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Abstract
This paper considers the classical newsvendor model when, (a) demand is autocorrelated, (b) the parameters of the marginal distribution of demand are unknown, and (c) historical data for demand are available for a sample of successive periods. An estimator for the optimal order quantity is developed by replacing in the theoretical formula which gives this quantity the stationary mean and the stationary variance with their corresponding maximum likelihood estimators. The statistical properties of this estimator are explored and general expressions for prediction intervals for the optimal order quantity are derived in two cases: (a) when the sample consists of two observations, and (b) when the sample is considered as sufficiently large. Regarding the asymptotic prediction intervals, specifications of the general expression are obtained for the time-series models AR(1), MA(1), and ARMA(1,1). These intervals are estimated in finite samples using in their theoretical expressions, the sample mean, the sample variance, and estimates of the theoretical autocorrelation coefficients at lag one and lag two. To assess the impact of this estimation procedure on the optimal performance of the newsvendor model, four accuracy implication metrics are considered which are related to: (a) the mean square error of the estimator, (b) the accuracy and the validity of prediction intervals, and (c) the actual probability of running out of stock during the period when the optimal order quantity is estimated. For samples with more than two observations, these metrics are evaluated through simulations, and their values are presented to appropriately constructed tables. The general conclusion is that the accuracy and the validity of the estimation procedure for the optimal order quantity depends upon the critical fractile, the sample size, the autocorrelation level, and the convergence rate of the theoretical autocorrelation function to zero.

Keywords: Newsvendor model; accuracy implication metrics; time-series models; prediction intervals; Monte-Carlo simulations.

JEL Codes: C13; C22; C53; M11; M21.

1. Introduction

In the majority of papers in stock control, the optimal inventory policies are derived under two conditions: (a) the parameter values of the stochastic law which generates the demand are known, and (b) the demand in successive periods is formed independently. In practice, the first condition does not hold. One solution to this problem is the substitution of the true moments of the demand distribution in the theoretical formulae determining the target inventory measures with values which are obtained through certain estimation procedures (e.g. Syntetos & Boylan, 2008; Janssen et al., 2009). Then, in the context of managerial aspects of inventories, the combined estimation – stock control operation should be evaluated through specific accuracy implication metrics which are usually related to service levels and inventory costs (Boylan & Syntetos 2006; Syntetos et al., 2010).

Regarding the second condition, for the last three decades, an increasing number of works has been starting to appear in the literature aiming to study the effect of a serially correlated demand on the behavior of target inventory measures in stock control and supply chain management (Zhang, 2007). In this context, a variety of time-series models, including ARIMA processes and linear state space models (Aviv, 2003), have been used to describe the evolution of demand. Adopting these time-series demand models, the research has been expanded to resolve several problems in inventory management including the determination of safety stocks and optimal policies in continuous and periodic review systems, as well as, the study of the bullwhip effect and the value of information sharing.

For the classical newsvendor model, the research on determining the order quantity when the demand in successive periods is autocorrelated and the parameters of demand distribution are unknown is very limited. Although a number of works offer solutions to the problem of not knowing the parameters of demand distribution (e.g. Ritchken & Sankar, 1984; Liyanage & Shanthikumar, 2005; Kevork, 2010, Akcay et al., 2011; Halkos & Kevork, 2012a), these works assume that demand in successive periods is formed independently. To the extent of our knowledge, the work of Akcay et al. (2012) is the only one which addresses in the classical newsvendor model the issues of both the correlated demand and the demand parameters estimation. In particular, using a simulation-based sampling algorithm, this work quantifies the expected cost due to parameter uncertainty when the demand process is an autoregressive-to-anything time series, and the marginal demand distribution is represented by the Johnson translation system with unknown parameters.

In the current paper, we study the classical newsvendor model (e.g. Silver et al., 1998; Khouja, 1999) when it operates under optimal conditions, and the demand for each period (or inventory cycle) is generated by the non-zero mean linear process with independent normal errors which have zero mean and the same variance. Assuming that historical data on demand are available for the most recent n successive periods, we determine for period $n+1$ the order quantity, by replacing in the theoretical expression which holds under optimality the unknown true stationary mean and the unknown true stationary variance with their corresponding Maximum Likelihood (ML) estimates.

This process leads to deviations between the computed order quantity and the corresponding optimal one. These deviations are not systematic since they are caused by the variability of the sample mean and the sample variance. Therefore, we consider the computed order quantity as an estimate for the optimal order quantity. This estimate belongs to the sampling distribution of the estimator which has been constructed after replacing in the theoretical expression (which gives the optimal order quantity) the true mean and the true variance with their corresponding ML estimators.

The distribution of this estimator for the optimal order quantity is derived for $n = 2$ and when n is sufficiently large. Then, general expressions of the exact (for $n = 2$) and the asymptotic prediction interval for the optimal order quantity are obtained. Regarding the exact prediction interval, apart from the sample mean and the sample variance, its formula contains also the theoretical autocorrelation coefficient at lag one. Using the estimate of this coefficient, we take the corresponding estimated exact prediction interval whose performance is evaluated for different autocorrelation levels over a variety of choices for the critical fractile. The latter quantity is the probability not to experience a stock out during the period when the newsvendor model operates at optimal conditions. Although the case of $n = 2$ could be considered as an extreme case, and possibly not realistic, the examination of the properties of the exact prediction interval for such a very small sample size gives considerable insights in the process of estimating the optimal order quantity. Besides, as it will be clearer below, it is too difficult to give for $n > 2$ analogous general expressions for exact prediction intervals.

As it is not possible to obtain exact prediction intervals for any $n > 2$, to carry on with the estimation of the optimal order quantity at any finite sample, we use the general expression of the asymptotic prediction interval. To evaluate its performance in finite samples, we consider three special cases of the linear process, which are the time-series models AR(1), MA(1), and ARMA(1,1). For each model, the specification of the general expression of the asymptotic prediction interval is obtained. In the models AR(1) and MA(1), the specified formula contains, apart from the sample mean and the sample variance, the unknown true variance and the theoretical autocorrelation coefficient at lag one. The corresponding formula in the ARMA(1,1) includes also the theoretical autocorrelation coefficient at lag two. Replacing the variance and the two autocorrelation coefficients with their corresponding sample estimates, we get the estimated prediction intervals whose performance is also evaluated for different sample sizes and again for different autocorrelation levels over a variety of choices for the critical fractile.

To assess the impact of the aforementioned estimation procedure for the order quantity on the optimal performance of the newsvendor model, we consider four accuracy implication metrics which are related to:

(a) the accuracy of the prediction intervals,

(b) the validity of the prediction intervals,

(c) the mean square error of the estimator of the optimal order quantity, and

(d) the actual probability not to have a stock-out during the period when the optimal order quantity is estimated.

Exact values for the four metrics are obtained for $n = 2$. For larger sample sizes, the metrics are obtained through Monte Carlo simulations. The values of these four metrics enable us to trace at different autocorrelation levels the minimum required sample size so that the estimation procedure to have a negligible impact on the optimal performance of the newsvendor model.

To derive the prediction intervals for the optimal order quantity, we studied the conditions under which the sample mean and the sample variance are uncorrelated and independent. For the general ARMA model, Kang and Goldsman (1990) showed that the correlation between the sample mean and several variance estimators is zero. These variance estimators are based on the techniques of the non-overlapping/overlapping batched means and the standardized time series. Bayazit et al. (1985) offered an expression for the covariance of the sample mean and the sample variance of a skewed AR(1).

Extending these findings, we prove in our work two further results. First, for any sample with two observations being drawn from the general linear process with independent normal errors, which have zero mean and constant variance, the sample mean and the sample variance are independent. Second, for the same linear process, when the theoretical autocorrelation function is positive and the autocorrelation coefficients are getting smaller as the lag increases, in any sample with more than two observations, the sample mean and the sample variance are uncorrelated but not independent.

Given the above arguments and remarks, the rest of the paper is organized as follows. In the next section we give a brief literature review of studies which adopted time series models to describe the evolution of demand in continuous review and periodic review inventory systems. In section 3 we present the newsvendor model with the demand in each period to follow the non-zero mean linear process, and we derive the theoretical expression which determines the optimal order quantity. In section 4 we derive the general expressions of the exact for $n = 2$ and the asymptotic prediction interval for the optimal order quantity. The evaluation of the estimated prediction intervals is performed and presented in section 5. Finally, section 6 concludes the paper summarizing the most important findings.

2. A brief review of the relevant literature

In the context of continuous review systems, the AR(1) and MA(1) processes have been adopted as demand models for studying customer service levels and deriving safety stocks and reorder points. Zinn et al. (1992) explained and quantified through simulations the effect of correlated demand on pre-specified levels of customer service when lead-time distribution is discrete uniform. Fotopoulos et al. (1988) offered a new method to find an upper bound of the safety stock when the lead time follows an arbitrary distribution. Ray (1982) derived the variance of the lead-time demand under fixed and random lead times when the parameters of the AR(1) and MA(1) are known, and when the expected demand during lead time is forecast. With fixed lead times, Urban (2000) derived variable reorder levels using for the demand during lead time appropriate forecasts and time-varying forecast errors variance/covariance, which are updated every period conditional upon the most recent observed demand.

For periodic review systems, Johnson and Thompson (1975) showed that when demand is generated by the stationary general autoregressive process, the myopic policy for the one period is optimal for any period of an infinite time horizon. To prove it they showed that in any period it is always possible to order up to the optimal order quantity. Assuming that demand is normal and covariance-stationary with known autocovariance function, Charnes et al. (1995) derived the safety stock required to achieve the desired stock-out probability with an order-up-to an initial inventory level. Urban (2005) developed a periodic review model when demand is AR(1) and depends on the amount of inventory displayed to the customer. Zhang (2007) quantified the effect of a temporal heterogeneous variance on the performance of a periodic review system using an AR(1) and a GARCH(1,1) to describe the dynamic changes in the level and the variance of demand respectively. Adopting the ARIMA (0,1,1) as the demand generating process in a periodic review system, Strijbosch et al. (2011) studied the effect of single exponential smoothing and simple moving average estimates on the fill rate conducting appropriate simulations.

Apart from the classical continuous and periodic review systems, there are also other active research areas on inventories where time series processes have been adopted as demand models. For instance, Zhang (2007) provides a list of works which use time series models to study the bullwhip effect, the value of information sharing, and the evolution of demand in supply chains. Ali et al. (2011) also provide a relevant literature for those works which by adopting time series processes as demand models explore the interaction between forecasting performance and inventory implications.

3. Background

Suppose that the demand size for period t (or inventory cycle t) of the classical newsvendor model is generated by the non- zero mean linear process

$$Y_t = \mu + \sum_{k=0}^{\infty} \psi_j \varepsilon_{t-k} , \tag{1}$$

where $\sum_{k=0}^{\infty} |\psi_k| < \infty$, and ε_t**s** are independent normal variables with $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$. Denote also by Q_t the order quantity for period t, ***p*** the selling price, ***c*** the purchase cost per unit, ***v*** the salvage value, and ***s*** the loss of goodwill per unit of product. To satisfy the demand of period t, the newsvendor has available stock at the start of the period only the order quantity Q_t. This means that any excess inventory at the end of period $t-1$ was disposed of through either consignment stocks or buyback arrangements and the salvage value was used to settle such arrangements. Further, by receiving this order quantity, no fixed costs are charged to the newsvendor.

Under the aforementioned notation and assumptions, and providing that the coefficient of variation of the marginal distribution of Y_t is not large (e.g. see Halkos and Kevork, 2012b), the expected profit of the newsvendor at the end of period t is derived from expression (1) of Kevork (2010) and is given by

$$E(\pi_t) = (p-c)Q_t - (p-v)[Q_t - E(Y_t | Q_t - Y_t \geq 0)]\Pr(Q_t - Y_t \geq 0) +$$

$$+ s[Q_t - E(Y_t | Q_t - Y_t < 0)]\Pr(Q_t - Y_t < 0).$$

Let $\varphi(z)$ and $\Phi(z)$ be respectively the probability density function and the distribution function of the standard normal evaluated at $z = (Q_t - \mu)/\sqrt{\gamma_o}$, where γ_o is the variance of the marginal distribution of Y_t. Since

$$\Pr(Q_t - Y_t \geq 0) = \Pr\left(\sum_{k=0}^{\infty} \psi_k \varepsilon_{t-k} \leq Q_t - \mu\right) = \Phi(z),$$

$$E(Y_t | Q_t - Y_t \geq 0) = \mu + E\left(\sum_{k=0}^{\infty} \psi_k \varepsilon_{t-k} \middle| \sum_{k=0}^{\infty} \psi_k \varepsilon_{t-k} \leq Q_t - \mu\right) =$$

$$= \mu + \sqrt{\gamma_o} E\left(Z | Z \leq \frac{Q_t - \mu}{\sqrt{\gamma_o}} \right) = \mu - \sqrt{\gamma_o} \frac{\varphi(z)}{\Phi(z)},$$

and

$$E(Y_t | Q_t - Y_t < 0) = \mu + \sqrt{\gamma_o} E\left(Z | Z > \frac{Q_t - \mu}{\sqrt{\gamma_o}} \right) = \mu + \sqrt{\gamma_o} \frac{\varphi(z)}{1 - \Phi(z)},$$

the expected profit becomes

$$E(\pi_t) = (p-c)Q_t + s(Q_t - \mu) - (p - v + s)\left\{(Q_t - \mu)\Phi(z) + \varphi(z)\sqrt{\gamma_o}\right\}.$$

The first and second order conditions obtained by differentiating $E(\pi_t)$ with respect to Q_t are

$$\frac{dE(\pi_t)}{dQ_t} = (p - c + s) - (p - v + s)\Phi(z) = 0$$

and

$$\frac{d^2E(\pi_t)}{dQ_t^2} = -(p - c + s)\frac{\varphi(z)}{\sqrt{\gamma_o}} < 0 .$$

Setting $R = (p - c + s)/(p - v + s)$, the first order condition leads to the following equation, which is known as the ctitical fractile equation,

$$\Phi_{z_R} = \Pr\left(Y_t \le Q_t^*\right) = \Pr\left(Z \le z_R\right) = \frac{p - c + s}{p - v + s} = R,$$

where R is the critical fractile, Q_t^* is the optimal order quantity, and $z_R = \left(Q_t^* - \mu\right)/\sqrt{\gamma_o}$. Thus the optimal order quantity for period t is determined from

$$Q_t^* = \mu + z_R\sqrt{\gamma_o}\,. \tag{2}$$

In the classical newsvendor model, no stock is carried from previous periods to the current. So, for a time horizon consisting of a number of periods, if the distribution of demand in each period remained the same with the same mean and the same variance, the optimal order quantity would depend upon only the critical fractile R. And R is function of the overage and underage costs. In the analysis which follows, to simplify notations and symbols, we shall assume that in each period of the time horizon for which demand data are available, the critical fractile does not change. Thus for each period of the time horizon, the optimal order quantity remains the same, and so it is legitimate to drop out the subscript t from the symbol of the optimal order quantity.

4. Prediction intervals for the optimal order quantity Q^*

Suppose that the linear process given in (1) has generated the realization $Y_1, Y_2, \ldots, Y_n$, which represents demand for the most recent n successive periods in the newsvendor model. Further, let $\overline{Y} = \sum_{t=1}^{n} Y_t / n$ and $\hat{\gamma}_o = \sum_{t=1}^{n}\left(Y_t - \overline{Y}\right)^2 / n$ be ML estimators of μ and γ_o respectively. Since in practice μ and γ_o are unknown quantities, replacing the ML estimators into (2), in the places of μ and γ_o, the resulting estimator for the optimal order quantity takes the form

$$\hat{Q}^* = f(\overline{Y}, \hat{\gamma}_o) = \overline{Y} + z_R \sqrt{\hat{\gamma}_o}\,. \qquad (3)$$

Given the estimator $\hat{Q}^*$, the rest of this section is organized as follows. At first, we derive the general expression for the exact prediction interval for Q^* when the sample consists of two observations. On the other hand, it is too difficult to give for $n > 2$ analogous general expressions for the exact prediction interval for two reasons. The first is that the sample variance consists of correlated chi-squared random variables and the second reason is that, as we shall show later, for the time series models AR(1), MA(1), and ARMA(1,1), the sample mean and the sample variance are not independent.

Despite the dependency of the sample mean and the sample variance, in the second part of this section we prove that for any $n > 2$ their covariance is zero. So, with the asymptotic distributions of $\overline{Y}$ and $\hat{\gamma}_o$ to be available in the literature, this allows us to construct the asymptotic variance-covariance matrix of $\overline{Y}$ and $\hat{\gamma}_o$, and then, by applying the multivariate Delta method, to derive the general expression of the asymptotic distribution of $\hat{Q}^*$.

4.1 Exact prediction interval for Q^* when $n = 2$

If Y_t is determined by the linear process given in (1) with $\sum_{k=0}^{\infty} |\psi_k| < \infty$ and ε_t's to be i.i.d. normal random variables with zero mean and constant variance, the sample Y_1, Y_2 follows the bivariate normal with marginal mean μ and variance γ_o. In this case, $\hat{\gamma}_o = X^2$, where $X = (Y_1 - Y_2)/2$, with $E(X) = 0$, $\mathrm{Var}(X) = \gamma_o (1 - \rho_1)/2$ and ρ_1 to be the autocorrelation coefficient at lag one. Then the statistic

$\sqrt{2}X/\sqrt{\gamma_o(1-\rho_1)}$ follows the standard normal, and hence $2\hat{\gamma}_o/(1-\rho_1) \sim \gamma_o\chi_1^2$. It also holds that the sample mean $\overline{Y} = (Y_1 + Y_2)/2$ is normally distributed with mean μ and variance $\gamma_o(1+\rho_1)/2$.

Proposition 1: If Y_1, Y_2 is a sample drawn from the linear process given in (1), with $\sum_{k=0}^{\infty}|\psi_k| < \infty$, and ε_t's to be i.i.d. random variables with $\varepsilon_t \sim N(0,\sigma_\varepsilon^2)$, then $\overline{Y}$ and $\hat{\gamma}_o$ are distributed independently.

Proof: See in the appendix.

Using the result of proposition 1, together with the distributions of $\overline{Y}$ and $\hat{\gamma}_o$, for $n = 2$ we derive the following statistic

$$T' = \frac{\frac{\sqrt{2}(\overline{Y}-\mu)}{\sqrt{\gamma_o(1+\rho_1)}} + \left(-z_R\sqrt{\frac{2}{1+\rho_1}}\right)}{\sqrt{\frac{2\hat{\gamma}_o}{(1-\rho_1)\gamma_o}}} = \sqrt{\frac{1-\rho_1}{1+\rho_1}}\frac{\overline{Y}-Q^*}{\sqrt{\hat{\gamma}_o}} \sim t_1'(\lambda), \tag{4}$$

where $t_1'(\lambda)$ is the non-central student-t distribution with one degree of freedom and non-centrality parameter

$$\lambda = -z_R\sqrt{\frac{2}{1+\rho_1}}. \tag{5}$$

So the interval

$$\left[\overline{Y} - t'_{1,1-\frac{\alpha}{2}}(\lambda)\sqrt{\hat{\gamma}_o\frac{1+\rho_1}{1-\rho_1}}\ ,\ \overline{Y} - t'_{1,\frac{\alpha}{2}}(\lambda)\sqrt{\hat{\gamma}_o\frac{1+\rho_1}{1-\rho_1}}\right] \tag{6}$$

is the exact $(1-\alpha)100\%$ prediction interval (P.I.) for Q^* for the special case where $n = 2$.

Remark 1: The exact distribution of $\hat{\gamma}_o$ for $n = 2$ allows the exact computation of the Bias of $\hat{Q}_t^*$. The statistic $\sqrt{2\hat{\gamma}_o / [\gamma_o(1-\rho_1)]}$ follows the chi-distribution with one degree of freedom, and so we have $E\left(\sqrt{\hat{\gamma}_o}\right) = \sqrt{\gamma_o(1-\rho_1)/\pi}$ and

$$\text{Bias}\left(\hat{Q}^*\right) = E\left(\hat{Q}^*\right) - Q^* = z_R\sqrt{\gamma_o}\left(\sqrt{\frac{1-\rho_1}{\pi}} - 1\right).$$

The expression within the parentheses containing ρ_1 and π is always negative. So, for any $R < 0.5$ the *bias* of $\hat{Q}^*$ is positive, while for $R > 0.5$ the *bias* is negative.

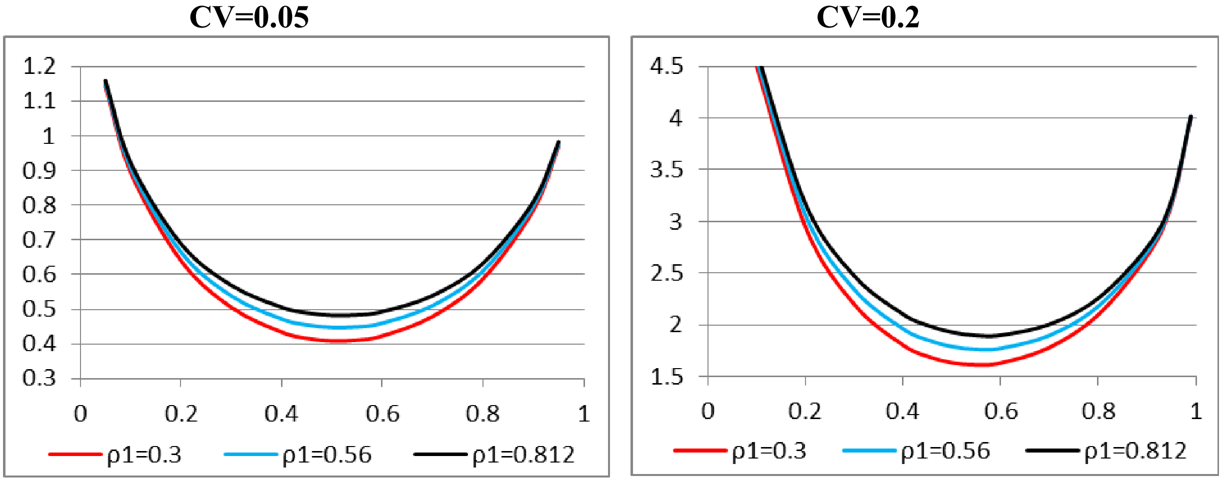
Remark 2: To evaluate the performance of the exact P.I. of (6), we define its relative expected half-length (REHL) as

$$\text{REHL} = \sqrt{\frac{1+\rho_1}{1-\rho_1}}\,\frac{t'_{1,1-\frac{\alpha}{2}}(\lambda) - t'_{1,\frac{\alpha}{2}}(\lambda)}{2Q^*}\,E\left(\sqrt{\hat{\gamma}_o}\right) = \sqrt{\frac{1+\rho_1}{\pi}}\,\frac{t'_{1,1-\frac{\alpha}{2}}(\lambda) - t'_{1,\frac{\alpha}{2}}(\lambda)}{2\left(CV^{-1} + z_R\right)}, \tag{7}$$

where $CV = \sqrt{\gamma_o}/\mu$ is the coefficient of variation, and Q^* is given in (2). Dividing by Q^* ensures the comparability of REHLs evaluated at different Rs, since increasing the critical fractile R, we get larger optimal order quantities.

Figure 1 illustrates the graph of the REHL versus R for different values of CV and ρ_1. The choice of values for ρ_1 is explained in the next section. By setting also the maximum CV at 0.2, we avoid to take a negative demand (especially in the simulations which are described in the next section), as we give a negligible probability (less than 0.00003%) to take a negative value from the marginal distribution of Y_t. Finally, the critical values of the non-central t were obtained through the statistical package MINITAB.

Figure 1: Graph of REHL as a function of R; n=2 and nominal confidence level 95%.



Looking at the two graphs of Figure 1, we first observe the increase of the REHL as R is getting closer either to zero or to one. Given CV and R, as ρ_1 increases the REHL is also increasing, while given ρ_1 and R, a higher CV results in larger REHLs. Finally, we observe that as CV is getting larger, the minimum of REHL is slightly shifted to the right of R=0.5.

4.2 Asymptotic prediction intervals for Q^*

To derive the asymptotic distribution of $\hat{Q}^*$, we shall use the asymptotic distributions of $\overline{Y}$ and $\hat{\gamma}_o$ stated in Priestley (1981, pp. 338, 339). Especially, when demand follows the non-zero mean linear process given in (1) with $\sum_{k=0}^{\infty}|\psi_k| < \infty$, and ε_t's to be independent normal variables with zero mean and constant variance, it holds that,

(a) $\sqrt{n}\left(\overline{Y} - \mu\right)$ has a limiting normal distribution with zero mean and variance

$$\gamma_o \sum_{k=-\infty}^{+\infty} \rho_k \text{ , and}$$

(b) $\sqrt{n}\left(\hat{\gamma}_o - \gamma_o\right)$ is asymptotically normal with mean zero and variance $2\gamma_o^2 \sum_{k=-\infty}^{+\infty} \rho_k^2$,

where ρ_k is the autocorrelation coefficient at lag k.

Proposition 2: Let $Y_1, Y_2, ..., Y_n$ be a sample drawn from the linear process given in (1) with $\sum_{k=0}^{\infty}|\psi_k| < \infty$, and ε_t's to be i.i.d. random variables with $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$. Then, for any sample size the covariance of the sample mean and the sample variance is zero.

Proof: See in the appendix.

The result of proposition 2, together with the asymptotic distributions of $\overline{Y}$ and $\hat{\gamma}_o$ lead us to state that

$$\sqrt{n}\begin{bmatrix} \overline{Y} - \mu \\ \hat{\gamma}_o - \hat{\gamma}_o \end{bmatrix} \xrightarrow{D} N_2(\mathbf{0}, \mathbf{\Sigma}),$$

where
$$\mathbf{\Sigma} = \begin{bmatrix} \gamma_o \sum_{k=-\infty}^{+\infty} \rho_k & 0 \\ 0 & 2\gamma_o^2 \sum_{k=-\infty}^{+\infty} \rho_k^2 \end{bmatrix},$$

"*D*" stands for convergence in distribution, and N_2 is the bivariate normal distribution. It also holds that

$$\text{p}\lim \hat{Q}^* = \textit{plim}\overline{Y} + z_R (\text{p}\lim \hat{\gamma}_o)^{0.5} = \mu + z_R \sqrt{\gamma_o} = Q^*.$$

So, by applying the multivariate Delta Method (e.g. Knight, 2000 pp. 149) we take

$$\sqrt{n}\{f(\overline{Y}, \hat{\gamma}_o) - f(\mu, \gamma_o)\} = \sqrt{n}(\hat{Q}^* - Q^*) \xrightarrow{D} N(\mathbf{0}, \mathbf{L}' \cdot \mathbf{\Sigma} \cdot \mathbf{L}),$$

$$\mathbf{L}' = \begin{bmatrix} \left.\frac{\partial f}{\partial \overline{Y}}\right|_{\substack{\overline{Y}=\mu \\ \hat{\gamma}_o = \gamma_o}} & \left.\frac{\partial f}{\partial \hat{\gamma}_o}\right|_{\substack{\overline{Y}=\mu \\ \hat{\gamma}_o = \gamma_o}} \end{bmatrix} = \begin{bmatrix} 1 & \frac{z_R}{2\sqrt{\gamma_o}} \end{bmatrix},$$

and

$$\mathbf{L}' \cdot \mathbf{\Sigma} \cdot \mathbf{L} = \gamma_o \left\{ \sum_{k=-\infty}^{+\infty} \rho_k + \frac{z_R^2}{2} \sum_{k=-\infty}^{+\infty} \rho_k^2 \right\}.$$

Thus,

$$\frac{\sqrt{n}\left(\hat{Q}^* - Q^*\right)}{\sqrt{\gamma_o\left(\sum_{k=-\infty}^{+\infty}\rho_k + \frac{z_R^2}{2}\sum_{k=-\infty}^{+\infty}\rho_k^2\right)}} \xrightarrow{D} N(0,1), \tag{8}$$

and so the asymptotic $(1-\alpha)100\%$ prediction interval for Q^* will have the form

$$\hat{Q}^* \pm z_{\alpha/2}\sqrt{\frac{\gamma_o}{n}\left(\sum_{k=-\infty}^{+\infty}\rho_k + \frac{z_R^2}{2}\sum_{k=-\infty}^{+\infty}\rho_k^2\right)}. \tag{9}$$

Example 1: Consider the stationary AR(1) model given by $Y_t = \mu + \phi(Y_{t-1} - \mu) + \varepsilon_t$, where $|\phi| < 1$, $\gamma_o = \sigma_\varepsilon^2/(1-\phi^2)$, and $\rho_k = \phi^k$ (k=0, 1, 2,…). Considering that the process has been started at some time in the remote past, and substituting repeatedly for Y_{t-1}, Y_{t-2}, Y_{t-3}, …, the AR(1) takes the form of process (1) with $\psi_j = \phi^j$. Then we have

$$\sum_{k=-\infty}^{\infty}\rho_k = 1 + 2\sum_{k=0}^{\infty}\rho_k = 1 + \frac{2\phi}{1-\phi} = \frac{1+\phi}{1-\phi}$$

and

$$\sum_{k=-\infty}^{\infty}\rho_k^2 = 1 + 2\sum_{k=0}^{\infty}\rho_k^2 = 1 + \frac{2\phi^2}{1-\phi^2} = \frac{1+\phi^2}{1-\phi^2}.$$

Hence the asymptotic prediction interval for Q^* is specified as

$$\hat{Q}^* \pm z_{\alpha/2}\sqrt{\frac{\gamma_o}{n}\left(\frac{1+\rho_1}{1-\rho_1} + \frac{z_R^2}{2}\frac{1+\rho_1^2}{1-\rho_1^2}\right)}, \tag{10}$$

since $\rho_1 = \phi$.

Example 2: Consider the invertible MA(1) model, $Y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1}$, with $|\theta| < 1$, $\gamma_o = \sigma_\varepsilon^2(1+\theta^2)$, $\rho_1 = \theta/(1+\theta^2)$, and $\rho_k = 0$ for $k \geq 2$. This model takes the form of process (1) by setting $\psi_o = 1$, $\psi_1 = \theta$, and $\psi_k = 0$ for $k \geq 2$. Hence we take

$\sum_{k=-\infty}^{\infty}\rho_k = 1 + 2\rho_1$, $\sum_{k=-\infty}^{\infty}\rho_k^2 = 1 + 2\rho_1^2$, and so the asymptotic prediction interval for Q^* is given by

$$\hat{Q}^* \pm z_{\alpha/2}\sqrt{\frac{\gamma_o}{n}\left(1 + 2\rho_1 + \frac{z_R^2}{2}\left(1 + 2\rho_1^2\right)\right)}. \tag{11}$$

Example 3: Consider the stationary and invertible ARMA(1,1) model which is given by $Y_t = \mu + \phi(Y_{t-1} - \mu) + \varepsilon_t + \theta\varepsilon_{t-1}$, $|\phi| < 1$, $|\theta| < 1$, $\gamma_o = \frac{1 + \theta^2 + 2\phi\theta}{1 - \phi^2}\sigma_\varepsilon^2$, $\rho_1 = \frac{(1 + \phi\theta)(\phi + \theta)}{1 + \theta^2 + 2\phi\theta}$, and $\rho_k = \phi^{k-1}\rho_1$ for $k \geq 2$. Given that the process has been started at some time in the remote past, Harvey (1993, pp. 26) shows that this model takes the form of process (1) with $\psi_o = 1$, $\psi_1 = \phi + \theta$, and $\psi_k = \phi\psi_{k-1}$ for $k \geq 2$. Thus, we take $\sum_{k=-\infty}^{\infty}\rho_k = 1 + 2\rho_1/(1 - \phi)$ and $\sum_{k=-\infty}^{\infty}\rho_k^2 = 1 + 2\rho_1^2/(1 - \phi^2)$. So the asymptotic prediction interval for Q^* is specified as

$$\hat{Q}^* \pm z_{\alpha/2}\sqrt{\frac{\gamma_o}{n}\left(1 + \frac{2\rho_1^2}{\rho_1 - \rho_2} + \frac{z_R^2}{2}\left(1 + \frac{2\rho_1^4}{\rho_1^2 - \rho_2^2}\right)\right)}, \tag{12}$$

after replacing ϕ by the ratio ρ_2/ρ_1.

We are closing this section by noting that for the three aforementioned time series models and for $n > 2$ the sample mean and the sample variance are not independent random variables. This is proved in proposition 3. So, it is required these intervals to be evaluated when they are applied to finite samples after replacing γ_o, ρ_1 and ρ_2 with their sample estimates. The results from this exercise and the relevant discussion are given in the next section.

Proposition 3: Let $Y_1, Y_2, ..., Y_n$ be a sample from the linear process given in (1) with $\sum_{k=0}^{\infty} |\psi_k| < \infty$, and ε_t's to be i.i.d. random variables with $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$. Suppose also that appropriate values are assigned to the ψ_j's weights such that $0 \le \rho_{n-1} < \rho_{n-2} < ... < \rho_1 < 1$. Then for any $n > 2$, the ML estimators $\overline{Y}$ and $\hat{\gamma}_o$ are not independent.

Proof: See in the appendix.

5. Prediction Interval Estimation

In this section we assess the performance of prediction intervals (6) and (9) when the demand in each period of the newsvendor model is generated by the three time-series models AR(1), MA(1), and ARMA(1,1). The evaluation is performed over a variety of values for the critical fractile R, and choices of number of observations in the sample n, when in the expressions (6), (10), (11) and (12) the unknown population parameters are replaced respectively with the sample mean $\overline{Y}$, the sample variance $\hat{\gamma}_o$, and the estimates of the theoretical autocorrelation coefficients ρ_1 and ρ_2, which are obtained from (e.g. see Harvey, 1993, page 11)

$$\hat{\rho}_k = \frac{\hat{\gamma}_k}{\hat{\gamma}_o} = \frac{\sum_{t=k+1}^{n} (Y_t - \overline{Y})(Y_{t-k} - \overline{Y})}{\sum_{t=1}^{k} (Y_t - \overline{Y})^2}.$$

For ease of exposition we divided this section into three parts. In the first part, we justify the choice of the parameter values for the three models, describe the evaluation criteria, and present the process of generating different realizations (or replications) for each model through Monte-Carlo simulations. In the second part, we derive and present some exact results for the evaluation criteria when the sample consists of only two observations. Finally, in the third part, we present and discuss simulation results for the evaluation criteria, which are computed for different sample sizes drawn from the generated replications of each model.

5.1 Design of the experimental framework

The choice of values for the parameters ϕ, θ and σ_ε^2 of the three models AR(1), MA(1), and ARMA(1,1) was made up under the following three principles:

(a) to produce different forms and levels of autocorrelation,

(b) for all the models under consideration the marginal distribution of Y_t to have the same stationary mean, μ, and the same variance, γ_o, ensuring in that way the same coefficient of variation, and

(c) the theoretical autocorrelation coefficient at lag one to be the same for the pairs AR(1) with MA(1), and AR(1) with ARMA(1,1).

The specifications of the three models which form the basis of our experimental framework are described in Table 1.

Table 1: Parameter values for the time series models

$\mu = 100$, $\gamma_o = 400$, CV = 0.2	
$\rho_1 = 0.3$	
AR(1)	: $\phi = 0.3$, $\sigma_\varepsilon^2 = 364$
$\rho_1 = 0.48$	
AR(1)	: $\phi = 0.48$, $\sigma_\varepsilon^2 = 307.84$
MA(1)	: $\theta = 0.75$, $\sigma_\varepsilon^2 = 256$
$\rho_1 = 0.56$	
AR(1)	: $\phi = 0.56$, $\sigma_\varepsilon^2 = 274.56$
ARMA(1,1)	: $\phi = 0.3$, $\theta = 0.4$, $\sigma_\varepsilon^2 = 260$
$\rho_1 = 0.812$	
AR(1)	: $\phi = 0.812$, $\sigma_\varepsilon^2 = 136.2624$
ARMA(1,1)	: $\phi = 0.68$, $\theta = 0.44$, $\sigma_\varepsilon^2 = 120$

After replacing in (6), (10), (11), (12), the population parameters μ, γ_o, ρ_1 and ρ_2 by their corresponding sample estimates, the performance of the estimated prediction intervals is assessed through four Accuracy Implication Metrics (AIM). The first AIM is the actual probability the estimated prediction interval to include (or otherwise to cover) the unknown population parameter which in our case is the optimal order quantity Q^*. We call this actual probability as coverage (CVG). The next two AIMs are related to the precision of the estimated prediction intervals. Particularly, we consider the Relative Mean Square Error (RMSE) of the estimator $\hat{Q}^*$ and the relative expected half length (REHL) of the prediction interval for Q^*. These two metrics are computed by dividing respectively the mean square error and the expected half-length of the interval by Q^*. The justification of dividing by Q^* has been already explained in the previous section.

The last AIM is related to the actual probability R_{act} not to experience a stock-out during the period. The use of this metric is imposed since by replacing in (2) the unknown quantities μ, γ_o with their corresponding estimates, it is very likely the order quantity to differ from Q^*. Then, when the newsvendor model operates at the optimal situation, the probability of not experiencing a stock-out during the period is not equal to the critical fractile R. The last AIM, therefore, gives the difference $R - R_{act}$.

For the time series models of Table 1, we showed in the previous section that for any $n > 2$, $\overline{Y}$ and $\hat{\gamma}_o$ are not independent. Due to the dependency of $\overline{Y}$ and $\hat{\gamma}_o$, it is extremely difficult, or even impossible, to derive for $n > 2$ the exact distribution of the estimator $\hat{Q}^*$, and so to obtain exact values for the four aforementioned AIMs. To overcome this problem, we organized and conducted appropriate Monte-Carlo

Simulations. In particular, for each model of Table 1, 20000 independent replications of maximum size 2001 observations were generated. To achieve stationarity in each AR(1) and ARMA(1,1), Y_o was generated from the stationary distribution $N(\mu, \gamma_o)$, with $\mu = 100$ and $\gamma_o = 400$. These values for μ and γ_o were also used in the MA(1).

Furthermore, in each replication of ARMA(1,1) and MA(1), ε_o was generated from the distribution $N(0, \sigma_\varepsilon^2)$, with the values of σ_ε^2 to be given in Table 1. We found out that with ε_o randomly generated, for $n = 2$, the simulated results for the CVG and the REHL were very close to the corresponding exact ones. On the contrary, starting each replication with $\varepsilon_o = 0$, the observed discrepancies among simulated and exact results of CVGs and REHLs were considerable.

For each model of Table 1, and in each one of the 20000 replications, the estimates $\overline{Y}$, $\hat{\gamma}_o$, $\hat{\rho}_1$ and $\hat{\rho}_2$ were obtained for different combinations of values of R and sample sizes n. Then, in each replication, having available these four estimates, for each combination of R and n, $\hat{Q}^*$ was computed using formula (2), and the corresponding prediction interval was constructed using respectively (10), (11), (12), after replacing in the variance of $\hat{Q}^*$ the unknown quantities γ_o, ρ_1 and ρ_2 with their corresponding estimates.

Using, therefore, for each model and for each combination of R and n, the 20000 different estimates from $\hat{Q}^*$, and the 20000 different estimated prediction intervals for Q^*, the four AIMs were obtained as follows:

(a) The CVG was computed as the percentage of the 20000 prediction intervals which included Q^*.

(β) The REHL was obtained dividing the average half-length of the 20000 prediction intervals by Q^*.

(c) For the RMSE, first we computed the MSE as the sum of the variance of the 20000 estimates from $\hat{Q}^*$ plus the squared of the difference of the average of the 20000 estimates from $\hat{Q}^*$ minus Q^*. Then the RMSE was computed dividing the MSE by Q^*.

(d) The difference $R - R_{act}$ was obtained by computing R_{act} as the percentage of the estimates from $\hat{Q}^*$ which were greater than the corresponding Y_{n+1} values.

Finally, let us mention that the random number generator which was used in this study is described in Kevork (2010), while details about its validity are found in Kevork (1990). To generate values from the normal distribution, we adopted the traditional method of Box and Muller which is described in Law (2007).

5.2 Exact results for CVG and REHL when $n = 2$

For $n = 2$, the estimate of ρ_1 is

$$\hat{\rho}_1 = \frac{\left(Y_2 - \frac{Y_1 + Y_2}{2}\right)\left(Y_1 - \frac{Y_1 + Y_2}{2}\right)}{\left(Y_1 - \frac{Y_1 + Y_2}{2}\right)^2 + \left(Y_2 - \frac{Y_1 + Y_2}{2}\right)^2} = -0.5 .$$

The lack of variability in $\hat{\rho}_1$ when $n = 2$ allows the exact computation of the CVG and the REHL. The process of obtaining the exact results for these two metrics is illustrated below.

Remark 3: Replacing in (5) and (6) ρ_1 with its estimate $\hat{\rho}_1 = -0.5$, the coverage of the estimated exact prediction interval (P.I.) when $n = 2$ is derived as

$$\mathrm{CVG} = \Pr\left\{ \overline{Y} - \sqrt{\frac{\hat{\gamma}_o}{3}} t'_{1,1-\alpha/2}(-2z_R) \le Q^* \le \overline{Y} - \sqrt{\frac{\hat{\gamma}_o}{3}} t'_{1,\alpha/2}(-2z_R) \right\} =$$

$$= \Pr\left\{ \sqrt{\frac{1}{3}} t'_{1,\alpha/2}(-2z_R) \le \frac{\overline{Y} - Q^*}{\sqrt{\hat{\gamma}_o}} \le \sqrt{\frac{1}{3}} t'_{1,1-\alpha/2}(-2z_R) \right\} =$$

$$= \Pr\left\{ \sqrt{\frac{1-\rho_1}{3(1+\rho_1)}} t'_{1,\alpha/2}(-2z_R) \le t'_1(\lambda) \le \sqrt{\frac{1-\rho_1}{3(1+\rho_1)}} t'_{1,1-\alpha/2}(-2z_R) \right\}, \qquad (13)$$

where $t'_1(\lambda)$ is given in (4) and λ in (5).

Then the corresponding REHL of the estimated P.I. will be

$$\mathrm{REHL}_e = \sqrt{\frac{1}{3}} \frac{t'_{1,1-\alpha/2}(-2z_R) - t'_{1,\alpha/2}(-2z_R)}{2Q^*} E\left(\sqrt{\hat{\gamma}_o}\right) = \sqrt{\frac{1-\rho_1}{3\pi}} \frac{t'_{1,1-\alpha/2}(-2z_R) - t'_{1,\alpha/2}(-2z_R)}{2\left(CV^{-1} + z_R\right)} \qquad (14)$$

Remark 4: For the AR(1), using in (10) the estimate $\hat{\gamma}_o$ and $\hat{\rho}_1 = -0.5$ instead of the true values γ_o and ρ_1, the expression inside the square root becomes $\hat{\gamma}_o\left(1 + 2.5z_R^2\right)/6$. Then, for $n = 2$ the CVG and the REHL of the corresponding estimated asymptotic P.I. are obtained as

$$\mathrm{CVG} = \Pr\left\{ \hat{Q}^* - z_{\alpha/2}\sqrt{\frac{\hat{\gamma}_o\left(1+2.5z_R^2\right)}{6}} \le Q^* \le \hat{Q}^* + z_{\alpha/2}\sqrt{\frac{\hat{\gamma}_o\left(1+2.5z_R^2\right)}{6}} \right\} =$$

$$= \Pr\left\{ -\left(z_R + z_{\alpha/2}\sqrt{\frac{\hat{\gamma}_o\left(1+2.5z_R^2\right)}{6}} \right) \le \frac{\overline{Y} - Q^*}{\sqrt{\hat{\gamma}_o}} \le -\left(z_R - z_{\alpha/2}\sqrt{\frac{\hat{\gamma}_o\left(1+2.5z_R^2\right)}{6}} \right) \right\} =$$

$$= \Pr\left\{ -\left(z_R + z_{\alpha/2}\sqrt{\frac{\hat{\gamma}_o\left(1+2.5z_R^2\right)}{6}} \right)\sqrt{\frac{1-\rho_1}{1+\rho_1}} \le t'_1(\lambda) \le -\left(z_R - z_{\alpha/2}\sqrt{\frac{\hat{\gamma}_o\left(1+2.5z_R^2\right)}{6}} \right)\sqrt{\frac{1-\rho_1}{1+\rho_1}} \right\} \quad (15)$$

and $$\mathrm{REHL}_e = z_{\alpha/2}\frac{\sqrt{1+2.5z_R^2}}{\sqrt{6}Q^*} E\left(\sqrt{\hat{\gamma}_o}\right) = z_{\alpha/2}\frac{\sqrt{(1-\rho_1)\left(1+2.5z_R^2\right)}}{\sqrt{6\pi}\left(CV^{-1} + z_R\right)}. \qquad (16)$$

Remark 5: For the MA(1), replacing in (11) γ_o and ρ_1 with their corresponding estimates, the expression inside the square root becomes $0.375\hat{\gamma}_o z_R^2$. So, for the estimated asymptotic prediction interval we have

$$\mathrm{CVG} = \Pr\left\{\hat{Q}^* - z_{\alpha/2}\sqrt{0.375\hat{\gamma}_o z_R^2} \le Q^* \le \hat{Q}^* + z_{\alpha/2}\sqrt{0.375\hat{\gamma}_o z_R^2}\right\} =$$

$$= \Pr\left\{-\left(z_R + z_{\alpha/2}\sqrt{0.375 z_R^2}\right) \le \frac{\overline{Y} - Q^*}{\sqrt{\hat{\gamma}_o}} \le -\left(z_R - z_{\alpha/2}\sqrt{0.375 z_R^2}\right)\right\} =$$

$$= \Pr\left\{-\left(z_R + z_{\alpha/2}\sqrt{0.375 z_R^2}\right)\sqrt{\frac{1-\rho_1}{1+\rho_1}} \le t_1'(\lambda) \le -\left(z_R - z_{\alpha/2}\sqrt{0.375 z_R^2}\right)\sqrt{\frac{1-\rho_1}{1+\rho_1}}\right\} \quad (17)$$

And $$\mathrm{REHL}_e = \frac{z_{\alpha/2}\sqrt{0.375 z_R^2}}{Q^*} E\left(\sqrt{\hat{\gamma}_o}\right) = z_{\alpha/2}\frac{\sqrt{0.375(1-\rho_1) z_R^2}}{\sqrt{\pi}\left(CV^{-1} + z_R\right)}. \quad (18)$$

Remark 6: For the ARMA(1,1), setting in (12) $\rho_2 = 0$ and $n = 2$ the expression inside the square root is the same as that one of the MA(1), namely $0.375\hat{\gamma}_o z_R^2$. Thus, for the two models CVGs and REHLs are the same when $n = 2$. The only difference among the two models is the range of ρ_1. For the MA(1) it holds $|\rho_1| < 0.5$, while for the ARMA(1,1) we have $|\rho_1| < 1$.

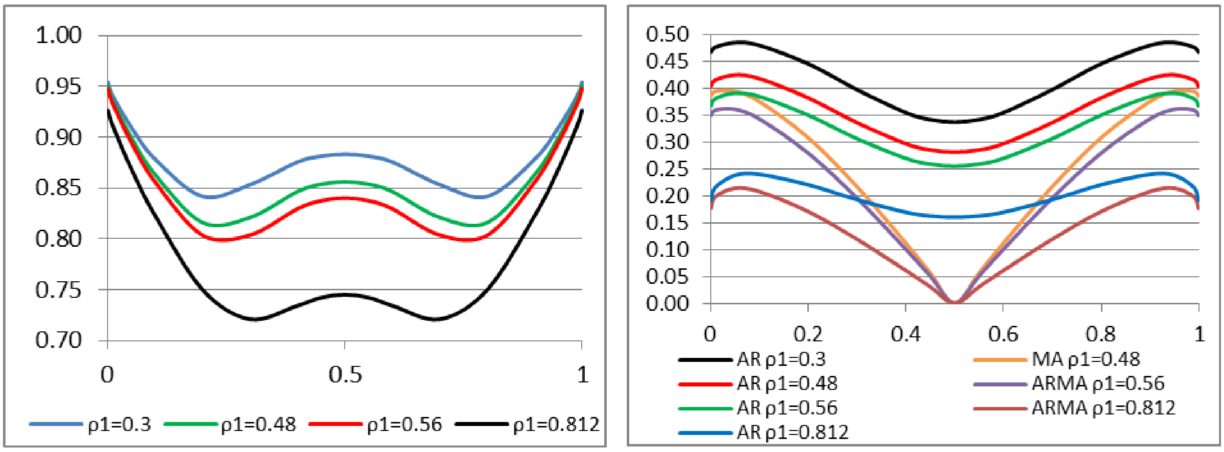
In Figure 2, for $n = 2$ we plot the CVGs of the estimated exact and the estimated asymptotic P.I.s versus the critical fractile R. The CVGs for the exact P.I.s were computed from (13), while the CVGs for the asymptotic P.I.s were obtained from (15) or (17). For any pair of values R and $1 - R$, the CVGs are the same. We observe from graph (a) that the CVGs are approaching the nominal confidence level 0.95, and in some cases they exceed it, when R is relatively close either to zero or to one. From graph (b), all the CVGs are poor as they are considerably lower than 0.95.

Also for the pairs, AR with MA, and, AR with ARMA, for which ρ_1 is the same, the CVGs in the corresponding ARs are greater.

Figure 2: Graph of CVG as a function of R for $n = 2$ and nominal confidence level 95%.

(a) Estimated exact Prediction Intervals

(b) Estimated asymptotic Prediction Intervals



In conjunction with Figure 2, Table 2 gives the CVGs of the estimated asymptotic P.I.s for some selected values of R. Together with the exact values, we also give the corresponding simulated ones, namely, the CVGs as these have been resulted in using the 20000 independent replications generated from running Monte-Carlo simulations. When the simulation run in each replication starts with Y_o and/or ε_o to be randomly chosen from their stationary normal distributions, the exact and the simulated CVGs are very close to each other, verifying the validity of the simulation results which follow in the next part. For the MA, we also give the simulated CVGs when the simulation run in each replication starts with $\varepsilon_o = 0$. In this case, all the simulated CVGs (apart from R=0.5) are lower than their exact values.

The REHLs of the estimated P.I.s are displayed in Table 3. Their exact values have been obtained from (14), (16) and (18) setting the nominal confidence level at 0.95. At this point, let us remind that the true REHLs which ensure equality between

CVGs and the nominal confidence level increase as ρ_1 is getting larger (see Figure 1). Unfortunately, such pattern of changes is not met in Table 3. Particularly, given R, the REHLs are

(a) decreasing when ρ_1 is getting larger, and

(b) greater in the exact P.I.s.

Table 2: Comparison between exact and simulated results for the coverage (CVG) which is attained by the estimated asymptotic prediction intervals, when $n = 2$ and the nominal confidence level is set at 95%. The simulated results are based on 20000 independent replications starting the simulation run in each replication with Y_o and/or ε_o to be randomly chosen from their stationary normal distributions.

		Critical Fractile							
ρ_1=0.3		**R=0.5**	**R=0.6**	**R=0.7**	**R=0.8**	**R=0.9**	**R=0.95**	**R=0.99**	**R=0.999**
AR	Exact	0.338	0.356	0.398	0.447	0.481	0.486	0.478	0.469
	Simulated	0.332	0.349	0.392	0.440	0.472	0.479	0.470	0.464
ρ_1=0.48		**R=0.5**	**R=0.6**	**R=0.7**	**R=0.8**	**R=0.9**	**R=0.95**	**R=0.99**	**R=0.999**
MA	Exact	0	0.112	0.218	0.309	0.376	0.395	0.395	0.387
	Simulated	0	0.111	0.218	0.306	0.372	0.394	0.393	0.386
		0	0.089*	0.175*	0.247*	0.294*	0.301*	0.293*	0.283*
AR	Exact	0.282	0.298	0.337	0.383	0.419	0.426	0.416	0.405
	Simulated	0.277	0.291	0.330	0.374	0.409	0.418	0.409	0.399
ρ_1=0.56		**R=0.5**	**R=0.6**	**R=0.7**	**R=0.8**	**R=0.9**	**R=0.95**	**R=0.99**	**R=0.999**
ARMA	Exact	0	0.101	0.196	0.280	0.344	0.362	0.360	0.350
	Simulated	0	0.098	0.193	0.272	0.325	0.341	0.334	0.327
AR	Exact	0.256	0.270	0.307	0.351	0.386	0.392	0.382	0.368
	Simulated	0.253	0.265	0.302	0.342	0.376	0.385	0.375	0.364
ρ_1=0.812		**R=0.5**	**R=0.6**	**R=0.7**	**R=0.8**	**R=0.9**	**R=0.95**	**R=0.99**	**R=0.999**
ARMA	Exact	0	0.061	0.120	0.171	0.208	0.215	0.199	0.177
	Simulated	0	0.063	0.123	0.170	0.200	0.204	0.184	0.164
AR	Exact	0.161	0.170	0.193	0.221	0.241	0.239	0.216	0.192
	Simulated	0.162	0.168	0.192	0.218	0.236	0.237	0.216	0.192

*: The simulation run in each replication started with $\varepsilon_o = 0$

The latter two remarks fully justify the size and the pattern of changes of the CVGs in Figures 2a and 2b. Furthermore, regarding the asymptotic P.I.s, for the pairs AR with MA and AR with ARMA, the estimated REHLs are greater in the corresponding AR models. This justifies why in Figure 2, for each pair of models with the same ρ_1, the AR gives higher CVGs.

Table 3: Exact results for the REHLs of the estimated prediction intervals, when $n = 2$ and the nominal confidence level is set at 95%.

ρ_1=0.3	Critical Fractile							
	R=0.2	**R=0.4**	**R=0.5**	**R=0.55**	**R=0.6**	**R=0.8**	**R=0.95**	**R=0.99**
Exact P.I.	1.793	0.821	0.693	0.697	0.742	1.277	2.131	2.728
AR asymptotic P.I.	0.151	0.086	0.076	0.075	0.077	0.108	0.158	0.197
ρ_1=0.48	**R=0.2**	**R=0.4**	**R=0.5**	**R=0.55**	**R=0.6**	**R=0.8**	**R=0.95**	**R=0.99**
Exact P.I.	1.546	0.708	0.597	0.601	0.640	1.100	1.836	2.351
MA asymptotic P.I.	0.099	0.026	0.000	0.012	0.024	0.070	0.121	0.155
AR asymptotic P.I.	0.130	0.074	0.065	0.065	0.067	0.093	0.137	0.169
ρ_1=0.56	**R=0.2**	**R=0.4**	**R=0.5**	**R=0.55**	**R=0.6**	**R=0.8**	**R=0.95**	**R=0.99**
Exact P.I.	1.422	0.651	0.549	0.552	0.588	1.012	1.689	2.162
ARMA asymptotic P.I.	0.091	0.024	0.000	0.011	0.022	0.065	0.111	0.143
AR asymptotic P.I.	0.120	0.068	0.060	0.060	0.061	0.085	0.126	0.156
ρ_1=0.812	**R=0.2**	**R=0.4**	**R=0.5**	**R=0.55**	**R=0.6**	**R=0.8**	**R=0.95**	**R=0.99**
Exact P.I.	0.929	0.426	0.359	0.361	0.385	0.662	1.104	1.413
ARMA asymptotic P.I.	0.059	0.016	0.000	0.007	0.014	0.042	0.073	0.093
AR asymptotic P.I.	0.078	0.044	0.039	0.039	0.040	0.056	0.082	0.102

5.3 Results from Monte-Carlo Simulations

In this part, we present simulated results for the four Accuracy Implication Metrics (AIMs), which have been obtained using for each model of Table 1 the 20000 generated independent replications after running the Monte-Carlo Simulations.

For the MA(1) and ARMA(1,1), Table 4 gives the number of replications for which in small samples the estimated asymptotic variance of $\hat{Q}^*$ was negative. This number becomes smaller when R approaches either 0 or 1. Nonetheless, with at least 20 observations in the sample, the number of negative values becomes negligible compared to the total of 20000 replications. For example, for the ARMA with $\rho_1 = 0.812$, when n=20 and R=0.99, the percentage of negative values ranges below 0.7%. In Tables 5, 6, 7, and 8 which follow, in small samples from the MA and the ARMA models the AIMs were computed using only those replications for which the estimated asymptotic variance of $\hat{Q}^*$ was positive.

In Table 5, all the CVGs are poor for small n, but approach the nominal confidence level 95% as n increases. The rate of convergence to 0.95 depends upon the autocorrelation level expressed by the size of ρ_1 and the rate of convergence of the autocorrelation function (ACF) to zero. The AR and the ARMA models of Table 1 have ACF of the same form. But in the ARMA the ACF converges to zero faster. So we observe that the CVGs in the ARMA approach 0.95 faster than those of the AR. Regarding the MA with $\rho_1 = 0.48$, since its ACF has a "cut-off" at lag 1, its CVGs are almost of the same size as those of the AR with $\rho_1 = 0.3$.

Table 4 : Number of replications with negative estimated asymptotic variance of $\hat{Q}^*$ for the MA(1) and ARMA(1,1) models. Results are based on 20000 independent replications generated from running Monte-Carlo simulations.

		Sample Sizes								
ρ_1=0.48		**n=5**	**n=10**	**n=20**	**n=30**	**n=40**	**n=50**	**n=60**	**n=80**	**n=100**
MA(1)	R=0.2	0	0	0	0	0	0	0	0	0
	R=0.3	216	1	0	0	0	0	0	0	0
	R=0.4	488	8	0	0	0	0	0	0	0
	R=0.5	624	15	0	0	0	0	0	0	0
	R=0.6	488	8	0	0	0	0	0	0	0
	R=0.7	216	1	0	0	0	0	0	0	0
	R=0.8	0	0	0	0	0	0	0	0	0
	R=0.99	0	0	0	0	0	0	0	0	0
ρ_1=0.56		**n=5**	**n=10**	**n=20**	**n=30**	**n=40**	**n=50**	**n=60**	**n=80**	**n=100**
ARMA(1,1)	R=0.2	650	119	14	3	0	0	0	0	0
	R=0.3	806	100	8	0	0	0	0	0	0
	R=0.4	921	87	3	0	0	0	0	0	0
	R=0.5	997	85	2	0	0	0	0	0	0
	R=0.6	921	87	3	0	0	0	0	0	0
	R=0.7	806	100	8	0	0	0	0	0	0
	R=0.8	650	119	14	3	0	0	0	0	0
	R=0.99	367	178	39	5	1	0	0	0	0
ρ_1=0.812		**n=5**	**n=10**	**n=20**	**n=30**	**n=40**	**n=50**	**n=60**	**n=80**	**n=100**
ARMA(1,1)	R=0.2	958	279	85	38	18	8	3	3	0
	R=0.3	1228	291	72	18	10	5	2	1	0
	R=0.4	1465	299	60	11	5	2	0	0	0
	R=0.5	1611	302	56	8	4	1	0	0	0
	R=0.6	1465	299	60	11	5	2	0	0	0
	R=0.7	1228	291	72	18	10	5	2	1	0
	R=0.8	958	279	85	38	18	8	3	3	0
	R=0.99	344	276	135	84	35	14	12	4	0

To make general recommendations for the required sample size to attain acceptable sizes of CVG, we consider that a CVG equal to 0.90 is a satisfactory approximation to the 95% nominal confidence level. So, looking at the entries of Table 5:

(a) For the AR with $\rho_1 = 0.3$ and the MA with $\rho_1 = 0.48$, a sample of at least 30 observations should be available.

(b) For the AR with $\rho_1 = 0.48$ and for the pair AR, ARMA with $\rho_1 = 0.56$ we need a sample of 50 observations or more.

(c) For the pair AR, ARMA with $\rho_1 = 0.812$ a sample of more than 100 observations is necessary.

Table 5: Coverage (CVG) of asymptotic prediction intervals for the AR(1), ARMA(1,1), and MA(1) models at nominal confidence level 0.95. Results are based on 20000 independent replications generated from running Monte-Carlo simulations.

		Sample Sizes									
ρ_1=0.3		**n=5**	**n=10**	**n=20**	**n=30**	**n=50**	**n=100**	**n=200**	**n=500**	**n=1000**	**n=2000**
R=0.5	AR	0.66	0.81	0.88	0.90	0.92	0.93	0.94	0.94	0.95	0.95
R=0.6	AR	0.67	0.81	0.88	0.90	0.92	0.93	0.94	0.94	0.95	0.95
R=0.8	AR	0.71	0.82	0.88	0.90	0.92	0.93	0.94	0.95	0.95	0.95
R=0.95	AR	0.74	0.83	0.88	0.90	0.92	0.93	0.94	0.95	0.95	0.95
R=0.99	AR	0.73	0.82	0.88	0.90	0.92	0.93	0.94	0.95	0.95	0.95
ρ_1=0.48		**n=5**	**n=10**	**n=20**	**n=30**	**n=50**	**n=100**	**n=200**	**n=500**	**n=1000**	**n=2000**
R=0.5	MA	0.67	0.83	0.89	0.91	0.93	0.94	0.94	0.95	0.95	0.95
	AR	0.60	0.76	0.85	0.88	0.91	0.93	0.94	0.94	0.95	0.95
R=0.6	MA	0.67	0.83	0.89	0.91	0.93	0.94	0.94	0.95	0.95	0.95
	AR	0.60	0.76	0.85	0.88	0.90	0.93	0.94	0.94	0.95	0.95
R=0.8	MA	0.68	0.82	0.89	0.91	0.92	0.93	0.94	0.95	0.95	0.95
	AR	0.64	0.77	0.85	0.88	0.90	0.92	0.94	0.94	0.95	0.95
R=0.95	MA	0.68	0.81	0.87	0.90	0.92	0.93	0.94	0.95	0.95	0.95
	AR	0.66	0.77	0.84	0.88	0.90	0.92	0.94	0.94	0.95	0.95
R=0.99	MA	0.67	0.79	0.87	0.89	0.91	0.93	0.94	0.95	0.95	0.95
	AR	0.65	0.76	0.84	0.87	0.90	0.92	0.94	0.94	0.95	0.95
ρ_1=0.56		**n=5**	**n=10**	**n=20**	**n=30**	**n=50**	**n=100**	**n=200**	**n=500**	**n=1000**	**n=2000**
R=0.5	ARMA	0.64	0.78	0.86	0.89	0.91	0.93	0.94	0.94	0.95	0.95
	AR	0.56	0.73	0.83	0.87	0.90	0.92	0.93	0.94	0.95	0.95
R=0.6	ARMA	0.64	0.78	0.86	0.89	0.91	0.93	0.94	0.94	0.95	0.95
	AR	0.56	0.73	0.83	0.87	0.90	0.92	0.94	0.94	0.95	0.95
R=0.8	ARMA	0.64	0.78	0.86	0.89	0.91	0.92	0.94	0.95	0.95	0.95
	AR	0.59	0.74	0.83	0.87	0.89	0.92	0.94	0.94	0.95	0.95
R=0.95	ARMA	0.62	0.76	0.85	0.88	0.91	0.92	0.94	0.94	0.95	0.95
	AR	0.61	0.73	0.82	0.86	0.89	0.91	0.93	0.94	0.95	0.95
R=0.99	ARMA	0.61	0.74	0.84	0.87	0.90	0.92	0.94	0.94	0.95	0.95
	AR	0.60	0.72	0.81	0.85	0.88	0.91	0.93	0.94	0.95	0.95
ρ_1=0.812		**n=5**	**n=10**	**n=20**	**n=30**	**n=50**	**n=100**	**n=200**	**n=500**	**n=1000**	**n=2000**
R=0.5	ARMA	0.45	0.62	0.76	0.82	0.86	0.90	0.93	0.94	0.95	0.95
	AR	0.39	0.56	0.70	0.77	0.83	0.88	0.92	0.93	0.94	0.95
R=0.6	ARMA	0.45	0.62	0.76	0.82	0.86	0.90	0.93	0.94	0.95	0.95
	AR	0.39	0.55	0.70	0.76	0.83	0.88	0.92	0.93	0.94	0.95
R=0.8	ARMA	0.45	0.62	0.76	0.81	0.86	0.90	0.93	0.94	0.95	0.95
	AR	0.40	0.55	0.69	0.76	0.82	0.88	0.91	0.93	0.94	0.95
R=0.95	ARMA	0.43	0.60	0.74	0.80	0.85	0.89	0.92	0.94	0.94	0.95
	AR	0.39	0.53	0.67	0.74	0.81	0.87	0.91	0.93	0.94	0.95
R=0.99	ARMA	0.40	0.58	0.72	0.78	0.84	0.88	0.92	0.94	0.94	0.95
	AR	0.36	0.51	0.65	0.72	0.79	0.85	0.90	0.93	0.94	0.94

From Tables 6 and 7, the REHL and the RMSE exhibit the same behavior for each model of Table 1. As R is getting larger, these two metrics decrease when $R < 0.5$, reach a minimum at some $R > 0.5$ and then start to increase again as R approaches one. Increasing either ρ_1 or n, the minimum REHLs are attained at values of R which are closer to 1. For example, with $\rho_1 = 0.48$ and $n = 100$, the minimum REHL is attained at some R around 0.7, while for $\rho_1 = 0.812$ and $n = 500$ the minimum occur for R around 0.8.

On the contrary, for any model and sample size of Table 7, the minimum RMSEs are observed when R ranges between 0.55 and 0.7. Regarding their sizes, given n and R, both REHLs and RMSEs become larger when ρ_1 increases. To relate also the CVGs of Table 5 to the REHLs and the RMSEs of Tables 6 and 7, in any of the three pairs of models under consideration, we observe that the CVGs of the estimated prediction intervals in the AR are always accompanied by larger REHLs and larger RMSEs compared to those of the corresponding MA or ARMA models.

Finally, from Table 8, the size of differences $R - R_{act}$ declines as n is getting larger. For $R > 0.5$ and small samples these differences are positive. The differences are negative for $R < 0.5$, but we do not report them in order (a) to reduce the length of the table, and (b) because for any pair of values R and $1 - R$, the absolute value of the differences is approximately the same. Given ρ_1 and n, $R - R_{act}$ becomes larger when R ranges between 0.8 and 0.95. In the same range of R, for samples neither too small nor too large, $R - R_{act}$ is larger in the AR than in the model which belongs to the same pair and has the same ρ_1. Considering also that an $R - R_{act}$ below 1.5% is negligible from the management practice point of view, we make the following recommendations for the required sample sizes to attain such small differences: (a) at least 30 observations for $\rho_1 = 0.3$, (b) at least 50 observations for ρ_1 equal to 0.48 or 0.56, and (c) more than 100 observations when $\rho_1 = 0.812$.

Table 6: Relative Expected Half-Length (REHL) of the asymptotic confidence intervals for the AR(1), ARMA(1,1), and MA(1) models at nominal confidence level 0.95, and coefficient of variation equal to 0.2. Results are based on 20000 independent replications generated from running Monte-Carlo simulations.

		Sample Sizes								
ρ_1=0.3		**n=10**	**n=20**	**n=30**	**n=50**	**n=100**	**n=200**	**n=500**	**n=1000**	**n=2000**
R=0.2	AR	0.1778	0.1420	0.1206	0.0961	0.0695	0.0498	0.0317	0.0225	0.0159
R=0.6	AR	0.1242	0.1011	0.0863	0.0690	0.0500	0.0359	0.0228	0.0162	0.0115
R=0.7	AR	0.1235	0.0998	0.0851	0.0679	0.0492	0.0352	0.0224	0.0159	0.0113
R=0.8	AR	0.1266	0.1011	0.0859	0.0684	0.0495	0.0354	0.0225	0.0160	0.0113
R=0.9	AR	0.1349	0.1061	0.0898	0.0713	0.0514	0.0368	0.0234	0.0166	0.0117
R=0.99	AR	0.1616	0.1242	0.1044	0.0825	0.0593	0.0423	0.0269	0.0191	0.0135
ρ_1=0.48		**n=10**	**n=20**	**n=30**	**n=50**	**n=100**	**n=200**	**n=500**	**n=1000**	**n=2000**
R=0.2	MA	0.1830	0.1472	0.1253	0.1000	0.0724	0.0518	0.0330	0.0234	0.0166
	AR	0.1886	0.1617	0.1407	0.1142	0.0837	0.0604	0.0386	0.0274	0.0194
R=0.6	MA	0.1293	0.1046	0.0892	0.0712	0.0515	0.0369	0.0235	0.0167	0.0118
	AR	0.1343	0.1169	0.1021	0.0831	0.0610	0.0441	0.0282	0.0200	0.0142
R=0.7	MA	0.1280	0.1033	0.0880	0.0703	0.0509	0.0364	0.0232	0.0164	0.0116
	AR	0.1325	0.1148	0.1001	0.0813	0.0597	0.0431	0.0276	0.0196	0.0139
R=0.8	MA	0.1303	0.1048	0.0892	0.0712	0.0515	0.0369	0.0235	0.0167	0.0118
	AR	0.1342	0.1151	0.1002	0.0813	0.0596	0.0430	0.0275	0.0195	0.0138
R=0.9	MA	0.1379	0.1102	0.0938	0.0748	0.0541	0.0388	0.0247	0.0175	0.0124
	AR	0.1408	0.1192	0.1034	0.0837	0.0613	0.0442	0.0282	0.0200	0.0142
R=0.99	MA	0.1636	0.1297	0.1101	0.0878	0.0635	0.0455	0.0289	0.0205	0.0145
	AR	0.1645	0.1363	0.1175	0.0947	0.0692	0.0498	0.0318	0.0226	0.0160
ρ_1=0.56		**n=10**	**n=20**	**n=30**	**n=50**	**n=100**	**n=200**	**n=500**	**n=1000**	**n=2000**
R=0.2	ARMA	0.1855	0.1601	0.1383	0.1116	0.0814	0.0587	0.0374	0.0266	0.0188
	AR	0.1915	0.1716	0.1516	0.1246	0.0923	0.0669	0.0429	0.0305	0.0216
R=0.6	ARMA	0.1316	0.1141	0.0991	0.0802	0.0586	0.0423	0.0270	0.0191	0.0136
	AR	0.1371	0.1246	0.1105	0.0910	0.0674	0.0489	0.0314	0.0223	0.0158
R=0.7	ARMA	0.1300	0.1126	0.0976	0.0789	0.0576	0.0415	0.0265	0.0188	0.0133
	AR	0.1351	0.1221	0.1081	0.0890	0.0659	0.0478	0.0306	0.0218	0.0154
R=0.8	ARMA	0.1321	0.1140	0.0985	0.0795	0.0580	0.0418	0.0266	0.0189	0.0134
	AR	0.1363	0.1221	0.1079	0.0887	0.0657	0.0476	0.0305	0.0217	0.0154
R=0.9	ARMA	0.1393	0.1195	0.1029	0.0828	0.0603	0.0434	0.0277	0.0196	0.0139
	AR	0.1423	0.1259	0.1110	0.0910	0.0673	0.0488	0.0313	0.0222	0.0157
R=0.99	ARMA	0.1647	0.1395	0.1194	0.0957	0.0695	0.0499	0.0318	0.0226	0.0160
	AR	0.1647	0.1428	0.1252	0.1023	0.0755	0.0547	0.0350	0.0249	0.0176
ρ_1=0.812		**n=10**	**n=20**	**n=30**	**n=50**	**n=100**	**n=200**	**n=500**	**n=1000**	**n=2000**
R=0.2	ARMA	0.1765	0.1864	0.1759	0.1523	0.1171	0.0865	0.0560	0.0399	0.0284
	AR	0.1736	0.1949	0.1921	0.1744	0.1397	0.1055	0.0692	0.0496	0.0353
R=0.6	ARMA	0.1259	0.1347	0.1276	0.1107	0.0852	0.0630	0.0408	0.0291	0.0207
	AR	0.1259	0.1428	0.1409	0.1281	0.1026	0.0775	0.0509	0.0364	0.0259
R=0.7	ARMA	0.1242	0.1322	0.1251	0.1085	0.0834	0.0616	0.0399	0.0285	0.0202
	AR	0.1234	0.1394	0.1375	0.1250	0.1001	0.0756	0.0496	0.0356	0.0253
R=0.8	ARMA	0.1256	0.1327	0.1252	0.1084	0.0833	0.0616	0.0398	0.0284	0.0202
	AR	0.1235	0.1388	0.1367	0.1242	0.0994	0.0751	0.0493	0.0353	0.0251
R=0.9	ARMA	0.1317	0.1374	0.1292	0.1117	0.0857	0.0633	0.0409	0.0292	0.0207
	AR	0.1275	0.1419	0.1396	0.1266	0.1013	0.0765	0.0502	0.0360	0.0256
R=0.99	ARMA	0.1539	0.1572	0.1469	0.1265	0.0969	0.0714	0.0462	0.0329	0.0234
	AR	0.1449	0.1585	0.1554	0.1407	0.1125	0.0849	0.0557	0.0399	0.0284

Table 7: Relative Mean Square Error (RMSE) of the estimator $\hat{Q}^*$ when the coefficient of variation equals to 0.2. Results are based on 20000 independent replications generated from running Monte-Carlo simulations.

		Sample Sizes								
ρ_1=0.3		**n=10**	**n=20**	**n=30**	**n=50**	**n=100**	**n=200**	**n=500**	**n=1000**	**n=2000**
R=0.2	AR	1.0576	0.5387	0.3644	0.2199	0.1097	0.0559	0.0223	0.0109	0.0054
R=0.55	AR	0.6802	0.3518	0.2385	0.1457	0.0737	0.0368	0.0148	0.0073	0.0036
R=0.6	AR	0.6752	0.3488	0.2363	0.1444	0.0732	0.0364	0.0147	0.0072	0.0036
R=0.7	AR	0.6906	0.3552	0.2400	0.1465	0.0744	0.0367	0.0148	0.0073	0.0036
R=0.8	AR	0.7482	0.3818	0.2570	0.1565	0.0796	0.0390	0.0158	0.0078	0.0039
R=0.99	AR	1.4238	0.7031	0.4671	0.2799	0.1419	0.0689	0.0280	0.0140	0.0070
ρ_1=0.48		**n=10**	**n=20**	**n=30**	**n=50**	**n=100**	**n=200**	**n=500**	**n=1000**	**n=2000**
R=0.2	MA	1.1726	0.5920	0.3983	0.2394	0.1192	0.0606	0.0242	0.0119	0.0059
	AR	1.4766	0.7781	0.5326	0.3251	0.1634	0.0833	0.0334	0.0164	0.0081
R=0.55	MA	0.7324	0.3751	0.2537	0.1545	0.0780	0.0389	0.0156	0.0077	0.0038
	AR	0.9772	0.5222	0.3578	0.2206	0.1122	0.0561	0.0226	0.0111	0.0055
R=0.6	MA	0.7280	0.3726	0.2519	0.1535	0.0777	0.0385	0.0155	0.0076	0.0038
	AR	0.9681	0.5168	0.3537	0.2182	0.1111	0.0554	0.0224	0.0110	0.0055
R=0.7	MA	0.7501	0.3825	0.2581	0.1571	0.0798	0.0393	0.0159	0.0078	0.0039
	AR	0.9820	0.5220	0.3562	0.2195	0.1121	0.0555	0.0225	0.0111	0.0055
R=0.8	MA	0.8233	0.4171	0.2807	0.1703	0.0867	0.0424	0.0172	0.0085	0.0043
	AR	1.0482	0.5531	0.3760	0.2311	0.1183	0.0582	0.0237	0.0117	0.0058
R=0.99	MA	1.6536	0.8162	0.5430	0.3238	0.1649	0.0801	0.0325	0.0163	0.0082
	AR	1.8735	0.9564	0.6410	0.3878	0.1989	0.0966	0.0395	0.0197	0.0099
ρ_1=0.56		**n=10**	**n=20**	**n=30**	**n=50**	**n=100**	**n=200**	**n=500**	**n=1000**	**n=2000**
R=0.2	ARMA	1.4366	0.7444	0.5056	0.3066	0.1535	0.0781	0.0313	0.0153	0.0076
	AR	1.7470	0.9416	0.6495	0.3993	0.2017	0.1029	0.0413	0.0203	0.0100
R=0.55	ARMA	0.9194	0.4841	0.3302	0.2028	0.1029	0.0514	0.0207	0.0102	0.0050
	AR	1.1649	0.6370	0.4395	0.2727	0.1391	0.0698	0.0282	0.0138	0.0069
R=0.6	ARMA	0.9129	0.4802	0.3272	0.2009	0.1021	0.0508	0.0205	0.0101	0.0050
	AR	1.1536	0.6302	0.4343	0.2695	0.1377	0.0688	0.0278	0.0137	0.0068
R=0.7	ARMA	0.9350	0.4895	0.3326	0.2040	0.1040	0.0514	0.0208	0.0102	0.0051
	AR	1.1677	0.6352	0.4364	0.2705	0.1386	0.0688	0.0279	0.0137	0.0069
R=0.8	ARMA	1.0154	0.5272	0.3568	0.2180	0.1114	0.0547	0.0222	0.0110	0.0055
	AR	1.2419	0.6705	0.4588	0.2837	0.1458	0.0719	0.0293	0.0144	0.0072
R=0.99	ARMA	1.9515	0.9794	0.6527	0.3914	0.1999	0.0969	0.0395	0.0198	0.0099
	AR	2.1829	1.1385	0.7677	0.4677	0.2410	0.1173	0.0481	0.0240	0.0120
ρ_1=0.812		**n=10**	**n=20**	**n=30**	**n=50**	**n=100**	**n=200**	**n=500**	**n=1000**	**n=2000**
R=0.2	ARMA	2.6566	1.5318	1.0786	0.6761	0.3456	0.1770	0.0713	0.0350	0.0172
	AR	3.3529	2.1202	1.5516	1.0064	0.5273	0.2727	0.1105	0.0542	0.0268
R=0.55	ARMA	1.7169	1.0185	0.7198	0.4557	0.2352	0.1188	0.0482	0.0237	0.0118
	AR	2.2089	1.4441	1.0619	0.6936	0.3651	0.1865	0.0762	0.0375	0.0186
R=0.6	ARMA	1.7049	1.0095	0.7124	0.4509	0.2329	0.1173	0.0476	0.0234	0.0116
	AR	2.1900	1.4292	1.0494	0.6851	0.3608	0.1838	0.0753	0.0370	0.0184
R=0.7	ARMA	1.7444	1.0247	0.7200	0.4546	0.2353	0.1177	0.0479	0.0236	0.0118
	AR	2.2268	1.4407	1.0533	0.6861	0.3617	0.1833	0.0753	0.0371	0.0184
R=0.8	ARMA	1.8895	1.0944	0.7640	0.4804	0.2490	0.1236	0.0505	0.0249	0.0124
	AR	2.3863	1.5197	1.1037	0.7161	0.3779	0.1905	0.0784	0.0387	0.0193
R=0.99	ARMA	3.5832	1.9532	1.3325	0.8205	0.4244	0.2073	0.0851	0.0423	0.0211
	AR	4.3303	2.5632	1.8126	1.1534	0.6088	0.3032	0.1255	0.0624	0.0310

Table 8: Values for $R - R_{act}$ using the estimator $\hat{Q}^*$ instead of the optimal order quantity Q^* when the coefficient of variation equals to 0.2. Results are based on 20000 independent replications generated from running Monte-Carlo simulations.

		Sample Sizes								
ρ_1=0.3		n=5	n=10	n=20	n=30	n=50	n=100	n=200	n=500	n=1000
R=0.6	AR	2.5%	1.9%	0.6%	0.4%	0.4%	0.4%	-0.5%	-0.2%	0.1%
R=0.7	AR	5.3%	3.3%	1.5%	1.1%	0.3%	0.6%	0.1%	-0.5%	-0.3%
R=0.8	AR	7.9%	4.6%	2.2%	1.4%	0.9%	0.9%	0.0%	-0.4%	0.0%
R=0.9	AR	9.4%	4.9%	2.1%	1.4%	0.7%	0.5%	0.2%	0.0%	-0.2%
R=0.95	AR	9.1%	4.5%	2.1%	1.3%	0.5%	0.5%	0.1%	0.0%	-0.2%
R=0.99	AR	6.6%	2.6%	1.3%	0.7%	0.2%	0.1%	0.1%	0.1%	0.0%
ρ_1=0.48		n=5	n=10	n=20	n=30	n=50	n=100	n=200	n=500	n=1000
R=0.6	MA	3.1%	1.8%	1.0%	0.5%	0.1%	0.6%	-0.3%	-0.2%	0.4%
	AR	3.1%	2.4%	1.2%	0.7%	0.6%	0.4%	-0.3%	-0.3%	-0.2%
R=0.7	MA	6.3%	3.5%	1.9%	1.0%	0.4%	0.6%	-0.2%	-0.2%	-0.3%
	AR	6.4%	4.1%	1.9%	1.3%	0.8%	0.8%	0.1%	-0.2%	-0.2%
R=0.8	MA	9.2%	5.0%	2.4%	1.5%	0.7%	0.7%	-0.1%	-0.1%	-0.3%
	AR	9.0%	5.5%	2.9%	1.8%	1.0%	1.0%	0.2%	-0.1%	0.0%
R=0.9	MA	10.7%	5.4%	2.5%	1.7%	0.9%	0.5%	0.2%	0.3%	0.0%
	AR	10.5%	5.9%	2.8%	2.0%	1.0%	0.5%	0.4%	0.0%	-0.1%
R=0.95	MA	10.2%	4.8%	2.1%	1.6%	0.8%	0.3%	0.1%	0.3%	-0.1%
	AR	10.2%	5.4%	2.5%	1.8%	0.7%	0.4%	0.2%	0.1%	0.0%
R=0.99	MA	7.4%	2.9%	1.2%	0.8%	0.3%	0.1%	0.1%	0.1%	0.0%
	AR	7.4%	3.1%	1.5%	0.9%	0.3%	0.2%	0.1%	0.1%	0.0%
ρ_1=0.56		n=5	n=10	n=20	n=30	n=50	n=100	n=200	n=500	n=1000
R=0.6	ARMA	3.5%	2.5%	1.1%	0.7%	0.5%	0.5%	-0.3%	-0.2%	0.1%
	AR	3.5%	2.7%	1.3%	0.9%	0.6%	0.4%	-0.4%	-0.5%	-0.2%
R=0.7	ARMA	7.1%	4.1%	2.2%	1.3%	0.7%	0.8%	-0.1%	-0.2%	0.0%
	AR	6.9%	4.7%	2.5%	1.6%	1.0%	0.8%	0.1%	-0.3%	-0.2%
R=0.8	ARMA	10.1%	5.6%	3.0%	1.8%	1.0%	0.9%	0.1%	0.0%	-0.2%
	AR	9.5%	6.1%	3.4%	2.1%	1.3%	1.1%	0.3%	0.0%	-0.1%
R=0.9	ARMA	11.6%	6.5%	2.9%	2.1%	1.1%	0.6%	0.3%	0.2%	-0.1%
	AR	11.1%	6.6%	3.3%	2.4%	1.3%	0.6%	0.4%	0.0%	0.0%
R=0.95	ARMA	11.2%	5.5%	2.6%	1.8%	0.7%	0.3%	0.2%	0.2%	0.0%
	AR	10.8%	6.0%	2.8%	2.1%	1.0%	0.5%	0.2%	0.1%	0.0%
R=0.99	ARMA	8.2%	3.4%	1.5%	1.0%	0.3%	0.2%	0.2%	0.1%	0.0%
	AR	7.8%	3.4%	1.6%	1.1%	0.4%	0.2%	0.1%	0.1%	0.1%
ρ_1=0.812		n=5	n=10	n=20	n=30	n=50	n=100	n=200	n=500	n=1000
R=0.6	ARMA	4.9%	3.5%	2.0%	1.8%	0.9%	0.6%	-0.1%	0.0%	0.1%
	AR	4.1%	3.4%	2.5%	2.0%	1.3%	1.0%	0.0%	0.1%	0.4%
R=0.7	ARMA	9.4%	6.7%	4.0%	2.9%	1.9%	1.2%	0.3%	-0.1%	0.2%
	AR	8.1%	6.9%	4.7%	3.7%	2.4%	1.7%	0.2%	0.0%	0.1%
R=0.8	ARMA	12.9%	9.3%	5.2%	3.7%	2.3%	1.3%	0.8%	0.2%	-0.1%
	AR	11.9%	9.6%	6.0%	4.8%	3.1%	1.7%	0.9%	0.3%	0.1%
R=0.9	ARMA	15.0%	9.6%	5.3%	4.1%	2.2%	1.1%	0.8%	0.4%	0.0%
	AR	13.5%	10.2%	6.3%	5.0%	3.1%	1.6%	1.2%	0.2%	0.3%
R=0.95	ARMA	14.4%	8.6%	4.4%	3.3%	1.6%	0.7%	0.6%	0.4%	0.1%
	AR	12.8%	8.6%	5.0%	4.2%	2.3%	1.0%	0.8%	0.4%	0.1%
R=0.99	ARMA	10.2%	5.0%	2.4%	1.6%	0.7%	0.4%	0.2%	0.2%	0.1%
	AR	9.1%	4.9%	2.5%	1.8%	1.1%	0.4%	0.2%	0.2%	0.1%

6. Conclusions

For the classical newsvendor model operating under optimal conditions we have developed a procedure to determine the order quantity when (a) demand in successive periods is autocorrelated, (b) the parameters of the stochastic law which generates the demand are unknown, and (c) data for the demand are available for a number of recent successive periods.

Using estimates for the stationary mean, the stationary variance and the theoretical autocorrelation coefficients at lags one and two, we illustrated how to estimate the optimal order quantity and to construct the corresponding prediction interval. General expressions for two types of predictions intervals were derived. The exact when the sample consists of two observations, and the asymptotic when the sample is considered as sufficiently large. Specifications of the asymptotic prediction interval were obtained for the stationary time series models AR(1), MA(1), and ARMA(1,1).

To study the impact of the estimation procedure on the optimal performance of the newsvendor model, we have considered four accuracy implication metrics. The first is the coverage of the estimated prediction intervals, that is, the actual probability the interval to include the optimal order quantity. The second is the expected half-length of the estimated prediction interval divided by the optimal order quantity. The third is the mean square error of the estimator for the optimal order quantity divided by the optimal order quantity. Finally, the last implication metric is the difference between the critical fractile and the actual probability of not running out of stock during the period when the optimal order quantity is estimated. Exact values for the first two metrics were obtained only when the sample size was two. In any other sample size greater than two, the four metrics were evaluated through Monte Carlo simulations.

Although the case of a sample with only two observations could be considered as extreme and unrealistic, the evaluation for such a small sample of the performance of both the exact and the asymptotic prediction intervals for the three time series models under consideration gave useful insights in the estimation process of the optimal order quantity. For instance, the analysis showed that it is too difficult to obtain exact prediction intervals for samples with more than two observations. Regarding the asymptotic prediction intervals, when they are estimated using a sample of size two, we verified the validity of the simulation results since the discrepancies between the exact and the simulated values of the coverage were negligible.

By estimating the exact and the asymptotic prediction intervals using a sample of two observations, we illustrated that only the exact prediction interval gave acceptable coverage in relation to the nominal confidence level, providing that the critical fractile was quite close either to zero or to one. The last remark cannot be taken as promising for using a sample of size two, since the actual probability not to experience a stock out during the period differed considerably from the critical fractile, especially when the critical fractile was close to zero or to one. Furthermore, the differences between the two probabilities were getting larger when the theoretical autocorrelation coefficient at lag one was approaching one.

The estimation of the asymptotic prediction intervals in finite samples of size greater than two gave some promising and acceptable results. For the three time series models under consideration the coverage was approaching to the nominal confidence level as the sample was getting larger. The rate of convergence, however, differed accordingly (a) of how fast the autocorrelation function decays to zero, and (b) the size of the theoretical autocorrelation coefficient at lag one. So, the convergence rate was slower for heavy autocorrelation levels and autocorrelation functions decaying to zero quite slowly. With a nominal confidence level of 0.95, a coverage of at least 0.90 was attained

(a) for low autocorrelation levels when the sample size was at least 30 observations,

(b) for moderate autocorrelation levels with a sample size of at least 50 observations, and

(c) for high autocorrelation levels when the sample exceeded 100 observations.

Furthermore, only for quite large samples the coverage was almost the same in the whole range of values of the critical fractile which we considered. For very small, or moderate, sample sizes the coverage was declining as the critical fractile was approaching one (or zero).

Increasing the critical fractile, the relative precision of the prediction intervals and the relative mean square error of the estimator for the optimal order quantity exhibited the same behavior. Depending upon the sample size and the size of the theoretical autocorrelation coefficient at lag one, the minimum values of these two accuracy implication metrics were attained at a critical fractile ranging between 0.5 and one.

Regarding the actual probability of not experiencing a stock-out during the period when the optimal order quantity is estimated, this probability was approaching the critical fractile as the sample size was increasing. For the autocorrelations levels and the sample sizes which we considered in this work, the differences between the critical fractile and this actual probability became larger when the critical fractile was ranging between 0.8 and 0.95. Nonetheless, having at least available the three aforementioned minimum required sample sizes for the three different autocorrelation levels for which an acceptable coverage was attained, the differences between these two probabilities were ranging below 1.5%.

Summarizing, therefore, for certain autocorrelation forms we give in the current paper guidelines for the minimum required sample size in order the prediction interval of the optimal order quantity to attain an acceptable coverage. But, even with this minimum required sample size, the researcher faces a dilemma. For that critical fractile where the precision is relatively large, for the same critical fractile the actual probability of not experiencing a stock out during the period has a relatively large distance from the critical fractile. We consider that the tables which we offer can help the practitioner to give his own priority and eventually to decide upon the size of the critical fractile that he will be aiming at. There is also the case the available sample size to be smaller than the required minimum. Again the tables which we offer can help the practitioner to trace the losses in the coverage and in the precision of the prediction interval for the optimal order quantity, as well as, to know a-priori the actual probability of not running out of stock during the period.

Appendix

Proof of Proposition 1

Let $\mathbf{Y}' = [Y_1 \quad Y_2 \quad \ldots \quad Y_n]$ and $\boldsymbol{\mu}' = \mu[1 \quad 1 \quad \ldots \quad 1]$. If Y_t is generated by the linear process given in (1) with ε_t's to be i.i.d. normal random variables with mean zero and constant variance, then $\mathbf{Y}$ follows the n-variate normal distribution with mean $\boldsymbol{\mu}$ and variance-covariance matrix

$$\boldsymbol{\Sigma} = \gamma_o \begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{n-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{n-2} \\ \rho_2 & \rho_1 & 1 & \cdots & \rho_{n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \rho_{n-1} & \rho_{n-2} & \rho_{n-3} & \cdots & 1 \end{bmatrix}.$$

Rewrite also $\overline{Y} = \boldsymbol{\beta}' \cdot \mathbf{Y}$, where $\boldsymbol{\beta}' = (1/n)[1 \quad 1 \quad \ldots \quad 1]$ and

$$\hat{\gamma}_o = \frac{1}{n}\left\{\left(1 - \frac{1}{n}\right)\sum_{j=1}^{n} Y_j^2 - \frac{2}{n}\sum_{i=1}^{n}\sum_{j=i+1}^{n} Y_{ij}\right\} = \mathbf{Y}' \cdot \mathbf{G} \cdot \mathbf{Y}, \text{ where } \mathbf{G} = \frac{1}{n^2}\begin{bmatrix} n-1 & -1 & \ldots & -1 \\ -1 & n-1 & \ldots & -1 \\ \ldots & \ldots & \ldots & \ldots \\ -1 & -1 & \ldots & n-1 \end{bmatrix}.$$

Theorem 2 of section 2.5 of Searle (1971) says that $\overline{Y}_n$ and $\hat{\gamma}_o$ are distributed independently when $\boldsymbol{\beta}' \cdot \boldsymbol{\Sigma} \cdot \mathbf{G} = \mathbf{0}'$. For n=2 this condition is met, and the proof is completed.

Proof of Proposition 2

This proof requires a set of prerequisite results. Setting $y_t = Y_t - \mu$ and using (1) we have

$$E(y_t y_{t+k}^2) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \psi_i \psi_j^2 E(\varepsilon_{t-i}\varepsilon_{t+k-j}^2) + 2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=j+1}^{\infty} \psi_i \psi_j \psi_r E(\varepsilon_{t-i}\varepsilon_{t+k-j}\varepsilon_{t+k-r}) = 0, \qquad \text{(A1)}$$

and

$$E(y_t y_{t-k}^2) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \psi_i \psi_j^2 E(\varepsilon_{t-i}\varepsilon_{t-k-j}^2) + 2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=j+1}^{\infty} \psi_i \psi_j \psi_r E(\varepsilon_{t-i}\varepsilon_{t-k-j}\varepsilon_{t+k-r}) = 0, \qquad \text{(A2)}$$

since $E(\varepsilon_t^3) = 0$, $E(\varepsilon_t^2 \varepsilon_r) = E(\varepsilon_t \varepsilon_r^2) = 0$ for $t \neq r$, and $E(\varepsilon_t \varepsilon_r \varepsilon_u) = 0$ for $t \neq r \neq u$.

Using (A1) and (A2),

$$E\left(Y_t Y_{t+k}^2\right) = E\left((y_t+\mu)(y_{t+k}+\mu)^2\right) = 2\mu\gamma_k + \mu\gamma_o + \mu^3 = E\left(Y_t^3\right) - 2\mu\gamma_o(1-\rho_k) \quad \text{(A3)}$$

and

$$E\left(Y_t Y_{t-k}^2\right) = E\left((y_t+\mu)(y_{t-k}+\mu)^2\right) = 2\mu\gamma_k + \mu\gamma_o + \mu^3 = E\left(Y_t^3\right) - 2\mu\gamma_o(1-\rho_k) \quad \text{(A4)}$$

Using (A3) and (A4),

$$E\left(\sum_{t=2}^{n-1}\sum_{k=1}^{t-1} Y_t Y_{t-k}^2\right) = E\left(\sum_{t=1}^{n-1}\sum_{k=1}^{n-t} Y_t Y_{t+k}^2\right) =$$

$$= \frac{n(n-1)}{2} E\left(Y_t^3\right) - n\mu\gamma_o\left\{(n-1) - 2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\rho_k\right\}. \quad \text{(A5)}$$

Using (A5),

$$E\left[\left(\sum_{t=1}^{n} Y_t\right)\left(\sum_{t=1}^{n} Y_t^2\right)\right] = nE\left(Y_t^3\right) + E\left(\sum_{t=1}^{n-1}\sum_{k=1}^{n-t} Y_t Y_{t+k}^2\right) + E\left(\sum_{t=2}^{n-1}\sum_{k=1}^{t-1} Y_t Y_{t-k}^2\right) =$$

$$= n^2 E\left(Y_t^3\right) - 2n\mu\gamma_o\left\{(n-1) - 2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\rho_k\right\}. \quad \text{(A6)}$$

Further, as the vector $\mathbf{Y}' = \begin{bmatrix} Y_1 & Y_2 & \dots & Y_n \end{bmatrix}$ follows the n-variate Normal distribution with the same marginal mean μ and variance-covariance matrix given in proposition 1,

$$\overline{Y} \sim N\left(\mu, \frac{\gamma_o}{n}\left[1 + 2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\right]\rho_k\right), \quad \text{(A7)}$$

and so

$$E\left(\overline{Y}^3\right) = \mu^3 + \frac{3\mu\gamma_o}{n}\left[1 + 2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\right]\rho_k. \quad \text{(A8)}$$

Setting $E\left(Y_t^3\right) = \mu^3 + 3\mu\gamma_o$, and using (A6) and (A8) we obtain

$$E\left(\overline{Y}\cdot\hat{\gamma}_o\right) = E\left[\left(\frac{1}{n}\sum_{t=1}^{n} Y_t\right)\left(\frac{1}{n}\sum_{t=1}^{n} Y_t^2 - \frac{1}{n^2}\left(\sum_{t=1}^{n} Y_t\right)^2\right)\right] =$$

$$= \frac{1}{n^2} E\left[\left(\sum_{t=1}^{n} Y_t\right)\left(\frac{1}{n}\sum_{t=1}^{n} Y_t^2\right)\right] - E\left(\overline{Y}^3\right) =$$

$$= \frac{\mu\gamma_o}{n}\left\{n-1-2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\rho_k\right\}. \tag{A9}$$

Finally, Sutradhar (1994) showed that

$$E(\hat{\gamma}_o) = \frac{\gamma_o}{n}\left\{n-1-2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\rho_k\right\}. \tag{A10}$$

Hence, from (A7), (A9), and (A10) we obtain

$$\mathrm{Cov}(\overline{Y}, \hat{\gamma}_o) = E(\overline{Y}\cdot\hat{\gamma}_o) - E(\overline{Y})E(\hat{\gamma}_o) = 0,$$

which completes the proof.

Proof of Proposition 3

Using Theorem 2 of Searle (1971), which was stated in proposition 1, we have

$$\boldsymbol{\beta}'\cdot\boldsymbol{\Sigma}\cdot\mathbf{G} = \frac{\gamma_o}{n^3}\left[\sum_{k=0}^{n-1}\rho_k \quad \rho_1+\sum_{k=0}^{n-2}\rho_k \quad \sum_{k=1}^{2}\rho_k+\sum_{k=0}^{n-3}\rho_k \quad \cdots \quad \sum_{k=0}^{n-1}\rho_k\right]\begin{bmatrix} n-1 & -1 & \ldots & -1 \\ -1 & n-1 & \ldots & -1 \\ \ldots & \ldots & \ldots & \ldots \\ -1 & -1 & \ldots & n-1 \end{bmatrix}.$$

To prove that $\overline{Y}$ and $\hat{\gamma}_o$ are not independent random variables, it is enough to show that at least one element of the product $\boldsymbol{\beta}'\cdot\boldsymbol{\Sigma}\cdot\mathbf{G}$ is not zero. We choose the element in row 1 and column 1 of $\boldsymbol{\beta}'\cdot\boldsymbol{\Sigma}\cdot\mathbf{G}$ which is

$$\frac{\gamma_o}{n^3}\left\{(n-1)\sum_{k=0}^{n-1}\rho_k - \sum_{k=1}^{n-1}[2n-(3+2(k-1))]\rho_k\right\}. \tag{A11}$$

The proof is completed by noting that when n is odd, the expression inside the brackets of (A11) becomes

$$-\sum_{k=1}^{\frac{n-1}{2}}(n-2k)\rho_k + \sum_{k=1}^{\frac{n-1}{2}}(n-2k)\rho_{n-k} = \sum_{k=1}^{\frac{n-1}{2}}(n-2k)(\rho_{n-k}-\rho_k) < 0,$$

while for n even this is

$$-\sum_{k=1}^{\frac{n}{2}-1}(n-2k)\rho_k + \sum_{k=1}^{\frac{n}{2}-1}(n-2k)\rho_{n-k} = \sum_{k=1}^{\frac{n}{2}-1}(n-2k)(\rho_{n-k}-\rho_k) < 0.$$

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