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In the first section we recall Bak Sneppen model and its evolutions. In this paper we wish to present some simplified cases in which explicit computations via Markov chains are possible, hence reaching a better understanding of some rather hidden phenomena of the general case: in particular "avalanches" can be read in terms of average return times and in terms of transitions between structures. A short comparison of the behaviour of different model of graphs of interaction is contained in section 2. The simple models allow us to introduce new frames that do not seem to have been considered in the previous literature, namely the case of partitioned Bak-Sneppen frames, that appear more realistic from the point of view of speed of evolution and do not present a unique criticality level, but a staircase tending towards a final equilibrium level. This is the object of section 3, while section 4 shows how this model clearly explains the overtaking of competitors with respect to species that seem to be well assessed and recalls some of our experimental data.

1. Outlook on Bak Sneppen models

Models of biological evolution have proved to be of the greatest interest in the so called econophysics. The Bak-Sneppen model was originally introduced as a simple model of evolution by Per Bak and Kim Sneppen [2] (compare also [1], [22]). Their original model can be defined as follows. There are N species arranged on a circle, each of which has been assigned a random fitness. The fitness values are independent and uniformly distributed on (0; 1). At each discrete time step the system is updated by locating the lowest fitness and replacing this fitness, and those of its two neighbors, by independent and uniform (0; 1) random variables. Bak-Sneppen models can be defined on a wide range of graphs using the same update rule as above. What the B-S model illustrates is that even random processes can result in self-organization to a critical state. see [19] for a discussion. Threshold fitness rises, rapidly at first, then exponentially slows until it reaches about 0.66, the critical state–from which level extinction sweep back and forth through the ecosystem. None of the changes observed in the system are designed, however, to increase the critical threshold or lead to extinction, but the dynamics of the model lead inevitably to self organized criticality.
Although the Bak-Sneppen model is extremely simple, it has not yet been solved in spite of numerous analytical and numerical investigations ([13], [5], [15], [21], [17], [6], [18],[16], [9]).
Motivated by the difficulty of analyzing rigorously even the one-dimensional version of the Bak-Sneppen model, J. Barbay and C. Kenyon [18] propose a still simpler model with discrete fitness values[2]. In their model each species has fitness 0 or 1, and each new fitness is drawn from the Bernoulli distribution with parameter p. Since there are typically several least fit species, the process then repeatedly chooses a species at random for mutation among the least fit species. They prove bounds on the average numbers of ones in the stationary distribution and present experimental results. Parameter p can substitute up to a certain level a plurality of values, but it cannot explain the staircase phenomenon of section 3. Hence binary structure, though simple and appealing is not sufficient for a thorough description of what may happen.[3]
Applications of self-organized criticality to the social sciences are much more controversial than applications in the physical and biological sciences. Certainly the social sciences involve complex interactions, but it is unclear whether these interactions can be qualified beyond applying random statistics such as Gaussian distributions and Brownian motions. The most quantitative of the social sciences is economics and. a large bibliography can be found in Turcotte [23], where suitable references are to be found. We just recall a short summary taken from [23]: "power-law (fractal) distributions are often found (Mandelbrot 1982, Levy *et al* 1996). Stock markets are characterized by crashes, which certainly resemble avalanches (Scheinkman and

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[2] Also this case is by no means trivial, as it was shown by Meester and Znamenski ([20])
[3] A further case is dealt by C. Bandt in [4], who shows that the discrete Bak-Sneppen model behaves exactly like the contact process, on an arbitrary graph, thus all results which have been shown for CP will immediately extend to DBS.

Woodford 1994, Mantegna and Stanley 1997). Bak *et al* (1993, 1997) have considered whether the behaviour of stock markets is an example of self-organized criticality, but relatively little other work has been done. It has been argued that there are log-periodic (complex fractal) fluctuations prior to stock market crashes (Feigenbaum and Freund 1996, Sornette *et al* 1996, Sornette and Johansen 1997), but both their validity and interpretation remain uncertain."
The authors anyhow think that a sound basis for applying Bak-Sneppen model of contact with neighbors is given by Duesenberry's demonstration effect. Its first presentation can be found in Duesenberry [10], while many application in consumer's economy and sociology can be found for example in Cavalli [7].

2. Binary Bak Sneppen linear models

The simplest case of non trivial discrete Bak-Sneppen process seems to be the case of 6 knots and 2 values (say 0, 1). In this case it is easy to enumerate all the possible structures (apart from rotations they are 14) and to calculate the transition matrix of the Markov chain that derives from the structure. Since in discrete case there are usually many minima, we suppose that it is chosen at random among all the minima; other rule can change deeply the evolution and will be discussed in another paper. Manual computations are rather tiresome, hence we constructed a computer program for achieving this goal. A first generalization is to change the connections in Bak-Sneppen original model, where the two cells attained are the right and the left neighbor of the minimum. In table 1 we list the two links (according to positive rotation). In view of symmetries the number of different cases is reduced to 6. Bak-Sneppen classic case lies in the first row. In the third column we give, as a synthesis of the process, the average percentage of 1's.

1st link	2nd link	Average
1	5 (= -1)	0,637389
1	2	0,620732
1	3	0,620732
1	4	0,607729
2	3	0,620732
2	4	0,75
random	random	0,61229

Table 1. Binary Bak-Sneppen on a frame of 6 knots

The last row deals with what at first sight is not a Bak Sneppen process, since the two neighbors are chosen at random, not following any fixed rule. It is equivalent to use, as a transition matrix, the (weighted) average of all the structures listed in the table: hence it can be reasonably compared with Bak-Sneppen structured processes.
Usually it is expected that the average value in Bak-Sneppen processes (self organizing criticality) is higher than in a random process, since Bak-Sneppen lets arise long chains of adjoining high values; chains are in some sense stable, hence they let the average grow with respect to random processes. The table shows that it is possible that in a structured process we get an average lower than in a random process, namely the strongly asymmetric case 1- 4 in which the two operating cells are just opposite, hence tend to break long chain of maximums wherever they can be located.

3. Partitioned frames

We point out now a much more interesting phenomenon, that may arise only in the case of a non prime number of knots (hence the reason for choosing 6 knots). The average of the last case (2-4) is much greater than all the remaining averages. In the transition matrix of the associated Markov chain there exists a persistent group[4] that is smaller than the whole set of 14 structures. Namely it contains only the 4 structures 01 01 01, 01 01 11, 01 11 11, 11 11 11, while the remaining 10 form a transient group: this means that there exist sets of transformations that allow to pass from any of them to any other, but there is the possibility of falling outside into the persistent group without possibility of returning back. In this simple case the analysis is straightforward. Knots 2 and 4, together with the minimum conventionally placed at 0, form a subset that has no interference with knots 1-3-5, that on the contrary are activated when minimum is attained in one of

[4] We follow Hsu's nomenclature [14] Compare also Chang-Piccinini [8]

those knots. Whenever an operation is performed, one 3-element subset is left unchanged, while the other 3-element subset is totally changed at random. The trick is that in this case the two subsets do not change between different operations, what on the contrary happens in all the remaining cases. The process cannot anyhow be divided into two independent subprocesses, since the minimum must be looked for among both the subsets (for example a subset containing 011 enters in this search, while a subset 111 enters in the search only if also the other one is a 111 subset, in which case there 6 minima of value 1). A further interaction is given by the number of minima; for example in the case 000 011 the first subset has probability ¾ of being chosen, while the other one only ¼.
When one set reaches the configuration 111 (probability 1/8) it becomes stable, in the sense that it can be changed only if the global minimum is 1, that is the configuration is 111 111. In this case anyhow at least one of the two subsets will save the configuration 111, that can no longer be destroyed. The four structures above in fact can be read, keeping the two subsets divided, as 000 111, 001 111, 011 111, 111 111. All computations become very simple since there is no longer a Bak Sneppen structure.
In particular the average delay time for reaching the persistent group starting from any configuration of the transient group is 8, and does not depend on the initial structure. The average time for reaching the top configuration 111 111 starting from 000 000 (or any other transient) is 16.
Remark that in the standard Bak Sneppen process this last average time is 23.07572. The increase in time is due to the impossibility of protecting the structure 111 from decay.
A richer, but similar situation, arises in the binary case of 8 knots. The subsets are formed by four elements and the different structures are 0000, 0001, 0011, 0101, 0111, 1111; the persistent group is thus formed by a subset 1111 coupled with any of this six structures. The estimations become less trivial because the structure with four elements is already Bak-Sneppen non trivial, inasmuch one element is not changed at random and preserves memory of the past. The average percentage of 1's is therefore somewhat increased, up to 0.7757. Using the transition matrix one can compute the average waiting time. In particular in this case it depends on the starting configuration, in particular from 0000 0000 it is 14.79873, while it attains a minimum from 0011 0011 or 0111 0111 where anyhow the 1 is saved. In this case the average time is 10.98887.

4. Staircase of critical configurations and overtaking

We come now to a more general case of partition of the Bak-Sneppen process. The simplest case that shows the main features requires 3 subsets of 3 knots each, and we consider a ternary set of values, say 0,1,2 . This corresponds to 9 knots and displacements of 3-6. A simplified analysis is the following: we attach label 0 to any subset in which the minimum is 0 (not regarding their number), label 1 to any subset in which the minimum is 1 (not regarding their number), and label 2 to the set 222.
In view of the simplification we cannot keep track of the single subsets, but we are able to enumerate them, getting the states 000, 100, 110, 111, 200, 210, 211, 220, 221, 222.
Table 2 shows the transition matrix.

A \DA	000	100	110	111	200	210	211	220	221	222
000	17/27									
100	9/27	17/27								
110		9/27	17/27	17/27						
111			9/27	9/27						
200	1/27				17/27					
210		1/27			9/27	17/27	17/27			
211			1/27	1/27		9/27	9/27			
220					1/27			17/27	17/27	17/27
221						1/27	1/27	9/27	9/27	9/27
222								1/27	1/27	1/27

Table 2: Transitions in a 9 knot tripartite Bak Sneppen set with 3 values.

There are two transient groups (110, 111) and (210, 211) and one persistent group (220, 221, 222). The final average is thus 1,667, since two subsets have the form 222, and the third one is random on the three values 012. The average times are 54 from 000, 100, 110,111 to the persistent group, 27 from the groups 200, 210, 211 to the persistent group.

One interesting fact is the transition from 100 to 210 and from 110 to 211. The subset that is changed into the optimal label 2 is not one labelled with 1, but one that is labelled with 0, that is its minimum must be lower than that of the best subset. In Bak-Sneppen models, as sometimes also in real life, the principle is "*quieta non movere*" (Let quiet things stay). For example in a situation A=122, B=112, C=011, it is impossible that the top ones reach the state 222, while it is possible, even if unlikely, that this state is reached by C. The idea is that movement requires dissatisfaction, while further movements are caused by some form of nested neighborhood with unsatisfied people that can in turn generate new dissatisfaction: We can remind the "happiness paradox" of Easterlin ([11], [12]).
In our numerical simulations we experimented mainly the case of four values since for lower number of values the standard Bak Sneppen distribution is anyhow too concentrated on the top value, hence the different steps are confounded with the casual fluctuation. As soon as the dimension of the subsets is increased it becomes more and more difficult to reach the stable steps; for example already subsets of 18 knots (for a total of 36) very often require more than 100,000 iterations in order to reach the persistent group. Remark that the transition, when it happens, is very similar to an "avalanche". In some cases the process is not complete, hence it can be reverted, but finally it happens that the threshold is reached. Increasing the number of values multiple thresholds arise; the lower levels can be easily overcome, while the top levels can prove to be almost unreachable. This is for example the case of ten values and forty knots, in which the top level has never been reached in ten simulations of 1,000,000 iterations. In this case realistic upper limit on the total number of knots seems to be reduced to 12.

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