Phi^{-1}(\alpha)\,m_s\sigma_u + \rho\sigma^2\overline{V} \geq 0, m_s \leq 0\right\} \\
A_3 &= \left\{m_s\mu + \Phi^{-1}(\alpha)\,m_s\sigma_u + \rho\sigma^2\overline{V} \leq 0, m_s \geq 0\right\} \\
A_4 &= \left\{m_s\mu - \Phi^{-1}(\alpha)\,m_s\sigma_u + \rho\sigma^2\overline{V} \leq 0, m_s \leq 0\right\}
\end{aligned}$$

and 1_{A_i} is the indicator function taking the value 1 if condition A_i is true and 0 otherwise.

Proof: See Appendix

This solution includes three parts. The first component $(m_s / \rho\sigma^2)$ is the optimal portfolio when no constraint is imposed. The second and the third parts come from the VaR constraints. This solution will lead to the unrestricted case if we set the VaR level extremely large such that it has no effect. Or alternatively in practice, it may happen just if the market risk level is very low compared to the capital requirement level. Using the above solution, we can now derive the manager portfolio expected return and its overall valuation.

2.3 Expected Portfolio Excess-Return

To analyze the performance of the manager in different market conditions, we provide below the expected value of portfolio excess-return conditional to actual market excess-return.

Proposition 2 *Given the market excess-return ($R_m = r_m$), assuming that $\Phi^{-1}(\alpha) \leq 0$, and under the VaR constraint in problem (6), the manager portfolio expected excess-return is*

$$\begin{aligned}
E(R_p|_{R_m=r_m}) = \; & r_m \Bigg(\left[B_2\Phi(B_3) - \tfrac{B_1}{\rho\sigma_s}\left[1-\Phi(B_3)\right] + \tfrac{1}{\rho\sigma_s}\int_{B_3}^{\infty} z\phi(z)dz \right] 1_{\{\mu \geq -\Phi^{-1}(\alpha)\sigma_u\}} \\
& + \Big[B_2\Phi(B_3) + B_4\left(1-\Phi(B_5)\right) - \tfrac{B_1}{\rho\sigma_s}\left[\Phi(B_5)-\Phi(B_3)\right] \\
& \quad + \tfrac{1}{\rho\sigma_s}\int_{B_3}^{B_5} z\phi(z)dz \Big] 1_{\{\Phi^{-1}(\alpha)\sigma_u \leq \mu \leq -\Phi^{-1}(\alpha)\sigma_u\}} \\
& + \left[B_4\left(1-\Phi(B_5)\right) - \tfrac{B_1\Phi(B_5)}{\rho\sigma_s} + \tfrac{1}{\rho\sigma_s}\int_{-\infty}^{B_5} z\phi(z)dz \right] 1_{\{\mu \leq \Phi^{-1}(\alpha)\sigma_u\}} \Bigg)
\end{aligned}$$

$$\begin{aligned}
\textit{where, } B_1 &= -\sigma_s\left(\frac{\mu}{\sigma_u^2} + \frac{r_m}{\sigma_s^2}\right), \; B_2 = \frac{-\overline{V}}{(\mu - \Phi^{-1}(\alpha)\sigma_u)}, \\
B_3 &= B_1 + \rho\sigma_s B_2, \; B_4 = \frac{-\overline{V}}{(\mu + \Phi^{-1}(\alpha)\sigma_u)}, \\
B_5 &= B_1 + \rho\sigma_s B_4; \; \textit{and } \phi \textit{ is the pdf of the standard normal distribution.}
\end{aligned} \tag{8}$$

Proof: See Appendix

This result is more complex than the result in the case where the manager is not restricted in his investment. In this latter case, the relation between market excess-return and the expected excess-return for the manager portfolio is quadratic as follows

$$E(R_p|_{R_m=r_m}) = \left(\frac{\mu}{\rho\sigma_u^2}\right) r_m + \left(\frac{1}{\rho\sigma_s^2}\right) r_m^2 \tag{9}$$

In order to compare the result with the short-selling constraint, we recall below the result provided by Gendron and Genest (1990). They show that in the situation where the share θ invested in the market portfolio is between a lower bound k_1 and an upper bound k_2, the expected excess-return for the manager portfolio is given by[9]

$$E(R_p|_{R_m=r_m}) = r_m\left(1-\Phi\left(k_u\right)\right) + \left(ar_m + br_m^2\right)\int_{k_l}^{k_u}\phi\left(z\right)dz + b\sigma_s r_m\int_{k_l}^{k_u} z\phi\left(z\right)dz \tag{10}$$

where $k_l = \left(k_1 - a - br_m\right)/b\sigma_s$, $k_u = \left(k_2 - a - br_m\right)/b\sigma_s$, $a = \dfrac{\mu}{\rho\sigma_u^2}$ and $b = \dfrac{1}{\rho\sigma_s^2}$

We can notice from proposition 1, that when $\overline{V} \to +\infty$, the expected excess return of a manager optimizing his portfolio allocation under the VaR constraint, should converge to the solution of the unconstrained situation. Also when $k_l \to +\infty$ (i.e. $k_1 \to +\infty$) and $k_u \to -\infty$ (i.e. $k_2 \to -\infty$), the expected excess-return of a manager optimizing his portfolio allocation under the short-selling constraint should converge to the unconstrained solution.[10]

Proposition 3 *Under the VaR constraint, if the average market excess return is zero ($\mu = 0$), the relation between the optimal portfolio expected excess-return and the market excess-return is symmetric.*

Proof: See Appendix

We will see with numerical examples in the next section that if $\mu \neq 0$, this symmetric relationship is not necessary true. The importance of this result resides in the comparison of the relation in other setup. So when the average market excess return is zero the VaR constraint produce a similar shape as the one obtained in the constraint-free framework. The only effect of the VaR constraint is just to flatten the curve and reduce the portfolio excess return in both directions of the market. However, when the market excess return becomes positive there is an asymmetric relation between portfolio and market returns.

[9] Using k_1 and k_2 , provides a more general specification of the constraint. In our calibration exercise, we will focus on the short-selling restriction which corresponds to $k_1 = 0$. We can also notice that for a borrowing restriction we can set $k_2 = 1$. But instead we will use an arbitrary large value of k_2 (=100) as unlike Gendron and Genest (1990), we do not impose a borrowing constraint.

[10] Compared to Gendron and Genest (1990), it worth noticing that both optimization problems are different since Gendron and Genest (1990) put the restriction directly on the weight invested in the market, while here our restriction is put on a combination of first and second moments of portfolio excess-return. This explains why our solution is more complex and as we will see in the calibrations, both results are very different.

2.4 Performance Valuation

As stated by Brennan (1979), one convenient way of summarizing the curvature of a manager's payoff function is to regard the latter as a contingent claim, assume continuous trading, and compute its value using a risk-neutral valuation relationship. Gendron and Genest (1990) provide the below formulation for the performance valuation

$$VL = (1 + r_f)^{-1} \int_{-\infty}^{\infty} E(R_p | r_m) d\overline{F}(r_m), \qquad \overline{F}(r_m) \sim N(0, \sigma_u^2), \tag{11}$$

where r_f is the risk-free rate.

This measure of performance accounts for both return and risk and can be related to the Jensen alpha.

2.4.1 Jensen Alpha and Performance Valuation

As defined above the use of a risk neutral relationship implies that the performance valuation is a risk-adjusted measure of expected return. To get a better view of this intuition, let take the Jensen alpha, α defined in the CAPM by

$$R_p = \alpha + \beta R_m$$

Under the risk neutral distribution $\overline{F}$, $\int_{-\infty}^{\infty} E(R_m | r_m) d\overline{F}(r_m) = \int_{-\infty}^{\infty} r_m d\overline{F}(r_m) = 0$. Therefore $VL = (1 + r_f)^{-1}\alpha$. So the performance valuation is exactly the discounted value of Jensen alpha under the market model assumption.

2.4.2 Performance Valuation under different Setups

We provide below the analytical expression of the performance valuation under different frameworks.

Case of no constraint: In absence of any constraint, we can easily see as it is the case in Gendron and Genest (1990) that, the portfolio manager performance valuation is

$$VL = (1 + r_f)^{-1} \left[\sigma_u^2 / \rho \sigma_s^2 \right]. \tag{12}$$

Equation (12) shows that the risk aversion is negatively related to the valuation when there is no restriction. When the manager is too risk averse he reduces its portfolio valuation. Besides as expected, the quality of manager information improves its valuation. In fact the quality of information is extremely good when $\sigma_s^2 \to 0$, and since VL is a decreasing function of σ_s^2, we have this expected result.

Under VaR constraint: Let us recall that it follows from the above proposition 2 that $E(R_p|_{R_m = r_m})$ has different expressions depending on the interval where μ belongs to. The most realistic range for μ is $\left[\Phi^{-1}(\alpha)\sigma_u, -\Phi^{-1}(\alpha)\sigma_u\right]$.[11] We therefore focus on this case.

[11] As the expected return can be positive or negative, the only range that can satisfy with this requirement is the one we use here for performance valuation

Proposition 4 *The performance valuation under the VaR constraint and when*

$$\Phi^{-1}(\alpha)\sigma_u \leq \mu \leq -\Phi^{-1}(\alpha)\sigma_u$$

is given by

$$VL_{VaR} = \tfrac{(1-C)}{(1+r_f)} \left[\sigma_s \left(B_2 \int_{-\infty}^{b_2} y\phi_{A^2}(y)dy + B_4 \int_{b_4}^{\infty} y\phi_{A^2}(y)dy + \tfrac{\mu}{\rho\sigma_u^2} \int_{b_2}^{b_4} y\phi_{A^2}(y)dy \right) + \tfrac{1}{\rho} \int_{b_2}^{b_4} y^2\phi_{A^2}(y)dy \right].$$

where, $b_i = -\sigma_s \left(\frac{\mu}{\sigma_u^2}\right) + \rho\sigma_s B_i$; $C = \dfrac{\sigma_s^2}{\sigma_u^2 + \sigma_s^2}$; $A = \sqrt{1/C}$, B_2 *and* B_4 *defined as in (8)*

Proof: See Appendix

It is straightforward that when $\overline{V} \to +\infty, VL_{VaR} \to VL$, the valuation in the unrestricted framework

Under short-selling constraint: As derived by Gendron and Genest (1990), the performance valuation of the portfolio manager when facing this trading restriction is given by,

$$VL_{SS} = (1 + r_f)^{-1} [V_1 + V_2 + V_3] \tag{13}$$

where

$$\begin{aligned} V_1 &= k_l \frac{\sigma_u^2}{\sigma_u^2 + \sigma_s^2} \int_{-\infty}^{c} ydF(y); V_2 = k_u \frac{\sigma_u^2}{\sigma_u^2 + \sigma_s^2} \int_{d}^{\infty} ydF(y); \\ V_3 &= \frac{a\sigma_u^2}{\sigma_u^2 + \sigma_s^2} \int_{c}^{d} ydF(y) + \frac{b\sigma_u^2}{\sigma_u^2 + \sigma_s^2} \int_{c}^{d} y^2 dF(y); \\ c &= \frac{k_l - a}{b}, d = \frac{k_u - a}{b}, \text{ and } F(y) \sim N(0, \sigma_u^2 + \sigma_s^2). \end{aligned}$$

Then it follows that when $k_l \to -\infty$; and $k_u \to +\infty$; VL_{SS} has almost the same value as VL. They argue that for well informed manager the short-selling constraint reduces considerably its valuation.

3 Analysis of the Model Implications

We use calibration method to analyze the results.[12] In order to do that we use realistic values of the parameters derived from actual data or commonly used in the literature. Although implied results depends of parameter values, this exercise provides intuition for a more realistic framework.

[12] The complexity of the above formula does not allow for a complete analytical analysis. And furthermore, such complete analysis including unrealistic parameters' values may not provide consistent conclusions.

3.1 Calibration

We carry our calibration with the objective of matching the US stock market dynamics over the recent decades. To compute the average excess-return we subtract the nominal long-term annual return of the US long-term treasury bond which is 7.5% over the period 1965-2005 to the nominal long-term market annual return of the S&P500 which is estimated at about 11.5% over the same period.[13] Therefore the average excess-return used for calibration is $\mu = 0.04$. We use the real long-term annual rate on US bond as the proxy of the risk-free return r_f and we obtain approximately 3%.

We use as benchmark for the risk aversion coefficient $\rho = 5$.[14] The confidence level of the market VaR that we use is 99% which is the level used by many major banks in US and also imposed by the 1996 Basel Accord. It follows that the benchmark α is 1%. We use the S&P500 long-term annual standard deviation (15%) as the proxy of market volatility σ_u. For the parameter σ_s characterizing the quality of the information the benchmark is taken as 10% which is between the value used by Gendron and Genest (1990) and the market volatility. The value of the VaR is also an important element in our calibration. $\overline{V}$ is the maximum percentage loss of the portfolio that should not be exceeded with probability $1 - \alpha$, in other words, for a $\overline{V} = 0.1$, the manager is certain with a probability $1 - \alpha$ to lose less that 10% of the value of his portfolio. To calibrate $\overline{V}$, we use the estimate of the VaR reported for six big US banks in Berkowitz and O'Brien (2003). This leads to an approximated $\overline{V} = 0.518$.[15]

3.2 Analyzing Manager Portfolio Performances under Different Restrictions

We perform the analysis on two different dimensions: (i) the relation between the manager portfolio expected return and the market return, and (ii) the overall valuation which integrates over all possible values of the market return to provide a single value of risk-adjusted performance. Using above values of parameters, we derive numerical value of expected portfolio return and overall valuation. Therefore we graph corresponding relation to have a better understanding of constraints' implications.

3.2.1 Characterization of the Performance of a VaR Restricted Manager: The Form of the Curvature

The form of the relation between the portfolio return and the market return is central here to analyze the manager performance. We start by comparing the general curvature form of this relation in the context of VaR constraint with the form of curvature obtained by Gendron and

[13] We also tried different time periods and find results very stable. The final choice on the 1965-2005 time period was guided by the availability of all these time-series at the time we performed the calibration.

[14] This is a reasonable value considering the range of values usually used in the financial literature.

[15] Actually, the average of the VaRs Berkowitz and O'Brien (2003) report is 3.43. As our estimated volatility is 0.15, and since their VaR is standardized, to get our VaR limit value, we multiply both numbers. We can notice that although the report daily standardized VaR, by using annual volatility, we obtain an annualized VaR.

Genest (1990) in the case of short-selling constraint.[16]

[INSERT Figure 1 HERE]

Two main observations can be derived from Figure 1. First, unlike the short-selling constraint, the VaR constraint expected portfolio excess-return has a curvature close to a quadratic form. Furthermore, with VaR constraint the manager can get positive portfolio excess-return when market excess-return is negative, what is not possible under the short-selling constraint. Actually, due to the fact that with the short-selling restriction, the portfolio manager will always hold a non-negative share of market portfolio, he is constrained to obtain a negative excess-return when market return is negative. This result in the VaR restricted framework is more in line with Pfleiderer and Bhattacharya (1983) or Admati and Ross (1985) information models rather than Treynor and Mazuy (1966) conjectured curvature reproduced by Gendron and Genest (1990) in the short-selling restricted framework. Second, in an upturn market situation specially with high returns, under the short-selling constraint the manager performance is comparable to a situation without any constraint, while the VaR constrained manager is significantly outperformed by the unconstrained manager.

3.2.2 The Effect of VaR Restriction on Performance

Under a VaR restriction, there is obviously a positive relation between $\overline{V}$ and the conditional portfolio excess-return (see Figure 2) as well as the overall portfolio valuation. The overall performance valuation of a manager under VaR restriction is lower than the one of a manager under the short-selling restriction, for a strong restriction ($\overline{V}$ lower than 0.75), and larger when $\overline{V}$ is above.

[INSERT Figure 2 HERE]

We can notice from Figure 2 that, for our benchmark value of the VaR threshold (0.518), the valuation of the overall performance will be more affected under VaR restriction than under short-selling restriction.

3.2.3 Market Volatility and Performance

When the market is less volatile, i.e. when σ_u is smaller, the conditional expected portfolio return of the manager under VaR restriction is higher for positive market excess-returns as well as for extreme negative excess-returns (see Figure 3). This results on extreme negative returns is due to the asymmetry induced by the fact that the average market excess-return is positive ($\mu = 4\%$). This result rests on the following rationale. The average excess-return is slightly positive, therefore when the market is in positive range, lower volatility is better since in this case the probability of

[16] In a special case, Gendron and Genest (1990) find that the short-selling constraint produces a flat and negative portfolio excess-return when the market is largely negative, and a positive and strongly increasing portfolio excess-return for largely positive market excess-return.

obtaining a negative return is reduced. But, if the market is too volatile the possibility of obtaining a lower return is important. When the market is in slightly negative range, a higher volatility is more likely to bring it into positive range. This is the main explanation why in this market situation the conditional expected excess-return is higher in more volatile market. Finally, when the market is in extremely negative range a higher volatility can lead to the violation of the VaR constraint, that is why the conditional expected excess-return in this case is higher for less volatile market.

If we assume that $\mu = 0$, the conditional expected portfolio return is symmetric as we have proven in proposition 3. In this case, unlike the unrestricted framework where there is a shift of the curve to the right when the market volatility increases and surpasses the volatility of the signal, we will get that, a VaR restricted portfolio manager gets a lower conditional expected return in more volatile market for almost all market excess-return value.

[INSERT Figure 3 HERE]

From the first chart of Figure 4, we can notice that the portfolio valuation increases with market volatility. To put this result in line with the conditional expected excess-return it is worth noticing that the "weight" used for overall valuation is larger around the mean where the curve with large volatility is above the one with smaller volatility. For the portfolio valuation to have almost a flat relationship with market volatility under the VaR regulation, the volatility should be much larger for instance when $\sigma_u > 0.3$.

[INSERT Figure 4 HERE]

The comparison with the short-selling regulation stands like this. With conditional expected excess-return as performance measure, market volatility reduces the manager performance in positive range but increases its performance in negative range, even though the conditional expected excess-return will never be positive. Therefore in term of these measure VaR and short-selling restriction have different implications.

With performance valuation, one can see from the second chart of Figure 4 that manager under short-selling restriction will outperform the one under VaR when the market is too volatile, for instance, when $\sigma_u > 0.12$. In the third chart of figure 4, we can see that when $\sigma_u > 0.2$, the change in the market volatility has no additional effect on the relative value of the portfolio valuation of a short-selling restricted manager compared to the unrestricted manager. It stands at its minimum value around 53%. While this is not true for the relative portfolio valuation of a VaR regulated manager. Its relative portfolio will decrease to almost 0% of the unrestricted manager portfolio valuation.

As a by-product of our study, we can interpret this as an indication that VaR can actually work at reducing risk in the manager portfolio in the sense that, caring about its performance, a VaR restricted portfolio manager should avoid market with high volatility and this may have as a result of reducing investment in high volatile project. In this sense we are more in line with Yiu (2004) and Cuoco, He and Isaenko (2008) than with Basak and Shapiro (2001).

3.2.4 Quality of Information on Performance

For a coherent information model the information quality should improve the performance of the manager who possesses it. Here we check the effect of the quality of private information on the relation between portfolio expected excess-return and the market excess-return, and then we analyze this effect on the overall performance valuation.

Expected Excess-Return and Quality of Private Information. In any setup, the curvature of the relation between expected portfolio excess-return and the market excess-return is robust to the information quality. However, this effect is different depending on whether the manager is restricted or not.

In order to analyze the effect of the quality of private information on the performance of an unconstrained portfolio manager, let us consider the expression of the portfolio expected excess-return in this case,

$$E\left(R_p|_{R_m=r_m}\right) = \frac{1}{\rho}\left[\left(\frac{\mu}{\sigma_u^2}\right) r_m + \left(\frac{1}{\sigma_s^2}\right) r_m^2\right]. \tag{14}$$

The information quality affects only the second term which defines the concavity of the curve. There is a positive relation between the accuracy of the information $\left(\frac{1}{\sigma_s^2}\right)$ and the expected excess-return of the portfolio. Moreover, a well informed portfolio manager performs better and symmetrically around $\left[-\mu\frac{\sigma_s^2}{2\sigma_u^2}\right]$ in good and bad market condition.

[INSERT Figure 5 HERE]

When there is no constraint, the portfolio manager takes full advantage of his private information especially in extreme market conditions. The information "premium" $\frac{r_m^2}{\sigma_s^2\rho}$ comes from the ability to use private information to take a long position in market upturn and a short position in market downturn. As we can see in Figure 5, a VaR regulated manager can still take advantage of a higher quality of information and ends up with a better conditional expected portfolio return. A short-selling constrained manager however, can not take a full advantage of the quality of his information when the market is in bad condition.

Valuation and Quality of Private Information. The negative relation between valuation and the standard deviation σ_s characterizes a positive relationship between the quality of information and the portfolio performance. In fact, the first chart of Figure 6 can equivalently be interpreted as a positive relationship between the quality of information $\left(1/\sigma_s^2\right)$ and valuation.

[INSERT Figure 6 HERE]

Also, the valuation loss due to the VaR regulation is larger than the loss due to the short-selling regulation when the quality of information is very high (small σ_s). But for lower and intermediate quality of information (large σ_s), the valuation loss due to the VaR constraint is lower. It follows that manager with a great capacity of obtaining a higher quality of information such as sophisticated investors (big banks and financial institutions with a good forecasting department as well as risk management department, hedge funds) will prefer the short-selling regulation to a VaR restriction.

More specifically, for a manager with a better quality of information ($\sigma_s < 0.12$), short-selling regulation is better than the VaR regulation (see, chart 2 and chart 3 of Figures 6). This preference changes when the quality of information is bad, i.e., $\sigma_s > 0.12$. Another important result is that when the quality of information is too bad for instance for $\sigma_s > 0.35$, a VaR regulated manager has almost the same portfolio valuation than an unregulated manager. In other words, if the market is efficient, the VaR restriction will have very little effect on the portfolio manager performance. Again, a manager with very accurate information finds the VaR regulation too harmful. Their valuation relative to the unrestricted market is hampered heavily. For instance, for $\sigma_s = 0.10$, the ratio of the VaR restricted manager portfolio valuation over an unrestricted manager portfolio valuation is almost 40%, meaning that the VaR restriction reduces the portfolio manager performance by more than half (60%)"

3.2.5 Risk-Aversion and Manager Performance

We analyze below the importance of the risk aversion effect on conditional expected excess-return in different market conditions and we then use the valuation to assess the overall effect of the attitude toward risk of the portfolio manager on his performance.

Expected Excess-Return and Risk Aversion. Without any restriction, when a manager possesses a private information, more he is risk averse, less he takes advantage of this information. The same implication is valid under the VaR restriction. In fact, the more aggressive manager is able to out-perform the less aggressive one in any market conditions; (see, Figure 7). Basak and Shapiro (2001) argue that, as an agent becomes less risk averse, he responds more aggressively to change in the state variable, this affects his likelihood to end up violating the constraint. However, with an accurate private information this apparent large level of risk taken is actually a wise and aggressive way to take advantage of his information. When the manager faces a short-selling restriction his aggressiveness is punished when the market return turns to be negative.

[INSERT Figure 7 HERE]

Although these results are not obvious from expression of the conditional expected portfolio excess-return in the case of restricted manager, we can easily notice from equation (14) that in the unrestricted framework, there is a negative relationship between the risk aversion coefficient ρ and the expected excess-return of the portfolio.

Gendron and Genest (1990) compute the expected excess-return of the portfolio manager under the short-selling constraint. But they do not study its relation with risk aversion. The Figure 7 provides the typical relationship between risk aversion and expected excess-return.

Valuation and Risk Aversion. Following the results from the conditional expected portfolio excess-return in the cases of unrestricted and VaR restricted manager, it is consistent to have an overall performance valuation which is a decreasing function of risk aversion. The short-selling restriction which presents different results depending on the market situation, finally follows overall the same pattern as in the other framework, where the valuation decreases with the risk aversion coefficient.

[INSERT Figure 8 HERE]

When manager are too risk averse, the valuation loss due to the VaR regulation is lower than the one due to the short-selling regulation. Also with very risk-averse manager, VaR regulation ultimately has no incident on his valuation.

A less risk-averse manager enjoys a higher portfolio valuation (see Figure 8). For a less risk averse manager (i.e. lower ρ), the short-selling regulation is better than the VaR. In fact, from Figure 8, we can notice that when ρ is small (less than 7.5), the ratio of the restricted portfolio valuation over the unrestricted portfolio valuation is higher in case of short-selling regulation than in the case of the VaR regulation. The result is different when ρ is higher.

Also, while for a very large ρ, the portfolio valuation of a VaR constraint manager is almost the same as the one of an unregulated manager, a short-selling regulated manager will never achieve the value of the overall portfolio valuation of an unrestricted manager whatever is its attitude toward risk. From the third charts of Figure 8 we see that the maximum that a short-selling regulated manager can achieve is about 53.5%, of the unrestricted manager portfolio. The VaR restriction seems to strongly penalize an aggressive manager while a short-selling restriction moderately affects manager regardless his risk aversion level.

3.3 Robustness Check

The calibration exercise is carried out using plausible values of parameters. Here we check if the variation of parameters will affect the overall pattern of the curves. We specially analyze under VaR restriction how the curve representing the relationship between portfolio return and market return evolves inside a range of values for the VaR limit, the information quality, and the risk version coefficient. As it is shown in Figure 9, we can see that the U-shape of the relation is quite stable when these parameters change.

[INSERT Figure 9 HERE]

This U-shape tends to vanish with stronger restricted VaR limit (upper graph in Figure 9), poorer information quality (left lower graph in Figure 9), and more risk averse portfolio manager (right lower graph in Figure 9).

We also analyze the robustness of the overall valuation ratio of VaR restricted manager over the unrestricted manager (see Figure 10), by changing two parameters at the same time.

[INSERT Figure 10 HERE]

Analyzing how the VaR restriction affects the overall performance in relation with the information level, we vary successively parameters representing the risk aversion, the limit of VaR, and the market excess return. The conclusion that good quality information (lower σ_s) is more affected by VaR constraint is quite consistent regardless the risk aversion (upper graph in Figure 10), the limit of VaR (left lower graph in Figure 10), and the market excess return (right lower graph in Figure 10). When taking a cross-sectional look of these three dimension graphs, we see that all variations are consistent with our two-dimension findings.

4 Conclusion

In this paper we use a framework of a portfolio manager with private information to characterize and analyze the portfolio manager performances with the conditional expected portfolio excess-return and portfolio valuation in the case of a VaR restricted manager. Furthermore, we compare manager performances under VaR constraint with those under unrestricted market as well as under short-selling regulated market.

We obtain with VaR constraint a less asymmetric curvature than the one obtained with the short-selling constraint. We show that under VaR restricion, if the average excess-return is null, the conditional expected portfolio excess-return is a symmetric function of the market excess return as in the case of unrestricted market environment. This contrasts with the asymmetry curvature of the conditional expected excess-return in the short-selling constraint case. However, in a more realistic case where the average excess-return is positive, the curvature is asymmetry in a VaR regulated market, although, unlike in the short-selling regulated market, it allows the manager to enjoy positive expected conditional excess-return even during downturn market conditions since the VaR restriction allows the manager to short sell the market portfolio, while the short selling restriction does not.

Moreover, we found some interesting results about the importance of the market volatility, the quality of information, and the attitude toward risk on the manager performances. Using the conditional expected excess-return to analyze the manager performance, the effect of market volatility, information quality and risk aversion depends on the market conditions. However, the overall portfolio valuation provides a way to assess the implications of these restrictions. When the overall portfolio valuation is considered to assess the manager performance, we found that: i) in more volatile market, compared to the short-selling regulation, the VaR regulation will have a strong negative effect on the manager performance; ii) the VaR constraint overall strongly affects a manager with good quality of information, while the short-selling restriction moderately affects manager performance regardless the quality of their information; iii) also a too aggressive manager finds the VaR regulation more harmful. Their valuation relative to the unrestricted market is strongly affected. Therefore the VaR restriction may not be the more preferable type of regulation for informed and aggressive institutions such as hedge-funds and investment banks. Analyzing and comparing the welfare effects of VaR and short-selling regulation is another challenging issue which should be addressed in a future research.

Appendix: Proofs

Proof. of Proposition 1:

The objective function is concave, so the global optimum (without constraint effect) is given by

$$\widehat{\widehat{\theta}}=\frac{m_s}{\rho\sigma^2}$$

If the above value is inside the constrained set, then it is the solution, if not the solution is on the boundary (corner solution).

i) For $\left\{\begin{array}{l} m_s \geq 0 \\ m_s\mu + \Phi^{-1}(\alpha)\, m_s\sigma_u + \rho\sigma^2\overline{V} \geq 0 \end{array}\right.$, or $\left\{\begin{array}{l} m_s \leq 0 \\ m_s\mu - \Phi^{-1}(\alpha)\, m_s\sigma_u + \rho\sigma^2\overline{V} \geq 0 \end{array}\right.$, one can observe that $\dfrac{m_s}{\rho\sigma^2}$ belongs to the interior of the constrained set therefore

$$\widehat{\theta} = \widehat{\widehat{\theta}} = \frac{m_s}{\rho\sigma^2}$$

ii) For $\left\{\begin{array}{l} m_s \geq 0 \\ m_s\mu + \Phi^{-1}(\alpha)\, m_s\sigma_u + \rho\sigma^2\overline{V} \leq 0 \end{array}\right.$, we have the corner solution

$$\widehat{\theta}= -\frac{\overline{V}}{\mu + \Phi^{-1}(\alpha)\,\sigma_u}$$

iii) For $\left\{\begin{array}{l} m_s \leq 0 \\ m_s\mu - \Phi^{-1}(\alpha)\, m_s\sigma_u + \rho\sigma^2\overline{V} \leq 0 \end{array}\right.$, we have the corner solution

$$\widehat{\theta}= -\frac{\overline{V}}{\mu - \Phi^{-1}(\alpha)\,\sigma_u}.$$

We can then summarize the general solution as follows

$$\widehat{\theta} = \frac{m_s}{\rho\sigma^2}\left[1_{A_1} + 1_{A_2}\right] - \frac{\overline{V}}{\mu + \Phi^{-1}(\alpha)\,\sigma_u}1_{A_3} - \frac{\overline{V}}{\mu - \Phi^{-1}(\alpha)\,\sigma_u}1_{A_4}$$

where

$$\begin{array}{rcl} A_1 & = & \{m_s\mu + \Phi^{-1}(\alpha)m_s\sigma_u + \sigma^2\rho\overline{V} \geq 0, m_s \geq 0\}, \\ A_2 & = & \{m_s\mu - \Phi^{-1}(\alpha)m_s\sigma_u + \sigma^2\rho\overline{V} \geq 0, m_s \leq 0\}, \\ A_3 & = & \{m_s\mu + \Phi^{-1}(\alpha)m_s\sigma_u + \sigma^2\rho\overline{V} \leq 0, m_s \geq 0\}, \\ A_4 & = & \{m_s\mu - \Phi^{-1}(\alpha)m_s\sigma_u + \sigma^2\rho\overline{V} \leq 0, m_s \leq 0\}. \end{array}$$

Q.E.D. ■

Proof. of Proposition 2:

We know that $E(R_p|_{R_m=r_m}) = E(\widehat{\theta}R_m|_{R_m=r_m})$, since $R_p = \widehat{\theta}R_m$.

It follows that

$$E(R_p|_{R_m=r_m}) = r_m E(\widehat{\theta}|_{R_m=r_m}). \tag{15}$$

In order to compute $E(\widehat{\theta}|_{R_m=r_m})$ one should express $\widehat{\theta}$ as function of r_m. Therefore, the first step of this proof is to rewrite $\widehat{\theta}$

Let us depart from the solution of proposition 1.

$$\widehat{\theta} = \frac{m_s}{\rho\sigma^2}[1_{A_1} + 1_{A_2}] - \frac{\overline{V}}{\mu + \Phi^{-1}(\alpha)\sigma_u}1_{A_3} - \frac{\overline{V}}{\mu - \Phi^{-1}(\alpha)\sigma_u}1_{A_4} \tag{16}$$

Given that $\overline{V} \geq 0$ and $\Phi^{-1}(\alpha) \leq 0$, it is straightforward that we have thee different expressions of $\widehat{\theta}$ with respect of the relevant interval:
(i) If $\mu \geq -\Phi^{-1}(\alpha)\sigma_u$,

$$\widehat{\theta} = \begin{cases} \frac{-\overline{V}}{\mu - \Phi^{-1}(\alpha)\sigma_u} & if \quad m_s \leq \frac{-\sigma^2\rho\overline{V}}{\mu - \Phi^{-1}(\alpha)\sigma_u} \\ \frac{m_s}{\rho\sigma^2} & if \quad m_s \geq \frac{-\sigma^2\rho\overline{V}}{\mu - \Phi^{-1}(\alpha)\sigma_u} \end{cases} \tag{17}$$

(ii) If $\Phi^{-1}(\alpha)\sigma_u \leq \mu \leq -\Phi^{-1}(\alpha)\sigma_u$;

One can easily shows that (16) is equivalent to:

$$\widehat{\theta} = \begin{cases} \frac{-\overline{V}}{\mu - \Phi^{-1}(\alpha)\sigma_u} & if \quad m_s \leq \frac{-\overline{V}\rho\sigma^2}{\mu - \Phi^{-1}(\alpha)\sigma_u} \\ \frac{m_s}{\rho\sigma^2} & if \quad \frac{-\overline{V}\rho\sigma^2}{\mu - \Phi^{-1}(\alpha)\sigma_u} \leq m_s \leq \frac{-\overline{V}\rho\sigma^2}{\mu + \Phi^{-1}(\alpha)\sigma_u} \\ \frac{-\overline{V}}{\mu + \Phi^{-1}(\alpha)\sigma_u} & if \quad m_s \geq \frac{-\overline{V}\rho\sigma^2}{\mu + \Phi^{-1}(\alpha)\sigma_u} \end{cases} \tag{18}$$

(iii) When $\mu \leq \Phi^{-1}(\alpha)\sigma_u$, one can easily shows that (16) is equivalent to:

$$\widehat{\theta} = \begin{cases} \frac{m_s}{\rho\sigma^2} & if \quad m_s \leq \frac{-\overline{V}\rho\sigma^2}{\mu + \Phi^{-1}(\alpha)\sigma_u} \\ \frac{-\overline{V}}{\mu + \Phi^{-1}(\alpha)\sigma_u} & if \quad m_s \geq \frac{-\overline{V}\rho\sigma^2}{\mu + \Phi^{-1}(\alpha)\sigma_u} \end{cases} \tag{19}$$

Since $m_s = \frac{1}{\sigma}\left(\frac{\mu}{\sigma_u^2} + \frac{s}{\sigma_s^2}\right)$, and $s \sim N(r_m, \sigma_s^2)$ we can consider that

$$m_s = \frac{1}{\sigma}\left(\frac{\mu}{\sigma_u^2} + \frac{r_m}{\sigma_s^2} + \frac{z}{\sigma_s}\right),$$

with $z \sim N(0,1)$. This last expression will be used in the rest of this proof

The second step that we now carry is the actual computation of $E(\widehat{\theta}|_{R_m=r_m})$. We will do this in each interval.

Let us start with the case when $\Phi^{-1}(\alpha)\sigma_u \leq \mu \leq -\Phi^{-1}(\alpha)\sigma$. It follows that

$$\widehat{\theta} = \begin{cases} \frac{-\overline{V}}{\mu - \Phi^{-1}(\alpha)\sigma_u} & if \quad z \leq -\frac{\overline{V}\rho\sigma_s}{\mu - \Phi^{-1}(\alpha)\sigma_u} - \sigma_s\left[\frac{\mu}{\sigma_u^2} + \frac{r_m}{\sigma_s^2}\right] \\ \frac{1}{\rho}\left[\frac{\mu}{\sigma_u^2} + \frac{r_m}{\sigma_s^2}\right] + \frac{z}{\rho\sigma_s} & if \quad -\frac{\overline{V}\rho\sigma_s}{\mu - \Phi^{-1}(\alpha)\sigma_u} - \sigma_s\left[\frac{\mu}{\sigma_u^2} + \frac{r_m}{\sigma_s^2}\right] \leq z \leq -\frac{\overline{V}\rho\sigma_s}{\mu + \Phi^{-1}(\alpha)\sigma_u} - \sigma_s\left[\frac{\mu}{\sigma_u^2} + \frac{r_m}{\sigma_s^2}\right] \\ \frac{-\overline{V}}{\mu + \Phi^{-1}(\alpha)\sigma_u} & if \quad z \geq -\frac{\overline{V}\rho\sigma_s}{\mu + \Phi^{-1}(\alpha)\sigma_u} - \sigma_s\left[\frac{\mu}{\sigma_u^2} + \frac{r_m}{\sigma_s^2}\right] \end{cases} \tag{20}$$

one can realize that $\widehat{\theta} = g(z)$, therefore $E(\widehat{\theta}|_{R_m=r_m}) = \int\limits_{-\infty}^{\infty} g(z)\phi(z)dz$, where $\phi(z)$ is the *pdf* of the normal distribution $N(0,1)$

$$
\begin{aligned}
E(\widehat{\theta}) &= \int_{-\infty}^{-\frac{\overline{V}\rho\sigma_s}{\mu-\Phi^{-1}(\alpha)\sigma_u}-\sigma_s\left[\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right]} \frac{-\overline{V}}{\mu-\Phi^{-1}(\alpha)\sigma_u}\phi(z)dz \\
&+ \int_{-\frac{\overline{V}\rho\sigma_s}{\mu-\Phi^{-1}(\alpha)\sigma_u}-\sigma_s\left[\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right]}^{-\frac{\overline{V}\rho\sigma_s}{\mu+\Phi^{-1}(\alpha)\sigma_u}-\sigma_s\left[\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right]} \left(\frac{1}{\rho}\left[\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right]+\frac{z}{\rho\sigma_s}\right)\phi(z)dz \\
&+ \int_{-\frac{\overline{V}\rho\sigma_s}{\mu+\Phi^{-1}(\alpha)\sigma_u}-\sigma_s\left[\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right]}^{\infty} \frac{-\overline{V}}{\mu+\Phi^{-1}(\alpha)\sigma_u}\phi(z)dz \\
&= E_1+E_2+E_3
\end{aligned}
$$

In the following derivation we will use the following notations $B_1 = -\sigma_s\left(\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right)$, $B_2 = \frac{-\overline{V}}{(\mu-\Phi^{-1}(\alpha)\sigma_u)}$, $B_3 = B_1+\rho\sigma_s B_2$, $B_4 = \frac{-\overline{V}}{(\mu+\Phi^{-1}(\alpha)\sigma_u)}$, $B_5 = B_1+\rho\sigma_s B_4$.

Let us develop each of these expressions,

$$
\begin{aligned}
E_1 &= \int_{-\infty}^{-\frac{\overline{V}\rho\sigma_s}{\mu-\Phi^{-1}(\alpha)\sigma_u}-\sigma_s\left[\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right]} \frac{-\overline{V}}{\mu-\Phi^{-1}(\alpha)\sigma_u}\phi(z)dz \\
&= \int_{-\infty}^{B_3} B_2\phi(z)dz \\
&= B_2\Phi\left(B_3\right)
\end{aligned}
$$

$$
\begin{aligned}
E_2 &= \int_{B_3}^{B_5} \left(\frac{1}{\rho}\left[\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right]+\frac{z}{\rho\sigma_s}\right)\phi(z)dz \\
&= \int_{B_3}^{B_5} \left(\frac{1}{\rho}\left[\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right]\right)\phi(z)dz + \int_{B_3}^{B_5}\left(\frac{z}{\rho\sigma_s}\right)\phi(z)dz \\
&= \frac{1}{\rho\sigma_s}\int_{B_3}^{B_5} z\phi(z)dz + \frac{\left[\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right]}{\rho}\int_{B_3}^{B_5}\phi(z)dz \\
\text{Thus,}\quad E_2 &= \frac{1}{\rho\sigma_s}\int_{B_3}^{B_5} z\phi(z)dz - \frac{B_5}{\rho\sigma_s}\left(\Phi(B_5)-\Phi(B_3)\right)
\end{aligned}
$$

$$\begin{aligned}
E_3 &= \int\limits_{-\frac{\overline{V}\rho\sigma_s}{\mu+\Phi^{-1}(\alpha)\sigma_u}-\sigma_s\left[\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right]}^{\infty} \frac{-\overline{V}}{\mu+\Phi^{-1}(\alpha)\sigma_u}\phi(z)dz \\
&= \frac{-\overline{V}}{\mu+\Phi^{-1}(\alpha)\sigma_u}\int\limits_{B_5}^{\infty}\phi(z)dz \\
&= B_4\int\limits_{B_5}^{\infty}\phi(z)dz \\
Thus, \quad E_3 &= B_4\left(1-\Phi(B_5)\right)
\end{aligned}$$

It follows that

$$E(\widehat{\theta}|_{R_m=r_m}) = \left[B_2\Phi\left(B_3\right)+B_4\left(1-\Phi(B_5)\right)-\frac{B_1}{\rho\sigma_s}\left[\Phi(B_5)-\Phi(B_3)\right]+\frac{1}{\rho\sigma_s}\int\limits_{B_3}^{B_5}z\phi(z)dz\right]$$

We now consider the case where $\mu \geq -\Phi^{-1}(\alpha)\sigma_u$. In this case

$$\widehat{\theta} = \begin{cases} \frac{-\overline{V}}{\mu-\Phi^{-1}(\alpha)\sigma_u} & if \quad z \leq -\frac{\overline{V}\rho\sigma_s}{\mu-\Phi^{-1}(\alpha)\sigma_u}-\sigma_s\left[\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right] \\ \frac{1}{\rho}\left[\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right]+\frac{z}{\rho\sigma_s} & if \quad z \geq -\frac{\overline{V}\rho\sigma_s}{\mu-\Phi^{-1}(\alpha)\sigma_u}-\sigma_s\left[\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right] \end{cases}. \tag{21}$$

Therefore

$$\begin{aligned}
E(\widehat{\theta}) &= \int\limits_{-\infty}^{-\frac{\overline{V}\rho\sigma_s}{\mu-\Phi^{-1}(\alpha)\sigma_u}-\sigma_s\left[\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right]} \frac{-\overline{V}}{\mu-\Phi^{-1}(\alpha)\sigma_u}\phi(z)dz \\
&+ \int\limits_{-\frac{\overline{V}\rho\sigma_s}{\mu-\Phi^{-1}(\alpha)\sigma_u}-\sigma_s\left[\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right]}^{\infty} \left(\frac{1}{\rho}\left[\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right]+\frac{z}{\rho\sigma_s}\right)\phi(z)dz \\
&= E_1+F_1
\end{aligned}$$

As in the previous case let us develop each expression

From the previous case we know that $E_1 = B_2\Phi\left(B_3\right)$ we will then compute only F_1

$$\begin{aligned}
F_1 &= \int\limits_{B_3}^{\infty}\left(\frac{1}{\rho}\left[\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right]+\frac{z}{\rho\sigma_s}\right)\phi(z)dz \\
&= \int\limits_{B_3}^{\infty}\left(\frac{1}{\rho}\left[\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right]\right)\phi(z)dz + \int\limits_{B_3}^{B_1}\left(\frac{z}{\rho\sigma_s}\right)\phi(z)dz \\
&= \frac{1}{\rho\sigma_s}\int\limits_{B_3}^{\infty}z\phi(z)dz + \frac{\left[\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right]}{\rho}\int\limits_{B_3}^{\infty}\phi(z)dz \\
\text{Thus } F_1 &= \frac{1}{\rho\sigma_s}\int\limits_{B_3}^{\infty}z\phi(z)dz - \frac{B_1}{\rho\sigma_s}\left(1-\Phi(B_3)\right)
\end{aligned}$$

Therefore,

$$E(\widehat{\theta}|_{R_m=r_m}) = \left[B_2\Phi\left(B_3\right) - \frac{B_1}{\rho\sigma_s}\left[1-\Phi(B_3)\right] + \frac{1}{\rho\sigma_s}\int_{B_3}^{\infty} z\phi(z)dz\right]$$

Finally in the case where $\mu \leq \Phi^{-1}(\alpha)\sigma_u$

$$\widehat{\theta} = \begin{cases} \frac{1}{\rho}\left[\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right]+\frac{z}{\rho\sigma_s} & if \quad z \leq -\frac{\overline{V}\rho\sigma_s}{\mu+\Phi^{-1}(\alpha)\sigma_u} - \sigma_s\left[\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right] \\ \frac{-\overline{V}}{\mu+\Phi^{-1}(\alpha)\sigma_u} & if \quad z \geq -\frac{\overline{V}\rho\sigma_s}{\mu+\Phi^{-1}(\alpha)\sigma_u} - \sigma_s\left[\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right] \end{cases} \tag{22}$$

And it follows that

$$\begin{aligned} E(\widehat{\theta}|_{R_m=r_m}) &= \int_{-\infty}^{-\frac{\overline{V}\rho\sigma_s}{\mu+\Phi^{-1}(\alpha)\sigma_u}-\sigma_s\left[\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right]} \left(\frac{1}{\rho}\left[\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right]+\frac{z}{\rho\sigma_s}\right)\phi(z)dz \\ &+ \int_{-\frac{\overline{V}\rho\sigma_s}{\mu+\Phi^{-1}(\alpha)\sigma_u}-\sigma_s\left[\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right]}^{\infty} \frac{-\overline{V}}{\mu+\Phi^{-1}(\alpha)\sigma_u}\phi(z)dz \\ &= F_2 + E_3 \end{aligned}$$

The value of E_3 is already known and is given by $E_3 = B_4\left(1-\Phi(B_5)\right)$ we will then compute only F_2

$$\begin{aligned} F_2 &= \int_{-\infty}^{-\frac{\overline{V}\rho\sigma_s}{\mu+\Phi^{-1}(\alpha)\sigma_u}-\sigma_s\left[\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right]} \left(\frac{1}{\rho}\left[\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right]+\frac{z}{\rho\sigma_s}\right)\phi(z)dz \\ &= \int_{-\infty}^{B_5}\left(\frac{1}{\rho}\left[\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right]+\frac{z}{\rho\sigma_s}\right)\phi(z)dz \\ &= \int_{-\infty}^{B_5}\frac{z}{\rho\sigma_s}\phi(z)dz + \int_{-\infty}^{B_5}\left(\frac{1}{\rho}\left[\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right]\right)\phi(z)dz \\ &= \frac{1}{\rho\sigma_s}\int_{-\infty}^{B_5} z\phi(z)dz + \frac{1}{\rho}\left[\frac{\mu}{\sigma_u^2}+\frac{r_m}{\sigma_s^2}\right]\int_{-\infty}^{B_5}\phi(z)dz \\ \text{Thus } F_2 &= \frac{1}{\rho\sigma_s}\int_{-\infty}^{B_5} z\phi(z)dz - \frac{B_1}{\rho\sigma_s}\left(\Phi(B_5)\right) \end{aligned}$$

It then follows that

$$E(\widehat{\theta}|_{R_m=r_m}) = \left[B_4\left(1-\Phi(B_5)\right) - \frac{B_1\Phi(B_5)}{\rho\sigma_s} + \frac{1}{\rho\sigma_s}\int_{-\infty}^{B_5} z\phi(z)dz\right]$$

By putting all three cases, the general expression of $E(R_p|_{R_m=r_m}) \equiv r_m E(\widehat{\theta}|_{R_m=r_m})$ is exactly what we have in the proposition 2.

Q.E.D. ■

Proof. of the Proposition 3

Let us first set a new notation $E(R_p|R_m = r_m) = E_{R_p}(r_m)$. We want to show that if the average market excess return is zero then the relation between the optimal portfolio expected return and the market return is symmetric, i.e.,

$$\mu = 0 \Rightarrow \forall r_m \in IR^+, E_{R_p}(-r_m) = E_{R_p}(r_m).$$

If $\mu = 0$, we have $\Phi^{-1}(\alpha)\sigma_u \leq \mu \leq -\Phi^{-1}(\alpha)\sigma_u$, therefore,

$$E_{R_p}(r_m) = r_m \left[B_2\Phi\left(B_3\right) + B_4\left(1 - \Phi(B_5)\right) - \frac{B_1}{\rho\sigma_s}\left[\Phi(B_5) - \Phi(B_3)\right] + \frac{1}{\rho\sigma_s}\int_{B_3}^{B_5} z\phi(z)dz \right]$$

and in this case $B_1 = \frac{r_m}{\sigma_s}, B_2 = \frac{\overline{V}}{\Phi^{-1}(\alpha)\sigma_u}; B_4 = -B_2;\ B_3 = \frac{r_m}{\sigma_s} + \rho\sigma_s B_2; B_5 = \frac{r_m}{\sigma_s} - \rho\sigma_s B_2$. Therefore $E_{R_p}(r_m)$ can be rewritten as

$$E_{R_p}(r_m) = r_m \left[\begin{array}{c} B_2\Phi\left(\frac{r_m}{\sigma_s} + \rho\sigma_s B_2\right) - B_2\left[1 - \Phi(\frac{r_m}{\sigma_s} - \rho\sigma_s B_2)\right] \\ -\frac{r_m}{\rho\sigma_s^2}\left[\Phi\left(\frac{r_m}{\sigma_s} - \rho\sigma_s B_2\right) - \Phi(\frac{r_m}{\sigma_s} + \rho\sigma_s B_2)\right] + \frac{1}{\rho\sigma_s}\int_{\frac{r_m}{\sigma_s}+\rho\sigma_s B_2}^{\frac{r_m}{\sigma_s}-\rho\sigma_s B_2} z\phi(z)dz \end{array} \right]. \tag{23}$$

Let us now show that $E_{R_p}(r_m) = E_{R_p}(-r_m)$.

From (23)

$$E_{R_p}(-r_m) = -r_m \left[\begin{array}{c} B_2\Phi\left(\frac{-r_m}{\sigma_s} + \rho\sigma_s B_2\right) - B_2\left(1 - \Phi(\frac{-r_m}{\sigma_s} - \rho\sigma_s B_2)\right) \\ +\frac{r_m}{\rho\sigma_s^2}\left[\Phi(\frac{-r_m}{\sigma_s} - \rho\sigma_s B_2) - \Phi(\frac{-r_m}{\sigma_s} + \rho\sigma_s B_2)\right] + \frac{1}{\rho\sigma_s}\int_{\frac{-r_m}{\sigma_s}+\rho\sigma_s B_2}^{\frac{-r_m}{\sigma_s}-\rho\sigma_s B_2} z\phi(z)dz \end{array} \right]$$

Using the relationships $\Phi\left(-x\right) = 1 - \Phi\left(x\right),\ \phi(x) = \phi(-x)$ for any x , we can set that

$$\begin{aligned} \Phi\left(\frac{-r_m}{\sigma_s} + \rho\sigma_s B_2\right) &= 1 - \Phi\left(\frac{r_m}{\sigma_s} - \rho\sigma_s B_2\right); \\ 1 - \Phi(\frac{-r_m}{\sigma_s} - \rho\sigma_s B_2) &= \Phi(\frac{r_m}{\sigma_s} + \rho\sigma_s B_2);\ \text{and} \\ \int_{\frac{-r_m}{\sigma_s}+\rho\sigma_s B_2}^{\frac{-r_m}{\sigma_s}-\rho\sigma_s B_2} z\phi(z)dz &= \int_{\frac{r_m}{\sigma_s}+\rho\sigma_s B_2}^{\frac{r_m}{\sigma_s}-\rho\sigma_s B_2} -y\phi(y)dy. \end{aligned}$$

Then,

$$\begin{aligned}
E_{R_p}(-r_m) &= -r_m \left[\begin{array}{c} B_2\left(1-\Phi\left(\frac{r_m}{\sigma_s}-\rho\sigma_s B_2\right)\right) - B_2\left(\Phi(\frac{r_m}{\sigma_s}+\rho\sigma_s B_2)\right) \\ +\frac{r_m}{\rho\sigma_s^2}\left[\Phi\left(\frac{r_m}{\sigma_s}-\rho\sigma_s B_2\right)-\Phi(\frac{r_m}{\sigma_s}+\rho\sigma_s B_2)\right] - \frac{1}{\rho\sigma_s}\int\limits_{\frac{r_m}{\sigma_s}+\rho\sigma_s B_2}^{\frac{r_m}{\sigma_s}-\rho\sigma_s B_2} y\phi(y)dy \end{array} \right] \\
&= -r_m \left[\begin{array}{c} -B_2\Phi(\frac{r_m}{\sigma_s}+\rho\sigma_s B_2) + B_2\left(1-\Phi\left(\frac{r_m}{\sigma_s}-\rho\sigma_s B_2\right)\right) \\ +\frac{r_m}{\rho\sigma_s^2}\left[\Phi\left(\frac{r_m}{\sigma_s}-\rho\sigma_s B_2\right)-\Phi(\frac{r_m}{\sigma_s}+\rho\sigma_s B_2)\right] - \frac{1}{\rho\sigma_s}\int\limits_{\frac{r_m}{\sigma_s}+\rho\sigma_s B_2}^{\frac{r_m}{\sigma_s}-\rho\sigma_s B_2} y\phi(y)dy \end{array} \right] \\
&= r_m \left[\begin{array}{c} B_2\Phi(\frac{r_m}{\sigma_s}+\rho\sigma_s B_2) - B_2\left(1-\Phi\left(\frac{r_m}{\sigma_s}-\rho\sigma_s B_2\right)\right) \\ -\frac{r_m}{\rho\sigma_s^2}\left[\Phi\left(\frac{r_m}{\sigma_s}-\rho\sigma_s B_2\right)-\Phi(\frac{r_m}{\sigma_s}+\rho\sigma_s B_2)\right] + \frac{1}{\rho\sigma_s}\int\limits_{\frac{r_m}{\sigma_s}+\rho\sigma_s B_2}^{\frac{r_m}{\sigma_s}-\rho\sigma_s B_2} y\phi(y)dy \end{array} \right] \\
&= E_{R_p}(r_m).
\end{aligned}$$

So $E_{R_p}(-r_m) = E_{R_p}(r_m)$. or more precisely, $E(-R_p|R_m = r_m) = E(R_p|R_m = r_m)$, and the proposition 3 follows Q.E.D. ■

Proof. of Proposition 4:

The performance valuation is given by

$$VL = (1+r_f)^{-1}\int\limits_{-\infty}^{\infty} E(R_p|r_m)d\overline{F}(r_m), \quad \overline{F}(r_m) \sim N(0,\sigma_u^2), \tag{24}$$

Since $\Phi^{-1}(\alpha)\sigma_u \leq \mu \leq -\Phi^{-1}(\alpha)\sigma_u$, we know that

$$E(R_p|_{R_m=r_m}) = r_m\left[B_2\Phi(B_3) + B_4(1-\Phi(B_5)) - \frac{B_1}{\rho\sigma_s}\left[\Phi(B_5)-\Phi(B_3)\right] + \frac{1}{\rho\sigma_s}\int\limits_{B_3}^{B_5} z\phi(z)dz\right]$$

It follows that

$$\begin{aligned}
VL = (1+r_f)^{-1} \quad & \left[\int\limits_{-\infty}^{\infty} r_m B_2(\Phi(B_3))\, d\overline{F}(r_m) + \int\limits_{-\infty}^{\infty} r_m B_4\left[1-\Phi(B_5)\right] d\overline{F}(r_m) \right. \\
& \left. - \int\limits_{-\infty}^{\infty} \frac{r_m[\Phi(B_5)-\Phi(B_3)]B_1}{\rho\sigma_s} d\overline{F}(r_m) + \int\limits_{-\infty}^{\infty} \frac{r_m}{\rho\sigma_s}\int\limits_{B_3}^{B_5} z\phi(z)dz d\overline{F}(r_m)\right]
\end{aligned}$$

Let us consider the following notation

$$VL = (1+r_f)^{-1}\left[V_1 + V_2 - V_3 + V_4\right]. \tag{25}$$

It is convenient to compute each of the V_i

Computation of V_1

$$
\begin{aligned}
V_1 &= \int_{-\infty}^{\infty} r_m B_2 \left(\Phi\left(B_3\right)\right) d\overline{F}(r_m) \\
&= B_2 \int_{-\infty}^{\infty} r_m B_2 \left(\Phi\left(B_3\right)\right) \phi_{\sigma_u^2}(r_m) dr_m \\
&= \frac{-\overline{V}}{(\mu - \Phi^{-1}(\alpha)\sigma_u)} \int_{-\infty}^{\infty} r_m \left[\Phi\left[-\sigma_s\left(\frac{\mu}{\sigma_u^2} + \frac{r_m}{\sigma_s^2} + \frac{\overline{V}\rho}{(\mu - \Phi^{-1}(\alpha)\sigma_u)}\right)\right]\right] \phi_{\sigma_u^2}(r_m) dr_m \\
&= \frac{-\overline{V}}{(\mu - \Phi^{-1}(\alpha)\sigma_u)} \int_{-\infty}^{\infty} r_m \left[\int_{-\infty}^{-\sigma_s\left(\frac{\mu}{\sigma_u^2} + \frac{r_m}{\sigma_s^2} + \frac{\overline{V}\rho}{(\mu - \Phi^{-1}(\alpha)\sigma_u)}\right)} \phi(u) du\right] \phi_{\sigma_u^2}(r_m) dr_m
\end{aligned}
$$

or

$$
\begin{aligned}
V_1 &= \frac{-\overline{V}}{(\mu - \Phi^{-1}(\alpha)\sigma_u)} \int_{-\infty}^{\infty} r_m \left[\int_{-\infty}^{\infty} 1_{u \leq \left[-\sigma_s\left(\frac{\mu}{\sigma_u^2} + \frac{r_m}{\sigma_s^2} + \frac{\overline{V}\rho}{(\mu - \Phi^{-1}(\alpha)\sigma_u)}\right)\right]} \phi(u) du\right] \phi_{\sigma_u^2}(r_m) dr_m \\
&= \frac{-\overline{V}}{(\mu - \Phi^{-1}(\alpha)\sigma_u)} \int_{-\infty}^{\infty} r_m \left[\int_{-\infty}^{\infty} 1_{\left(u + \frac{r_m}{\sigma_s}\right) \leq \left[-\sigma_s\left(\frac{\mu}{\sigma_u^2} + \frac{\overline{V}\rho}{(\mu - \Phi^{-1}(\alpha)\sigma_u)}\right)\right]} \phi(u) du\right] \phi_{\sigma_u^2}(r_m) dr_m
\end{aligned}
$$

i.e.

$$
V_1 = \frac{-\overline{V}}{(\mu - \Phi^{-1}(\alpha)\sigma_u)} \int_{-\infty}^{\infty} r_m \left[\int_{-\infty}^{\infty} 1_{\left(u + \frac{r_m}{\sigma_s}\right) \leq \left[-\sigma_s\left(\frac{\mu}{\sigma_u^2} + \frac{\overline{V}\rho}{(\mu - \Phi^{-1}(\alpha)\sigma_u)}\right)\right]} \phi(u) du\right] \phi_{\sigma_u^2}(r_m) dr_m \qquad (26)
$$

In order to proceed we will consider the following linear transformation of variables

$$
\begin{cases}
y &= u + \frac{r_m}{\sigma_s} \\
v &= u
\end{cases}
$$

The Jacobian matrix of this transformation is equal to σ_s.

It follows that

$$
V_1 = \frac{-\overline{V}}{(\mu - \Phi^{-1}(\alpha)\sigma_u)} \int_{-\infty}^{\infty} \left(\sigma_s\left(y - v\right)\right) \left[\int_{-\infty}^{\infty} 1_{y \leq \left[-\sigma_s\left(\frac{\mu}{\sigma_u^2} + \frac{\overline{V}\rho}{(\mu - \Phi^{-1}(\alpha)\sigma_u)}\right)\right]}\right] \phi_{\sigma_u^2}(\sigma_s\left(y - v\right)) \phi(u) du dr_m
$$

Besides,

$$
\begin{aligned}
\phi(v)\phi_{\sigma_u^2}(\sigma_s(y-v)) &= \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}v^2\right]\frac{1}{\sqrt{2\pi\sigma_u^2}}\exp\left[-\frac{\sigma_s^2}{2\sigma_u^2}(y-v)^2\right] \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi\sigma_u^2}}\exp\left[-\frac{1}{2}\left(\left(1+\frac{\sigma_s^2}{\sigma_u^2}\right)v^2+\frac{\sigma_s^2}{\sigma_u^2}y^2-\frac{2\sigma_s^2}{\sigma_u^2}yv\right)\right] \\
&= \frac{1}{\sqrt{2\pi\left(1+\frac{\sigma_u^2}{\sigma_s^2}\right)}}\exp\left[-\frac{1}{2\left(1+\frac{\sigma_u^2}{\sigma_s^2}\right)}y^2\right] \\
&\quad \times\frac{1}{\sqrt{2\pi\left(\frac{\sigma_u^2\sigma_s^2}{\sigma_u^2+\sigma_s^2}\right)}}\exp\left[-\frac{1}{2\left(\frac{\sigma_u^2\sigma_s^2}{\sigma_u^2+\sigma_s^2}\right)}\left(\sigma_s\left(v-\frac{\sigma_s^2}{\sigma_u^2+\sigma_s^2}y\right)\right)^2\right] \\
&= \phi_{\left(1+\frac{\sigma_u^2}{\sigma_s^2}\right)}(y)\phi_{\left(\frac{\sigma_u^2\sigma_s^2}{\sigma_u^2+\sigma_s^2}\right)}\left(\sigma_s\left(v-\frac{\sigma_s^2}{\sigma_u^2+\sigma_s^2}y\right)\right) \\
&= \phi_{A^2}(y)\phi_{B^2}(\sigma_s(v-Cy))
\end{aligned}
$$

with $\phi_{\alpha^2}(x)$ means $X \sim N(0,\alpha^2)$, and where $A^2=\left(1+\frac{\sigma_u^2}{\sigma_s^2}\right)$; $B^2=\left(\frac{\sigma_u^2\sigma_s^2}{\sigma_u^2+\sigma_s^2}\right)$; $C=\frac{\sigma_s^2}{\sigma_u^2+\sigma_s^2}$

i.e.

$$
\phi(v)\phi_{\sigma_s^2}(\sigma_s(y-v)) = \phi_{A^2}(y)\phi_{B^2}(\sigma_s(v-Cy)) \tag{27}
$$

It follows that

$$
\begin{aligned}
V_1 &= \frac{-\overline{V}}{(\mu-\Phi^{-1}(\alpha)\sigma_u)}\int_{-\infty}^{\infty}\left(\sigma_s^2(y-v)\right)\left[\int_{-\infty}^{\infty}1_{y\le\left[-\sigma_s\left(\frac{\mu}{\sigma_u^2}+\frac{\overline{V}\rho}{(\mu-\Phi^{-1}(\alpha)\sigma_u)}\right)\right]}\right]\phi_{A^2}(y)\phi_{B^2}(\sigma_s(v-Cy))\,dvdy \\
&= \frac{-\overline{V}}{(\mu-\Phi^{-1}(\alpha)\sigma_u)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}1_{y\le\left[-\sigma_s\left(\frac{\mu}{\sigma_u^2}+\frac{\overline{V}\rho}{(\mu-\Phi^{-1}(\alpha)\sigma_u)}\right)\right]}\sigma_s^2(y-v)\,\phi_{A^2}(y)\phi_{B^2}(\sigma_s(v-Cy))\,dvdy \\
&= \frac{-\overline{V}}{(\mu-\Phi^{-1}(\alpha)\sigma_u)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}1_{y\le\left[-\sigma_s\left(\frac{\mu}{\sigma_u^2}+\frac{\overline{V}\rho}{(\mu-\Phi^{-1}(\alpha)\sigma_u)}\right)\right]}\sigma_s^2(y-v)\,\phi_{A^2}(y)\phi_{B^2}(\sigma_s(v-Cy))\,dvdy \\
&= \frac{-\overline{V}}{(\mu-\Phi^{-1}(\alpha)\sigma_u)}\int_{-\infty}^{\infty}1_{y\le\left[-\sigma_s\left(\frac{\mu}{\sigma_u^2}+\frac{\overline{V}\rho}{(\mu-\Phi^{-1}(\alpha)\sigma_u)}\right)\right]}\sigma_s^2\phi_2(y)\left[\int_{-\infty}^{\infty}(y-v)\,\phi_{B^2}(\sigma_s(v-Cy))\,dv\right]dy
\end{aligned}
$$

One can show that $\int_{-\infty}^{\infty}(y-v)\,\phi_{B^2}(\sigma_s(v-Cy))\,dv=\frac{(1-C)y}{\sigma_s}$

Actually, using the following transformation $x=\sigma_s(v-Cy)$

The Jacobian of the distribution is given by $\frac{dv}{dx}=\frac{1}{\sigma_s}$, thus

$$\begin{aligned}
\int_{-\infty}^{\infty}(y-v)\,\phi_{B^2}\left(\sigma_s\left(v-Cy\right)\right)dv &= \int_{-\infty}^{\infty}\left(\left(1-C\right)y-\frac{x}{\sigma_s}\right)\phi_{B^2}\left(x\right)\frac{1}{\sigma_s}dx \\
&= \frac{1}{\sigma_s}\int_{-\infty}^{\infty}\left(\left(1-C\right)y-\frac{x}{\sigma_s}\right)\phi_{B^2}\left(x\right)dx \\
&= \frac{1}{\sigma_s}\left[\int_{-\infty}^{\infty}\left(\left(1-C\right)y\right)\phi_{B^2}\left(x\right)dx+\int_{-\infty}^{\infty}\left(-\frac{x}{\sigma_s}\right)\phi_{B^2}\left(x\right)dx\right] \\
&= \frac{1}{\sigma_s}\left[\left(\left(1-C\right)y\right)-\frac{1}{\sigma_s}0\right] \\
&= \frac{\left(1-C\right)y}{\sigma_s}
\end{aligned}$$

Therefore,

$$\begin{aligned}
V_1 &= \frac{-\overline{V}}{(\mu-\Phi^{-1}(\alpha)\sigma_u)}\int_{-\infty}^{\infty}1_{y\leq\left[-\sigma_s\left(\frac{\mu}{\sigma_u^2}+\frac{\overline{V}\rho}{(\mu-\Phi^{-1}(\alpha)\sigma_u)}\right)\right]}\sigma_s^2\phi_{A^2}(y)\frac{\left(1-C\right)y}{\sigma_s}dy \\
V_1 &= \frac{-\overline{V}\left(1-C\right)\sigma_s}{(\mu-\Phi^{-1}(\alpha)\sigma_u)}\int_{-\infty}^{-\sigma_s\left(\frac{\mu}{\sigma_u^2}+\frac{\overline{V}\rho}{(\mu-\Phi^{-1}(\alpha)\sigma_u)}\right)}y\phi_{A^2}(y)dy \\
V_1 &= \left(1-C\right)\sigma_s B_2\int_{-\infty}^{-\sigma_s\left(\frac{\mu}{\sigma_u^2}-\rho B_2\right)}y\phi_{A^2}(y)dy
\end{aligned}$$

Computation of V_2

$$V_2=\int_{-\infty}^{\infty}r_m B_4\left[1-\Phi\left(B_5\right)\right]d\overline{F}(r_m)$$

After replacing B_1, B_4, and B_5 by their values, and using the same transformation as in V_1 we obtain

$$V_2=B_4\sigma_s(1-C)\int_{\left[-\sigma_s\rho\left(\frac{\mu}{\sigma_u^2}\right)+\sigma_s\rho B_4\right]}^{\infty}y\phi_{A^2}(y)dy$$

Computation of V_3

$$V_3=\int_{-\infty}^{\infty}\frac{r_m B_1[\Phi(B_5)-\Phi(B_3)]}{\rho\sigma_s}d\overline{F}(r_m)=\int_{-\infty}^{\infty}\int_{B_3}^{B_5}\phi(u)du\frac{r_m B_1}{\rho\sigma_s}d\overline{F}(r_m)$$

After replacing B_1, B_3, and B_5 by their values, and by using the same linear transformation as in V_1, we obtain

$$V_3 = \frac{-1}{\rho} \left[\frac{\mu(1-C)\sigma_s}{\sigma_u^2} \int\limits_{-\sigma_s\left(\frac{\mu}{\sigma_u^2}\right)+\rho\sigma_s B_2}^{-\sigma_s\left(\frac{\mu}{\sigma_u^2}\right)+\rho\sigma_s B_4} y\phi_{A^2}(y)dy \right.$$
$$\left. + \int\limits_{-\sigma_s\left(\frac{\mu}{\sigma_u^2}\right)+\rho\sigma_s B_2}^{-\sigma_s\left(\frac{\mu}{\sigma_u^2}\right)+\rho\sigma_s B_4} \left[(1-C)^2y^2 + \frac{B^2}{\sigma_s^2}\right] \phi_{A^2}(y)dy \right]$$

Computation of V_4

$V_4 = \int\limits_{-\infty}^{\infty} \frac{r_m}{\rho\sigma_s} \int\limits_{B_3}^{B_5} z\phi(z)dzd\overline{F}(r_m)$

After replacing B_3, and B_5 by their values,and follow the same steps as in V_1 we obtain

$$V_4 = \frac{1}{\rho} \left(\int\limits_{-\sigma_s\left(\frac{\mu}{\sigma_u^2}\right)+\rho\sigma_s B_2}^{-\sigma_s\left(\frac{\mu}{\sigma_u^2}\right)+\rho\sigma_s B_4} \left[C(1-C)y^2 - \frac{B^2}{\sigma_s^2}\right] \phi_{A^2}(y)dy \right)$$

By putting all together, the general expression of $VL \equiv (1+r_f)^{-1}[V_1 + V_2 - V_3 + V_4]$ is exactly what we have in the proposition 4.

Q.E.D. ■

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Figure 1 The relation between the portfolio expected excess return (Conditional Expected Portfolio Return) and the market excess-return (Market Return) under unrestricted, Short-Selling restriction and VaR restriction (parameters $\sigma_s = 0.1$; $\sigma_u = 0.15$; $\rho = 5$; $\mu = 0.04$; $\alpha = 0.01$; $k_1 = 0$; $k_2 = 100$; $\bar{V} = 0.518$).

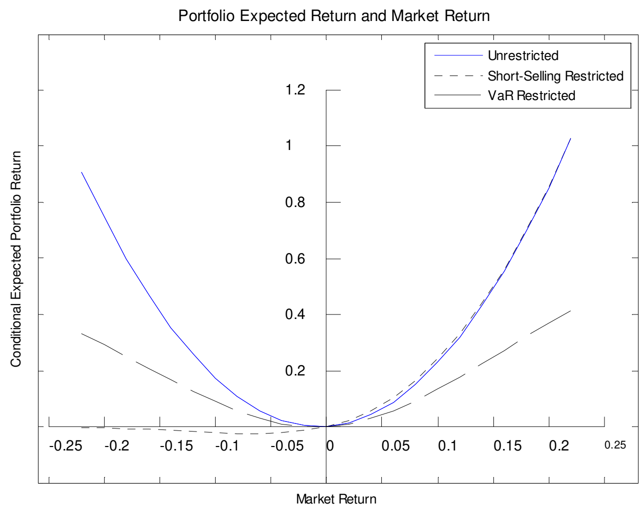

Figure 2. The upper graph represents the expected return in case of the VaR Restriction for three different levels of the VaR limit (with parameters $\sigma_s = 0.1$; $\sigma_u = 0.15$; $\rho = 5$; $\mu = 0.04$; $\alpha = 0.01$) The lower graph shows the ratio of the valuation of restricted portfolio manager over the valuation of unrestricted portfolio manager (with parameters $\sigma_s = 0.1$; $\sigma_u = 0.15$; $\rho = 5$; $\mu = 0.04$; $\alpha = 0.01$; $k_1 = 0$; $k_2 = 100$; $\bar{V} = 0.518$).

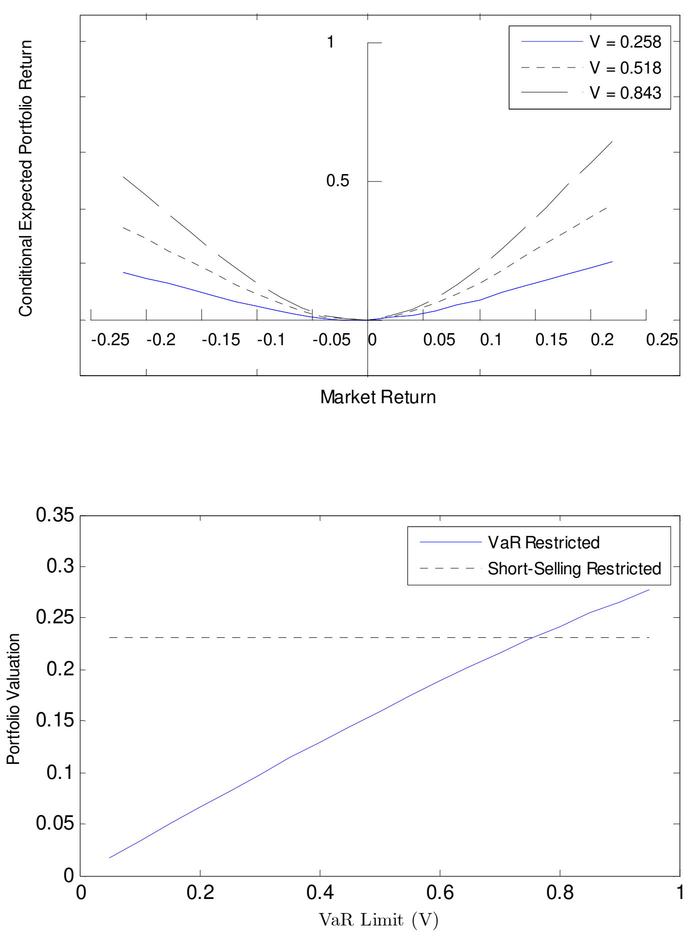

Figure 3. The upper graph represents the expected portfolio excess-return in the case of unrestricted manager for two different level of market volatility (with parameters $\sigma_s = 0.1$; ρ=5; $\mu = 0.04$). The left lower graph shows the expected return in the case of VaR restricted manager (with parameters σ_s =0.1; $\rho = 5$; $\mu = 0.04$; $\alpha = 0.01$; $\overline{V} = 0.518$). The right lower graph represents the expected return in the case of short-selling restricted manager (with parameters $\sigma_s = 0.1$; $\rho = 5$; $\mu = 0.04$; $k_1 = 0$; k_2 =100)

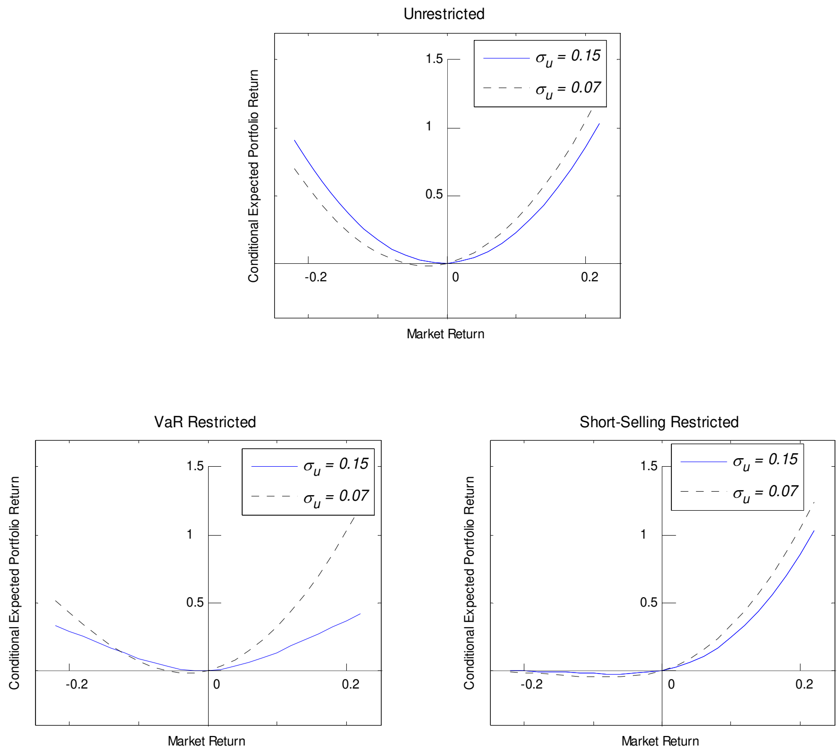

Figure 4. The upper graph represents the valuation in case of VaR Restriction in function of the market volatility (with parameters $\sigma_s = 0.10$; $\mu = 0.04$; $\overline{V} = 0.518$; $\alpha = 0.01$; $\rho = 5$). The left lower graph shows the valuation of restricted portfolio manager (with parameters $\sigma_s = 0.10$; $\rho = 5$; $\overline{V}$=0.518; $k_1 = 0$; $k_2 = 100$; $\mu = 0.04$; α=0.01). The right lower graph represents the ratio of the valuation of restricted portfolio manager over the valuation of unrestricted portfolio manager (with parameters $\sigma_s = 0.10$; $\rho = 5$; $\overline{V} = 0.518$; $k_1 = 0$; $k_2 = 100$; $\mu = 0.04$; α=0.01).

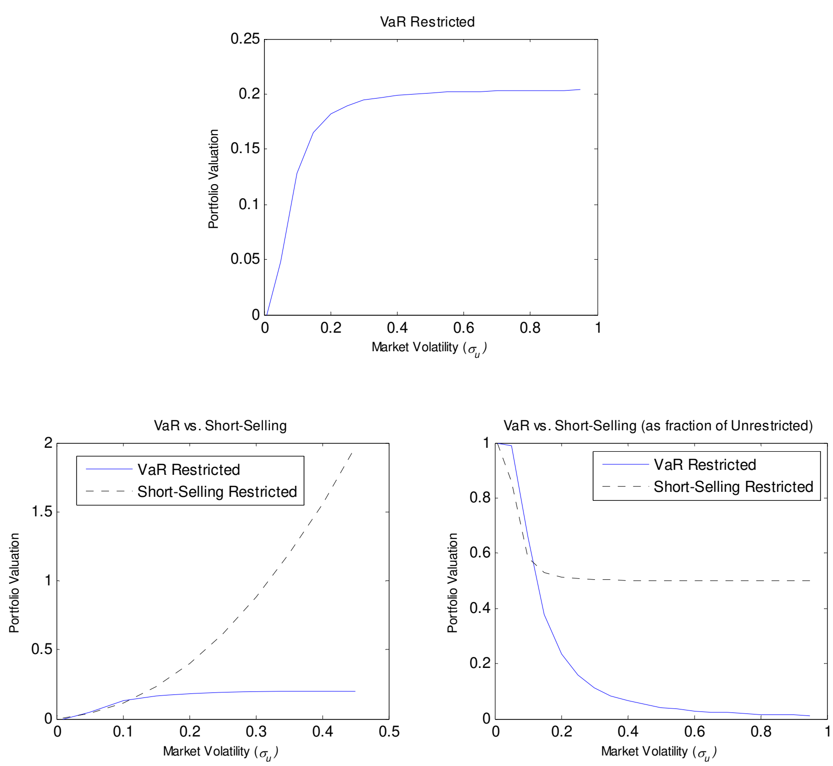

Figure 5. The upper graph represents the expected portfolio excess-return in the case of unrestricted manager for two different level of information accuracy (with parameters $\sigma_u = 0.15$; $\rho = 5$; $\mu = 0.04$). The left lower graph shows the expected return in the case of VaR restricted manager (with parameters $\sigma_u = 0.15$; $\rho = 5$; $\mu = 0.04$; $\alpha = 0.01$; $\overline{V} = 0.518$). The right lower graph represents the expected return in the case of short-selling restricted manager (with parameters $\sigma_u = 0.15$; $\rho = 5$; $\mu = 0.04$; k_1=0; $k_2 = 100$)

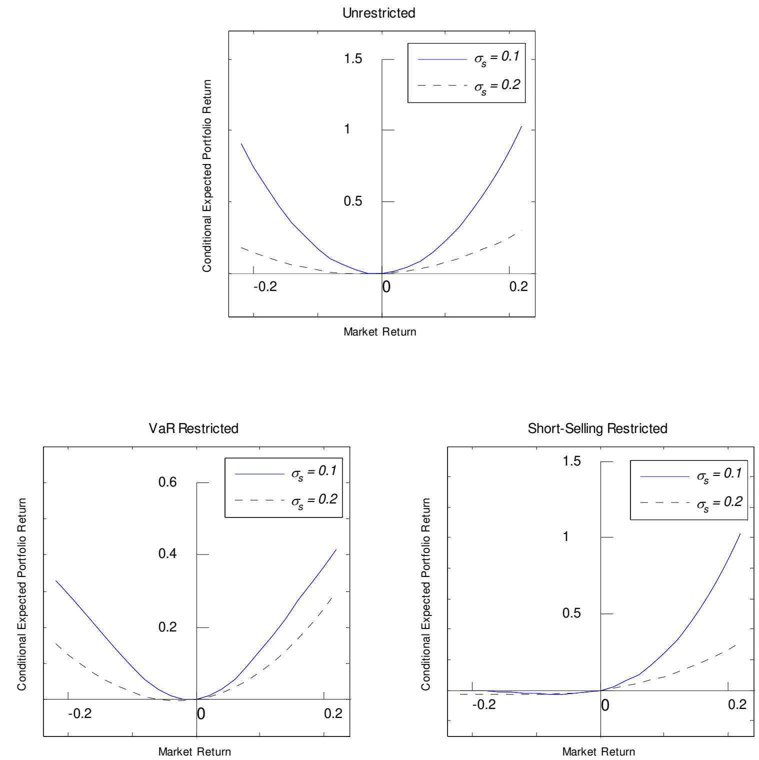

Figure 6. The upper graph represents the valuation in case of VaR Restriction in function of the information quality (with parameters $\sigma_u = 0.15$; $\mu = 0.04$; $\overline{V} = 0.518$; $\alpha = 0.01$; $\rho = 5$). The left lower graph shows the valuation of restricted portfolio manager (with parameters $\sigma_u = 0.15$; $\rho = 5$; $\overline{V} = 0.518$; $k_1 = 0$; $k_2 = 100$; $\mu = 0.04$; $\alpha = 0.01$). The right lower graph represents the ratio of the valuation of restricted portfolio manager over the valuation of unrestricted portfolio manager (with parameters $\sigma_u = 0.15$; $\rho = 5$; $\overline{V} = 0.518$; $k_1 = 0$; $k_2 = 100$; $\mu = 0.04$; $\alpha = 0.01$).

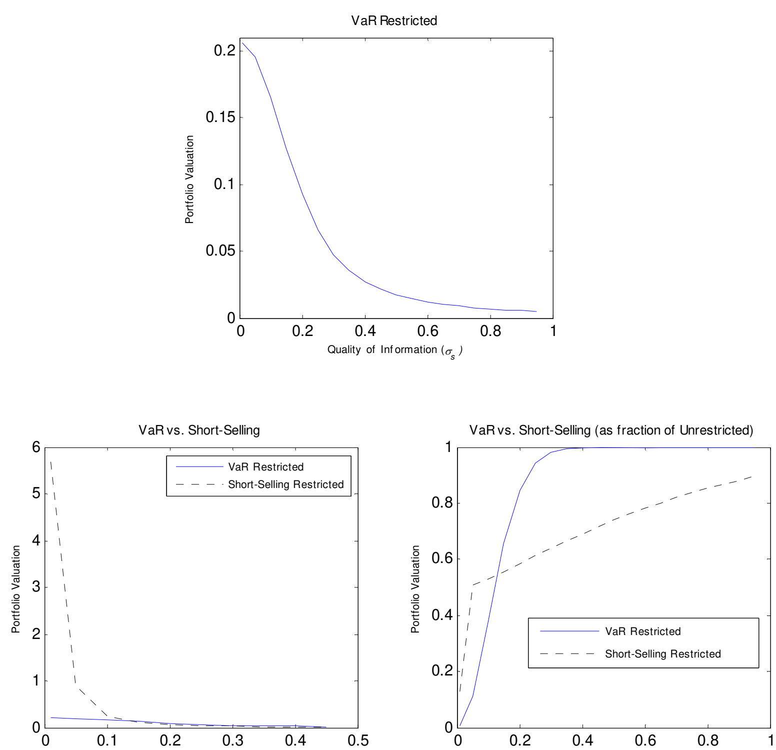

Figure 7. The upper graph represents the expected portfolio excess-return in the case of unrestricted manager for two different level of risk aversion (with parameters $\sigma_u = 0.15$; $\sigma_s = 0.10$; $\mu = 0.04$). The left lower graph shows the expected return in the case of VaR restricted manager (with parameters $\sigma_u = 0.15$; $\sigma_s = 0.10$; $\mu = 0.04$; $\alpha = 0.01$; $\overline{V} = 0.518$). The right lower graph represents the expected return in the case of short-selling restricted manager (with parameters $\sigma_u = 0.15$; $\sigma_s = 0.10$; $\mu = 0.04$; $k_1 = 0$; $k_2 = 100$)

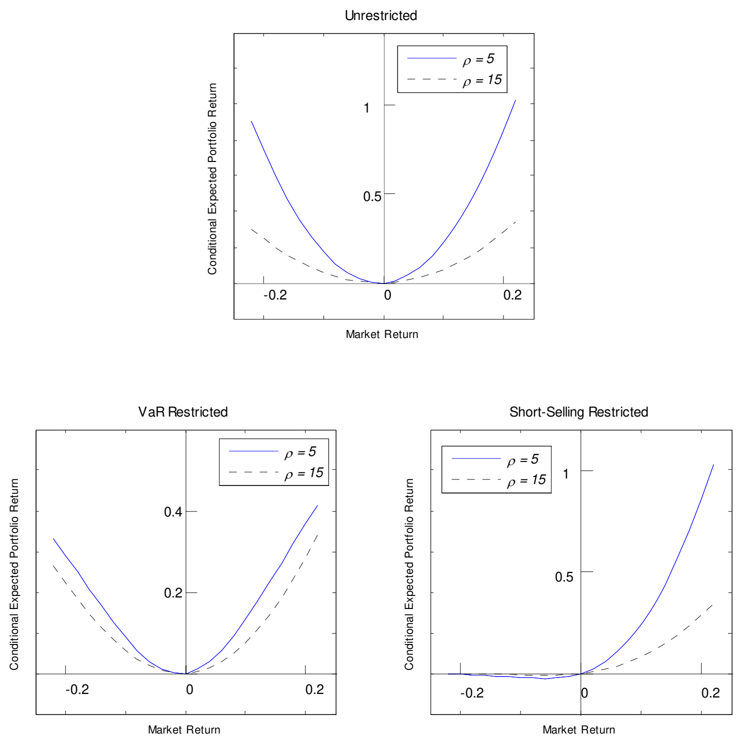

Figure 8. The upper graph represents the valuation in case of VaR Restriction in function of the risk aversion (with parameters $\sigma_u = 0.15$; $\sigma_s = 0.10$; $\mu = 0.04$; $\overline{V} = 0.518$; $\alpha = 0.01$). The left lower graph shows the valuation of restricted portfolio manager (with parameters $\sigma_u = 0.15$; $\sigma_s = 0.10$; $\overline{V} = 0.518$; $k_1 = 0$; $k_2 = 100$; $\mu = 0.04$; $\alpha = 0.01$). The right lower graph represents the ratio of the valuation of restricted portfolio manager over the valuation of unrestricted portfolio manager (with parameters $\sigma_u = 0.15$; $\sigma_s = 0.10$; $\overline{V} = 0.518$; $k_1 = 0$; $k_2 = 100$; $\mu = 0.04$; $\alpha = 0.01$).

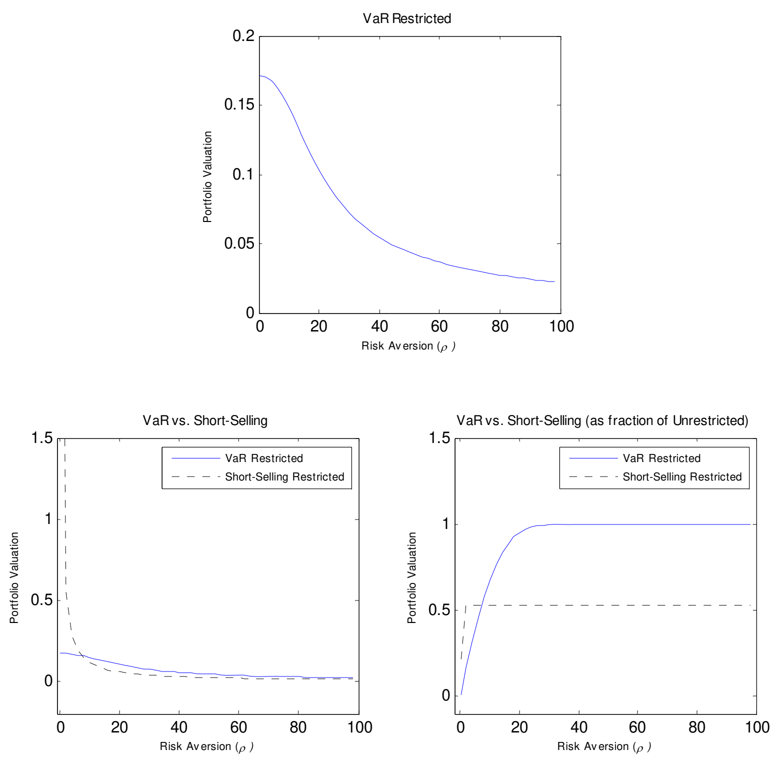

Figure 9. The upper graph represents the expected return in case of the VaR Restriction as a function of both the market return and the VaR limit (with parameters $\sigma_s = 0.1$; $\sigma_u = 0.15$; $\rho = 5$; $\mu = 0.04$; $\alpha = 0.01$). The left lower graph shows the expected return in the case of VaR restricted manager Restriction as a function of both the market return and the information quality (with parameters $\sigma_u = 0.15$; $\rho = 5$; $\mu = 0.04$; $\alpha = 0.01$; $\bar{V} = 0.518$). The right lower graph represents the expected return in the case of VaR restricted manager as a function of both the market return and the risk aversion (with parameters $\sigma_u = 0.15$; $\sigma_s = 0.10$; $\mu = 0.04$; $\alpha = 0.01$; $\bar{V} = 0.518$).

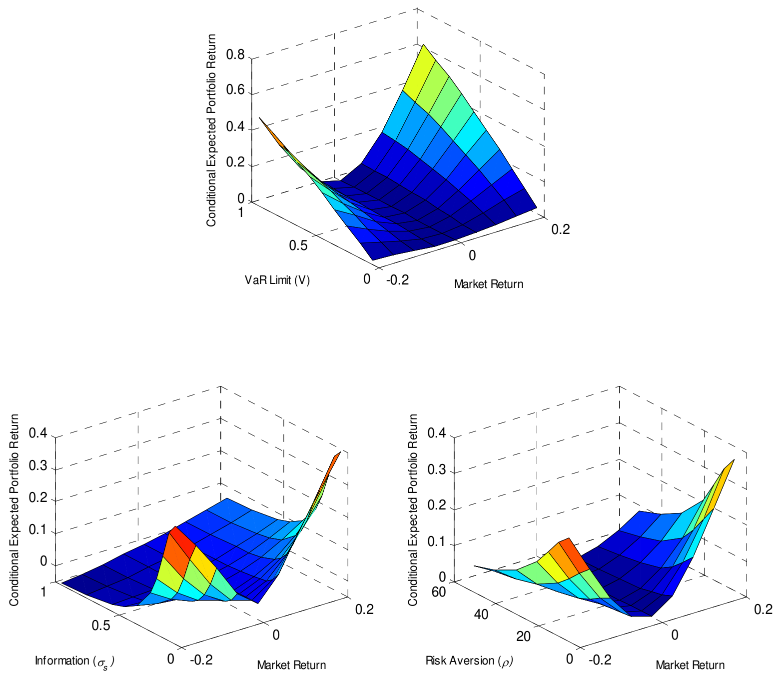

Figure 10. The upper graph represents the valuation in case of VaR Restriction in function of both the information quality and the risk aversion (with parameters $\sigma_u = 0.15$; $\mu = 0.04$; $\bar{V} = 0.518$; $\alpha = 0.01$). The left lower graph represents the valuation in case of VaR Restriction in function of both the information quality and the VaR Limit (with parameters $\sigma_u = 0.15$; $\mu = 0.04$; $\alpha = 0.01$; $\rho = 5$). The right lower graph represents the valuation in case of VaR Restriction in function of both the information quality and the average market excess return (with parameters $\sigma_u = 0.15$; $\bar{V} = 0.518$; $\alpha = 0.01$; $\rho = 5$).