Nuclear Arms Race and Environment

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Abstract

This paper introduces a new factor, environment, into a nuclear arms race model. In this model, nuclear weapons produce larger defense power compared with conventional arms, but hurt the environment meanwhile. In the global welfare maximum level, both conventional and nuclear weapons budget are zero. However, the competitive equilibrium may not achieve the optimum. I give the condition to jump out of the prisoner's dilemma.

Keywords. nuclear arms race, environment, prisoner's dilemma, endogenous Richardson model.

I. Introduction

In 1945, the U.S. threw two atomic bombs at Hiroshima and Nagasaki, which opened the Pandora's Box, that is, began the nuclear age. In 1947, the first atomic bomb of the Soviet Union successfully exploded, which marked the official start of the US-Soviet nuclear arms race that ended with the collapse of the Soviet Union in 1991. Nuclear power accumulated in the protracted nuclear arms race could destroy the earth for several times, which has endangered the human race seriously. Nuclear weapons have been the Sword of Damocles above all over the world. Albert Einstein, the most important promoter of the Manhattan Project devoted the rest of his life being a champion of nuclear disarmament campaign. The campaign have swept the globe in the past decades. Now people want more. The international Global Zero movement, which fight for eliminating all nuclear weapons finally, has been supported widely. Moreover, the movement produced at least one Nobel Peace Prize Laureate.[1]

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[1]Barack Obama won the price at 2010.

This paper focuses on following three key questions about nuclear arms race. First, when will nuclear weapons converge to a steady state, or grow infinitely? Second, what is the optimal level of nuclear armaments holding under global welfare maximization? Third, how far can we go in the nuclear disarmament movement?

The developing of the game theory is almost simultaneous with and the Cold War. Therefore, the topic of my paper has always been attracting the attention of many economists. For the first question, Richardson (1939,1960) established the first arms race model using two exogenous differential equations, which has become a benchmark for analyzing arms race.Brito (1972) developed the endogenous Richardson model for the first time, and discussed the existence and stability of equilibrium. The second question is a well-discussed class issue in the literature. Brito and Intriligator (1995) proposed a summary of the theoretical works on the first two problems. The third problem remains a open question. Brzoska (2007) discussed on the effects of disarmament conversion after the Cold War. Another foundation work in this field is the arms race model based on second-strike deterrence proposed by Intriligator (1975), which has been widely used in the nuclear game. To simplify the analysis, this paper ignores the production process, and will not discuss the arms race and the mutual influence of economic growth. Readers can refer to the articles by Zou (1995), Gong and Zou (2003) for this topic if interested.

The paper constructs a bilateral military game model for former questions. The environmental damages caused by nuclear weapons is introduced into the utility function of the model as a negative term. Home country's military power will influence the enemy country's security. A country needs to trade off defense security, daily consumption and environment in order to determine the optimal level of military. In this model, nuclear weapons can built the defence power more effectively, but pollute the environment potentially. The government should allow for the two effects of its nuclear weapons budget.

In this framework, both nuclear and conventional weapons are zero at global welfare optimal level, but the Nash equilibrium of great power nuclear game leads to a different results. Here is a prisoner's dilemma in each period. In dynamic view, the paper builds an endogenous Richardson model, and gives the condition to guarantee the existence of a steady state. In the process of convergence towards the steady state, both substitution and complementary effect are possible between conventional weapons and nuclear weapons, yet there always exists crowding-out effect between daily consumption and nuclear armaments budget. At last, I discuss the condition that serves to resolve the prisoner's dilemma.

The paper is organized as follows. Section II will establish the model, and section III embodies all the functional form. Section IV deduces the social optimum through a central planner model. Section V calculates the competitive equilibrium, and compares it with the global optimal level. Section VI discusses the dynamic characteristics of the arms race. Section VII gives the condition that serves to bring it back to the global optimum. In the end, Part VIII will summarize the main conclusions of the paper.

II. Model setup

There are two countries in the model, ij, and $i, j \in \{1, 2\}$ and $i \neq j$. It is true that more than two nuclear states exist in the real world. However, the two countries assumption is not special, due to nuclear arms race always happens between two countries well matched in strength, such as the Cold War between the United States and the Soviet Union, or the arms race between India and Pakistan. Given t period of national wealth $w_{i,t}$, the government of country i select domestic consumption$c_{i,t}$, conventional weapons budget $m^c_{i,t}$ and nuclear weapons budget $m^n_{i,t}$ to maximize their utility $U_{i,t}$. $U_{i,t}$ is a function of consumption $c_{i,t}$; national defense security $D_i(m_{i,t}, m_{j,t})$ and environment $S_i(m^n_{i,t}, m^n_{j,t})$, which form is as below:

$$\mathbb{E}U_{i,t}[(c_{i,t}, D_i(m_{i,t}, m_{j,t}), S_i(m^n_{i,t}, m^n_{j,t})]. \tag{1}$$

I introduce defense security into utility function, because governments can use the weapons protecting daily consumptions from the threat of robbery, as Grossman and Kim (1995) hold. I impose following assumptions on the two countries' preference:

$$\begin{gathered}\frac{\partial U_{i,t}}{\partial c_{i,t}} \geq 0, \quad \frac{\partial^2 U_{i,t}}{\partial c^2_{i,t}} \leq 0, \quad \frac{\partial U_{i,t}}{\partial D_i} \geq 0, \quad \frac{\partial^2 U_{i,t}}{\partial D^2_i} \leq 0, \quad \frac{\partial U_{i,t}}{\partial S_i} \geq 0, \quad \frac{\partial^2 U_{i,t}}{\partial S^2_i} \leq 0, \\ \frac{\partial^2 U_{i,t}}{\partial c_{i,t}\partial D_i} \geq 0, \quad \frac{\partial^2 U_{i,t}}{\partial D_i \partial S^2_i} \geq 0, \quad \frac{\partial^2 U_{i,t}}{\partial S_i \partial c_{i,t}} \geq 0.\end{gathered} \tag{2}$$

National defense security $D_i(m_{i,t}, m_{j,t})$ is positively related to their military strength $m_{i,t}$, while negatively is related to the military strength of their enemy $m_{j,t}$, $m_{i,t}$ marginal utility descends.

$$\frac{\partial D_i}{\partial m_{i,t}} \geq 0, \quad \frac{\partial^2 D_i}{\partial m^2_{i,t}} \leq 0, \quad \frac{\partial D_i}{\partial m_{j,t}} \leq 0, \tag{3}$$

Military forces m_i are made up of nuclear and conventional weapons, which budgets are m_c and m_n. m_i is a function of m_n and m_c,

$$m_{i,t} = g_i(m^c_{i,t}, m^n_{i,t}). \tag{4}$$

Suppose that

$$\frac{\partial g_i}{\partial m^n_{i,t}} \geq \frac{\partial g_i}{\partial m^c_{i,t}} \geq 0. \tag{5}$$

Then, comparing to conventional weapon, the same budget used in nuclear weapons will generate greater deterrent power.

Different with conventional weapons, that nuclear weapons could produce catastrophic damage on the country's own or its enemy's environment. Therefore, the article assumes environment of country i, S_i as a function of its own nuclear weapons $m^n_{i,t}$ and its enemy $m^n_{j,t}$.

As analyzed above, following assumptions are imposed on the derivative of $S_i(m^n_{i,t}, m^n_{j,t})$

$$\frac{\partial \mathbb{E} S_i}{\partial m^n_{i,t}} \leq 0, \quad \frac{\partial \mathbb{E} S_i}{\partial m^n_{j,t}} \leq 0. \tag{6}$$

The expectation operator $\mathbb{E}$ is because of that the malignant environmental events triggered by the nuclear leakage are random events, which are always difficult to control, such as Chernobyl and the Fukushima nuclear accident. Although political decision can determine nuclear wars, there still have uncertainty when they product the weapons. Precisely because of these uncertainties, the utility function of formula (1) also contains the desired operator. In later account case, the various effects of its own and enemy's nuclear weapons on the environment will have a specific considerations, the uncertainty will be treated accordingly.

The current wealth of country i, $w_{i,t}$, can be used for consumption of daily consumer goods $c_{i,t}$, it can also use budget $m^c_{i,t}$ to buy conventional weapons, or to buy the nuclear weapons by using budget $m^n_{i,t}$. The right end of the budget constraints shown as equation (7).

$$w_{i,t} = c_{i,t} + m^c_{i,t} + m^n_{i,t}. \tag{7}$$

In this article, the daily consumer goods are used for valuating wealth and weaponry. The effect of growth to arms race and the arms race for growth are very worthy of attention, but this is not the goal of this study. Thus, production process is not considered in this paper, the outputs $Y_{i,t}$, $Y_{j,t}$. Another source of wealth is from legacy of last period nuclear weapons. Different from conventional weapons and daily consumer goods, nuclear weapons would generally have a longer tenure of use. This paper assumes that nuclear weapons as durable, period t of nuclear weapons will be retained to the next phase of in the form of depreciation durable goods, the depreciation rate as δ that is $(1-\delta)m^n_{i,t}$part of period t's nuclear weapons budget will still be used in period $t+1$. The model constructs the dynamic process of wealth accumulation by this way. In order to simplify the analysis dimensions, assuming the daily consumptions c_i and conventional weapons $m_{c,i}$are both non-durable. This assumption will not affect the results. The left hand of the budget constraint shown as formula (8),

$$w_{i,t} = Y_{i,t} + (1-\delta)m^n_{i,t-1}. \tag{8}$$

If each country has different government in different period, each government only concerned about the optimization problem within the single period. Here is the myopic government assumption, which is commonly used in political economics. As dynamic game structure on control variables is difficult to clearly portray in the standard dynamic macroeconomics models (Brito and Intriligator, 1995), this assumption is always used to simplify the analysis the traditional model of the arms race, such as Brito (1972) assumes that the decisions are made based solely on the behavior of the current. This paper follows the literature; but I introduce a different form of the assumption.

As far as concerned, country i needs to solve the following optimization problem in period t.

$$\begin{aligned} \max_{c_{i,t},m_{i,t},m^c_{i,t},m^n_{i,t}} \quad & \mathbb{E}U_{i,t}[(c_{i,t}, D_i(m_{i,t}, m_{j,t}), s_i(m^n_{i,t}, m^n_{j,t})], \\ s.t. \quad & Y_{i,t} + (1-\delta)m^n_{i,t-1} = c_{i,t} + m^c_{i,t} + m^n_{i,t}, \\ & m_{i,t} = g_i(m^c_{i,t}, m^n_{i,t}), \\ & m^c_{i,t}, \; m^n_{i,t} \geq 0. \end{aligned} \tag{9}$$

At first, this paper calculates a central planner problem to find global optimum as a foundation of later analysis. Once a country is doing decision-making, the nuclear power and overall military forces of another country is seen as given. Simultaneous solving the two countries' reaction function, which is to get the country's nuclear Nash equilibrium. These calculations will be given in the specific examples in Section III.

III. Specialization

As in van der Ploeg and Zeeuw(1990), Zou(1995), I assume that the utility function is separately additive forms. Each item is constant absolute risk aversion form (CARA). Then utility function in function (1) is changed into the specific form in function (10,

$$U_{i,t} = -e^{-\rho c_{i,t}} - \theta_m e^{-\rho D_i(m_{i,t}, m_{j,t})} - \theta_s e^{-\rho S_i(m^c_{i,t}, m^n_{j,t})}, \tag{10}$$

where ρ is coefficient of relative risk aversion. Suppose that ρ's value remains the same for three different utility sources.

Suppose that national defense D_i of country i is a linear function of military force m_i of native country and military force $m_j$of enemy country,

$$D_i(m_{i,t}, m_{j,t}) = m_{i,t} - K m_{j,t}. \tag{11}$$

Let

$$1 < K < 1 + \min\left\{ \frac{A + A^*}{A + A^* + N + 1 - \delta}, \; \frac{A + A^*}{1 + N} \right\}, \tag{12}$$

assuming that deterrent caused by the enemy unit increasing weapons is greater than the security brought by the native country increasing weapons in the same extent. An extreme example is who have the first atomic bomb, although having nuclear weapons can bring great advantages of national defense for the native country, such advantages cannot offset the enormous fear when the enemy state has nuclear weapons. To simplify the analysis, I assume that upper limit exists in value K, which is shown in formula (12), which meets the requirements of formula (3).

Suppose that the function form of the environment S_i is as function (13),

$$S_i(m^n_{i,t}, m^n_{j,t}) = s_0 - s_l \tilde{\varepsilon_l}(m^n_{i,t}) - s_w \tilde{\varepsilon_j}(\overline{m}^n_{i,t}) - s_d \tilde{\varepsilon_i}(\overline{m}^n_{j,t}), \tag{13}$$

where s_0 is the initial environment level, s_l is the environment damage caused by the outbreak of nuclear leaks, $\tilde{\varepsilon_l}$is the possibility of nuclear leaks in country i, which is related with the nuclear weapons budget $m_{i,t}^n$. s_w is the environmental disaster caused by the nuclear strike from country j on country i, $\tilde{\varepsilon_j}$ is the possibility of nuclear strike from country j to country i, which is related with second-strike capacity $\overline{m}_{i,t}^n$ of country i. The definition of $\overline{m}_{i,t}^n$ is shown as function (14),

$$\overline{m}_{i,t}^n = m_{i,t}^n - Bm_{j,t}^n, \tag{14}$$

where B represents the striking capability of the nuclear war launching country, which can destroy the nuclear weapons of enemy state. Intriligator (1975) offered a classic analysis of possibility of broke out of nuclear war influenced by nuclear deterrence. To simplify the analysis, I assume that the second-strike capacity of country i negatively correlates with the possibility of a nuclear war launched by country j. s_d in the fourth item is the retaliatory of country j on country i for nuclear strike, $\tilde{\varepsilon_j}$ is the possibility of nuclear strike of country i on country j, the mean value and the second-strike capacity, , which mean value is negatively correlated with the second-strike capacity $\overline{m}_{j,t}^n$ of country j. Suppose that

$$\tilde{\varepsilon_l} \sim N(m_{i,t}^n, \sigma^2) \quad \tilde{\varepsilon_j} \sim N(M - \overline{m}_{i,t}^n, \sigma^2) \quad \tilde{\varepsilon_i} \sim N(M - \overline{m}_{j,t}^n, \sigma^2), \tag{15}$$

Given $m_{i,t}^n$, $m_{j,t}^n$, $\tilde{\varepsilon_l}$, $\tilde{\varepsilon_j}$, $\tilde{\varepsilon_i}$ are independent of each other. Let

$$s_0 = \frac{1}{2}\rho\left(s_l^2 + s_w^2 + s_d^2\right)\sigma^2 + s_w M + s_d M. \tag{16}$$

The original uncertain issue is equivalent of the following certainty problem,

$$\mathbb{E}\left\{-\theta_s e^{-\rho s(m_{i,t}^n, m_{j,t}^n)}\right\} = -\theta_s e^{-\rho(-Am_{i,t}^n - A^* m_{j,t}^n)}, \tag{17}$$

where A is the disaster to the environment of native country brought by the nuclear weapons of native country, A^* is the disaster to the environment of native country brought by the nuclear weapons of other countries, both definitions are shown as formula (18),

$$A = s_l - s_w + s_d, \quad A^* = Bs_w - s_d. \tag{18}$$

Let

$$A, \quad A^* \geq 0, \tag{19}$$

Assumption in formula (6) can be satisfied.

Specified function 4 as

$$m_{i,t} = m_{i,t}^c + (1+N)m_{i,t}^n, \tag{20}$$

where

$$N \geq 0. \tag{21}$$

It is easy to demonstrate that formula (20) meets the assumptions in formula (5).

Now, the optimization issue of country i in the period t can be transformed into function (22),

$$
\begin{aligned}
\max_{c_{i,t},m_{i,t},m^c_{i,t},m^n_{i,t}} \quad & -e^{-\rho c_{i,t}} - \theta_m e^{-\rho(m_{i,t}-Km_{j,t})} - \theta_s e^{-\rho(-Am^n_{i,t}-A^* m^n_{j,t})}, \\
s.t. \quad & Y_{i,t} + (1-\delta)m^n_{i,t-1} = c_{i,t} + m^c_{i,t} + m^n_{i,t}, \\
& m_{i,t} = m^c_{i,t} + (N+1)m^n_{i,t}, \\
& m^c_{i,t},\ m^n_{i,t} \geq 0.
\end{aligned}
\tag{22}
$$

In this section, we have assumed that the utility function, deterrent capability and the degree of vulnerability of the environment of the two countries are exactly the same. The goal of the article is not to discuss the impact of economic growth. To simplify the analysis, further assume that $Y_{i,t} = Y_{j,t} \equiv Y$, $m^n_{i,0} = m^n_{j,0}$ in later calculations. As a result, the two countries are completely symmetrical in the game. In the real world, both parties in the nuclear arms race have the close offensive strength, which provides the conclusions of the text with greater applicability.

IV. Global welfare maximum

This section assumes a central planner optimizing the overall welfare function of the world in order to solve the global optimum. The two countries are completely symmetrical in the model; assuming that the weights of the utilities of the two countries are the same in the social welfare function. According to the assumption in (2), the utility function meets concavity. Hence, the daily consumption, the level of military deterrence and nuclear weapons budget of the two countries are the same at global optimum. Thus $m_{j,t} = m_{i,t}$, $m^n_{j,t} = m^n_{i,t}$. Let $w_{i,t} = w_{j,t} \equiv w_t$, according to (22), the global optimum problem of the central planner about the country i can be simplified as (22),

$$
\begin{aligned}
\max_{c_i,m_i,m^c_{i,t},m^n_{i,t}} \quad & -e^{-\rho c_i} - \theta_m e^{-\rho(1-K)m_{i,t}} - \theta_s e^{-\rho(-A-A^*)m^n_{i,t}}, \\
s.t. \quad & w_t = c_{i,t} + m^c_{i,t} + m^n_{i,t}, \\
& m_{i,t} = m^c_{i,t} + (N+1)m^n_{i,t}, \\
& m^c_{i,t},\ m^n_{i,t} \geq 0.
\end{aligned}
\tag{23}
$$

Proposition 1 gives the optimum solution to the problem of the central planner.

Proposition 1 *Conventional and nuclear weapons budget are zero in both the two countries when maximizing global welfare.*

Proof: Substitute the budget constraints in (23) into the objective function, and the optimization problem of the central planners becomes

$$\begin{aligned} \max_{m_{i,t},m^n_{i,t}} \quad & -e^{-\rho(w_t-m_{i,t}+m^n_{i,t})}-\theta_m e^{-\rho(1-K)m_{i,t}}-\theta_s e^{-\rho(-A-A^*)m^n_{i,t}}, \\ s.t. \quad & m_{i,t}-m^n_{i,t},\ m^n_{i,t}\geq 0. \end{aligned} \tag{24}$$

According to assumptions in (12), $K>1$. Thus,

$$\frac{\partial U_i}{\partial m_i}=-\rho e^{-\rho(w_t-m_i+m_{n,i})}-\theta_m\rho(K-1)e^{\rho(K-1)m_i}<0, \tag{25}$$

The optimal value of the defense deterrence m_i is the lower bound 0. And because

$$m^c_{i,t}=m_{i,t}-m^n_{i,t}\geq 0 \tag{26}$$

$m^n_{i,t}$ is zero too. ■

As is discussed earlier, as the two countries are completely symmetric, their military levels are surely identical at global optimum. The increase of their deterrence level will only simultaneously reduce their utility of the national defense and security. In turn, if the military budget is cut down, the wealth saved can be used to increase daily consumption, which can thus enhance the utility. Thus, when the global overall welfare is the maximum, the optimal military forces must be zero, as is described in Proposition 1. At this point, the two countries' nuclear weapon budgets are also zero, and their living environment can be exempt from the threat of nuclear disaster.

V. Nash equilibrium

This part will solve the real game of great powers. In period t, given the level of enemy military deterrence $m_{j,t}$ and nuclear weapon holdings $m^n_{j,t}$, country i's reaction function is solved as is shown in (27):

$$(N+2A)\begin{bmatrix} m_{i,t} \\ m^n_{i,t}\end{bmatrix}=\begin{bmatrix}(N+A)K & -NA^* \\ K & -2A^*\end{bmatrix}\begin{bmatrix} m_{j,t} \\ m^n_{j,t}\end{bmatrix}+\begin{bmatrix} A \\ -1\end{bmatrix}w_{i,t}+\frac{1}{\rho}\begin{bmatrix} N+A & -N \\ 1 & -2\end{bmatrix}\begin{bmatrix}\ln\theta_m \\ \ln\theta_s+\ln A-\ln N\end{bmatrix}. \tag{27}$$

Based on the assumptions in (19) and (21), $N+2A>0$. when the military deterrence of the hostile country, $m_{j,t}$, rises, the domestic nuclear weapon budget $m^n_{i,t}$ and military deterrence $m_{i,t}$ will increase both. Another interesting result is that when the military strength of the hostile country remains unchanged while the nuclear weapon budget $m^n_{j,t}$ rises, the domestic nuclear weapon budget and deterrent capability will reduce. This is because when the hostile country increases its nuclear deterrence, its expected environmental damage extent will also increase to the extent of A^*. The environmental marginal utility diminishes, which means that the expected environmental level of the home country is no longer bear the potential destruction of its existing nuclear weapon on the environment. Therefore, it will choose to reduce the nuclear weapon budget $m^n_{i,t}$. On the other side, the reduction of nuclear deterrence can not be completely replaced with commeasurable conventional weapon deterrent, and therefore the overall deterrence, $m_{i,t}$, will descend.

The coefficient $A \geq 0$ in front of w_i , which means the military forces will increase with the increase of its wealth level. However as for $m_{n,i}$, the coefficient is negative in front of w_i, so the increase of wealth level will cause reduction of the nuclear weapon budget. This is because that the nuclear weapons budget is actually a balance between the accumulated efficiency of military deterrence and the potential environmental threat: when the economic strength is stronger, the military accumulation efficiency becomes relatively less important, and environment becomes a problem that the decision-makers concern more about.

With the increase in the degree of preference for military security θ_m, $m_{i,t}$ and $m^n_{i,t}$ will both decrease. With the increase of the emphasis on environment θ_s, the nuclear weapon budget $m^n_{i,t}$ will naturally decrease, and as the conventional weapons can not fill the resulting decrease, $m^n_{i,t}$ will also be decrease.

Concerning the environment, the influence on the environment damage of the hostile country is similar with that on the nuclear weapon budget. When A^* increases, its nuclear weapon budget $m^n_{i,t}$ and military deterrence $m_{i,t}$ will both increase. Another naturally result is $m^n_{i,t}$ decreases with the increase of A.

According to the model setup, the game structures of the two countries are completely symmetrical. Then, the Nash equilibrium of the game is also completely symmetrical. Let $w_{i,t} = w_{j,t} \equiv w_t$, establish a simultaneous response function of the two countries, and determine the Nash equilibrium solution, as is shown in (28),

$$\begin{bmatrix} m_{i,t} \\ m^n_{i,t} \end{bmatrix} = \begin{bmatrix} m_{j,t} \\ m^n_{j,t} \end{bmatrix} = \frac{1}{\Gamma} \begin{bmatrix} A + A^* \\ K - 1 \end{bmatrix} w_t + \frac{1}{\rho\Gamma} \begin{bmatrix} A + A^* + N & -N \\ 1 & K - 2 \end{bmatrix} \begin{bmatrix} \ln\theta_m \\ \ln\theta_s + \ln A - \ln N \end{bmatrix}, \tag{28}$$

where

$$\Gamma = (A + A^*)(2 - K) - N(K - 1) > 0. \tag{29}$$

Here, the conventional weapons and daily consumptions of the two countries are as shown in formula (30),

$$\begin{aligned} \begin{bmatrix} m^c_{i,t} \\ c_{i,t} \end{bmatrix} = \begin{bmatrix} m^c_{j,t} \\ c_{j,t} \end{bmatrix} &= \frac{1}{\Gamma} \begin{bmatrix} A + A^* - (N+1)(K-1) \\ -(A + A^*)(K - 1) \end{bmatrix} w_t \\ &+ \frac{1}{\rho\Gamma} \begin{bmatrix} A + A^* - 1 & 1 - (K-1)(A + A^*) \\ -A - A^* & (K-1)(A + A^* - 1) \end{bmatrix} \begin{bmatrix} \ln\theta_m \\ \ln\theta_s + \ln A - \ln N \end{bmatrix}. \end{aligned} \tag{30}$$

Comparing with formula(28) and Proposition 1, it can be found that the equilibrium holdings of $m^n_{i,t}$, $m_{i,t}$, $m^n_{j,t}$, and $m_{j,t}$are different with those at global optimum, which Proposition 2 summarizes.

Proposition 2 *In the Nash equilibrium, conventional and nuclear weapons holdings of the two countries are not necessarily the same as global welfare optimum. Here is a prisoner's dilemma.*

VI. Dynamic system

According to formula (28), the accumulation equation of nuclear weapons $m_{n,t}$ is as shown in formula (31),

$$m_t^n = \left[\frac{K-1}{\Gamma}(1-\delta)\right] m_{t-1}^n + \left[\frac{K-1}{\Gamma}\overline{Y} + \frac{1}{\Gamma\rho}\ln\theta_m - (2-K)\frac{1}{\Gamma\rho}\ln\frac{A\theta_s}{N}\right]. \tag{31}$$

This in fact is the one-dimensional differential version of the Richardson Model. According to the dynamic system theory, if

$$\left|\frac{1}{\Gamma}(K-1)(1-\delta)\right| = \left|\frac{1-\delta}{(A+A^*)\frac{2-K}{K-1} - N}\right| < 1, \tag{32}$$

then there is a stable equilibrium in this nuclear arms race. Or else the nuclear arms race will not converge. The lemma 3 specifically gives the stability condition obtained from formula (32).

Proposition 3 *If the threat on the environment caused by nuclear weapons,*

$$A + A^* > (N + 1 - \delta)\frac{K-1}{2-K}, \tag{33}$$

the arms race will converge to a steady state. Or else, if the deterrent power of nuclear weapons,

$$N > (A + A^*)\frac{2-K}{K-1} + 1 - \delta, \tag{34}$$

the steady state will also exist.

This is because when the environmental threat $A + A^*$ is large enough, the environmental cost for adding one unit nuclear weapon will be too high for both the two countries. Then, the arms race tends to converge naturally. When the deterrent power N is large enough, a country can achieve deterrence purposes with only a small amount of nuclear weapons. So there is no need to pay the environmental costs for excess nuclear weapons. Then, the arms race also tends to converge. Only when the deterrence efficiency of nuclear weapons is not large enough, and the environmental threats it caused is small enough can the nuclear arms race no longer to converge to a stable equilibrium. The nuclear weapons budget m_n^* in the steady state is as shown in formula (35),

$$m_n^* = \frac{1}{\Theta}(K-1)\overline{Y} + \frac{1}{\Theta\rho}\ln\theta_m - (2-K)\frac{1}{\Theta\rho}\ln\frac{A\theta_s}{N}, \tag{35}$$

where

$$\Theta = (2-K)(A+A^*) - (K-1)(N+1-\delta) > 0. \tag{36}$$

The dynamic path of conventional weapons and daily consumptions are as shown in formula (37),

$$\begin{bmatrix} m_t^c \\ c_t \end{bmatrix} = \frac{1}{\Gamma} \begin{bmatrix} (A+A^*)(1-\delta)-(N+1)(K-1)(1-\delta) \\ -(A+A^*)(K-1)(1-\delta) \end{bmatrix} m_{t-1}^n + \frac{1}{\Gamma} \begin{bmatrix} A+A^*-(N+1)(K-1) \\ -(A+A^*)(K-1) \end{bmatrix} \overline{Y}$$
$$+ \frac{1}{\rho\Gamma} \begin{bmatrix} A+A^*-1 & 1-(K-1)(A+A^*) \\ -A-A^* & (K-1)(A+A^*-1) \end{bmatrix} \begin{bmatrix} \ln\theta_m \\ \ln\theta_s + \ln A - \ln N \end{bmatrix}. \tag{37}$$

Compare the coefficient in front of m_{t-1}^n in the row of m_t^c in formula (37 and the coefficient in front of m_{t-1}^n in formula (31), if

$$A + A^* > (K-1)(1+N), \tag{38}$$

then the conventional weapons budget and nuclear weapons budget change at same direction, showing a complementary relationship. This is because that the environmental threats $A + A^*$ of nuclear weapons is too large compared with the deterrence efficiency N; and when adjusting the level of deterrence, the government should always find a balance between them. Otherwise, if

$$N > \frac{A+A^*}{K-1} - 1, \tag{39}$$

Then the conventional weapons budget and nuclear weapons budget change reversely and show an substitutional relationship. This is because the benefit N of nuclear weapons in deterrence efficiency has a distinct advantage, compared with the environmental threats $A + A^*$ at this time. If a country wants to adjust the level of deterrence, nuclear weapons are the best choice.

Compare the second row of (37) and the coefficient in front of m_{t-1}^n in formula (31), according to the assumption in (19), here is

$$-A - A^* \leq 0. \tag{40}$$

The nuclear weapons budget and daily consumptions output indicate reverse movements. Here is always the crowding-out effect of them.

In the steady state, the conventional weapons budget m_c^* and daily consumptions c^* are as shown in (41),

$$\begin{bmatrix} m_c^* \\ c^* \end{bmatrix} = \frac{1}{\Gamma\Theta}((1-\delta)(K-1)+\Theta) \begin{bmatrix} \Xi \\ -\Omega \end{bmatrix} \overline{Y}$$
$$+ \frac{1}{\rho\Gamma} \begin{bmatrix} \frac{\Xi}{\Theta}(1-\delta)+A+A^*-1 & \frac{\Xi}{\Theta}(2-K)(1-\delta)-\Omega+1 \\ -\frac{\Omega}{\Theta}(1-\delta)-A-A^* & -\frac{\Omega}{\Theta}(2-K)(1-\delta)-\Omega+K-1 \end{bmatrix} \begin{bmatrix} \ln\theta_m \\ \ln\theta_s + \ln A - \ln N \end{bmatrix}. \tag{41}$$

where

$$\Xi = A + A^* - (N+1)(K-1) > 0, \quad \Omega = (A+A^*)(K-1) > 0. \tag{42}$$

VII. Back to global welfare maximum

The above analysis discusses the interior point solution. On the other hand, the following will discuss the conditions of angular point solution, which matches the global optimum level. Proposition 4 gives this condition of the angular point solution.

Proposition 4 *If the arms race satisfies*

$$(K-1)\overline{Y}+\frac{1}{\rho}\ln\theta_m-(2-K)\frac{1}{\rho}\ln\frac{A\theta_s}{N}<0, \tag{43}$$

the nuclear weapons holding is zero in the steady state, which is the same as the global welfare maximum level. However, the conventional weapons still exist.

As shown in formula (35), if θ_s, the emphasis on the environment of both the two countries, or A, the environmental threat to the home country, is large enough, the level of nuclear arms can reach the global optimum. On the other hand, when the deterrent power of nuclear weapons N is small enough, or the emphasis θ_m on national defense and security is small enough, the efficiency advantage of nuclear weapons in defense deterrence can not match with the potential damage caused to the environment, and thus global zero nuclear may be reached. If the wealth level $\overline{Y}$ is very small, and nuclear arms race can not be sustained, it is also possible to get rid of the prisoner's dilemma. However, the conditions of proposition 4 cannot guarantee that the holdings of conventional weapons are also zero, which is different from the global optimum.

VIII. Conclusions

This paper established a great power military game mode which contains two military means, namely conventional and nuclear weapons. A country needs to distribute its wealth on daily consumption, conventional weapons and nuclear weapons, and obtain utility from daily consumption, military security and the environment. The nuclear weapons can be used as military deterrence more efficiently and seek to national defense and security, and the budget saved can also be spent in consumption to improve the utility; however in the other hand, the nuclear arms race will bring the world into the threat of nuclear disaster, which is also a potential environmental threat that requires the considerations of the policy makers. Unlike the conventional weapons, the nuclear weapons, as the durable goods, can accumulate in the periods. On the basis of the above-mentioned structure, this thesis established an endogenous one-dimensional Richardson nuclear arms race model with a symmetrical game.

In the model, the number of conventional and nuclear weapons is zero at global optimum. However, this is unable to be achieved in the great power games, and there is a Prisoner's Dilemma in the nuclear game. When people's the attention on the environment, or the potential environmental threat of nuclear weapons on the country or its hostile country reaches a certain extent, the equilibrium holdings of nuclear weapons is likely to be the same with those at maximum global welfare, zero. When the potential threat caused by nuclear weapons on the environment is sufficiently large, the arms race will converge to a stable equilibrium. During the process of convergence to equilibrium level, there is a crowding-out effect between daily consumption and armaments budget.

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