Risk Parity Portfolios with Risk Factors

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Abstract

Portfolio construction and risk budgeting are the focus of many studies by academics and practitioners. In particular, diversification has spawn much interest and has been defined very differently. In this paper, we analyze a method to achieve portfolio diversification based on the decomposition of the portfolio's risk into risk factor contributions. First, we expose the relationship between risk factor and asset contributions. Secondly, we formulate the diversification problem in terms of risk factors as an optimization program. Finally, we illustrate our methodology with some real life examples and backtests, which are: budgeting the risk of Fama-French equity factors, maximizing the diversification of an hedge fund portfolio and building a strategic asset allocation based on economic factors.

Keywords: risk parity, risk budgeting, factor model, ERC portfolio, diversification, concentration, Fama-French model, hedge fund allocation, strategic asset allocation.

JEL classification: G11, C58, C60.

1 Introduction

While Markowitz's insights on diversification live on, practical limitations to direct implementations of his original approach have recently lead to the rise of heuristic approaches. Approaches such as equally-weighted, minimum variance, most diversified portfolio, equally-weighted risk contributions, risk budgeting or diversified risk parity strategies have become attractive to academic and practitioners alike (see e.g. Meucci, 2007; Choueifaty and Coignard, 2008; Meucci, 2009; Maillard *et al.*, 2010; Bruder and Roncalli, 2012; Lohre *et al.*, 2012) for they provide elegant and systematic methodologies to tackle the construction of diversified portfolios.

While explicitly pursuing diversification, portfolios constructed with the methodologies cited above may well lead to solutions with unfortunately hidden risk concentration. Meucci (2009)'s work on diversification across principal component factors provided a clue to resolving this unfortunate problem by focusing on underlying risk factors. In this paper, we build on this idea and combine it with the risk-budgeting approach of Bruder and Roncalli

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(2012) to develop a risk-budgeting methodology focused on risk factors. When used with the objective of maximizing risk diversification, our approach is tantamount to diversifying across the 'true' sources of risk and often leads to a solution with equally-weighted risk factor contributions. Hence, we dubbed it the 'risk factor parity' approach.

This methodology is however more versatile. In this paper, we make explicit the relations between any risk factor and asset contributions to portfolio risk. And, in particular, we show that the equally-weighted risk factor contributions approach is equivalent to a risk-budgeting solution on the assets of the portfolio with a specific risk budgets profile. We also examine different optimization programs to obtain the desired outcome.

The plan of this paper is thus as follows. Section 2 provides a couple of motivating examples. In Section 3, we derive the relations between asset and risk factors contributions to overall risk, and provide some illustrations. In Section 4, we consider two types of portfolio construction methodologies, first by matching risk contributions and second by minimizing a concentration index. Finally, Section 5 provides three applications of our portfolio construction methodologies: one in the context of risk budgeting on equity risk factors, a second on portfolios of hedge funds, and finally an application to strategic allocation.

2 Motivations

2.1 On the importance of the asset universe

Let's consider a universe of 4 assets with equal volatilities and a uniform correlation ρ. In this case, the equally-weighted risk contribution (ERC) portfolio $x^{(4)}$ corresponds to the equally-weighted portfolio. Let's add to this universe a fifth asset with equal volatility. We assume that this asset is perfectly correlated with the fourth asset. In that case, the composition of the ERC portfolio $x^{(5)}$ depends on the value of the correlation coefficient ρ. For example, if the cross-correlation ρ between the first four assets is zero, the ERC portfolio's composition is given by $x_i^{(5)} = 22.65\%$ for the three first assets, while $x_i^{(5)} = 16.02\%$ for the two last assets.

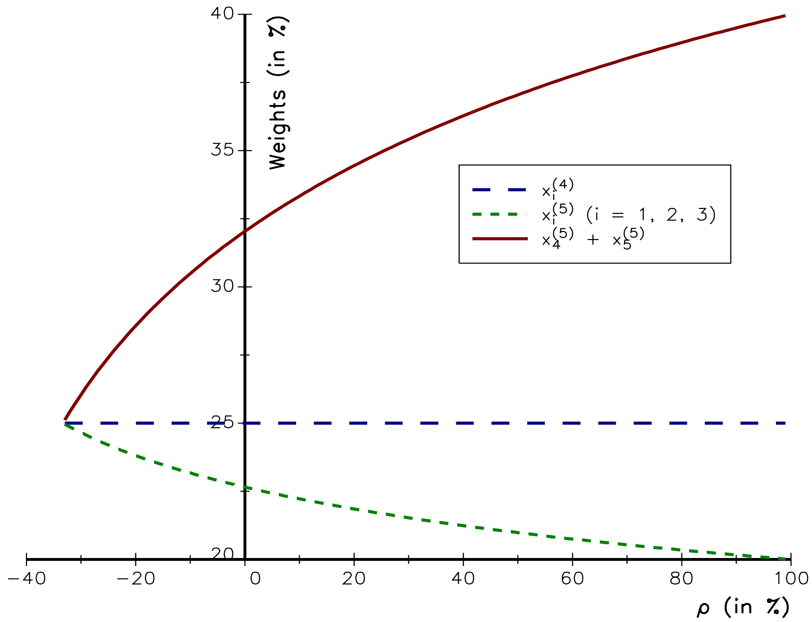
Such a solution could be confusing to a professional since in reality there are only four and not five assets in our universe[1]. A financial professional would expect the sum of the exposures to the fourth and fifth assets to equal the exposure to each of the first three assets, i.e $x_4^{(5)} + x_4^{(5)} = 25\%$. And, the weights of the assets four and five are therefore equal to the weight of the fourth asset in the 4-assets universe divided by $2 - x_5^{(5)} = x_4^{(5)} = x_4^{(4)}/2 = 12.5\%$. However, this solution is reached only when the correlation coefficient ρ is at its lowest value[2] as shown in Figure 1.

This example shows that the asset universe is an important factor when considering risk parity portfolios. Of course, this is a toy example, but similar issues arise in practice. Let's consider multi-asset classes universes. If the universe includes 5 equity indices and 5 bond indices, then the ERC portfolio will be well balanced between equity and bond in terms of risk. Conversely, if the universe includes 7 equity indices and 3 bond indices, the equity risk of the ERC portfolio represents then 70% of the portfolio's total risk, a solution very unbalanced between equity and bond risks.

[1] The fourth and fifth assets are in fact the same since $\rho_{4,5} = 1$. This problem is also known as the duplication invariance problem (Choueifaty *et al.*, 2011).

[2] ρ is then equal to -33.33%

Figure 1: Weights of the ERC portfolio with respect to ρ



2.2 Which risk would you like to diversify?

We consider a set of m primary assets $(\mathcal{A}'_1, \ldots, \mathcal{A}'_m)$ with a covariance matrix Ω. We now define n synthetic assets $(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ which are composed of the primary assets. We denote $W = (w_{i,j})$ the weight matrix such that $w_{i,j}$ is the weight of the primary asset $\mathcal{A}'_j$ in the synthetic asset $\mathcal{A}_i$. Indeed, the synthetic assets could be interpreted as portfolios of the primary assets. For example, $\mathcal{A}'_j$ may represent a stock whereas $\mathcal{A}_i$ may be an index. By construction, we have $\sum_{i=1}^{n} w_{i,j} = 1$. It comes that the covariance matrix of the synthetic assets Σ is equal to $W \Omega W^\top$.

We now consider a portfolio $x = (x_1, \ldots, x_n)$ defined with respect to the synthetic assets. The volatility of this portfolio is then $\sigma(x) = \sqrt{x^\top \Sigma x}$. We deduce that the risk contribution of the synthetic asset i is:

$$\mathcal{RC}(\mathcal{A}_i) = x_i \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}}$$

We could also defined the portfolio with respect to the primary assets. In this case, the composition is $y = (y_1, \ldots, y_m)$ where $y_j = \sum_{i=1}^{n} x_i w_{i,j}$. In a matrix form, we have $y = W^\top x$. In a similar way, we may compute the risk contribution of the primary assets j:

$$\mathcal{RC}(\mathcal{A}'_j) = y_j \frac{(\Omega y)_j}{\sqrt{y^\top \Omega y}}$$

Let's consider an example with 6 primary assets. The volatility of these assets is respectively 20%, 30%, 25%, 15%, 10% and 30%. We assume that the assets are not correlated.

We consider three equally-weighted synthetic assets with:

$$W = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 & & \\ & & 1/4 & 1/4 & 1/4 & 1/4 \\ & & 1/2 & 1/2 & & \end{pmatrix}$$

First, let's consider portfolio #1 with synthetic assets' weights of 36%, 38% and 26% respectively. The risk contributions of these positions to the volatility of portfolio #1 is provided in Table 1a. Notice that this portfolio is very close to the ERC portfolio. It is well-balanced in terms of risk with respect to the synthetic assets. However, if we analyze this portfolio in terms of primary assets (cf. Table 1b), about 80% of the risk of the portfolio is then concentrated on the third and fourth primary assets. We have here a paradoxical situation. Depending on the analysis, this portfolio is either well diversified or is a risk concentrated portfolio. Let's now consider portfolio #2 with synthetic weights 48%, 50% and 2%. The risk contributions for this portfolio are provided in Table 2. In this case, the portfolio is not very well balanced in terms of risk, because the two first assets represent more than 97% of the risk of the portfolio, whereas the risk contribution of the third asset is less than 3% (cf. Table 2a). This portfolio is thus far from the ERC portfolio. However, an analysis in terms of the primary assets, shows that this portfolio is less concentrated than the previous one (cf. Table 2b). The average risk contribution is 16.67%. Notice that primary assets which have a risk contribution above (resp. below) this level in the first portfolio have a risk contribution that decreases (resp. increases) in the second portfolio. In Figure 2, we have reported the Lorenz curve of the risk contributions of these two portfolios with respect to the primary and synthetic assets. We verify that the second portfolio is less concentrated in terms of primary risk.

Table 1: Risk decomposition of portfolio #1

(a) Along synthetic assets $\mathcal{A}_1, \ldots, \mathcal{A}_n$

$\sigma(x) = 10.19\%$	x_i	$\mathcal{MR}(\mathcal{A}_i)$	$\mathcal{RC}(\mathcal{A}_i)$	$\mathcal{RC}^\star(\mathcal{A}_i)$
$\mathcal{A}_1$	36.00%	9.44%	3.40%	33.33%
$\mathcal{A}_2$	38.00%	8.90%	3.38%	33.17%
$\mathcal{A}_3$	26.00%	13.13%	3.41%	33.50%

(b) Along primary assets $\mathcal{A}'_1, \ldots, \mathcal{A}'_m$

$\sigma(y) = 10.19\%$	y_i	$\mathcal{MR}(\mathcal{A}'_i)$	$\mathcal{RC}(\mathcal{A}'_i)$	$\mathcal{RC}^\star(\mathcal{A}'_i)$
$\mathcal{A}'_1$	9.00%	3.53%	0.32%	3.12%
$\mathcal{A}'_2$	9.00%	7.95%	0.72%	7.02%
$\mathcal{A}'_3$	31.50%	19.31%	6.08%	59.69%
$\mathcal{A}'_4$	31.50%	6.95%	2.19%	21.49%
$\mathcal{A}'_5$	9.50%	0.93%	0.09%	0.87%
$\mathcal{A}'_6$	9.50%	8.39%	0.80%	7.82%

This second example shows the importance of the risk factors. If we replace 'primary assets' by risk factors and 'synthetic assets' by 'investment assets', it highlights what the underlying risks we would like to manage are. For example, a commodity fund manager is perhaps less interested in the financial risks represented by individual commodities (like gas oil, cocoa, corn, gold, etc.) and more interested in economic risks like energy, inflation, etc.

Table 2: Decomposition of portfolio #2

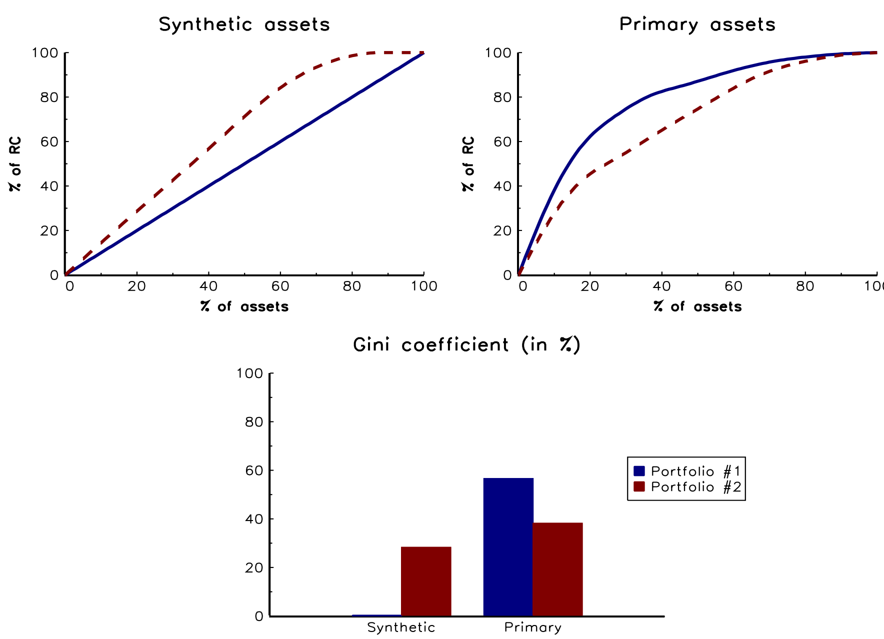
(a) Along synthetic assets $\mathcal{A}_1, \ldots, \mathcal{A}_n$

$\sigma(x) = 9.47\%$	x_i	$\mathcal{MR}(\mathcal{A}_i)$	$\mathcal{RC}(\mathcal{A}_i)$	$\mathcal{RC}^\star(\mathcal{A}_i)$
$\mathcal{A}_1$	48.00%	9.84%	4.73%	49.91%
$\mathcal{A}_2$	50.00%	9.03%	4.51%	47.67%
$\mathcal{A}_3$	2.00%	11.45%	0.23%	2.42%

(b) Along primary assets $\mathcal{A}'_1, \ldots, \mathcal{A}'_m$

$\sigma(y) = 9.47\%$	y_i	$\mathcal{MR}(\mathcal{A}'_i)$	$\mathcal{RC}(\mathcal{A}'_i)$	$\mathcal{RC}^\star(\mathcal{A}'_i)$
$\mathcal{A}'_1$	12.00%	5.07%	0.61%	6.43%
$\mathcal{A}'_2$	12.00%	11.41%	1.37%	14.46%
$\mathcal{A}'_3$	25.50%	16.84%	4.29%	45.35%
$\mathcal{A}'_4$	25.50%	6.06%	1.55%	16.33%
$\mathcal{A}'_5$	12.50%	1.32%	0.17%	1.74%
$\mathcal{A}'_6$	12.50%	11.88%	1.49%	15.69%

Figure 2: Lorenz curve of risk contributions



Remark 1. *This second example is also related to the first example*[3]*. In the first example, the right way to diversity is not to use the ERC portfolio, but different risk budgets* $b = (25\%, 25\%, 25\%, 12.5\%, 12.5\%)$*. In this case, we verify that we obtain the desired solution. The central question is then how to define properly these risk budgets.*

3 Computing the risk decomposition with risk factors

3.1 The linear factor model

We consider a set of n assets $\{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$ and a set of m risk factors $\{\mathcal{F}_1, \ldots, \mathcal{F}_m\}$. We denote by R_t the $(n \times 1)$ vector of asset returns at time step t and Σ its associated covariance matrix; and $\mathcal{F}_t$ the $(m \times 1)$ vector of factor returns at t and Ω its associated covariance matrix. We assume the following linear factor model:

$$R_t = A\mathcal{F}_t + \varepsilon_t \tag{1}$$

with $\mathcal{F}_t$ and ε_t two uncorrelated random vectors. ε_t is an i.i.d. $(n \times 1)$ centered random vector of variance D. A is the $(n \times m)$ 'loadings' matrix. Using (1), it is easy to deduce the following second-order relationship:

$$\Sigma = A\Omega A^\top + D$$

At the portfolio level, we denote the portfolio's asset exposures by the $(n \times 1)$ vector x and the portfolio's risk factors exposures by the $(m \times 1)$ vector y. These vectors are related through the P&L function Π_t of the portfolio at time t:

$$\Pi_t = x^\top R_t = x^\top A\mathcal{F}_t + x^\top \varepsilon_t = y^\top \mathcal{F}_t + \eta_t$$

with[4] $y = A^\top x$ and $\eta_t = x^\top \varepsilon_t$. Let's note $B = A^\top$ and B^+ the Moore-Penrose inverse of B. We have therefore

$$x = B^+ y + e$$

where $e = (I_n - B^+ B)\, x$ is a $(n \times 1)$ vector in the kernel of B.

3.2 Defining the marginal risk contribution of factors

Let's consider a convex risk measure $\mathcal{R}$. We have $\mathcal{R}(x) = \mathcal{R}(y, e)$. We notice that the idiosyncratic variable e that represents the specific risks of each asset position in the portfolio is a function of the portfolio composition x. Thus, the marginal risk contribution $\mathrm{d}\mathcal{R}(x) / \mathrm{d}x_i$ of the i^{th} asset exposure is given as a function of the marginal risk contribution $\mathrm{d}\mathcal{R}(y, e) / \mathrm{d}y$ of the factor exposures and the marginal risk contribution of the specific or idiosyncratic risk of asset i $\mathrm{d}\mathcal{R}(y, e) / \mathrm{d}e$ by[5]:

$$\frac{\partial \mathcal{R}(x)}{\partial x_i} = \left(\frac{\partial \mathcal{R}(y, e)}{\partial y} B\right)_i + \left(\frac{\partial \mathcal{R}(y, e)}{\partial e} \left(I_n - B^+ B\right)\right)_i$$

[3] In the first example, the problem comes from the fact that we have 5 assets, but only 4 factors.

[4] By definition, we have $\operatorname{cov}(\mathcal{F}_t, \eta_t) = \mathbf{0}$.

[5] We have:

$$\frac{\mathrm{d}\mathcal{R}(x)}{\mathrm{d}x} = \frac{\partial \mathcal{R}(y, e)}{\partial y} \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial \mathcal{R}(y, e)}{\partial e} \frac{\mathrm{d}e}{\mathrm{d}x}$$

This decomposition is not very convenient however since it decomposes the $(n \times 1)$ vector x into a $(m \times 1)$ vector y and a $(n \times 1)$ error vector e. Another route is to write the following decomposition (Meucci, 2007):

$$x = \begin{pmatrix} B^{+} & \tilde{B}^{+} \end{pmatrix} \begin{pmatrix} y \\ \tilde{y} \end{pmatrix} = \bar{B}^{\top} \bar{y} \tag{2}$$

where $\tilde{B}^{+}$ is any $n \times (n-m)$ matrix that spans the left nullspace of B^{+}. In this case, we could state the following theorem:

Theorem 1. *The marginal risk contribution of assets are related to the marginal risk contribution of factors in the following way:*

$$\frac{\partial \mathcal{R}(x)}{\partial x} = \frac{\partial \mathcal{R}(x)}{\partial y} B + \frac{\partial \mathcal{R}(x)}{\partial \tilde{y}} \tilde{B}$$

We deduce that the marginal risk contribution of the j^{th} factor exposure is given by:

$$\frac{\partial \mathcal{R}(x)}{\partial y_j} = \left(A^{+} \frac{\partial \mathcal{R}(x)}{\partial x^{\top}} \right)_j$$

For the additional factors, we have:

$$\frac{\partial \mathcal{R}(x)}{\partial \tilde{y}_j} = \left(\tilde{B} \frac{\partial \mathcal{R}(x)}{\partial x^{\top}} \right)_j$$

Proof. See Appendix A.2 page 29. □

3.3 Euler decomposition of the risk measure

It is easy to verify that for any convex risk measure that verifies the Euler decomposition $\mathcal{R} = \partial_x \mathcal{R}(x) \cdot x = \sum_{i=1}^{n} x_i \cdot \partial_{x_i} \mathcal{R}(x)$, the Euler decomposition is verified in the new coordinate system $(y, \tilde{y})$:

$$\mathcal{R}(x) = \frac{\partial \mathcal{R}(x)}{\partial y} y + \frac{\partial \mathcal{R}(x)}{\partial \tilde{y}} \tilde{y}$$

Let's note $\mathcal{RC}(\mathcal{F}_j) = y_j \cdot \partial_{y_j} \mathcal{R}(x)$ the risk contribution of the factor j with respect to the risk $\mathcal{R}$. Using Theorem 1, we find that the sum of the risk contributions of the factor exposures is simply expressed as:

$$\sum_{j=1}^{m} \mathcal{RC}(\mathcal{F}_j) = y^{\top} \frac{\partial \mathcal{R}(x)}{\partial y^{\top}} = x^{\top} (AA^{+}) \frac{\partial \mathcal{R}(x)}{\partial x^{\top}} \tag{3}$$

where $AA^{+} = (B^{+}B)^{\top}$ is the orthogonal projector onto the range of $B = A^{\top}$. We notice then that we do not retrieve all the risk measure if we only consider the risk contributions of the factors:

$$\sum_{j=1}^{m} \mathcal{RC}(\mathcal{F}_j) \leq \mathcal{R}(x)$$

The difference is due to the additional factors:

$$\begin{aligned} \mathcal{R}(x) - \sum_{j=1}^{m} \mathcal{RC}(\mathcal{F}_j) &= \sum_{j=1}^{n-m} \mathcal{RC}\left(\tilde{\mathcal{F}}_j\right) \\ &= x^{\top} (\tilde{B}^{\top} \tilde{B}) \frac{\partial \mathcal{R}(x)}{\partial x^{\top}} \\ &= x^{\top} (I_n - AA^{+}) \frac{\partial \mathcal{R}(x)}{\partial x^{\top}} \end{aligned}$$

If we consider the volatility of the portfolio, we obtain this result:

Theorem 2. *When the risk measure $\mathcal{R}(x)$ is the volatility of the portfolio $\sigma(x) = \sqrt{x^\top \Sigma x}$, the risk contribution of the j^{th} factor is:*

$$\mathcal{RC}(\mathcal{F}_j) = \frac{\left(A^\top x\right)_j \cdot \left(A^+ \Sigma x\right)_j}{\sigma(x)}$$

For the risk factors $\tilde{\mathcal{F}}_t$, the results become:

$$\mathcal{RC}\left(\tilde{\mathcal{F}}_j\right) = \frac{\left(\tilde{B}x\right)_j \cdot \left(\tilde{B}\Sigma x\right)_j}{\sigma(x)}$$

Proof. See Appendix A.3 page 30. □

Remark 2. *Theorems 1 and 2 are valid if the number of assets n is larger than the number of factors m. In the case where $n \leq m$, we obtain the same results, but the additional factors $\tilde{\mathcal{F}}_j$ vanish.*

Remark 3. *Using the previous results, we could define the risk contribution of the asset i with respect to the risk factors. We have:*

$$\mathcal{RC}(\mathcal{A}_i) = \frac{\partial\,\mathcal{R}(x)}{\partial\, y} B e_i \left(e_i^\top B^+ y + e_i^\top \tilde{B}^+ \tilde{y}\right) + \frac{\partial\,\mathcal{R}(x)}{\partial\, \tilde{y}} \tilde{B} e_i \left(e_i^\top B^+ y + e_i^\top \tilde{B}^+ \tilde{y}\right)$$

In the case $\tilde{y} = 0$, this formula reduces to:

$$\mathcal{RC}(\mathcal{A}_i) = \left(\frac{\partial\,\mathcal{R}(x)}{\partial\, y} B + \frac{\partial\,\mathcal{R}(x)}{\partial\, \tilde{y}} \tilde{B}\right) E_i B^+ y$$

where E_i is a null matrix of dimension $(n \times n)$ except for the entry (i,i) which takes the value one.

3.4 Some illustrations

3.4.1 The case $n \geq m$

Let us consider an example with 4 assets and 3 factors. The loadings matrix is:

$$A = \begin{pmatrix} 0.9 & 0 & 0.5 \\ 1.1 & 0.5 & 0 \\ 1.2 & 0.3 & 0.2 \\ 0.8 & 0.1 & 0.7 \end{pmatrix}$$

The three factors are uncorrelated and their volatilities are equal to 20%, 10% and 10%. We consider a diagonal matrix D with specific volatilities 10%, 15%, 10% and 15%. The corresponding correlation matrix of asset returns is (in %):

$$\rho = \begin{pmatrix} 100.0 & & & \\ 69.0 & 100.0 & & \\ 79.5 & 76.4 & 100.0 & \\ 66.2 & 57.2 & 66.3 & 100.0 \end{pmatrix}$$

and their volatilities are respectively 21.19%, 27.09%, 26.25% and 23.04%. The risk decomposition of the equally-weighted portfolio is given in Table 3. We notice that the equally-weighted portfolio produces a risk-balanced portfolio in terms of assets' risk contributions, but not in terms of factors' risk contributions. Indeed, the first factor represents more than 80% of the risk. In the next section, we present a method in order to obtain portfolios which are more balanced with respect to factors.

Table 3: Risk decomposition of the equally-weighted portfolio

(a) Along assets $\mathcal{A}_1, \ldots, \mathcal{A}_n$

$\sigma(x) = 21.40\%$	x_i	$\mathcal{MR}(\mathcal{A}_i)$	$\mathcal{RC}(\mathcal{A}_i)$	$\mathcal{RC}^\star(\mathcal{A}_i)$
$\mathcal{A}_1$	25.00%	18.81%	4.70%	21.97%
$\mathcal{A}_2$	25.00%	23.72%	5.93%	27.71%
$\mathcal{A}_3$	25.00%	24.24%	6.06%	28.32%
$\mathcal{A}_4$	25.00%	18.83%	4.71%	22.00%

(b) Along factors $\mathcal{F}_1, \ldots, \mathcal{F}_m$ and $\tilde{\mathcal{F}}_1, \ldots, \tilde{\mathcal{F}}_{n-m}$

$\sigma(y) = 27.45\%$	y_i	$\mathcal{MR}(\mathcal{F}_i)$	$\mathcal{RC}(\mathcal{F}_i)$	$\mathcal{RC}^\star(\mathcal{F}_i)$
$\mathcal{F}_1$	100.00%	17.22%	17.22%	80.49%
$\mathcal{F}_2$	22.50%	9.07%	2.04%	9.53%
$\mathcal{F}_3$	35.00%	6.06%	2.12%	9.91%
$\tilde{\mathcal{F}}_1$	2.75%	0.52%	0.01%	0.07%

The case $n \geq m$ corresponds also to our example in Section 2.1. We have 4 factors[6] with:

$$A = \begin{bmatrix} I_4 \\ \mathbf{e}_4^\top \end{bmatrix}$$

Ω is the covariance matrix of the first four assets while D is a matrix of zeros. If we use the framework developed in this section, we retrieve all the results. In particular, if we consider the ERC portfolio, we find that the three first factors have a contribution equal to 20% whereas the fourth factor has a contribution of 40%. In Table 4, we have reported the risk decomposition of the ERC portfolio with respect to the risk factors when the cross-correlation between the four first factors is equal to 0%. If we use another value for the correlation, results are different in terms of factor' weights and risk contributions. But the relative risk contributions $\mathcal{RC}^\star(\mathcal{F}_i)$ remain the same.

Table 4: Risk decomposition of the ERC portfolio with respect to risk factors

$\sigma(y) = 10.13\%$	y_i	$\mathcal{MR}(\mathcal{F}_i)$	$\mathcal{RC}(\mathcal{F}_i)$	$\mathcal{RC}^\star(\mathcal{F}_i)$
$\mathcal{F}_1$	22.65%	8.94%	2.03%	20.00%
$\mathcal{F}_2$	22.65%	8.94%	2.03%	20.00%
$\mathcal{F}_3$	22.65%	8.94%	2.03%	20.00%
$\mathcal{F}_4$	32.04%	12.65%	4.05%	40.00%
$\tilde{\mathcal{F}}_1$	0.00%	0.00%	0.00%	0.00%

[6]These 4 factors are implicitly the first four assets.

3.4.2 The case $n < m$

This case corresponds to the example with primary and synthetic assets in Section 2.2. Indeed, the primary assets represent the risk factors. The loading matrix A is then equal to the weight matrix W, the covariance matrix of the factors is the covariance matrix of the primary assets whereas D is a matrix of zeros. In Table 5, we have reported the risk decomposition of the portfolio #1. We notice that the weight of the factor $\mathcal{F}_j$ is equal to the weight of the primary asset $\mathcal{A}'_j$, but we don't obtain the same risk decomposition: $\mathcal{RC}(\mathcal{F}_i) \neq \mathcal{RC}(\mathcal{A}'_i)$. The main reason is that the risk decomposition is not unique when $m > n$. We face then an identification problem, which is a well-known problem in statistics. If we analyze the loading matrix A, we conclude that there is in fact three factors, because the primary assets $\mathcal{A}'_1$ and $\mathcal{A}'_2$ (resp. $\mathcal{A}'_5$ and $\mathcal{A}'_6$) influence together only the first synthetic asset $\mathcal{A}'_1$ (resp. the second synthetic asset $\mathcal{A}'_2$). In a same way, we notice a perfect symmetry between the primary assets $\mathcal{A}'_3$ and $\mathcal{A}'_4$. This is why we deduce that the risk contributions of the factors satisfies this following system of equations:

$$\left\{\begin{array}{l}\mathcal{RC}(\mathcal{A}'_1) + \mathcal{RC}(\mathcal{A}'_2) = \mathcal{RC}(\mathcal{F}_1) + \mathcal{RC}(\mathcal{F}_2) \\ \mathcal{RC}(\mathcal{A}'_3) + \mathcal{RC}(\mathcal{A}'_4) = \mathcal{RC}(\mathcal{F}_3) + \mathcal{RC}(\mathcal{F}_4) \\ \mathcal{RC}(\mathcal{A}'_5) + \mathcal{RC}(\mathcal{A}'_6) = \mathcal{RC}(\mathcal{F}_5) + \mathcal{RC}(\mathcal{F}_6)\end{array}\right.$$

It explains that only combination of factors could be identified, and not the factors themselves.

Table 5: Risk decomposition of the portfolio #1 with respect to risk factors

$\sigma(y) = 10.13\%$	y_i	$\mathcal{MR}(\mathcal{F}_i)$	$\mathcal{RC}(\mathcal{F}_i)$	$\mathcal{RC}^{\star}(\mathcal{F}_i)$
$\mathcal{F}_1$	9.00%	5.74%	0.52%	5.07%
$\mathcal{F}_2$	9.00%	5.74%	0.52%	5.07%
$\mathcal{F}_3$	31.50%	13.13%	4.14%	40.59%
$\mathcal{F}_4$	31.50%	13.13%	4.14%	40.59%
$\mathcal{F}_5$	9.50%	4.66%	0.44%	4.34%
$\mathcal{F}_6$	9.50%	4.66%	0.44%	4.34%

4 Portfolio construction with factor risk budgeting

In this section, we consider two types of portfolio construction. In the first one, the allocation problem consists in matching some risk budgets with respect to risk factors. We show that the solution may not exist. This is why we consider a second approach which minimizes the risk concentration between the factors.

4.1 Matching the risk budgets

We would like to build a risk budgeting portfolio such that the risk contributions match a set of given risk budgets $\{b_1, \ldots, b_m\}$:

$$\mathcal{RC}(\mathcal{F}_j) = b_j \mathcal{R}(x) \tag{4}$$

This general problem can be formulated as a quadratic problem in a fashion similar to Bruder and Roncalli (2012):

$$\begin{aligned}
\left(y^{\star},\hat{y}^{\star}\right) &= \arg\min \sum_{j=1}^{m} \left(\mathcal{RC}\left(\mathcal{F}_j\right) - b_j \mathcal{R}\left(y,\tilde{y}\right)\right)^2 \\
&\quad \text{u.c.} \quad \left\{ \begin{array}{l} \mathbf{1}^{\top} B^{+} y + \mathbf{1}^{\top} \tilde{B}^{+} \tilde{y} = 1 \\ \mathbf{0} \preceq B^{+} y + \tilde{B}^{+} \tilde{y} \preceq \mathbf{1} \end{array} \right.
\end{aligned} \tag{5}$$

where $\preceq$ denotes element-wise inequalities. If there is a solution to the optimization problem (5) and if the objective function is equal to zero at the optimum, it implies that it is also the solution of the matching problem (4). This problem (5) is however difficult to solve analytically as it involves PDE as its first-order conditions[7]. The following paragraphs examine a related set of problems in which cases one can find an equivalent optimization problem in a more convenient form.

4.1.1 Non-negative risk factor allocation constraint

Bruder and Roncalli (2012) show that a problem of the form:

$$\begin{aligned}
x^{\star} &= \arg\min \mathcal{R}\left(x\right) \\
&\quad \text{u.c.} \quad \left\{ \begin{array}{l} \mathcal{RC}\left(\mathcal{A}_i\right) = b_i \mathcal{R}\left(x\right) \\ \mathbf{1}^{\top} x = 1 \\ x \succeq \mathbf{0} \end{array} \right.
\end{aligned} \tag{6}$$

can be solved by considering the alternative problem:

$$\begin{aligned}
z^{\star} &= \arg\min \mathcal{R}\left(z\right) \\
&\quad \text{u.c.} \quad \left\{ \begin{array}{l} \sum_{i=1}^{n} b_i \ln z_i \geq c \\ z \succeq \mathbf{0} \end{array} \right.
\end{aligned} \tag{7}$$

There exists then a **unique** unnormalized solution $z^{\star}$ and posing $x^{\star} = z^{\star} / \left(\mathbf{1}^{\top} z^{\star}\right)$ provides the optimal portfolio's asset exposures.

Similarly, using the decomposition of the portfolio asset exposures x into factor risk exposure y given by Equation (2), the optimization problem with risk factor budgeting constraints can be formulated as:

$$\begin{aligned}
\left(y^{\star},\tilde{y}^{\star}\right) &= \arg\min \mathcal{R}\left(y,\tilde{y}\right) \\
&\quad \text{u.c.} \quad \left\{ \begin{array}{l} \sum_{j=1}^{m} b_j \ln y_j \geq c \\ y \succeq \mathbf{0} \end{array} \right.
\end{aligned} \tag{8}$$

We notice however that this problem induces a positivity constraint[8] on the risk factor y.

The separation principle Note that $\tilde{y}$ is related to the idiosyncratic (or specific) risk of the portfolio. Indeed, we have:

$$x^{\top} \varepsilon_t = \eta_t = e^{\top} R_t = \tilde{y}^{\top} \tilde{\mathcal{F}}_t$$

[7] Nevertheless, we could solve it numerically using the SQP algorithm.

[8] If $y_j \leq 0$, we could replace the factor $\mathcal{F}_j$ by a new factor $\mathcal{F}'_j = -\mathcal{F}_j$ meaning that $y'_j \geq 0$. This is why we could always impose that $y_j \geq 0$.

Since we focus in this paper on portfolio building controlling for the risk budgets associated with the risk factors, we will consider for now that we have an unconstrained minimization problem on the variable $\tilde{y}$, and will focus instead on solving the constrained optimization problem on y. Furthermore, it is always possible to solve a problem of the form (8) in two steps as long as the constraints on y and $\tilde{y}$ are separate. We obtain:

$$\begin{aligned} y^\star &= \arg\min \tilde{\mathcal{R}}(y) \\ &\text{u.c.} \quad \begin{cases} \sum_{j=1}^m b_j \ln y_j \geq c \\ y \succeq \mathbf{0} \end{cases} \end{aligned} \tag{9}$$

where $\tilde{\mathcal{R}}(y) = \inf_{\tilde{y}} \mathcal{R}(y, \tilde{y})$. The solution will be of the form $(y^\star, \varphi(y^\star))$ where $\tilde{y}^\star = \varphi(y^\star)$ is solution to the first step of the optimization problem. The optimal portfolio allocation $x^\star$ is then recovered simply as:

$$x^\star = B^+ y^\star + \tilde{B}^+ \varphi(y^\star)$$

By construction, we also have[9]:

$$\frac{\partial \tilde{\mathcal{R}}}{\partial y}(y) = \frac{\partial \mathcal{R}}{\partial y}(y, \varphi(y)) \qquad \text{and} \qquad \frac{\partial \mathcal{R}}{\partial \tilde{y}}(y, \varphi(y)) = 0$$

Application to the volatility risk measure For instance, if we measure the risk using the P&L volatility $\mathcal{R}(y, \tilde{y}) = \sigma(y, \tilde{y})$, the problem is now equivalent to a quadratic optimization problem with constraints on y only. Let us denote $\bar{\Omega}$ the covariance matrix between factors. We have:

$$\begin{aligned} \bar{\Omega} &= \operatorname{cov}\left(\mathcal{F}_t, \tilde{\mathcal{F}}_t\right) \\ &= \begin{pmatrix} (B^+)^\top \Sigma B^+ & (B^+)^\top \Sigma \tilde{B}^+ \\ \left(\tilde{B}^+\right)^\top \Sigma B^+ & \left(\tilde{B}^+\right)^\top \Sigma \tilde{B}^+ \end{pmatrix} \\ &= \begin{pmatrix} \Omega & \Gamma^\top \\ \Gamma & \tilde{\Omega} \end{pmatrix} \end{aligned}$$

Hence, we solve the augmented quadratic problem:

$$\begin{aligned} &\min \quad y^\top \Omega y + \tilde{y}^\top \tilde{\Omega} \tilde{y} + 2\tilde{y}^\top \Gamma^\top y \\ &\text{u.c.} \quad \begin{cases} \sum_{j=1}^m b_j \ln y_j \geq c \\ y \succeq \mathbf{0} \end{cases} \end{aligned}$$

in two steps, by first minimizing with respect to $\tilde{y}$, and then with respect to y. The solution of the first step is given by $\tilde{y} = \varphi(y) = -\tilde{\Omega}^{-1}\Gamma^\top y$. The problem is thus reduced to:

$$\begin{aligned} &\min \quad y^\top S y \\ &\text{u.c.} \quad \begin{cases} \sum_{j=1}^m b_j \ln y_j \geq c \\ y \succeq \mathbf{0} \end{cases} \end{aligned}$$

with $S = \Omega - \Gamma\tilde{\Omega}^{-1}\Gamma^\top$ the Schur complement of $\tilde{\Omega}$. Because we have $\Gamma^\top = (B^+)^\top \Sigma \tilde{B}^+$, we deduce that the optimal portfolio asset allocation is thus given by:

$$\begin{aligned} x^\star &= B^+ y^\star + \tilde{B}^+ \varphi(y^\star) \\ &= \left(B^+ - \tilde{B}^+ \tilde{\Omega}^{-1} (B^+)^\top \Sigma \tilde{B}^+\right) y^\star \end{aligned}$$

[9]The second equality results from the definition of φ as an infimum. The key to the first equality is being able to apply the chain rule, i.e. φ being regular enough to be differentiable. If φ is a strict minimum and if $\mathcal{R}$ is differentiable, then an application of the implicit function theorem to $\partial_{\tilde{y}} \mathcal{R}(y, \tilde{y}) = 0$ shows that φ is differentiable.

Remark 4. *So long as the factors $\mathcal{F}_t$ and $\tilde{\mathcal{F}}_t$ are uncorrelated ($\Gamma = \mathbf{0}$), a solution of the form $(y^\star, 0)$ exists. In this case, the optimization problem could also be solved separately on each of the orthogonal spaces:* $\operatorname{col} A$, *as a constrained problem in y of the form (7), and* $\ker A$, *as an unconstrained minimization problem. The (un-normalized) asset allocation is given by $x^\star = B^+ y^\star$.*

4.1.2 Adding long-only constraints

Additionally, if we want to consider long-only allocations x, we must also include the following constraint:

$$x = B^+ y + \tilde{B}^+ \tilde{y} \succeq \mathbf{0}$$

The corresponding optimization problem is modified as follows:

$$\begin{aligned} (y^\star, \tilde{y}^\star) &= \arg\min \mathcal{R}(y, \tilde{y}) \\ &\text{u.c.} \quad \begin{cases} \sum_{j=1}^{m} b_j \ln y_j \geq c \\ y \succeq \mathbf{0} \\ B^+ y + \tilde{B}^+ \tilde{y} \succeq \mathbf{0} \end{cases} \end{aligned} \tag{10}$$

We can again apply the separation principle[10] as in Section 4.1.1 and obtain this formulation:

$$\begin{aligned} (y^\star, \tilde{y}^\star) &= \arg\min \tilde{\mathcal{R}}(y) \\ &\text{u.c.} \quad \begin{cases} \sum_{j=1}^{m} b_j \ln y_j \geq c \\ y \succeq \mathbf{0} \\ B^+ y + \tilde{B}^+ \varphi(y) \succeq \mathbf{0} \end{cases} \end{aligned} \tag{11}$$

which is a convex optimization problem with a unique solution (if it exists) if, and only if, φ is convex. It is easy to see that this condition is verified if we choose $\mathcal{R}$ to be the P&L volatility. It is also true if $\mathcal{R}$ is additively separable between y and $\tilde{y}$, in which case $\varphi(y)$ is constant. This later situation should normally arise for some classic risk measures[11] if the factors $\mathcal{F}_t$ and $\tilde{\mathcal{F}}_t$ are uncorrelated[12] (which depends only on the choice of $\tilde{B}^+$).

Remark 5. *The solution may not exist even if φ is convex.*

Remark 6. *An analysis presented in Appendix A.4 in the case where $\mathcal{R}$ is the P&L volatility shows that the solution to formulation (9) will be solution to (11) as long as there exists $\lambda = (\lambda_x, \lambda_y) \succeq \mathbf{0}$ such that:*

$$\left(A^+ - \left(\tilde{B}^+\right)^\top \Sigma \tilde{\Omega}^{-1} \left(\tilde{B}^+\right)^\top\right) \lambda_x + \lambda_y = 0$$

According to this analysis, this condition is likely to be verified for some non trivial $\lambda \in \mathbb{R}_+^{n+m}$. In such case, there exists $\zeta > 0$ such that $0 \leq \min y_j \leq \zeta$.

[10] In general, the separation principle cannot be applied in cases where the constraints on $(y, \tilde{y})$ are not separate. However, in this case, if φ is a strict minimum of $\mathcal{R}$, which is the case for the P&L volatility, then the two problems (10) and (11) are equivalent. Indeed, if $y^\star$ is solution to (11), then $(y^\star, \varphi(y^\star))$ is obviously a feasible point for (10). Conversely, if $(y^{\star\star}, \tilde{y}^{\star\star})$ is solution to (10), then $\mathcal{R}$ being convex, any local optimum has to be a global optimum. Therefore, we must have:

$$\tilde{\mathcal{R}}(y^{\star\star}) = \mathcal{R}(y^{\star\star}, \tilde{y}^{\star\star}) = \mathcal{R}(y^{\star\star}, \varphi(y^{\star\star})).$$

And, since φ is a strict minimum, $(y^{\star\star}, \tilde{y}^{\star\star}) = (y^{\star\star}, \varphi(y^{\star\star}))$ and thus, $B^+ y^{\star\star} + \tilde{B}^+ \varphi(y^{\star\star}) \succeq 0$.

[11] It is for example the case of value-at-risk and expected shortfall considering elliptic distributions or kernel estimation.

[12] See Appendix A.2 in Meucci (2007).

4.1.3 An illustration

We consider the example with 4 assets and 3 factors presented in Section 3.4.1 page 8. We have seen that the equally-weighted portfolio concentrates the risk on the first factor. We would like to build a portfolio with more balanced risk across the factors. For instance, if $b = (49\%, 25\%, 25\%)$, we obtain the results given in Table 6. We notice that the corresponding portfolio presents positive weights.

Table 6: Matching the risk budgets $(49\%, 25\%, 25\%)$

(a) Optimal solution $(y^\star, \tilde{y}^\star)$

$\sigma(y) = 21.27\%$	y_i	$\mathcal{MR}(\mathcal{F}_i)$	$\mathcal{RC}(\mathcal{F}_i)$	$\mathcal{RC}^\star(\mathcal{F}_i)$
$\mathcal{F}_1$	93.38%	11.16%	10.42%	49.00%
$\mathcal{F}_2$	24.02%	22.14%	5.32%	25.00%
$\mathcal{F}_3$	39.67%	13.41%	5.32%	25.00%
$\tilde{\mathcal{F}}_1$	16.39%	1.30%	0.21%	1.00%

(b) Corresponding portfolio $x^\star$

$\sigma(x) = 21.27\%$	x_i	$\mathcal{MR}(\mathcal{A}_i)$	$\mathcal{RC}(\mathcal{A}_i)$	$\mathcal{RC}^\star(\mathcal{A}_i)$
$\mathcal{A}_1$	15.08%	17.44%	2.63%	12.36%
$\mathcal{A}_2$	38.38%	23.94%	9.19%	43.18%
$\mathcal{A}_3$	0.89%	21.82%	0.20%	0.92%
$\mathcal{A}_4$	45.65%	20.29%	9.26%	43.54%

We suppose now that $b = (19\%, 40\%, 40\%)$. Results are reported in Table 7. We then obtain a long/short portfolio with a short position in the first asset. In Table 8, we report the solution to the optimization problem (5) if we impose that the weights of the portfolio are positive. At the optimum, the objective function is not equal to zero, meaning that there is no solution to the matching problem (4).

Table 7: Matching the risk budgets $(19\%, 40\%, 40\%)$

(a) Optimal solution $(y^\star, \tilde{y}^\star)$

$\sigma(y) = 23.41\%$	y_i	$\mathcal{MR}(\mathcal{F}_i)$	$\mathcal{RC}(\mathcal{F}_i)$	$\mathcal{RC}^\star(\mathcal{F}_i)$
$\mathcal{F}_1$	92.90%	4.79%	4.45%	19.00%
$\mathcal{F}_2$	28.55%	32.79%	9.36%	40.00%
$\mathcal{F}_3$	45.21%	20.71%	9.36%	40.00%
$\tilde{\mathcal{F}}_1$	−23.57%	−0.99%	0.23%	1.00%

(b) Corresponding portfolio $x^\star$

$\sigma(x) = 23.41\%$	x_i	$\mathcal{MR}(\mathcal{A}_i)$	$\mathcal{RC}(\mathcal{A}_i)$	$\mathcal{RC}^\star(\mathcal{A}_i)$
$\mathcal{A}_1$	−26.19%	14.13%	−3.70%	−15.81%
$\mathcal{A}_2$	32.69%	21.21%	6.94%	29.63%
$\mathcal{A}_3$	14.28%	20.41%	2.91%	12.45%
$\mathcal{A}_4$	79.22%	21.79%	17.26%	73.73%

Table 8: Imposing the long-only constraint with $b = (19\%, 40\%, 40\%)$

(a) Optimal solution $(y^\star, \tilde{y}^\star)$

$\sigma(y) = 21.82\%$	y_i	$\mathcal{MR}(\mathcal{F}_i)$	$\mathcal{RC}(\mathcal{F}_i)$	$\mathcal{RC}^\star(\mathcal{F}_i)$
$\mathcal{F}_1$	89.85%	6.89%	6.19%	28.37%
$\mathcal{F}_2$	23.13%	28.67%	6.63%	30.40%
$\mathcal{F}_3$	47.02%	19.12%	8.99%	41.20%
$\tilde{\mathcal{F}}_1$	2.53%	0.26%	0.01%	0.03%

(b) Corresponding portfolio $x^\star$

$\sigma(x) = 21.82\%$	x_i	$\mathcal{MR}(\mathcal{A}_i)$	$\mathcal{RC}(\mathcal{A}_i)$	$\mathcal{RC}^\star(\mathcal{A}_i)$
$\mathcal{A}_1$	0.00%	15.90%	0.00%	0.00%
$\mathcal{A}_2$	32.83%	22.03%	7.23%	33.15%
$\mathcal{A}_3$	0.00%	20.51%	0.00%	0.00%
$\mathcal{A}_4$	67.17%	21.72%	14.59%	66.85%

Remark 7. *The existence problem of long-only portfolios could be easily illustrated with a portfolio of bonds. It is commonly argued that the three factors of the yield curve are the general level of interest rates, the slope of the yield curve and its convexity. Building a bond portfolio, for instance, where the slope and convexity factors have the same magnitude than the level factor is not possible if we impose that the weights are positive. In the long-only case, risk contributions of the slope and convexity factors are then bounded. We do not face this problem if the long-only constraint vanishes.*

4.2 Minimizing the risk concentration between the risk factors

In the previous section, we have shown that the solution to the matching problem (4) does not necessarily exist. Of course, we could always optimize the problem (5) and consider the optimal solution without verifying that the objective function is equal to zero. In this case, we obtain the portfolio for which the risk contributions are the closest to the risk budgets in the sense of the $\mathcal{L}_2$ absolute norm[13]. We consider now a related problem:

$$\mathcal{RC}(\mathcal{F}_j) \simeq \mathcal{RC}(\mathcal{F}_k)$$

The idea is to find a portfolio which is well balanced in terms of risk contributions with respect to the common factors. A first idea is to set $b_j = b_k$ and to use use the previous framework. Another way is to minimize the concentration between the risk contributions $(\mathcal{RC}(\mathcal{F}_1), \ldots, \mathcal{RC}(\mathcal{F}_m))$.

4.2.1 Concentration indexes

Let $p \in \mathbf{R}_+^n$ such that $\mathbf{1}^\top p = 1$. p is then a probability distribution. A probability distribution p^+ is perfectly concentrated if there exists one observation i_0 such that $p_{i_0}^+ = 1$ and

[13] If we prefer to use a $\mathcal{L}_2$ relative norm, we could replace the objective function as follows:

$$(y^\star, \tilde{y}^\star) = \arg\min_y \sum_{j=1}^m \sum_{k=1}^m \left(\frac{\mathcal{RC}(\mathcal{F}_j)}{b_j} - \frac{\mathcal{RC}(\mathcal{F}_k)}{b_k} \right)^2$$

$p_i^+ = 0$ if $i \neq i_0$. As the opposite, a probability distribution p^- such that $p_i^- = 1/n$ for all $i = 1, \ldots, n$ has no concentration. A concentration index is a mapping function $\mathcal{C}(p)$ such that $\mathcal{C}(p)$ increases with concentration and verifies $\mathcal{C}(p^-) \leq \mathcal{C}(p) \leq \mathcal{C}(p^+)$. Eventually, this index could be normalized such that $\mathcal{C}(p^-) = 0$ and $\mathcal{C}(p^+) = 1$.

In our context, the vector p represents the risk contributions of the portfolio. $\mathcal{C}(p)$ will then measure the risk concentration of the portfolio with respect to the risk factors. The most popular methods to measure the concentration are the Herfindahl index and the Gini index. Another interesting statistic is the Shannon entropy which measures diversity, that is opposite of the concentration. Their definition is given below.

The Herfindahl index The Herfindahl index associated to p is defined as:

$$\mathcal{H}(p) = \sum_{i=1}^{n} p_i^2$$

This index takes the value 1 for a probability distribution p^+ and $1/n$ for a distribution with uniform probabilities. To scale the statistics onto $[0, 1]$, we consider the normalized index $\mathcal{H}^\star(p)$ defined as follows:

$$\mathcal{H}^\star(p) = \frac{n\mathcal{H}(p) - 1}{n - 1}$$

The Gini index The Gini index measures the distance between the Lorenz curve of p and the Lorenz curve of p^-. Its analytical expression is:

$$\mathcal{G}(p) = \frac{2\sum_{i=1}^{n} i p_{i:n}}{n\sum_{i=1}^{n} p_{i:n}} - \frac{n+1}{n}$$

with $\{p_{1:n}, \ldots, p_{n:n}\}$ the ordered statistics of $\{p_1, \ldots, p_n\}$. We verify that $\mathcal{G}(p^-) = 0$ and $\mathcal{G}(p^+) = 1 - 1/n$.

The Shannon entropy It is defined as follows:

$$\mathcal{I}(p) = -\sum_{i=1}^{n} p_i \ln p_i$$

When the Shannon entropy is used to measure the diversity, we prefer to consider the statistic $\mathcal{I}^\star(p) = \exp(\mathcal{I}(p))$. We notice that $\mathcal{I}^\star(p^-) = n$ and $\mathcal{I}^\star(p^+) = 1$.

Remark 8. *The entropy measure $\mathcal{I}^\star$ is sometimes interpreted as the degree of diversification, because it represents the true number of bets of the portfolio (Meucci, 2009). Another equivalent measure is the inverse of the Herfindal index.*

4.2.2 An illustration

We continue our example by minimizing the risk concentration between the three risk factors. Results are given in Table 9. We notice that this portfolio satisfies $\mathcal{H}^\star = 0$, $\mathcal{G} = 0$ and $\mathcal{I}^\star = 3$ if we consider the risk contributions of the three factors as the statistics of interest p. In this case, these three criteria are equivalent. If we impose now some constraints, the optimal portfolios will differ. For instance, if we assume that the weights are larger than 10%, we obtain the optimal portfolios given in Table 10. In this case, the weights depend on the criterion and the three optimization problems are not equivalent. Therefore, the Gini index puts more weight on the fourth asset than the Herfindahl index or the entropy measure.

Table 9: The lowest risk concentrated portfolio

(a) Optimal solution $(y^\star, \tilde{y}^\star)$

$\sigma(y) = 21.88\%$	y_i	$\mathcal{MR}(\mathcal{F}_i)$	$\mathcal{RC}(\mathcal{F}_i)$	$\mathcal{RC}^\star(\mathcal{F}_i)$
$\mathcal{F}_1$	91.97%	7.91%	7.28%	33.26%
$\mathcal{F}_2$	25.78%	28.23%	7.28%	33.26%
$\mathcal{F}_3$	42.22%	17.24%	7.28%	33.26%
$\tilde{\mathcal{F}}_1$	6.74%	0.70%	0.05%	0.21%

(b) Corresponding portfolio $x^\star$

$\sigma(x) = 21.88\%$	x_i	$\mathcal{MR}(\mathcal{A}_i)$	$\mathcal{RC}(\mathcal{A}_i)$	$\mathcal{RC}^\star(\mathcal{A}_i)$
$\mathcal{A}_1$	0.30%	16.11%	0.05%	0.22%
$\mathcal{A}_2$	39.37%	23.13%	9.11%	41.63%
$\mathcal{A}_3$	0.31%	20.93%	0.07%	0.30%
$\mathcal{A}_4$	60.01%	21.09%	12.66%	57.85%

Table 10: Optimal portfolios with $x_i \geq 10\%$

Criterion	$\mathcal{H}(x)$	$\mathcal{G}(x)$	$\mathcal{I}(x)$
x_1	10.00%	10.00%	10.00%
x_2	22.08%	18.24%	24.91%
x_3	10.00%	10.00%	10.00%
x_4	57.92%	61.76%	55.09%
$\mathcal{H}^\star$	0.0436	0.0490	0.0453
$\mathcal{G}$	0.1570	0.1476	0.1639
$\mathcal{I}^\star$	2.8636	2.8416	2.8643

4.3 Solving some invariance problems

In this paragraph, we show that optimized risk parity portfolios with risk factors solve two important invariance problems.

4.3.1 The duplication invariance property

Let us come back to the first problem that has motivated this research. Choueifaty *et al.* (2011) show that the ERC portfolio does not verify the duplication invariance property, meaning that the ERC portfolio changes if we duplicate one asset.

Let $\Sigma^{(n)}$ be the covariance matrix of the n assets. We consider the RB portfolio $x^{(n)}$ corresponding to the risk budgets $b^{(n)}$. We suppose now that we duplicate the last asset. In this case, the covariance matrix of the $n+1$ assets is:

$$\Sigma^{(n+1)} = \begin{pmatrix} \Sigma^{(n)} & \Sigma^{(n)}\mathbf{e}_n \\ \mathbf{e}_n^\top \Sigma^{(n)} & 1 \end{pmatrix}$$

We associate the factor model with $\Omega = \Sigma^{(n)}$, $D = \mathbf{0}$ and:

$$A = \left[\begin{array}{c} I_n \\ \mathbf{e}_n^\top \end{array} \right]$$

We consider the portfolio $x^{(n+1)}$ such that the risk contribution of the factors match the risk budgets $b^{(n)}$. It is then easy to show that the matching portfolio verifies $y^\star = x^{(n)}$ in terms of factor weights. We have then $x_i^{(n+1)} = x_i^{(n)}$ if $i < n$ and $x_n^{(n+1)} + x_{n+1}^{(n+1)} = x_n^{(n)}$. This result shows that an ERC portfolio verifies the duplication invariance property if the risk budgets are expressed with respect to factors and not to assets.

4.3.2 The polico invariance property

Choueifaty *et al.* (2011) suggest that a diversified portfolio should verify another desirable property called the polico invariance property:

> "*The addition of a positive linear combination of assets already belonging to the universe should not impact the portfolio's weights to the original assets, as they were already available in the original universe. We abbreviate 'positive linear combination' to read Polico.*"

We use the previous framework and introduce an asset $n+1$ which is a linear (normalized) combination α of the first n assets. In this case, the covariance matrix of the $n+1$ assets is:

$$\Sigma^{(n+1)} = \left(\begin{array}{cc} \Sigma^{(n)} & \Sigma^{(n)}\alpha \\ \alpha^\top \Sigma^{(n)} & \alpha^\top \Sigma^{(n)} \alpha \end{array} \right)$$

We associate the factor model with $\Omega = \Sigma^{(n)}$, $D = \mathbf{0}$ and:

$$A = \left[\begin{array}{c} I_n \\ \alpha^\top \end{array} \right]$$

We consider the portfolio $x^{(n+1)}$ such that the risk contribution of the factors match the risk budgets $b^{(n)}$. It is then easy to show that the factor weights of the matching portfolio satisfy:

$$y^\star = \left[\begin{array}{c} A^+ x^{(n)} \\ 0 \end{array} \right]$$

We have then $x^{(n+1)} = Ay^\star$, which implies that $x_i^{(n)} = x_i^{(n+1)} + \alpha_i x_{n+1}^{(n+1)}$ if $i \leq n$. This result shows that a RB portfolio (and so an ERC portfolio) verifies the polico invariance property if the risk budgets are expressed with respect to factors and not to assets.

Remark 9. *This result is related to our example with primary and synthetic assets.*

5 Applications

In this section, we consider different applications. The first one concerns the risk budgeting of equity factors. In the second application, we build portfolios of hedge funds with a better diversification between the principal component factors. Finally, we compare the risk parity approach based on risk factors and asset classes in a strategic asset allocation perspective.

5.1 Budgeting the Fama-French-Carhart factors

Let R_i be the return of the i^{th} asset and R_f be the risk-free rate. In the capital asset pricing model (CAPM) of Sharpe (1964), we have:

$$\mathbb{E}\left[R_i\right] = R_f + \beta_i \left(\mathbb{E}\left[R_{\text{MKT}}\right] - R_f\right)$$

where R_{MKT} is the return of the market portfolio and β_i is the measure of the systematic risk defined by:

$$\beta_i = \frac{\text{cov}\left(R_i, R_{\text{MKT}}\right)}{\text{var}\left(R_{\text{MKT}}\right)}$$

This model is also called the one-factor pricing model, because the stock return is entirely explained by one common risk factor represented by the market. According to Fama and French (2004), this model is the centerpiece of MBA investment courses and is widely used in finance by practitioners despite a large body of evidence in the academic literature of the invalidation of this model:

> "*The attraction of the CAPM is that it offers powerful and intuitively pleasing predictions about how to measure risk and the relation between expected return and risk. Unfortunately, the empirical record of the model is poor – poor enough to invalidate the way it is used in applications.*" (Fama and French, 2004, p. 25)

In 1992, Fama and French studied several factors to explain average returns (size, E/P, leverage and book-to-market equity). In 1993, Fama and French extended their empirical work and proposed a three factor-model:

$$\mathbb{E}\left[R_i\right] - R_f = \beta_i^{\text{MKT}} \left(\mathbb{E}\left[R_{\text{MKT}}\right] - R_f\right) + \beta_i^{\text{SMB}} \mathbb{E}\left[R_{\text{SMB}}\right] + \beta_i^{\text{HML}} \mathbb{E}\left[R_{\text{HML}}\right]$$

where R_{smb} is the return of small stocks minus the return of large stocks and R_{hml} is the return of stocks with high book-to-market values minus the return of stocks with low book-to-market values. In a key paper, Carhart (1997) uses a four-factor model, adding a one-year momentum factor to the Fama-French three-factor model. Over the years, this model has become the standard model in the asset management industry.

We consider a universe of 6 equity indexes: MSCI USA large growth (LG), MSCI USA large value (LV), MSCI USA mid growth (MG), MSCI USA mid value (MV), MSCI USA small growth (SG) and MSCI USA small value (SV). We specify the factor model as follows:

$$R_t^i - \left(R_{f,t} + R_{\text{MKT},t}\right) = \beta_i^{\text{SMB}} R_{\text{SMB},t} + \beta_i^{\text{HML}} R_{\text{HML},t} + \beta_i^{\text{MOM}} R_{\text{MOM},t} + \varepsilon_{i,t}$$

where i represents one of the previous equity indexes. We estimate the different parameters of the model by using maximum likelihood. If we assume that the specific factors are uncorrelated and if we consider a one-year rolling observations, we obtain the following figures at the end of June 2011:

$$\hat{A} = \begin{pmatrix} -0.09 & -0.32 & -0.03 \\ -0.22 & 0.24 & -0.14 \\ 0.18 & -0.24 & 0.22 \\ 0.16 & 0.29 & -0.07 \\ 0.67 & -0.07 & 0.18 \\ 0.62 & 0.31 & -0.10 \end{pmatrix}, \quad \hat{D} = \text{diag} \begin{pmatrix} 4.68 \\ 4.10 \\ 12.03 \\ 9.93 \\ 5.70 \\ 5.21 \end{pmatrix} \times 10^{-4}$$

and:

$$\hat{\Omega} = \begin{pmatrix} 77.26 & 1.25 & 34.09 \\ 1.25 & 33.18 & -9.35 \\ 34.09 & -9.35 & 58.02 \end{pmatrix} \times 10^{-4}$$

We consider three sets of risk budgets with respect to the factors SMB, HML and MOM:

1. With $b = (25\%, 25\%, 25\%)$, we build a portfolio very well balanced between the three common factors;
2. With $b = (10\%, 60\%, 10\%)$, the HML factor represents 60% of the portfolio risk whereas the risk contribution of the two other common factors is only 20%;
3. With $b = (10\%, 10\%, 60\%)$, the MOM factor is the main contributor of the portfolio risk.

Moreover, for each set of risk budgets, we estimated the unconstrained portfolio and the long-only portfolio[14]. Results are given in Table 11. With the risk budgets $(25\%, 25\%, 25\%)$, we find two portfolios which match perfectly these figures. The main weighted assets are the LG, LV, MV and SV indexes. With the second set of risk budgets, we notice that the exposure on the value indexes (LV, MV and SV) increases whereas the exposure in the growth indexes (LG, MG, and SG) decreases. It is coherent with the Fama-French model, because HML is a typical value versus growth factor. If we were to build a portfolio with more risk on the momentum factor, the main weighted assets would be the large and small value indexes. We notice also that it is not possible to find a long-only portfolio that matches the risk budgets $(10\%, 10\%, 60\%)$.

Table 11: RB portfolios with Fama-French-Carhart factors (June 2011)

	#1	#1*	#2	#2*	#3	#3*
x_{SMB}	11.51%	10.64%	11.90%	10.25%	6.96%	5.03%
x_{HML}	14.63%	13.73%	29.30%	25.32%	10.92%	7.94%
x_{MOM}	−9.19%	−8.72%	−11.13%	−9.88%	−10.76%	−9.01%
$\mathcal{RC}(\mathcal{F}_{\mathrm{SMB}})$	25.00%	25.00%	10.00%	10.00%	10.00%	6.49%
$\mathcal{RC}(\mathcal{F}_{\mathrm{HML}})$	25.00%	25.00%	60.00%	60.00%	10.00%	4.78%
$\mathcal{RC}(\mathcal{F}_{\mathrm{MOM}})$	25.00%	25.00%	10.00%	10.00%	60.00%	53.21%
$\sum_{j=1}^{3} \mathcal{RC}\left(\tilde{\mathcal{F}}_j\right)$	25.00%	25.00%	20.00%	20.00%	20.00%	35.52%
x_{LG}	24.93%	23.15%	1.56%	3.20%	33.93%	31.92%
x_{LV}	25.10%	29.09%	33.61%	37.55%	33.60%	37.45%
x_{MG}	−5.07%	0.00%	−6.00%	0.00%	−9.26%	0.00%
x_{MV}	30.21%	22.44%	50.37%	38.43%	14.71%	5.84%
x_{SG}	2.35%	0.00%	0.97%	0.00%	2.43%	0.00%
x_{SV}	22.49%	25.32%	19.48%	20.83%	24.60%	24.78%

Remark 10. *The previous results are sensitive to the study date. For instance, if we choose the end of June 2012, we obtain different figures as reported in Table 12. The main differences come from the weight of the mid-cap indexes, whereas the most coherent results are for the large-cap indexes.*

[14] This case is represented by a star *.

Table 12: RB portfolios with Fama-French-Carhart factors (June 2012)

	#1	#1*	#2	#2*	#3	#3*
x_{SMB}	7.94%	7.94%	6.18%	6.18%	4.99%	5.03%
x_{HML}	−8.01%	−8.01%	−17.57%	−17.57%	10.58%	10.71%
x_{MOM}	1.56%	1.56%	0.97%	0.97%	2.95%	2.97%
$\mathcal{RC}\left(\mathcal{F}_{\mathrm{SMB}}\right)$	25.00%	25.00%	10.00%	10.00%	10.00%	10.00%
$\mathcal{RC}\left(\mathcal{F}_{\mathrm{HML}}\right)$	25.00%	25.00%	60.00%	60.00%	10.00%	10.00%
$\mathcal{RC}\left(\mathcal{F}_{\mathrm{MOM}}\right)$	25.00%	25.00%	10.00%	10.00%	60.00%	60.00%
$\sum_{j=1}^{3}\mathcal{RC}\left(\tilde{\mathcal{F}}_j\right)$	25.00%	25.00%	20.00%	20.00%	20.00%	20.00%
x_{LG}	42.32%	42.32%	47.45%	47.45%	34.93%	34.54%
x_{LV}	23.31%	23.31%	17.50%	17.50%	39.94%	40.15%
x_{MG}	8.25%	8.25%	14.67%	14.67%	−1.75%	0.00%
x_{MV}	4.24%	4.24%	3.01%	3.01%	2.11%	0.58%
x_{SG}	11.17%	11.17%	11.40%	11.40%	6.45%	4.82%
x_{SV}	10.70%	10.70%	5.97%	5.97%	18.33%	19.91%

5.2 Diversifying a portfolio of hedge funds

We consider the Dow Jones Credit Suisse AllHedge index. This index is composed of 10 subindexes: (1) convertible arbitrage, (2) dedicated short bias, (3) emerging markets, (4) equity market neutral, (5) event driven, (6) fixed income arbitrage, (7) global macro, (8) long/short equity, (9) managed futures and (10) multi-strategy. For the global index and all subindexes, we could obtain the monthly NAV from the web site `www.hedgeindex.com`.

We consider a volatility risk measure and statistical factors based on the principal component analysis of the two-year covariance matrix of asset returns. We do not try to interpret the PCA factors. We could build factors, which are perhaps more pertinent to analyse the hedge fund industry. Nevertheless, PCA is frequently used to classify dynamic strategies (Fung and Hsieh, 1997). The use of PCA holds great interest for our application, because it produces independent factors. We could then easily characterize the degree of diversification of the portfolio by using concentration indexes.

We compare three portfolios: the asset-weighted portfolio[15], the ERC portfolio and the factor-weighted portfolio. For the latter, the weights are computed such that the risk budget assigned to each of the first four PCA factors is equal to 25%. Each portfolio is rebalanced at the end of each month.

The risk decomposition with respect to the PCA factors is given in Figure 3. Most of the risk is concentrated on the first PCA factor for the asset-weighted portfolio. For the ERC portfolio, we obtain a more diversified allocation than the asset-weighted portfolio, but the ERC weights' ranges for a specific factor are large. It is not the case with the factor-weighted portfolio. Most of the times, we succeed in targeting the assigned budgets. The corresponding weights are given in Figure 4. We notice of course that the ERC-weighted portfolio or factor-weighted portfolio induce more turnover. The simulation of the performance is reported in Figure 5.

[15]The data for assets are no longer publicly available from the web site `www.hedgeindex.com`. Therefore, we consider them as constant and use an estimate of the average holdings.

Figure 3: Risk decomposition of the portfolios in terms of factors

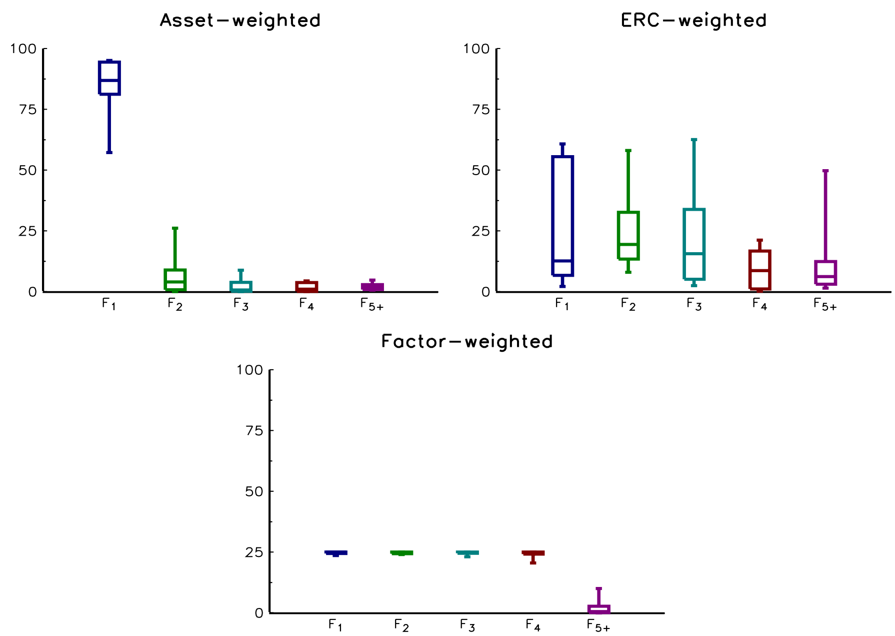


Figure 4: Weights of the portfolios

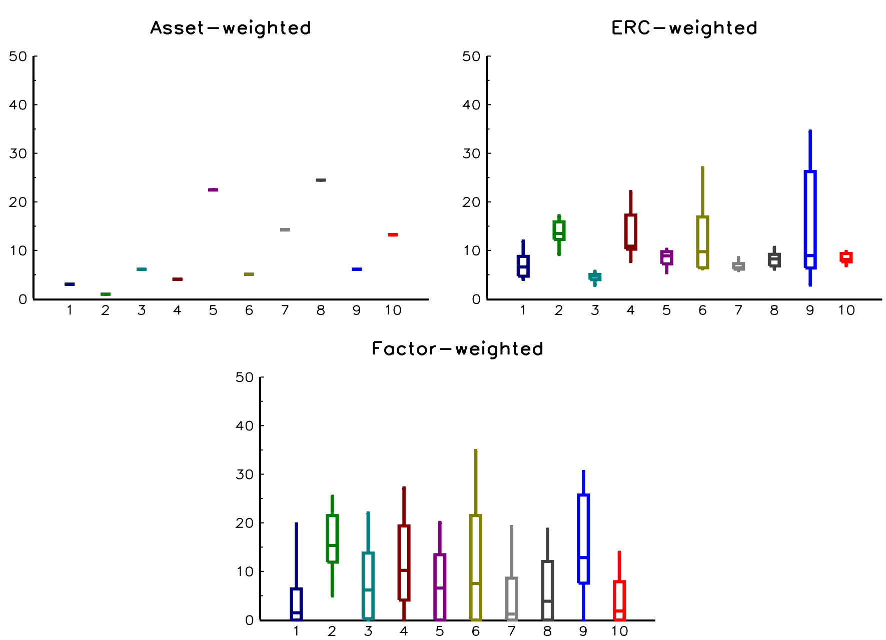
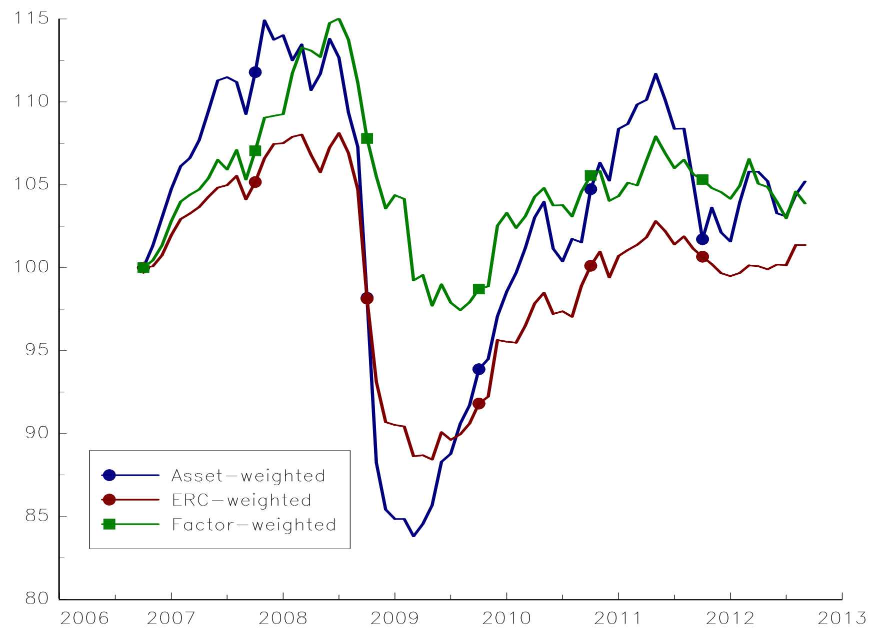


Figure 5: Performance simulation of the portfolios



In Table 13, we report for each portfolio statistics of performance and risk and measures of concentration using the risk contributions with respect to PCA factors[16]. We notice that the ERC-weighted portfolio and factor-weighted portfolio have smaller risks (volatility, drawdown and kurtosis) than the asset-weighted portfolio. Moreover, the factor-weighted portfolio improves significantly the risk diversification of the ERC portfolio. According to the Herfindahl index, the ERC portfolio plays less than three independent factors, whereas the factor-weighted portfolio is exposed to four independent factors.

Table 13: Statistics of hedge fund portfolios (Sep. 2009 - Aug. 2012)

	Asset-weighted	ERC-weighted	Factor-weighted
$\hat{\mu}_{1\text{Y}}$ (in %)	0.86	0.23	0.64
$\hat{\sigma}_{1\text{Y}}$ (in %)	7.93	4.85	4.58
$\mathcal{MDD}$ (in %)	-27.08	-18.22	-15.30
γ_1	-2.04	-1.84	-0.60
γ_2	6.24	6.88	1.68
$\mathcal{H}^\star$	0.72	0.30	0.14
$\mathcal{N}^\star$	1.40	2.96	4.16
$\mathcal{G}$	0.83	0.67	0.52
$\mathcal{I}^\star$	1.75	3.81	4.34

[16] $\hat{\mu}_{1\text{Y}}$ is the annualized performance, $\hat{\sigma}_{1\text{Y}}$ is the the yearly volatility and $\mathcal{MDD}$ is the maximum drawdown observed for the entire period. These statistics are expressed in %. Skewness and excess kurtosis correspond to γ_1 and γ_2. The concentration measures are computed using the risk contributions of the first four PCA factors: $\mathcal{H}^\star$ is the normalized Herfindahl index, $\mathcal{N}^\star = \mathcal{H}^{-1}$ is the effective number of independent factors, $\mathcal{G}$ is the Gini index and $\mathcal{I}^\star$ is the diversity measure based on the Shannon entropy.

5.3 Building a strategic asset allocation

Strategic asset allocation (SAA) is the choice of equities, bonds, and alternative assets that an investor wishes to hold in the long-run, usually from 10 to 50 years. Combined with tactical asset allocation (TAA) and constraints on liabilities, it defines the long-term investment policy of pension funds. By construction, SAA requires long-term assumptions involving asset risk/return characteristics as a key input. It could be done using macroeconomic models and forecasts of structural factors such as population growth, productivity and inflation (Eychenne *et al.*, 2011). Using these inputs, one may obtain a SAA portfolio using a mean-variance optimization procedure. However, due to the uncertainty of these inputs and the instability of mean-variance portfolios, many institutional investors prefer to use these long-run figures as a selection criterion for the asset classes they would like to have in their strategic portfolio and define the corresponding risk budgets.

This approach has been largely studied by Bruder and Roncalli (2012), who present, for instance, an example of the risk budgeting policy of a pension fund. Another example of such approach is the SAA policy adopted by Danish pension fund ATP. Indeed, the fund ATP defines its SAA using a risk parity approach. According[17] to Henrik Gade Jepsen, CIO of ATP:

> "*Like many risk practitioners, ATP follows a portfolio construction methodology that focuses on fundamental economic risks, and on the relative volatility contribution from its five risk classes. [...] The strategic risk allocation is 35% equity risk, 25% inflation risk, 20% interest rate risk, 10% credit risk and 10% commodity risk*".

These risk budgets are then transformed into asset classes' weights. At the end of Q1 2012, the asset allocation of ATP was also 52% in fixed-income, 15% in credit, 15% in equities, 16% in inflation and 3% in commodities[18].

In this paragraph, we explore a similar approach by combining the risk budgeting approach to define the asset allocation, and the economic approach to define the factors. This approach has been already proposed by Kaya *et al.* (2011) who use two economic factors: growth and inflation. As explained by Eychenne *et al.* (2011), these factors are the two main pillars of strategic asset allocation models. Using their long-run path, we could then define the long-run path for short rates, bonds, equities, high yield, etc. This approach is adequately suited for pension funds with liabilities which are indexed on some economic factors like inflation.

Following Eychenne *et al.* (2011), we consider 7 economic factors grouped into four categories:

1. activity: gdp & industrial production;
2. inflation: consumer prices & commodity prices;
3. interest rate: real interest rate & slope of the yield curve;
4. currency: real effective exchange rate.

[17] *Source*: Investment & Pensions Europe, June 2012, Special Report Risk Parity.
[18] *Source*: FTfm, June 10, 2012.

We collect quarterly data from Datastream. We estimate a model using YoY relative variations for the study period Q1 1999 – Q2 2012. We consider 13 asset classes classified as follows: equity (US, Euro, UK and Japan), sovereign bonds (US, Euro, UK and Japan), corporate bonds (US, Euro), High yield (US, Euro) and TIPS (US). We then build four portfolios (cf. Table 14). The first portfolio is a balanced stock/bond asset mix, the second portfolio represents a defensive allocation with only 20% invested in equities, and the third portfolio represents an aggressive allocation with 80% invested in equities. The fourth portfolio is calibrated such that activity, inflation, interest rates and currency represent respectively 34%, 20%, 40% and 5%. In this scenario, the overall weight on equities sums to 49%, while the weight on bonds sums to 51% with a large position on corporate bonds.

Table 14: Weights of the four SAA portfolios

	Equity				Sovereign Bonds				Corp. Bonds		High Yield		TIPS
	US	Euro	UK	Japan	US	Euro	UK	Japan	US	Euro	US	Euro	US
#1	20%	20%	5%	5%	10%	5%	5%	5%	5%	5%	5%	5%	5%
#2	10%	10%			20%	15%	5%	5%	5%	5%	5%	5%	15%
#3	30%	30%	10%	10%	10%	10%							
#4[a]	19.0%	21.7%	6.2%	2.3%		5.9%			24.1%	10.7%	2.6%	7.5%	

[a]Weights are estimated using the risk budgets of factors.

In Table 15, we report the risk contributions of these allocations with respect to our four categories and an additional grouping representing specific risk not explained by the economic factors. We obtain results coherent with financial and economic theories. For example, activity explains a large part of the risk of the aggressive portfolio (#3). The defensive portfolio (#2) concentrates most of the risk on interest rates. Holding a portfolio more exposed to inflation risk implies de-leveraging the exposure on sovereign bonds and TIPS (cf. Portfolio #4). We believe these results to be very appealing to pension funds. And, it explains why some are exploring this route in order to aline their strategic asset allocation with their liability constraints.

Table 15: Risk contributions of SAA portfolios with respect to economic factors

Factor	#1	#2	#3	#4
Activity	36.91%	19.18%	51.20%	34.00%
Inflation	12.26%	4.98%	9.31%	20.00%
Interest rate	42.80%	58.66%	32.92%	40.00%
Currency	7.26%	13.04%	5.10%	5.00%
Specific factors	0.77%	4.14%	1.47%	1.00%

6 Conclusion

This paper generalizes the risk parity approach of Bruder and Roncalli (2012) to consider risk factors instead of assets. It appears that the problem becomes trickier as multiple solutions can exist, and the existence of the RB portfolio is not guaranteed when we impose long-only constraints. We propose therefore to formulate the diversification problem in terms of risk factors as an optimization program.

We illustrate our methodology with real life examples. Our first application deals with risk budgeting of the Fama-French equity factors. Commonly, one uses regression models to measure the exposure of an equity portfolio with respect to these factors. The problem with such approach is that the real signification of the estimated beta coefficients remains unclear. Thus, we propose to replace the regression approach by a risk contribution approach which is more intuitive to portfolio managers. Our second application considers diversifying a hedge fund portfolio. Using PCA factors as the underlying sources of 'true' risk, we obtain some interesting results. Yet, we are cautious to claim better performance as the portfolio construction is very sensitive to these PCA factors, which are not always stable through time.

Finally, our last application studies the strategic asset allocation problem of pension funds that face liability constraints. The underlying idea is to build a portfolio in order to hedge some risk factors, like activity, inflation, interest rate or currency. Our preliminary results are promising. They open a door toward rethinking the long-term investment policy of pension funds.

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A Technical results

A.1 Notations and relations

Variable	Dimension	Expression
A	$(n \times m)$	$B^\top$
B	$(m \times n)$	$A^\top$
A^+	$(m \times n)$	$\left(B^\top\right)^+ = \left(B^+\right)^\top$
B^+	$(n \times m)$	$\left(A^\top\right)^+ = \left(A^+\right)^\top$
$\tilde{B}^+$	$(n \times r)$	$\ker\left(\left(B^+\right)^\top\right) = \ker\left(A^+\right)$
		$\left(B^+\right)^\top = \left(B^\top\right)^+ = A^+$
		$\left(B^+\right)^\top \tilde{B}^+ = A^+\tilde{B}^+ = \mathbf{0}$
$\left(\tilde{B}^+\right)^\top$	$(r \times n)$	$\ker\left(A^\top\right)^\top = \tilde{B}$
		$r = m - n$
$\tilde{B}$	$(r \times n)$	$\left(\tilde{B}^+\right)^+ \left(I - B^+A^\top\right) = \ker\left(B\right)^\top = \ker\left(A^\top\right)^\top$
		$\tilde{B}^\top \tilde{B} = I_n - AA^+$
x	$(n \times 1)$	$B^+ y + \tilde{B}^+ \tilde{y}$
y	$(m \times 1)$	$A^\top x$
$\tilde{y}$	$(r \times 1)$	$\left(\tilde{B}^+\right)^+ \left(x - B^+ y\right) = \tilde{B} x$
Π_t	(1×1)	$x^\top R_t = x^\top A \mathcal{F}_t + x^\top \varepsilon_t = y^\top \mathcal{F}_t + \tilde{y}^\top \tilde{\mathcal{F}}_t$
$\mathcal{F}_t$	$(m \times 1)$	$\left(B^+\right)^\top R_t$
$\tilde{\mathcal{F}}_t$	$(r \times 1)$	$\left(\tilde{B}^+\right)^\top R_t$
$\begin{pmatrix} \mathcal{F}_t \\ \tilde{\mathcal{F}}_t \end{pmatrix}$	$(n \times 1)$	$\bar{B} R_t$
$\bar{B}$	$(n \times n)$	$\begin{pmatrix} \left(B^+\right)^\top \\ \left(\tilde{B}^+\right)^\top \end{pmatrix}$
$\bar{B}^\top$	$(n \times n)$	$\begin{pmatrix} B^+ & \tilde{B}^+ \end{pmatrix}$
$\left(\bar{B}^\top\right)^{-1}$	$(n \times n)$	$\begin{pmatrix} B \\ \tilde{B} \end{pmatrix}$

A.2 Decomposition of the marginal risk contribution

We consider the following decomposition[19]:

$$x = \begin{pmatrix} B^{+} & \tilde{B}^{+} \end{pmatrix} \begin{pmatrix} y \\ \tilde{y} \end{pmatrix} = \bar{B}^{\top} \bar{y} \tag{12}$$

where $\tilde{B}^{+} = \operatorname{col}(B^{+})$ is any $n \times (n-m)$ matrix that spans the left nullspace of B^{+} which is also the left nullspace of A. Using results of Meucci (2007), $\bar{B}^{\top}$ is invertible by construction and we have:

$$\begin{pmatrix} \mathcal{F}_t \\ \tilde{\mathcal{F}}_t \end{pmatrix} = \bar{B} R_t$$

We have therefore $\Pi_t = y^{\top} \mathcal{F}_t + \tilde{y}^{\top} \tilde{\mathcal{F}}_t$. Using Equation (12) and the chain rule of calculus, it comes that:

$$\begin{aligned} \frac{\partial \mathcal{R}(x)}{\partial x} &= \frac{\partial \mathcal{R}(y, \tilde{y})}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial \mathcal{R}(y, \tilde{y})}{\partial \tilde{y}} \frac{\partial \tilde{y}}{\partial x} \\ &= \frac{\partial \mathcal{R}(y, \tilde{y})}{\partial y} B + \frac{\partial \mathcal{R}(y, \tilde{y})}{\partial \tilde{y}} \tilde{B} \\ &= \frac{\partial \mathcal{R}(\bar{y})}{\partial \bar{y}} \left(\bar{B}^{\top}\right)^{-1} \end{aligned} \tag{13}$$

with:

$$\left(\bar{B}^{\top}\right)^{-1} = \begin{pmatrix} B \\ \tilde{B} \end{pmatrix}$$

for some $(n-m) \times n$ matrix $\tilde{B}$. Thus, we deduce:

$$\frac{\partial \mathcal{R}(x)}{\partial x} = \frac{\partial \mathcal{R}(x)}{\partial y} B + \frac{\partial \mathcal{R}(x)}{\partial \tilde{y}} \tilde{B}$$

Using Equation (13), it comes that:

$$\frac{\partial \mathcal{R}(\bar{y})}{\partial \bar{y}} = \frac{\partial \mathcal{R}(x)}{\partial x} \bar{B}^{\top}$$

If we consider the risk factors $\mathcal{F}_t$, we have:

$$\frac{\partial \mathcal{R}(x)}{\partial y} = \frac{\partial \mathcal{R}(x)}{\partial x} B^{+}$$

Since $\left(B^{+}\right)^{\top} = \left(B^{\top}\right)^{+} = A^{+}$ by property of the Moore-Penrose inverse, we finally obtain that:

$$\frac{\partial \mathcal{R}(x)}{\partial y^{\top}} = \left(B^{+}\right)^{\top} \frac{\partial \mathcal{R}(x)}{\partial x^{\top}} = A^{+} \frac{\partial \mathcal{R}(x)}{\partial x^{\top}}$$

For the risk factors $\tilde{\mathcal{F}}_t$, the results become:

$$\frac{\partial \mathcal{R}(x)}{\partial \tilde{y}} = \frac{\partial \mathcal{R}(x)}{\partial x} \tilde{B}^{+}$$

and:

$$\frac{\partial \mathcal{R}(x)}{\partial \tilde{y}^{\top}} = \left(\tilde{B}^{+}\right)^{\top} \frac{\partial \mathcal{R}(x)}{\partial x^{\top}} = \tilde{B} \frac{\partial \mathcal{R}(x)}{\partial x^{\top}}$$

[19] We assume implicitly that the number of assets is larger than the number of factors ($n > m$).

A.3 Computing the risk contribution of factors

We assume that the risk measure $\mathcal{R}(x)$ is the volatility of the portfolio $\sigma(x)=\sqrt{x^\top \Sigma x}$. We obtain the following expression:

$$
\begin{aligned}
\mathcal{R}^2(x) &= \left(B^+ y+\tilde{B}^+\tilde{y}\right)^\top \Sigma \left(B^+ y+\tilde{B}^+\tilde{y}\right) \\
&= y^\top \left(B^+\right)^\top \Sigma B^+ y+\tilde{y}^\top \left(\tilde{B}^+\right)^\top \Sigma B^+ y+y^\top \left(B^+\right)^\top \Sigma \tilde{B}^+\tilde{y}+\tilde{y}^\top \left(\tilde{B}^+\right)^\top \Sigma \tilde{B}^+\tilde{y} \\
&= y^\top \left(B^+\right)^\top \Sigma B^+ y+\tilde{y}^\top \left(\tilde{B}^+\right)^\top \Sigma \tilde{B}^+\tilde{y}+2y^\top \left(B^+\right)^\top \Sigma \tilde{B}^+\tilde{y}
\end{aligned}
$$

It comes that:

$$
\begin{aligned}
\mathcal{R}(x) &= y^\top \frac{\left(B^+\right)^\top \Sigma B^+ y}{\sigma(x)}+\tilde{y}^\top \frac{\left(\tilde{B}^+\right)^\top \Sigma \tilde{B}^+\tilde{y}}{\sigma(x)}+2\frac{y^\top \left(B^+\right)^\top \Sigma \tilde{B}^+\tilde{y}}{\sigma(x)} \\
&= \sum_{j=1}^{m} \mathcal{RC}\left(\mathcal{F}_j\right)+\sum_{j=1}^{n-m} \mathcal{RC}\left(\tilde{\mathcal{F}}_j\right)
\end{aligned}
$$

A first idea is to assume that $\mathcal{F}_t$ and $\tilde{\mathcal{F}}_t$ are uncorrelated, and to identify the risk contribution of the j^{th} factor as:

$$
\mathcal{RC}\left(\mathcal{F}_j\right)=\frac{y_j\cdot\left(\left(B^+\right)^\top \Sigma B^+ y\right)_j}{\sigma(x)}
$$

But the problem is that even if $\left(B^+\right)^\top \tilde{B}^+=\mathbf{0}_{m\times(n-m)}$, $\left(B^+\right)^\top \Sigma\tilde{B}^+\neq\mathbf{0}_{m\times(n-m)}$. A second idea it to apply directly Theorem 1. In this case, we have:

$$
\mathcal{RC}\left(\mathcal{F}_j\right)=\frac{\left(A^\top x\right)_j\cdot\left(A^+\Sigma x\right)_j}{\sigma(x)}
$$

and:

$$
\mathcal{RC}\left(\tilde{\mathcal{F}}_j\right)=\frac{\left(\tilde{B}x\right)_j\cdot\left(\tilde{B}\Sigma x\right)_j}{\sigma(x)}
$$

A.4 Analysis of the optimisation problem (11) when $\mathcal{R}$ is the volatility risk measure

Recall that $\varphi(y)=-\tilde{\Omega}^{-1}\Gamma^\top y$ if $\mathcal{R}$ is the volatility risk measure. The positivity constraint on x in terms of y can then be written as:

$$
\left(B^+-\tilde{B}^+\tilde{\Omega}^{-1}\left(B^+\right)^\top \Sigma\tilde{B}^+\right)y\succeq\mathbf{0}
$$

The Lagrangian function of the problem (11) is:

$$\begin{aligned}
\mathcal{L}(y, \lambda_c, \lambda_y, \lambda_x) &= \mathcal{R}(y) - \lambda_c \left(\sum_{j=1}^{m} b_j \ln y_j - c \right) - \\
& \quad \lambda_y^\top y - \lambda_x^\top \left(B^+ y + \tilde{B}^+ \varphi(y) \right) \\
&= \mathcal{R}(y) - \lambda_c \left(\sum_{j=1}^{m} b_j \ln y_j - c \right) - \\
& \quad \left(\lambda_y^\top + \lambda_x^\top \left(B^+ - \tilde{B}^+ \tilde{\Omega}^{-1} \left(B^+ \right)^\top \Sigma \tilde{B}^+ \right) \right) y
\end{aligned}$$

The associated Karush-Kuhn-Tucker (KKT) conditions are then:

$$\begin{cases}
\left(\lambda_y^\top + \lambda_x^\top \left(B^+ - \tilde{B}^+ \tilde{\Omega}^{-1} \left(B^+ \right)^\top \Sigma \tilde{B}^+ \right) \right) y = 0 \\
\min \left(\lambda_c, \sum_{j=1}^{m} b_j \ln y_j - c \right) = 0 \\
\lambda \succeq \mathbf{0}
\end{cases}$$

with $\lambda = (\lambda_x, \lambda_y)$. Since, for *any* convex optimization problem with differentiable objective and constraints functions, any point that satisfy the KKT conditions are primal and dual optimal, and have zero duality gap (Boyd & Vandenberghe, 2004), we conclude that $y \succ \mathbf{0}$ and:

$$\left(\lambda_y^\top + \lambda_x^\top \left(B^+ - \tilde{B}^+ \tilde{\Omega}^{-1} \left(B^+ \right)^\top \Sigma \tilde{B}^+ \right) \right) = \mathbf{0}$$

Indeed, if the above equality is not verified then the Lagrange dual function is $-\infty$ and the primal problem has no solution because of strong duality. The first order conditions are given by:

$$y_j \frac{\partial \mathcal{R}(y)}{\partial y_j} = \lambda_c b_j$$

We verify as in Bruder & Roncalli (2012) that the risk contributions of each factor $\mathcal{F}_j$ are indeed proportional to the levels b_j.

A.4.1 Feasibility of linear inequality constraints

Boyd and Vandenberghe (2004) provide the condition to the feasibility of a set of linear inequalities $Mx \preceq N$. They show that the system of linear inequalities is feasible if, and only if, the following system is **not** feasible:

$$\begin{cases}
\lambda \succeq \mathbf{0} \\
M^\top \lambda = \mathbf{0} \\
N^\top \lambda < 0
\end{cases}$$

Similar conditions are also given for the set of strict linear inequalities $Mx \prec N$ as:

$$\begin{cases}
\lambda \succeq \mathbf{0} \\
\lambda \neq \mathbf{0} \\
M^\top \lambda = \mathbf{0} \\
N^\top \lambda \leq 0
\end{cases}$$

Using the above result, the conditions for the set of inequalities $y \succeq \mathbf{0}$ and $x \succeq \mathbf{0}$ to be feasible is obtained if the following alternative system is not feasible:

$$\begin{cases} \lambda \succeq \mathbf{0} \\ \left(A^+ - \left(\tilde{B}^+\right)^\top \Sigma\tilde{\Omega}^{-1} \left(\tilde{B}^+\right)^\top\right) \lambda_x + \lambda_y = \mathbf{0} \\ \mathbf{0}^\top \lambda_x + \mathbf{0}^\top \lambda_y < 0 \end{cases}$$

The alternative system above is however never feasible (because of the last inequality[20]). And there always exists a (possibly trivial) solution y satisfying $y \succeq \mathbf{0}$ and $x \succeq \mathbf{0}$. Note that the set of *strict* inequalities for $y \succ \mathbf{0}$ and $x \succ \mathbf{0}$ is however never satisfied for a solution of the optimization problem (11).

A.4.2 Existence of a non trivial solution to the set $y \succeq \mathbf{0}$ and $x \succeq \mathbf{0}$

The existence of a (non trivial) solution can be investigated by examining the following system:

$$\exists \zeta > 0 \quad \text{s.t.} \quad \begin{cases} y \succeq \zeta \mathbf{1} \\ \left(B^+ - \tilde{B}^+ \tilde{\Omega}^{-1} \left(B^+\right)^\top \Sigma \tilde{B}^+\right) y \succeq \mathbf{0} \end{cases}$$

It's easy to see that its alternative system given by:

$$\begin{cases} \left(A^+ - \left(\tilde{B}^+\right)^\top \Sigma\tilde{\Omega}^{-1} \left(\tilde{B}^+\right)^\top\right) \lambda_x + \lambda_y = \mathbf{0} \\ \lambda \succeq \mathbf{0} \\ \lambda_y \neq \mathbf{0} \end{cases} \tag{14}$$

does not depend on ζ. Thus, a sufficient condition for the system $y \succ \mathbf{0}$ and $x \succeq \mathbf{0}$ to be feasible is that there does not exist $(\lambda_x, \lambda_y) \in \mathbb{R}_+^n \times \mathbb{R}_{+*}^m$ such that the system (14) is feasible. However, if $y^\star$ is solution to the optimization problem (11), then there exist (λ_x, λ_y) such that:

$$\begin{cases} \left(A^+ - \left(\tilde{B}^+\right)^\top \Sigma\tilde{\Omega}^{-1} \left(\tilde{B}^+\right)^\top\right) \lambda_x + \lambda_y = \mathbf{0} \\ (\lambda_x, \lambda_y) \succeq \mathbf{0} \end{cases}$$

Using the results above, there are two possible scenarios:

1. $\lambda_y = \mathbf{0}$ and $\lambda_x \in \ker \left(A^+ - \left(\tilde{B}^+\right)^\top \Sigma\tilde{\Omega}^{-1} \left(\tilde{B}^+\right)^\top\right)$;
2. $\lambda_y \neq \mathbf{0}$ and there exists $\zeta > 0$ such that $0 \leq \min y_j \leq \zeta$.

Since the intersection of $\ker \left(A^+ - \left(\tilde{B}^+\right)^\top \Sigma\tilde{\Omega}^{-1} \left(\tilde{B}^+\right)^\top \quad I_m\right)$ and $\{\lambda \in \mathbb{R}^{n+m} \mid \lambda \succeq \mathbf{0}\}$ is likely to be larger than the singleton $\{\mathbf{0}\}$, there could be several extrema of the Lagrangian function that verify the second scenario.

[20] This inequality is derived from the fact that the alternative system is feasible only if the lagrangian *dual* function of the set of equalities is strictly positive (Boyd and Vandenberghe, 2004).