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24 January 2013

Existence of Nash Equilibrium in Games with a Measure Space of Players and Discontinuous Payoff Functions*

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Abstract

Balder's (2002) model of games with a measure space of players is integrated with the line of research on finite-player games with discontinuous payoff functions which follows Reny (1999). Specifically, we extend the notion of continuous security, introduced by McLennan, Monteiro, and Tourky (2011) and Barelli and Meneghel (2012) for finite-players games, to games with a measure space of players and establish the existence of pure strategy Nash equilibrium for such games. A specification of our main existence result is provided which is ready to fit the needs of applications. As an illustration, we consider several optimal income tax problems in the spirit of Mirrlees (1971) and use our game-theoretic result to show the existence of an optimal income tax in each of these problems.

*Thanks to Paulo Barelli, Alex Gershkov, Michael Greinecker, Mário Páscoa, Phil Reny, Hamid Sabourian and Pedro Teles for helpful conversations. Financial support from Fundação para a Ciência e a Tecnologia is gratefully acknowledged.

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1 Introduction

The line of research initiated by Dasgupta and Maskin (1986) and continued, amongst others, by Reny (1999) has been successful in obtaining equilibrium existence results for finite-player games with discontinuous payoff functions. In this paper we extend this approach to the context of generalized games with a measure space of players, a class of games first considered by Schmeidler (1973), the state of the art now set by Balder (2002). In particular, concerning existence of Nash equilibrium, we bring the branch of game theory dealing with games with a measure space of players on par with that dealing in a systematic way with games with discontinuous payoff functions.

Besides of being of general game-theoretic interest, the motivation is that several economic problems which are addressed in the literature can be modeled as games with a continuum of players, but where payoff functions need neither be continuous nor satisfy the assumptions in Balder (2002). As an example, we will consider a version of Mirrlees's (1971) model of optimal taxation (see Section 3).

Our approach to deal with discontinuous payoff functions in the setting of games with a measure space of players is based on the notion of C-security, which was developed in the context of finite-player games by McLennan, Monteiro, and Tourky (2011). More precisely, we take a version of this notion, called continuous security, which was introduced by Barelli and Meneghel (2012), and adapt it to the particular measurability needs arising when there may be a continuum of players.[1] We remark that the notion of C-security generalizes that of better-reply security, which was introduced in the pioneering paper of Reny (1999).

Our notion of continuous security covers, in particular, games where, as in Balder (2002), payoff functions are assumed to be upper semi-continuous and the value functions of the players are assumed to be lower semi-continuous.[2] In fact, when value functions are assumed to be lower semi-continuous, it allows for payoff functions that are merely weakly upper semi-continuous (as defined in Carmona (2009)).

In addition to the pure strategy existence result of Balder (2002), our result covers that of Khan and Sun (1999). In this latter result, payoff functions are continuous but the entire distribution of the actions of players with non-convex action sets may be relevant for the payoff of each single player, whereas in Balder (2002) only a finite-dimensional vector of summary statistics of the actions of such players may matter for payoffs. We remark that the key assumption in Khan and Sun (1999) amounts to a strengthening of the hypothesis that the measure on a set of players with non-

[1] Actually, in Barelli and Soza (2009), which is an earlier version of Barelli and Meneghel (2012), continuous security is called "generalized B-security," in line with early versions of McLennan, Monteiro, and Tourky (2011) where C-security was called "B-security".

[2] See Section 2.2 for the formal definition of the value function of a player.

convex action sets be non-atomic. Our result will show, in particular, that such a strengthening of non-atomicity still allows to obtain pure-strategy Nash equilibria, with payoff dependence modeled as in Khan and Sun (1999) and non-convex action sets, when payoff functions may be discontinuous.[3,4]

The paper is organized as follows. In Section 2 we introduce some notation and terminology (Section 2.1), present the general model (Section 2.2), our definition of continuous security (Section 2.3), and our main existence results (Section 2.4). The relationship between these results and that of Balder (2002) is detailed in Section 2.5. In Section 2.6, we present a special case of the general model where, in particular, there are at most countably many atomic players, and where each player's payoff depends on his choice, on the choices of the atomic players, and on the vector of the joint distributions of the actions and players' attributes appearing in each one of countably many sub-populations of the atomless players. Sufficient conditions for continuous security are presented in Section 2.6 and Section 2.7, where generalized better-reply secure games are considered. Section 3 applies our results to several optimal income taxation problems. The proofs of our results, as well as the lemmas needed for them, are in Section 4. For convenience of the reader, our main results are restated in Section 4 before theirs proofs.

2 Definitions and results

2.1 General notation and terminology

(a) $\varphi\colon A \rightrightarrows B$ denotes a correspondence from the set A to the set B, i.e., a map from A to the power set of B.

(b) We use "usc" as abbreviation for "upper semi-continuous," "lsc" for "lower semi-continuous," and "uhc" for "upper hemi-continuous."

(c) If A and B are topological spaces, a correspondence $\varphi\colon A \rightrightarrows B$ is called *well-behaved* if it is uhc and takes non-empty and closed values.

(d) For a topological space X, $\mathcal{B}(X)$ denotes the Borel σ-algebra of X.

(e) If A and B are as in (c), and (T, Σ, ν) is a measure space, we call a correspondence $\varphi\colon T \times A \rightrightarrows B$ a *Caratheodory correspondence* if $\varphi(t, \cdot)$ is well behaved for

[3]Further papers on existence of equilibrium with a measure space of players include Barelli and Duggan (2012), Khan (1986), Khan, Rath, and Sun (1997), Martins da Rocha and Topuzu (2008), Mas-Colell (1984), Páscoa (1993), and Rath (1996).

[4]Recent papers on finite-player games with discontinuous payoff functions include Bagh and Jofre (2006), Balder (2011), Bich (2009), Bich and Laraki (2012), Carmona (2011), Carmona (2012), de Castro (2011), Nessah (2011), Prokopovych (2011), Prokopovych (2012), Reny (2009), Reny (2011a), and Reny (2011b).

each $t \in T$, and if for each $a \in A$, the graph of $\varphi(\cdot, a)$ is measurable, i.e., belongs to $\Sigma \otimes \mathcal{B}(B)$.

(f) Given functions $f\colon X \to Y$ and $g\colon X \to Z$, we denote by (f, g) the function $x \mapsto (f(x), g(x))\colon X \to Y \times Z$.

(g) co$E$ denotes the convex hull of a subset E of a topological linear space.

(h) A measure space (T, Σ, ν) is called *separable* if $L_1(\nu)$ (with its usual norm) is separable. We will also say that a measure ν' on a set T' is separable to mean that (T', Σ', ν') is separable if $\Sigma' \subseteq 2^{T'}$ is the domain of ν'.

(i) A measure space (T, Σ, ν) is called *super-atomless* if for every $E \in \Sigma$ with $\nu(E) > 0$, the subspace of $L^1(\nu)$ consisting of the elements of $L^1(\nu)$ vanishing off E is non-separable. (For equivalent definitions, see Podczeck (2009).) We also say that a measure ν' on a set T' is super-atomless to mean that (T', Σ', ν') is super-atomless if $\Sigma' \subseteq 2^{T'}$ is the domain of ν'.

We remark that atomless Loeb probability spaces are super-atomless. Also, as follows from Fremlin (2008, Proposition 521P(b)), Lebesgue measure on the unit interval can be extended to a super-atomless probability measure (see Podczeck 2009).

2.2 The general model

The model of games we consider is (the pure strategy part of) that of Balder (2002), with some modifications.

There is a measure space (T, Σ, ν) of players. The measure space (T, Σ, ν) may be non-atomic or purely atomic, or may have both an atomic part and a non-atomic part. This allows for non-atomic games as well as for finite-player games as special cases, but also covers situations where finitely many large players and players belonging to a continuum of negligible players interact. The following is supposed to hold.

(A1) (T, Σ, ν) is a complete non-trivial finite measure space ("non-trivial finite" meaning $0 < \nu(T) < \infty$).

In Balder (2002), it is assumed in addition that (T, Σ, ν) is separable. However, we will also consider spaces of players where this condition does not hold, and for this reason we will introduce this condition as part of a special assumption in Section 2.4.

The set T of players is grouped into two measurable subsets $\bar{T}$ and $\hat{T}$ with $\bar{T} \cap \hat{T} = \emptyset$ and $\bar{T} \cup \hat{T} = T$. The set $\bar{T}$ is the set of those players for which convexity assumptions will be made in (A4) below and in our notion of continuous security stated in the next section. It is assumed:

(A2) $\hat{T}$ is contained in the non-atomic part of (T, Σ, ν).

Action sets of players are subsets of a universe X where

(A3) X is a Souslin locally convex topological vector space.

Recall that a topological space X is called Souslin if it is Hausdorff and if there is a continuous surjection from a Polish space onto X. Thus any Polish space is a Souslin space. Examples of locally convex spaces that matter in several economic models and which are Souslin but not Polish are separable Banach spaces with the weak topology and duals of separable Banach spaces with the weak* topology.

The action set of player $t \in T$ is denoted by X_t, and by Γ_G we denote the graph of the action sets correspondence $t \mapsto X_t$. It is assumed:

(A4) (i) For each $t \in T$, the action set X_t is a non-empty compact subset of X.

(ii) Γ_G is a measurable subset of $T \times X$, i.e., belongs to $\Sigma \otimes \mathcal{B}(X)$.

(iii) For every $t \in \bar{T}$, X_t is convex.

A strategy profile (or, for short, a strategy) is a measurable function $f\colon T \to X$ such that $f(t) \in X_t$ for almost all $t \in T$. By S_G we denote the set of all strategies in the game G. Thus S_G is just the set of all measurable a.e. selections of the action sets correspondence $t \mapsto X_t$.

A player's payoff depends on his own action and on a so-called externality which reflects the choices of all players. This externality is modeled in the following way.

Let $\bar{S}_G = \{f|_{\bar{T}} \colon f \in S_G\}$. That is, $\bar{S}_G$ is the set of the restrictions of the elements of S_G to $\bar{T}$, or, in other words, the set of all strategy profiles of the players in $\bar{T}$. In addition, let $\hat{\mathcal{C}}$ be a countable set of functions $q\colon \Gamma_G \cap (\hat{T} \times X) \to \mathbb{R}$ such that (i) q is measurable, (ii) $q(t, \cdot)$ is continuous for each $t \in \hat{T}$, (iii) there is an integrable function $\theta_q \colon \hat{T} \to \mathbb{R}_+$ such that $\sup\{|q(t,x)| \colon x \in X_t\} \leq \theta_q(t)$ for each $t \in \hat{T}$. (It is understood in (i), as well as in the sequel, that, whenever it matters, products of measurable spaces are endowed with the product σ-algebra, and subsets of measurable spaces with the subspace σ-algebra.) Let $\bar{e}\colon S_G \to \bar{S}_G$ be given by $\bar{e}(f) = f|_{\bar{T}}$, and $\hat{e}\colon S_G \to \mathbb{R}^{\hat{\mathcal{C}}}$ by $\hat{e}(f) = \langle \int_{\hat{T}} q(t, f(t)) \mathrm{d}\nu(t) \rangle_{q \in \hat{\mathcal{C}}}$. Note that the integrals are indeed defined. Now define $e\colon S_G \to \bar{S}_G \times \mathbb{R}^{\mathcal{C}}$ by setting $e(f) = (\bar{e}(f), \hat{e}(f))$ for each $f \in S_G$. The map e is the externality map of the game. Together with the own actions, its values determine the payoff of a single player.

Actually, in Balder (2002) it is assumed that the set $\hat{\mathcal{C}}$ is finite. We will look at this condition in Section 2.4.

Let $E_G \subseteq \bar{S}_G \times \mathbb{R}^{\hat{\mathcal{C}}}$ denote the image of S_G under e, i.e., $E_G = e(S_G)$. The set E_G is given a topology specified as follows. First, the set $\bar{S}_G$ is given the feeble topology. Recall from Balder (2002) that the feeble topology on $\bar{S}_G$ is the coarsest topology such that the map $h \mapsto \int_{\bar{T}} q(t, h(t)) \mathrm{d}\nu(t) \colon \bar{S}_G \to \mathbb{R}$ is continuous for each $q \in \bar{\mathcal{G}}$, where $\bar{\mathcal{G}}$ is the set of all functions $q\colon \bar{T} \times X \to \mathbb{R}$ such that (i) q is measurable, (ii)

$q(t,\cdot)$ is linear and continuous for each $t \in \bar{T}$, (iii) there is an integrable function $\theta_q : \bar{T} \to \mathbb{R}_+$ such that $\sup\{|q(t,x)| : x \in X_t\} \le \theta_q(t)$ for each $t \in \bar{T}$. Secondly, $\mathbb{R}^{\hat{\mathcal{C}}}$ is given the product topology defined from the usual topology of $\mathbb{R}$. Now E_G is given the subspace topology defined from the product topology of $\bar{S}_G \times \mathbb{R}^{\hat{\mathcal{C}}}$.

Each player $t \in T$ has a payoff function $u_t : X_t \times E_G \to [-\infty, +\infty]$. Thus, given a strategy profile $f \in S_G$, player t's payoff is determined by his own action $f(t)$ and by the externality $e(f)$.

In addition to the payoff function, for each player $t \in T$ there is a constraint correspondence $A_t : E_G \rightrightarrows X_t$. The set $A_t(y)$ specifies the actions that are actually available for player t given the externality $y \in E_G$. As elements of E_G represent social outcomes given choices of all players, the set $A_t(y)$ can be viewed as a socially constrained action set of player t given $y \in E_G$.

We summarize a game as just outlined by a list $G = ((T,\Sigma,\nu), X, \langle X_t, u_t, A_t\rangle_{t\in T}, e)$. Given such a game G, we denote by w_t the value function of player $t \in T$; that is, $w_t : E_G \to [-\infty, +\infty]$ is the function defined by setting

$$w_t(y) = \sup\{u_t(x,y) : x \in A_t(y)\},\ y \in E_G.$$

A strategy f is called a Nash equilibrium (for short, an equilibrium) of the game G if $f(t) \in A_t(e(f))$ and $u_t(f(t), e(f)) = w_t(e(f))$ for almost all $t \in T$.

We note that several assumptions in addition to those presented above are made in the pure strategy Nash equilibrium existence result in Balder (2002). These additional assumptions will be listed in Section 2.5 below.

A couple of remarks are in order.

Remark 1. (a) Assuming that all the action sets are included in the same Souslin locally convex space X is not a big restriction. Indeed, suppose for instance that, for two Souslin locally convex spaces X_0 and X_1, we have $X_t \subseteq X_0$ for all $t \in \bar{T}$ and $X_t \subseteq X_1$ for all $t \in \hat{T}$, without imposing any relationship between X_0 and X_1. In this case, we can set $X = X_0 \times X_1$ and identify with X_0 with the subspace $X_0 \times \{0\}$ of X, and X_1 with the subspace $\{0\} \times X_1$, noting that the product of two locally convex Souslin spaces is again a space of this kind (directly from the definition of such space.)

(b) In fact, it suffices to assume that X_1 is just a completely regular Souslin space (without imposing any linear structure on X_1). The reasons are the following. First, no convexity assumptions are made with respect to the players in $\hat{T}$. Second, if X_1 is a completely regular Souslin space then, writing $M(X_1)$ for the space of all bounded signed Borel measures on X_1 with the narrow topology, $M(X_1)$ is a locally convex Souslin space (Schwartz (1973, p. 387, Corollary)) and the identification of the points in X_1 with the corresponding Dirac measures defines a homeomorphic embedding of X_1 into $M(X_1)$.

Remark 2. Note that for players in $\bar{T}$, action and externality are not independent of each other, as a given value of the externality determines the action of a player in $\bar{T}$. However, as pointed out by Balder (2002, Section 2.4), this does not imply any inconsistency in the way payoffs are modeled.

This is easy to see in the simple special case where $\bar{T}$ is countable and $\{t\} \in \Sigma$ for each $t \in \bar{T}$. Indeed, in this case, $\bar{S}_G$ is the same as $\prod_{t \in \bar{T}} X_t$. Thus, writing π_t for the projection of $\prod_{t' \in \bar{T}} X_{t'} \times \mathbb{R}^{\hat{\mathcal{C}}}$ onto $\prod_{t' \in \bar{T} \setminus \{t\}} X_{t'} \times \mathbb{R}^{\hat{\mathcal{C}}}$, we can assume that, for each $t \in \bar{T}$, the payoff function u_t is such that $u_t(x, y) = v_t(x, \pi_t(y))$ for any $x \in X_t$ and $y \in E_G$, where v_t is a function defined on $X_t \times \pi_t(E_G)$, and this resolves any consistency issues.

As for the general case, let $\bar{T}_1$ be the set of the non-negligible players in $\bar{T}$ (i.e., those $t \in \bar{T}$ for which the outer measure of $\{t\}$ is strictly positive), and let $\bar{T}_2$ be the set of negligible players in $\bar{T}$. For an element $g \in \bar{S}_G$, write $g^\bullet$ for the ν-equivalence class of g in the space of measurable functions from $\bar{T}$ to X, and let $\bar{S}_G^\bullet$ be the set of all these equivalence classes. Define $\pi \colon \bar{S}_G \times \mathbb{R}^{\hat{\mathcal{C}}} \to \bar{S}_G^\bullet \times \mathbb{R}^{\hat{\mathcal{C}}}$ by setting

$$\pi(y) = (g^\bullet, h), \ y = (g, h) \in \bar{S}_G \times \mathbb{R}^{\hat{\mathcal{C}}}.$$

We can assume that for players t belonging to $\bar{T}_2$, the payoff function u_t satisfies $u_t(x, y) = v_t(x, \pi(y))$ for every $x \in X_t$ and $y \in E_G$, where v_t is a function defined on $X_t \times \pi(E_G)$, and this resolves any consistency issues for the players in $\bar{T}_2$. Consider any $t \in \bar{T}_1$. As $\bar{T}$ is a measurable subset of T, there is an atom $F \subseteq \bar{T}$ such that $t \in F$. Let $\bar{S}_G^F$ be the set of restrictions of the elements of $\bar{S}_G$ to $\bar{T} \setminus F$, and define $\pi_F \colon \bar{S}_G \times \mathbb{R}^{\hat{\mathcal{C}}} \to \bar{S}_G^F \times \mathbb{R}^{\hat{\mathcal{C}}}$ by setting

$$\pi_F(y) = (g|_{\bar{T} \setminus F}, h), \ y = (g, h) \in \bar{S}_G \times \mathbb{R}^{\hat{\mathcal{C}}}.$$

We can assume that, for some function v defined on $X_t \times \pi_F(E_G)$, the payoff function $u_{t'}$ of each player $t' \in F$ satisfies $u_{t'}(x, y) = v(x, \pi_F(y))$ for every $x \in X_{t'}$ and $y \in E_G$, and this resolves any consistency issues for the players belonging to F.

Similarly as with the payoff functions, the fact that, for players in $\bar{T}$, action and externality are not independent of each other does not imply any inconsistency in the way the constraint correspondences are modeled.

However, this whole issue will not play any role in the arguments concerning existence of Nash equilibrium.

Remark 3. If $\bar{T}$ is countable and $\{t\} \in \Sigma$ for each $t \in \bar{T}$, then $\bar{S}_G$ is the same as $\prod_{t \in \bar{T}} X_t$. Moreover, if $\nu(\{t\}) > 0$ and the action set X_t is compact for each $t \in \bar{T}$, then the feeble topology on $\bar{S}_G$ is the same as the product topology on $\prod_{t\in\bar{T}} X_t$. This is so for two reasons. First, compactness of X_t means that the weak topology of the locally convex space X coincides on X_t with the given topology of X; thus a net $\langle x_\alpha\rangle$ in X_t converges to some $x \in X_t$ if and only if $p(x_\alpha) \to p(x)$ for each continuous linear function $p\colon X \to \mathbb{R}$. Second, if $\{t\} \in \Sigma$ and X_t is compact for each $t \in \bar{T}$, then, for any such p, the function $q\colon \bar{T} \times X \to \mathbb{R}$, where $q(t,\cdot) = p$ for one $t \in \bar{T}$ and $q(t,\cdot)$ is the zero functional elsewhere in $\bar{T}$, belongs to the set $\bar{\mathcal{G}}$ in the definition of the feeble topology. Thus, given that $\nu(\{t\}) > 0$ for each $t \in \bar{T}$, it follows that if a net $\langle h_\alpha\rangle$ in $\bar{S}_G$ converges to some $h \in \bar{S}_G$ for the feeble topology, then it converges to this h for the product topology of $\prod_{t\in\bar{T}} X_t$. In view of (iii) in the definition of $\bar{\mathcal{G}}$, it is clear that the reverse implication also holds, given that $\bar{T}$ is countable.

Remark 4. The model presented above contains the standard (normal-form) model of finite-player games as a special case (as long as action sets are contained in locally convex Souslin spaces). Indeed, suppose for the measure space (T, Σ, ν) of players that T is finite, $\Sigma = 2^T$, and ν is the counting measure. In accordance with (A2), set $\bar{T} = T$. Then $S_G = \bar{S}_G$ and thus, by what was noted in the previous remark, S_G is the same as $\prod_{t\in T} X_t$ and the feeble topology on S_G is the same as the product topology on $\prod_{t\in T} X_t$. Concerning the payoff functions, we refer to the second paragraph of Remark 2.

The model described in this section is intended as a general framework that can encompass several simpler models. This point has been emphasized in Balder (2002). In this line, we show in Section 2.6 that the class of games where each player's payoff depends on the own action and the distribution of the actions of all players (considered in Schmeidler (1973) and Mas-Colell (1984)) is included in the framework of this section. Example 1 shows this for a simple game and illustrates the general model of this section.

Example 1. *A non-atomic game where each player's payoff depends on his own action and on the distribution of the actions of all players.* Suppose the space of players is the unit interval with Lebesgue measure and that each player's action set is $\{a_1, a_2, a_3\}$, where a_1, a_2, a_3 are distinct points of $\mathbb{R}$. Then, writing Δ for the unit simplex in $\mathbb{R}^3$, a distribution of actions can be described by a point $y \in \Delta$, with the interpretation that the ith coordinate y^i is the fraction of players choosing action a_i. In this notation, payoff functions such that a player's payoff is determined by the own action and the distribution of the actions of all players are functions with domain $\{a_1, a_2, a_3\} \times \Delta$. Assume for simplicity that all players have the same payoff function $u\colon \{a_1, a_2, a_3\} \times \Delta \to \mathbb{R}$.

To represent this game in the setting of this section, let (T, Σ, μ) be the unit interval with Lebesgue measure, let $X = \mathbb{R}$, and for all $t \in T$, let $X_t = \{a_1, a_2, a_3\}$.

Thus $\Gamma_G = T \times \{a_1, a_2, a_3\}$. Define functions $q_i \colon \Gamma_G \to \mathbb{R}$, $i = 1, 2, 3$, by setting

$$q_i(t, x) = \begin{cases} 1 & \text{if } x = a_i \\ 0 & \text{otherwise.} \end{cases}$$

Let $\hat{T} = T$, and let $e \colon S_G \to \mathbb{R}^3$ be the externality map defined from $\hat{\mathcal{C}} = \{q_1, q_2, q_3\}$. Write e^i for the ith coordinate function of e, $i = 1, 2, 3$, and note that for any $f \in S_G$, $e^i(f) = \nu(\{t \in T \colon f(t) = a_i\})$. In particular, we have $E_G \equiv e(S_G) = \Delta$.

Finally, for each $t \in T$, let $u_t = u$ and $A_t(y) = \{a_1, a_2, a_3\}$ for all $y \in E_G$. It is clear that the game G defined in this way satisfies (A1)–(A4).

2.3 Continuous security

The notion of continuous security was introduced in the case of finite-player games by Barelli and Meneghel (2012) (see also Barelli and Soza (2009)), building on the notion of multiple security, which was developed by McLennan, Monteiro, and Tourky (2011). We first present the definition of continuous security for finite-player games and then extend this notion to games with a continuum of players.

Consider a game $G = \langle X_i, u_i \rangle_{i \in I}$ with finitely many players, where I is the set of players, X_i is player i's action space, and $u_i : \prod_{j \in I} X_i \to \mathbb{R}$ player i's payoff function. Assume that, for each $i \in I$, X_i is a nonempty, compact, and convex subset of a Hausdorff locally convex topological vector space and u_i is bounded. The game G is called *continuously secure* if for each $y \in \prod_{i \in I} X_i$ which is not a Nash equilibrium of G, there exists a neighborhood U of y in $\prod_{i \in I} X_i$, a vector $\alpha \in \mathbb{R}^I$ and, for every $i \in I$, a well-behaved correspondence $\varphi_i \colon U \rightrightarrows X_i$ such that:

(a) For every $i \in I$ and every $y' \in U$, $\varphi_i(y')$ is convex or included in a finite-dimensional subspace of X_i.

(b) For every $i \in I$, $u_i(x, y'_{-i}) \geq \alpha_i$ for all $y' \in U$ and $x \in \varphi_i(y')$.

(c) For each $y' \in U$ there is an $i \in I$ such that $y'_i \notin \operatorname{co}\{x \in X_i \colon u_i(x, y'_{-i}) \geq \alpha_i\}$.

Here y'_i is the projection of y' onto X_i, and y'_{-i} that on $\prod_{j \in I \setminus \{i\}} X_j$.[5]

[5] The definition of "continuously secure" in Barelli and Meneghel (2012) is not exactly equal to the one presented here. Actually, Barelli and Meneghel (2012) do not require (a). Unfortunately, the proof of Theorem 2.2 in Barelli and Meneghel (2012) does not go through without (a). The reason is that the correspondence Φ in that proof is not necessarily closed-valued, because the convex hull of a compact set need not be closed in an infinite-dimensional space. To solve this problem, one can, as we did here, require $\varphi_i(y')$ to be convex, as in Barelli and Soza (2009), or to be included in a finite-dimensional subspace of X_i, as in McLennan, Monteiro, and Tourky (2011).

Unlike in the finite-player case, measurability properties are not trivially satisfied when there is a continuum of players, but have to be assumed explicitly. In regard to an adaption of the above notion of continuous security to games as described in the previous section, this means the following. First, the analogs of the correspondences φ_i must, as correspondences taking values in the universal action space X, be linked together over the space of players in a measurable way, which can be done using the notion of Caratheodory correspondence as stated in Section 2.1. Second, the analog of the vector α must be a measurable function on the space players. Third, the single player i in (c) of the definition of continuous security must be replaced by a non-negligible set of players.

Another point concerns the notion of neighborhood of a strategy that is involved in the above definition. Now in the games we consider, the way the payoff of a single player is affected by the actions of the other players is modeled by the externality map e, and therefore it is natural to take, as analogs of the sets U in the above definition, subsets of the externality space E_G.

Summarizing this discussion leads to the following definition, where E_G is regarded as being endowed with the topology introduced in Section 2.2, and CS abbreviates "continuous security." For sake of generality, we allow for a restriction operator $\mathcal{X}$ in the spirit of McLennan, Monteiro, and Tourky (2011) (see Remark 5 for a discussion).

Definition 1. A game $G = ((T, \Sigma, \nu), X, \langle X_t, u_t, A_t \rangle_{t \in T}, e)$ is said to satisfy *CS* if there is a correspondence $\mathcal{X} : T \times E_G \rightrightarrows X$, with $\mathcal{X}(t, y) \subseteq X_t$ for all $(t, y) \in T \times E_G$, such that whenever $y \in E_G$ is such that there is no equilibrium strategy f with $e(f) = y$, there is a neighborhood U of y in E_G, a Caratheodory correspondence $\varphi : T \times U \rightrightarrows X$, and a measurable function $\alpha : T \to [-\infty, +\infty]$ such that:

(a) For each $y' \in U$, $\varphi(t, y') \subseteq \mathcal{X}(t, y')$ for all $t \in T$.

(b) For all $y' \in U$ and all $t \in \bar{T}$, $\varphi(t, y')$ is convex or included in a finite-dimensional subspace of X.

(c) For each $y' \in U$, $u_t(x, y') \geq \alpha(t)$ for almost all $t \in T$ and all $x \in \varphi(t, y')$.[6]

(d) If f is a strategy with $e(f) \in U$, $f(t) \in \mathcal{X}(t, e(f))$ for almost all $t \in \hat{T}$, and $f(t) \in \mathrm{co}\mathcal{X}(t, e(f))$ for almost all $t \in \bar{T}$, then there is a non-negligible set $T' \subseteq T$ such that for every $t \in T' \cap \hat{T}$, $u_t(f(t), e(f)) < \alpha(t)$, and for every $t \in T' \cap \bar{T}$, $f(t) \notin \mathrm{co}\{x \in \mathcal{X}(t, e(f)) : u_t(x, e(f)) \geq \alpha(t)\}$.

Concerning the players in $\bar{T}$, we note that the analogs of the projections y_{-i} in the definition of continuous security for finite-player games stated above are now involved implicitly in the sense of what was pointed out in Remark 2.

[6] Note that the exceptional set of measure zero may vary with y'.

Remark 5. A particular case of Definition 1 is obtained when $\mathcal{X}(t, y) = A_t(y)$ for all $(t, y) \in T \times E_G$. Actually, in Section 2.5 we show that, under assumptions as in Balder (2002), CS is satisfied for this specification of $\mathcal{X}$. However, even when no constraint correspondences in the sense of the A_t's of our model are specified in a game, it is useful to allow for general restriction operators $\mathcal{X}$ because, in regard to applications, it means, to quote McLennan, Monteiro, and Tourky (2011), "that the analyst is allowed to specify restrictions" on the strategy profiles that have to be considered in (d). In our context, this is illustrated in Example 3 in Section 2.6.

The following remark may be useful in applications of CS.

Remark 6. (i) If for each $t \in \bar{T}$, $u_t(\cdot, y)$ is quasi-concave and $\mathcal{X}(t, y)$ is convex for all $y \in E_G$, then (d) in this definition is equivalent to the simpler statement:

(d') If f is a strategy with $e(f) \in U$ and $f(t) \in \mathcal{X}(t, e(f))$ for almost all $t \in T$, then there is a $T' \subseteq T$ with $\nu(T') > 0$ such that $u_t(f(t), e(f)) < \alpha(t)$ for all $t \in T'$.

(ii) In the case where $\mathcal{X}(t, y) = X_t$ for all $t \in T$ and $y \in E_G$, (d) in the definition reduces to the statement:

(d") If f is a strategy with $e(f) \in U$, then there is a $T' \subseteq T$ with $\nu(T') > 0$ such that for all $t \in T' \cap \hat{T}$, $u_t(f(t), e(f)) < \alpha(t)$, and for all $t \in T' \cap \bar{T}$, $f(t) \notin \mathrm{co}\{x \in X_t \colon u_t(x, e(f)) \geq \alpha(t)\}$.

(iii) In particular, if for every $t \in T$ and $y \in E_G$, $\mathcal{X}(t, y) = X_t$, and for every $t \in \bar{T}$, X_t is convex and $u_t(\cdot, y)$ is quasi-concave for all $y \in E_G$, then (d) is equivalent to:

(d"') If f is a strategy with $e(f) \in U$, then there is a $T' \subseteq T$ with $\nu(T') > 0$ such that $u_t(f(t), e(f)) < \alpha(t)$ for all $t \in T'$.

Remark 7. As noted in Remark 4, a special case of the general model of Section 2.2 is that of finite-player games in normal-form (provided that the action spaces of the players are included in a Souslin locally convex space). For this special case, it may be seen that Definition 1 is exactly equivalent to the definition of continuous security presented earlier for finite-player games, given that payoff functions are real-valued and bounded, so that the $\alpha(t)$'s in Definition 1 can be assumed to be real numbers.

Remark 8. We note that the framework of Balder (2002) has been extended in Martins da Rocha and Topuzu (2008) by allowing players to have non-ordered preferences. However, the notion of continuous security requires players to have payoff functions, and this is the reason why we adopted Balder's (2002) model. Recently, two conditions were introduced to deal with discontinuous finite-player games when players may have non-ordered preferences. These are condition B (and B_g) in Barelli and Soza (2009) and the condition of point security in Reny (2011b). We leave it for future research whether our results extends to games in the framework of Martins da Rocha and Topuzu (2008) by using some adaptations of these conditions.

2.4 The main existence results

In this section we state our two main results on existence of Nash equilibrium. They correspond to two scenarios, described in assumptions (S1) and (S2) below.

As mentioned in Section 2.2, in Balder (2002) the set $\hat{\mathcal{C}}$ in the definition of the externality map e is assumed to be finite and the measure space of players is assumed to be separable. We gather these two conditions in the following assumption.

(S1) (T, Σ, ν) is separable, and the set $\hat{\mathcal{C}}$ in the definition of e is finite.

Theorem 1. *Let* $G = ((T, \Sigma, \nu), X, \langle X_t, u_t, A_t \rangle_{t \in T}, e)$ *be a game satisfying* (A1)-(A4), (S1)*, and* CS*. Then* G *has a Nash equilibrium.*

(See Section 4.4 for the proof.) Theorem 1 generalizes the pure strategy Nash equilibrium existence result of Balder (2002). Indeed, by Theorem 3 in Section 2.5, Theorem 1 implies Balder's result. By the example in that section, the converse fails.

We remark that Theorem 1 also implies the Nash equilibrium existence result for continuously secure finite-player games by Barelli and Meneghel (2012), provided the action sets of all the players are included in a Souslin locally convex space (however, recall footnote 5). Indeed, suppose for the space (T, Σ, ν) of players that T is finite and ν is the counting measure. Then, by what was pointed out in Remark 4, with $T = \bar{T}$ our setting of games reduces to that of standard normal-form finite-player games. In particular, the measurability assumption (A4)(ii) trivially holds, and so do (A1)-(A3), as well as (S1) (with $\hat{\mathcal{C}}$ being the empty set). The remaining assumptions of Theorem 1 are CS and (i) and (iii) of (A4), and these are the assumptions in the result by Barelli and Meneghel (2012) (concerning CS, see Remark 7).

Turning back to the setting of games which allows for a continuum of players, note that the requirement in (S1) for the set $\hat{\mathcal{C}}$ to be finite implies that the externality map cannot necessarily distinguish between two strategy profiles with different distributions of the actions of the players in $\hat{T}$. To cover the case where payoffs may depend on the entire distribution of the actions of the players in $\hat{T}$, we need to allow the set $\hat{\mathcal{C}}$ to be countably infinite, unless the action sets of the players in $\hat{T}$ are included in a common finite subset of X. However, in order for a convexifying effect of large numbers still to be present with a countably infinite $\hat{\mathcal{C}}$, we have to strengthen the non-atomicity assumption in (A2) by requiring the subspace measure on $\hat{T}$ defined from ν to be super-atomless. The following assumption formulates this requirement in a way that avoids any other hypothesis on the measure ν.

(S2) $\bar{T}$ is equal to the atomic part of (T, Σ, ν) and the subspace measure on $\hat{T}$ defined from ν is super-atomless.

Theorem 2. *Let* $G = ((T, \Sigma, \nu), X, \langle X_t, u_t, A_t \rangle_{t \in T}, e)$ *be a game satisfying* (A1), (A3), (A4), (S2), *and* CS. *Then* G *has a Nash equilibrium.*

(The proof may be found in Section 4.4.) As may be seen in Section 2.6, Theorem 2 indeed implies a pure strategy Nash equilibrium result for games where the payoff of each single player may depend on the entire distribution of the actions of the players in $\hat{T}$. In particular, Theorem 2 implies the existence results in Khan and Sun (1999), Carmona and Podczeck (2009), and Keisler and Sun (2009) (see Remark 13 in Section 2.7 below). We note that, in fact, Theorem 2 applies to situations where payoffs may depend on the vector of the distributions of the actions played in each one of countably many sub-populations of $\hat{T}$.

2.5 Connection to Balder (2002)

In the framework of Section 2.2, consider the following additional assumptions.

(A5) The map $(t, x) \mapsto u_t(x, y) \colon \Gamma_G \to [-\infty, +\infty]$ is measurable for each $y \in E_G$.

(A6) (i) For each $t \in T$, the correspondence A_t is well-behaved.
(ii) For each $y \in E_G$, the graph of the correspondence $t \mapsto A_t(y)$ is measurable, i.e., belongs $\Sigma \otimes \mathcal{B}(X)$.

(A7) For every $t \in T$, u_t is usc and w_t is lsc.

(A8) For every $t \in \bar{T}$, the set $\{x \in A_t(y) \colon u_t(x, y) = w_t(y)\}$ is convex for all $y \in E_G$.

In the existence result about pure strategy Nash equilibria in Balder (2002), these assumptions are made in addition to (A1)-(A4) and (S1). The following theorem shows that our notion of continuous security covers this case.

Theorem 3. *Let* $G = ((T, \Sigma, \nu), X, \langle X_t, u_t, A_t \rangle_{t \in T}, e)$ *be a game satisfying (A1)-(A4) and (S1) or (S2). If (A5)-(A8) hold in addition, then* G *satisfies* CS.

(See Section 4.5 for the proof.)

Note that (A7) is satisfied whenever all constraint correspondences are lower hemicontinuous and all payoff functions are continuous. Furthermore, if for each $t \in \bar{T}$, A_t takes convex values and $u_t(\cdot, y)$ is quasi-concave for all $y \in E_G$, then (A8) is satisfied. Thus Theorem 3 shows that for games satisfying (A1)-(A6) and (S1) or (S2), continuous security according to our notion holds whenever all constraint correspondences and all payoff functions are continuous, and for players belonging to $\bar{T}$, the constraint correspondences take convex values and the payoff functions are quasi-concave in the own action. In particular, if $T = \hat{T}$, and if there are no constraint correspondences (in other words, if $A_t(y) = X_t$ for all $t \in T$ and $y \in E_G$), then CS holds, as it should be, whenever all payoff functions are continuous.

It is well known that in finite-player games, continuous security does not imply that payoff functions are usc or that value functions are lsc (see, e.g., Carmona (2009)). The example below shows that this is so also in the context of the present paper. In particular, the example shows that the converse of the implication in Theorem 3 does not hold in general.

Example 2. *A continuously secure non-atomic game where payoff functions are not usc and value functions are not lsc.* Consider the game G defined in Example 1 and specify u by setting, for all $y \in E_G$ and some $\varepsilon > 0$,

$$u(a_1, y) = \begin{cases} y^1 & \text{if } y^1 \neq 1/2, \\ y^1 + \varepsilon & \text{if } y^1 = 1/2, \end{cases}$$

$$u(a_2, y) = \begin{cases} 1 - y^1 & \text{if } y^1 < 1/2, \\ 1 - y^1 - \varepsilon & \text{if } y^1 \geq 1/2, \end{cases}$$

and

$$u(a_3, y) = \begin{cases} 1/2 & \text{if } y^1 \neq 1/2, \\ 1/2 + \varepsilon & \text{if } y^1 = 1/2. \end{cases}$$

As already noted, the game G satisfies (A1)-(A4). Clearly, (A5), (A6), and (S1) are also satisfied, and so is (A8) as $\bar{T} = \emptyset$. Evidently u is not usc and the corresponding value function is not lsc. However, with $\mathcal{X}(t, y) = X_t$ for all $(t, y) \in T \times E_G$, the game satisfies CS. To see this, note first that if $f \in S_G$ satisfies $e^1(f) = 1/2$, then f is a Nash equilibrium of this game if and only if $e^3(f) = 1/2$.

Let $y \in E_G$ be such that there is no equilibrium strategy with $e(f) = y$. The case $y^1 \neq 1/2$ is easy because, in this case, there is a neighborhood V of y in E_G such that u is continuous on $K \times V$. Hence, assume $y^1 = 1/2$. Since $y \neq e(f)$ for any equilibrium strategy f, then $y^2 > 0$.

Let $U = \{y' \in E_G \colon y'^1 > 0 \text{ and } y'^2 > 0\}$, so that U is a neighborhood of y in E_G, and let $\alpha \colon T \to \mathbb{R}$ be given by $\alpha(t) = 1/2$ for all $t \in T$, and $\varphi \colon T \times U \rightrightarrows X$ by $\varphi(t, y') = \{a_3\}$ for all $(t, y') \in T \times U$, so that α is measurable and φ is a Caratheodory correspondence. Clearly, with this choice of φ, (a) of CS holds, and since $\bar{T} = \emptyset$, so does (b) of CS. Also, $u_t(x, y') \geq 1/2$ for all $t \in T$, $y' \in U$ and $x \in \varphi(t, y')$, so (c) of CS is satisfied for φ and α. But (d) of CS is satisfied as well: If f is a strategy with $e(f) \in U$, then $e^1(f) > 0$ and $e^2(f) > 0$; hence, as $e^i(f) = \nu(f^{-1}\{a_i\})$, setting

$$T' = \begin{cases} f^{-1}(\{a_1\}) \text{ if } e^1(f) < 1/2, \\ f^{-1}(\{a_2\}) \text{ if } e^1(f) \geq 1/2, \end{cases}$$

it follows that $\nu(T') > 0$ and that $u_t(f(t), e(f)) < \alpha(t)$ for all $t \in T'$. $\square$

The following definition places a weakening of the condition on payoff functions to be upper semi-continuous, introduced in Carmona (2009), in the context of our model.

Definition 2. A game $G = ((T, \Sigma, \nu), X, \langle X_t, u_t, A_t \rangle_{t \in T}, e)$ is said to be *weakly upper semi-continuous* (abbreviated "weakly usc" in the sequel) if the following holds for all $t \in T$: Whenever $(x_n, y_n) \to (x, y)$ in $X_t \times E_G$ and $\overline{\lim}\, u_t(x_n, y_n) \neq u_t(x, y)$, there is an $x' \in A_t(y)$ such that $u_t(x', y) > \overline{\lim}\, u_t(x_n, y_n)$.

Remark 9. As $x' \neq x$ is not required in this definition, a game with usc payoff functions and well-behaved constraint correspondences is weakly usc. On the other hand, it is easy to find examples showing that the converse need not hold.

Theorem 4. *Let $G = ((T, \Sigma, \nu), X, \langle X_t, u_t, A_t \rangle_{t \in T}, e)$ be a game satisfying (A1)-(A4) and (S1) or (S2). If, in addition, G is weakly usc, w_t is lsc for all $t \in T$, and (A5), (A6), and (A8) hold, then G is continuously secure.*

(See Section 4.6 for the proof.)

2.6 A concretization of the general model

In this section we will present a specification of the model laid out in Section 2.2, illustrating what is covered by the notion of externality map, and in particular aiming to provide a bridge to potential applications.

In typical applications with a measure space of players, large atomic players appear as singletons, and no convexity assumptions are made on the non-atomic part of the space of players. In view of this, we replace (A2) by the following condition.

(A9) The set $\bar{T}$ is countable, and for each $t \in \bar{T}$, $\{t\} \in \Sigma$ with $\nu(\{t\}) > 0$.

Note that by what was stated in Remark 3, (A9) implies that the set $\bar{S}_G$ of restrictions of strategy profiles to $\bar{T}$ is equal to $\prod_{t \in \bar{T}} X_t$ and that if (A4)(i) holds in addition, then the feeble topology on $\bar{S}_G$ is the same as the topology of pointwise convergence, i.e., the product topology of $\prod_{t \in \bar{T}} X_t$.

We are going to present a specification of the externality map so that, in an explicit way, the entire distribution of the actions of the players in $\hat{T}$ may matter for the payoff of each single player. However, in some contexts, such a specification is still too narrow. For example, payoffs of players may depend on both the distribution of actions chosen by men and that of the actions chosen by women. We will cover this kind of example by allowing the payoffs of players to depend on the distributions of the actions played in each one of countably many sub-populations of $\hat{T}$.

One may also think of examples of the following kind. Suppose the players in $\hat{T}$ are workers, which may be of different productivity. Now if the total output of workers is relevant for payoffs, then it is not just the distribution of actions, efforts say, of the players in $\hat{T}$ that matters for payoffs, but rather the joint distribution of actions and productivity attributes of these players. To capture this sort of example, we consider a space C of players' attributes (or characteristics) and a map $c\colon \hat{T} \to C$ assigning attributes to the players in $\hat{T}$. The following is supposed to hold.

(A10) (i) C is a completely regular Souslin space.
(ii) The map $c\colon \hat{T} \to C$ is measurable.

In most applications, C will be a Polish space. However, for sake of generality, and for symmetry with assumption (A3) on the actions universe X, we just assume (A10)(i).

At a first glance it may look odd to have the function c to be defined only on the subset $\hat{T}$ of T. However, this is not a restriction. In fact, the externality of the game will be defined in such a way that the payoff of any single player may depend on the entire action profile of the players in $\bar{T}$, and attributes of a player in $\bar{T}$ that are relevant for payoffs of other players may be considered as incorporated already in the identity of this player as point in $\bar{T}$.

Summing up, we want to give a specification of the externality of a game so that, in an explicit way, situations are described where each player's payoff may depend on the strategy profile of the players in $\bar{T}$ and on the vector of the joint distributions of the actions and players' attributes appearing in each one of countably many sub-populations of $\hat{T}$. To this end, let $M^1_+(X \times C)$ denote the set of all Borel probability measures on $X \times C$. Let J be a non-empty countable set and suppose that for each $j \in J$ a non-negligible measurable subset T_j of $\hat{T}$ is given. Finally, let $\tilde{e}\colon S_G \to \prod_{t\in\bar{T}} X_t \times \big(M^1_+(X \times C)\big)^J$ be the map given by setting

$$\tilde{e}(f) = \big(f|_{\bar{T}}, \big\langle (1/\nu(T_j))(\nu|_{T_j}) \circ (f|_{T_j}, c|_{T_j})^{-1} \big\rangle_{j\in J}\big)$$

for every $f \in S_G$. Note that if (A10)(ii) holds, then the distributions involved in the above expression are defined. The map $\tilde{e}$ is now taken to be the externality map of a game. Let $\tilde{E}_G$ denote the image of S_G under $\tilde{e}$, i.e., $\tilde{E}_G = \tilde{e}(S_G)$. Now the payoff function of player t is taken to be a function $u_t\colon X_t \times \tilde{E}_G \to [-\infty, +\infty]$, thus being of the form that was intended.

In the context of an externality map $\tilde{e}$ as defined here, we summarize a game by a list $G = ((T, \Sigma, \nu), X, \langle X_t, u_t, A_t\rangle_{t\in T}, \tilde{e})$, on the understanding that the constraint correspondences A_t are defined on $\tilde{E}_G$, and the payoff functions u_t on the respective sets $X_t \times \tilde{E}_G$. As may be seen from the proof of Theorem 5 below, the form of the map $\tilde{e}$ is just a concrete version of the form in which the externality map was defined in Section 2.2.

As payoff functions are now defined on the respective sets $X_t \times \tilde{E}_G$, and constraint correspondence on $\tilde{E}_G$, we have to adjust the definition of continuous security stated in Section 2.3, putting it into terms of $\tilde{e}$ and $\tilde{E}_G$. In particular, we have to choose a topology on the set $\tilde{E}_G$. With the following choice we will get a statement of continuous security, in terms of $\tilde{e}$ and $\tilde{E}_G$, which will turn out to be topologically equivalent to that in Section 2.3. Assuming that (A3) and (A10) hold, we regard $M^1_+(X \times C)$ as being endowed with the narrow topology,[7] and the action sets X_t of the players in $\bar{T}$ as being endowed with the subspace topology defined from the topology of X. Now we give the set $\tilde{E}_G$ the subspace topology defined from the product topology of $\prod_{t\in\bar{T}} X_t \times \big(M^1_+(X \times C)\big)^J$.

We use the abbreviation CS' to differentiate the following notion of continuous security from the version called CS in Section 2.3.

Definition 3. A game $G = ((T, \Sigma, \nu), X, \langle X_t, u_t, A_t\rangle_{t\in T}, \tilde{e})$ is said to satisfy *CS'* if there is a correspondence $\mathcal{X} : T \times \tilde{E}_G \rightrightarrows X$, with $\mathcal{X}(t, y) \subseteq X_t$ for all $(t, y) \in T \times \tilde{E}_G$, such that whenever $y \in \tilde{E}_G$ is such that there is no equilibrium strategy f with $\tilde{e}(f) = y$, there is a neighborhood U of y in $\tilde{E}_G$, a Caratheodory correspondence $\varphi\colon T \times U \rightrightarrows X$, and a measurable function $\alpha : T \to [-\infty, +\infty]$ such that:

(1) For each $y' \in U$, $\varphi(t, y') \subseteq \mathcal{X}(t, y')$ for all $t \in T$.

(2) For all $y' \in U$ and all $t \in \bar{T}$, $\varphi(t, y')$ is convex or included in a finite-dimensional subspace of X.

(3) For each $y' \in U$, $u_t(x, y') \geq \alpha(t)$ for almost all $t \in T$ and all $x \in \varphi(t, y')$.

(4) If f is a strategy with $\tilde{e}(f) \in U$, $f(t) \in \mathcal{X}(t, \tilde{e}(f))$ for almost all $t \in \hat{T}$, and $f(t) \in \mathrm{co}\mathcal{X}(t, \tilde{e}(f))$ for almost all $t \in \bar{T}$, then there is a non-negligible set $T' \subseteq T$ such that for every $t \in T' \cap \hat{T}$, $u_t(f(t), \tilde{e}(f)) < \alpha(t)$, and for every $t \in T' \cap \bar{T}$, $f(t) \notin \mathrm{co}\{x \in \mathcal{X}(t, \tilde{e}(f))\colon u_t(x, \tilde{e}(f)) \geq \alpha(t)\}$.

The following example illustrates the above notion, and in particular what is gained by allowing for restriction operators.

Example 3. Consider the static benchmark game in Angeletos, Hellwig, and Pavan (2007). The space of players is the unit interval with Lebesgue measure. Players simultaneously choose either to attack the status quo, represented by 1, or refrain

[7] Recall that the narrow topology on the set $M^1_+(Z)$ of Borel probability on a topological space Z is defined to be the smallest topology on $M^1_+(Z)$ for which all sets of the form $\{\mu \in M^1_+(Z)\colon \mu(G) > \alpha\}$ are open, where G is an open subset of Z, and α a real number, and note that if Z is completely regular, then this topology agrees with the topology of pointwise convergence on the bounded continuous functions on Z, evaluation being given by integration.

from attacking, represented by 0. Each player's payoffs is as follows: Refraining from attacking yields a payoff of zero. The payoff of attacking is $1-c$, where $0<c<1$, if the fraction of the players who attack is at least θ, where $\theta \in (0,1]$, and is $-c$ if the fraction of the players who attack is less than θ.

To represent this game in the setting of this section, let (T, Σ, ν) be the unit interval with Lebesgue measure, $\bar{T}=\emptyset$, $X=\mathbb{R}$, and, for all $t \in T$, $A_t(y)=X_t=\{0,1\}$ for all $y \in E_G$. Note that $\tilde{E}_G$ can be identified with $[0,1]$, with $y=e(f) \in [0,1]$ denoting the fraction of players choosing 1 in the strategy profile f. Finally, for all $t \in T$, let u_t be defined by

$$u_t(x,y)=\begin{cases} 0 & \text{if } x=0, \\ 1-c & \text{if } x=1 \text{ and } y \geq \theta, \\ -c & \text{if } x=1 \text{ and } y<\theta, \end{cases}$$

for all $(x,y) \in X_t \times \tilde{E}_G$.

Now this game satisfies CS' for the restriction operator $\mathcal{X}$ defined by setting $\mathcal{X}(t,y)=\{0\}$ for all $t \in T$ and $y \in [0,1]$. Indeed, let $y \in \tilde{E}_G$ be such that there is no equilibrium strategy f with $e(f)=y$. Then $y \in (0,1)$ as may easily be seen. Let $U=(0,1)$ and define $\varphi: T \times U \rightrightarrows \{0,1\}$ by $\varphi(t,y')=\{0\}$ for all $(t,y') \in T \times U$. Then φ is a Caratheodory correspondence such that (1) and (2) of CS' hold. Define $\alpha: T \to \mathbb{R}$ by setting $\alpha(t)=-c$ for all $t \in T$. Then (3) of CS' holds, and so does (4), vacuously, because there is no strategy profile f satisfying both $e(f) \in U$ and $f(t) \in \mathcal{X}(t, e(f))$ for almost all $t \in T$. Thus, CS' holds.

The next theorem will be proved as a consequence of Theorem 2. The proof will show, in particular, that CS' can be reduced to CS.

Theorem 5. *Let $G=((T,\Sigma,\nu), X, \langle X_t, u_t, A_t\rangle_{t\in T}, \tilde{e})$ be a game satisfying (A1), (A3), (A4), (A9), (A10), (S2), and CS'. Then G has a Nash equilibrium.*

(See Section 4.7 for the proof.)

Remark 10. The case where there is no attribute function can be regarded as a special case of the framework of this section, by simply letting the attribute space C be any singleton in this case. Thus Theorem 5, as well as Theorem 6 below and the theorems in the next section, continues to be true for a game where there is no attributes function c, and $\tilde{e}$ is defined just in terms of distributions of actions.

The next result provides sufficient conditions for CS' to hold. The two conditions in this result, presented in the following definitions, are versions of the notions of "generalized payoff security" and of "better-reply closed" game, introduced for finite-player games by Barelli and Soza (2009) and Carmona (2011), respectively.[8]

Definition 4. A game $G = ((T, \Sigma, \nu), X, \langle X_t, u_t, A_t \rangle_{t \in T}, \tilde{e})$ is said to satisfy *GPS* if for all $y \in \tilde{E}_G$ and $\varepsilon > 0$ there is a neighborhood U of y in $\tilde{E}_G$, a Caratheodory correspondence $\varphi : T \times U \rightrightarrows X$, and a measurable function $\alpha \colon T \to \mathbb{R}$ such that:

1. For each $y' \in U$, $\varphi(t, y') \subseteq A_t(y')$ for all $t \in T$.
2. For all $y' \in U$ and all $t \in \bar{T}$, $\varphi(t, y')$ is convex or included in a finite-dimensional subspace of X.
3. For each $y' \in U$, $u_t(x, y') \geq \alpha(t)$ for almost all $t \in T$ and all $x \in \varphi(t, y')$.
4. $\nu(\{t \in T \colon \alpha(t) \geq w_t(y) - \varepsilon\}) \geq 1 - \varepsilon$.

Definition 5. A game $G = ((T, \Sigma, \nu), X, \langle X_t, u_t, A_t \rangle_{t \in T}, \tilde{e})$ is said to satisfy *BRC* if the following holds for any strategy profile f: If there is a sequence $\langle f_n \rangle$ of strategy profiles with $\tilde{e}(f_n) \to \tilde{e}(f)$ such that, for almost all $t \in T$, (a) $f_n(t) \in A_t(\tilde{e}(f_n))$ for all $n \in \mathbb{N}$, (b) $f(t) \in \mathrm{LS}\, f_n(t)$, and (c) $\underline{\lim}_n u_t(f_n(t), \tilde{e}(f_n)) \geq w_t(\tilde{e}(f))$, then f is an equilibrium of G.

Note that under (A9) and (A4), $e(f_n) \to \tilde{e}(f)$ implies $f_n(t) \to f(t)$ for each $t \in \bar{T}$ (see the definition of $\tilde{e}$ and the paragraph after the statement of (A9)), so that (b) in this definition reduces to a condition on the restrictions of strategy profiles to $\hat{T}$.

It is common in applications to assume for an atomic player that his action set is convex, that his constraint correspondence takes convex values, and that his payoff function is quasi-concave in his action. Such convexity properties make the definition of CS' easier as we have noted in Remark 6, and therefore we will assume them in the following theorem as well as in the theorems in the next section. Convexity of the action sets of such players is already part of (A4). Thus we introduce here:

(A11) For every $t \in \bar{T}$, $A_t(y)$ is convex and $u_t(\cdot, y)$ is quasi-concave for all $y \in \tilde{E}_G$.

Theorem 6. *Let $G = ((T, \Sigma, \nu), X, \langle X_t, u_t, A_t \rangle_{t \in T}, \tilde{e})$ be a game satisfying (A1), (A3), (A4), (A9)-(A11), (S2), GPS and BRC. Then G also satisfies CS', and consequently, by Theorem 5, G has a Nash equilibrium.*

(See Section 4.8 for the proof.)

[8]For finite-player games, our definition of generalized payoff security is equivalent to the original definition of that notion in Barelli and Soza (2009). Note also that for finite-player games, the property of a game being better reply closed is equivalent to the property of "weak reciprocal upper semi-continuity at all non-equilibrium strategies;" see Carmona (2011, Theorem 5), and see Bagh and Jofre (2006) for the definition of weak reciprocal upper semi-continuity.

2.7 Better-reply secure games

In this section we consider games as specified in the previous section and look at the notion of generalized better-reply security, introduced by Barelli and Soza (2009) for finite-player games, based on the concept of better-reply security, which was developed by Reny (1999); see also Barelli and Meneghel (2012).

Fix a game $G = ((T, \Sigma, \nu), X, \langle X_t, u_t, A_t\rangle_{t\in T}, \tilde{e})$. To avoid technical complications, following Barelli and Soza (2009) we assume in this section:

(A12) For every $y \in \tilde{E}_G$, $u_t(\cdot, y)$ is bounded for all $t \in T$.

In addition, (A1), (A3)-(A6), and (A11) are assumed to hold in the following.

Now for any $y \in \tilde{E}_G$, let $\mathcal{P}_y$ be the set of all triples (U, φ, α) where U is a neighborhood of y in $\tilde{E}_G$, $\varphi\colon T \times U \rightrightarrows X$ a Caratheodory correspondence, and $\alpha\colon T \to \mathbb{R}$ a measurable function such that

(i) For each $y' \in U$, $\varphi(t, y') \subseteq A_t(y')$ for all $t \in T$.

(ii) For all $y' \in U$ and all $t \in \bar{T}$, $\varphi(t, y')$ is convex or included in a finite-dimensional subspace of X.

(iii) For each $y' \in U$, $u_t(x, y') \geq \alpha(t)$ for almost all $t \in T$ and all $x \in \varphi(t, y')$.

Let $\mathcal{Q}_y$ be the set of all measurable functions $\alpha\colon T \to \mathbb{R}$ which are a component of some $(U, \varphi, \alpha) \in \mathcal{P}_y$ and, for each $\alpha \in \mathcal{Q}_y$, write $\alpha^\bullet$ for the ν-equivalence class of α, i.e., the element of $L^0(\nu)$ determined by α. We regard $L^0(\nu)$ as endowed with its usual partial order. Note that by (A1), (A3)-(A5), and (A12), for each $y \in \tilde{E}_G$ there are measurable functions $\beta^1_y\colon T \to \mathbb{R}$ and $\beta^2_y\colon T \to \mathbb{R}$ such that $\beta^1_y(t) \leq u_t(x, y) \leq \beta^2_y(t)$ for each $t \in T$ and $x \in X_t$ (see Castaing and Valadier (1977, Lemma III.39)). Thus, by (A6) and (A11), the set $\{\alpha^\bullet\colon \alpha \in \mathcal{Q}_y\}$ is a non-empty bounded subset of $L^0(\nu)$ for each $y \in \tilde{E}_G$. Therefore, since $L^0(\nu)$ is Dedekind-complete (see Fremlin (2001, 241G)), for each $y \in \tilde{E}_G$ there is a $\underline{w}^\bullet_y \in L^0(\nu)$ such that $\underline{w}^\bullet_y = \sup\{\alpha^\bullet\colon \alpha \in \mathcal{Q}_y\}$.

The following definition provides a notion of generalized better reply security in our context.

Definition 6. A game $G = ((T, \Sigma, \nu), X, \langle X_t, u_t, A_t\rangle_{t\in T}, \tilde{e})$ is said to satisfy *GBRS*$^\bullet$ if the following holds for any strategy profile f: If there is a sequence $\langle f_n\rangle$ of strategy profiles with $\tilde{e}(f_n) \to \tilde{e}(f)$ such that for almost all $t \in T$, (a) $f_n(t) \in A_t(\tilde{e}(f_n))$, (b) $f(t) \in \mathrm{LS}\, f_n(t)$, and (c) $\underline{\lim}_n\, u_t(f_n(t), \tilde{e}(f_n)) \geq \underline{w}^\bullet_{\tilde{e}(f)}(t)$, then f is an equilibrium of G.

As in the context of GPS, if (A9) and (A4) hold then (b) in the definition of GBRS$^\bullet$ reduces to a condition on the restrictions of strategy profiles to $\hat{T}$.

Theorem 7. *Let $G = ((T, \Sigma, \nu), X, \langle X_t, u_t, A_t\rangle_{t\in T}, \tilde{e})$ be a game satisfying (A1), (A3)-(A6), (A9)-(A12), (S2), and GBRS$^\bullet$. Then G also satisfies CS', and consequently, by Theorem 5, G has a Nash equilibrium.*

(See Section 4.9 for the proof.)

In the rest of this section we consider games without constraint correspondences. In particular, we will relate Definition 6 to the notion of generalized better reply security in Barelli and Soza (2009). We indicate a game without constraint correspondences by a list $G = ((T, \Sigma, \nu), X, \langle X_t, u_t\rangle_{t\in T}, \tilde{e})$. Fix such a game.

For each $y \in \tilde{E}_G$ and each $t \in T$, let $\mathcal{P}^t_y$ be the set of all triples (U, φ, α) where U is a neighborhood of y in $\tilde{E}_G$, $\varphi\colon U \rightrightarrows X_t$ a well-behaved correspondence, and α a real number such that $u_t(x, y') \geq \alpha$ for all $y' \in U$ and $x \in \varphi(t, y')$, and such that if $t \in \bar{T}$, $\varphi(t, y')$ is convex or included in a finite-dimensional subspace of X for all $y' \in U$. Let $\mathcal{Q}^t_y$ be the set of all real numbers α which are a component of some $(U, \varphi, \alpha) \in \mathcal{P}^t_y$. Now for each $t \in T$, define a function $\underline{w}_t\colon E_G \to \mathbb{R}$ by setting $\underline{w}_t(y) = \sup\{\alpha\colon \alpha \in \mathcal{Q}^t_y\}$ for each $y \in \tilde{E}_G$.

Definition 7. A game $G = ((T, \Sigma, \nu), X, \langle X_t, u_t\rangle_{t\in T}, \tilde{e})$ is said to satisfy *GBRS* if the following holds for any strategy profile f: If there is a sequence $\langle f_n\rangle$ of strategy profiles such that (a) $\tilde{e}(f_n) \to \tilde{e}(f)$ and, for almost all $t \in T$, (b) $f(t) \in \mathrm{LS}\, f_n(t)$ and (c) $\underline{\lim}_n\, u_t(f_n(t), \tilde{e}(f_n)) \geq \underline{w}_t(\tilde{e}(f))$, then f is an equilibrium of G.

Clearly, if T is countable and (A9) holds, then for all $y \in \tilde{E}_G$ we have $\underline{w}^\bullet_y(t) = \underline{w}_t(y)$ for all $t \in T$, so that GBRS$^\bullet$ and GBRS are equivalent. By the characterization of generalized better reply security in Carmona (2011), this means in particular that for finite-player games (without constraint correspondences) the definition of generalized better reply security in Barelli and Soza (2009) is equivalent to GBRS$^\bullet$.

The equivalence between GBRS$^\bullet$ and GBRS also holds in the following setting of large games, considered, e.g., in Mas-Colell (1984, remarks after Theorem 1), Rath (1992), or Khan, Rath, and Sun (1997). These papers assume a common action set for all players in a game, continuous payoff functions, and that the map assigning payoff function to players is measurable as a map from the measure space of players to the set of payoff functions when this latter set is given the topology of uniform convergence. In the terminology of Section 2.6, this way of setting up a game can be captured with the following assumption, without requiring continuous payoff functions.

(A13) (i) For some $\hat{X} \subseteq X$, $X_t = \hat{X}$ for all $t \in \hat{T}$.
- (ii) u_t is bounded for all $t \in T$.
- (iii) The map $t \mapsto u_t : \hat{T} \to B(\hat{X} \times \tilde{E}_G)$ is measurable, where $B(\hat{X} \times \tilde{E}_G)$ denotes the space of real-valued bounded functions on $\hat{X} \times \tilde{E}_G$, endowed with the sup-norm.
- (iv) $\{u_t : t \in \hat{T}\}$ is separable as a subset of $B(\hat{X} \times \tilde{E}_G)$.

For the rest of this section, note that (A13) implies both (A5) and (A12), and that if there are no constraint correspondences specified in a game then (A6) reduces to (i) and (ii) of (A4), and (A11) reduces to the following statement:

(A11') For every $t \in \bar{T}$, $u_t(\cdot, y)$ is quasi-concave for all $y \in \tilde{E}_G$.

Theorem 8. *Let $G = ((T, \Sigma, \nu), X, \langle X_t, u_t \rangle_{t \in T}, \tilde{e})$ be a game satisfying (A1), (A3), (A4), (A9), (A10), (A11') and (A13). Then for each $y \in \tilde{E}_G$, $\underline{w}_y^\bullet(t) = \underline{w}_t(y)$ for almost all $t \in T$.*

(See Section 4.10 for the proof.)

Remark 11. Note that under (A13)(i), parts (ii) and (iv) of (A13) hold automatically if action and externality set of the game are compact and metrizable, and all payoff functions are continuous. We remark that if there does not exit a probability space (Ω, Σ, ν) with ν atomless and $\Sigma = 2^\Omega$ (which is consistent with ZFC, while the existence of such a probability space cannot be proved to be consistent with ZFC), then (A13)(iii) implies that (A13)(iv) holds essentially, i.e., for some null set N in $\hat{T}$, the set $\{u_t \colon t \in \hat{T} \setminus N\}$ is a separable subset of $B(\hat{X} \times \tilde{E}_G)$ (see Fremlin (2003, 438D)). Thus, given that (A13)(iii) is assumed to hold, (A13)(iv) should not be viewed as an additional restriction.

Remark 12. If (A13)(i) and (A3) hold, and all payoff functions are continuous, then, according to a well-known fact, (A13)(iii) and (A5) are equivalent. However, when payoff functions need not be continuous, then (A3)(iii) imposes more restrictions than (A5). E.g., take (T, Σ, ν) to be $[0, 1]$ with Lebesgue measure, let $\hat{T} = T$, $\hat{X} = [0, 1]$, and suppose that for each $t \in \hat{T}$ and each $y \in \tilde{E}_G$, $u_t(\cdot, y)$ is the characteristic function of $[t, 1]$. Then (A5) is satisfied, but as $t \mapsto u_t$ is an injection, and any $E \subseteq [0, 1]$ has a relatively open image in $\{u_t \colon t \in \hat{T}\}$ under this map, (A13)(iii) fails.

Combining Theorems 7 and 8 we get the following result.

Theorem 9. *Let $G = ((T, \Sigma, \nu), X, \langle X_t, u_t \rangle_{t \in T}, \tilde{e})$ be a game satisfying (A1), (A3), (A4), (A9), (A10), (A11'), (A13), (S2), and GBRS. Then G also satisfies CS', and consequently, by Theorem 5, G has a Nash equilibrium.*

Using Theorem 9, it is easy to relate our results to those of Khan and Sun (1999), Carmona and Podczeck (2009), and Keisler and Sun (2009):

Remark 13. In these papers, a game is given by a super-atomless complete probability space (T, Σ, ν) of players,[9] a finite partition $\langle T_i \rangle_{i \in I}$ of T into non-negligible

[9] Recall that an atomless Loeb probability space is super-atomless, and that the notion of saturated probability space in Keisler and Sun (2009) and that of super-atomless probability space are equivalent.

measurable sets, a common compact metric action space K for all players, and a payoff function $V(t)$ for each $t \in T$, where $V(t)$ is the value at t of a measurable function $V: T \to C(K \times M_+^1(K)^I)$, denoting by $M_+^1(K)$ the space of Borel probability measures on K, endowed with the narrow topology, and by $C(K \times M_+^1(K)^I)$ the space of real-valued continuous functions on $K \times M_+^1(K)^I$, endowed with the sup-norm. In terms of the present section, this yields a game $G = ((T, \Sigma, \nu), X, \langle X_t, u_t \rangle_{t \in T}, \tilde{e})$ specified as follows, so that Theorem 9 applies. For every $t \in T$, let $X_t = K$. By what was noted in Remark 1(b), K can be viewed as a subset of a Souslin locally convex space X, so that (A3) holds for G. Concerning the externality, let $\hat{T} = T$ and define $\tilde{e}: S_G \to M_+^1(X)^I$ by setting $\tilde{e}(f) = \big\langle (1/\nu(T_i))(\nu|_{T_i}) \circ (f|_{T_i})^{-1}) \big\rangle_{i \in I}$ for $f \in S_G$. Observe that for each $i \in I$, $(1/\nu(T_i)(\nu|_{T_i})$ is an atomless probability measure on T, so that for any $y \in M_+^1(K)$ we have $y = (1/\nu(T_i))(\nu|_{T_i}) \circ (f|_{T_i})^{-1})$ for some $f \in S_G$. Hence, because $\langle T_i \rangle_{i \in I}$ is a partition of T, we have $\tilde{E}_G \equiv \tilde{e}(S_G) = M_+^1(K)^I$. Thus for each $t \in T$, we may set $u_t = V(t)$. Evidently (A1), (A4), (A9), and (A11') hold for G (note that $\bar{T} = \emptyset$). Clearly (i)-(iii) of (A13) hold. Concerning (A13)(iv), just note that $C(K \times M_+^1(K)^I)$ is separable, because $K \times M_+^1(K)^I$ is a compact and metrizable by the fact that K is a compact metric space. The game G satisfies GBRS. Indeed, note first that the continuity of the payoff functions implies that $\underline{w}_t(y) = w_t(y)$ for each $t \in T$ and $y \in \tilde{E}_G$. Now another appeal to the continuity of the payoff functions shows that GBRS holds. Finally, as the space (T, Σ, ν) of players is super-atomless, (S2) holds for G. Thus, by Remark 10, Theorem 9 applies, showing that G has an equilibrium. Thus Theorem 9 implies Theorem 1 in Khan and Sun (1999), Corollary 4(4) in Carmona and Podczeck (2009), as well as the necessity part of Theorem 4.6 in Keisler and Sun (2009). In fact, these latter results are implied by Theorem 2, as existence of equilibrium in Theorem 9 follows from Theorem 5, and Theorem 5 is a consequence of Theorem 2 (see Section 4.7).

3 An Application: Optimal Income Taxation

We consider a version of the model of Mirrlees (1971) on optimal income taxation. Specifically, we use our main results to address the existence of an optimal income tax in several optimal taxation problems.

The economy consists of a continuum of individuals, described by a super-atomless complete probability space $(\hat{T}, \hat{\Sigma}, \hat{\nu})$, and a government. There is a single consumption good, which can be produced using labor. Each individual $t \in \hat{T}$ is endowed with one unit of time and is described by his skill level n_t, which is the quantity of labor provided by t per unit of time. We assume that there is an upper bound $\bar{n} > 0$ on the level of skills and an upper bound $\bar{m} \geq \bar{n}$ on consumption. Writing $M = [0, \bar{m}]$ and

$L = [0,1]$, an individual t is further characterized by a continuous utility function $\tilde{u}_t : M \times L \to \mathbb{R}_+$, so that his utility is $\tilde{u}_t(m,l)$ when his individual consumption is m and his effort level is l. Let $N = [0, \bar{n}]$ and let $\hat{n} \colon \hat{T} \to N$ denote the function $t \mapsto n_t$. We make the following assumptions:

(T1) The map $t \mapsto \tilde{u}_t \colon \hat{T} \to C(M \times L)$ is measurable, where $C(M \times L)$ denotes the space of real-valued continuous functions on $M \times L$ endowed with the sup-norm.

(T2) For every $t \in \hat{T}$, $\tilde{u}_t$ is strictly increasing in m, strictly decreasing in l, and $0 \leq \tilde{u}_t(m,l) \leq 1$ for all $(m,l) \in M \times L$.

(T3) The map $t \mapsto n_t \colon \hat{T} \to N$ is measurable.

(T4) The distribution $\hat{\nu} \circ \hat{n}^{-1}$ is atomless.

As in Golosov, Kocherlakota, and Tsyvinski (2003), one unit of labor is transformed into one unit of consumption. This assumption is made for simplicity since, normalizing the price of consumption to one, it implies that the equilibrium price of labor is equal to one, too.[10]

The government chooses an income tax, which, as in Mirrlees (1971), is described by a function $\lambda : [0, \bar{n}] \to \mathbb{R}_+$, with the interpretation that someone with income z cannot consumer more that $\lambda(z)$ after tax (note that $[0, \bar{n}]$ is the set of possible incomes). As in Mirrlees (1971), income taxes are non-decreasing and right-continuous (see Proposition 2 in Mirrlees (1971)). In addition, we assume that $\lambda(\bar{n}) \leq \bar{m}$.

The underlying assumption, here as well as in Mirrlees (1971), is that the government can observe the income level of an individual but neither her skill nor her effort level. Thus, we assume that the government observes neither the function $\hat{n}$ assigning skills to individuals nor the effort level chosen by individuals. Specifically, the government observes only the joint distribution of skills, utility functions, consumption and effort levels.

Let Λ be the set of all non-decreasing right-continuous functions $\lambda : [0, \bar{n}] \to \mathbb{R}_+$ satisfying $\lambda(\bar{n}) \leq \bar{m}$. To get a suitable topology on Λ, we identify Λ with the space of Borel measures μ on $[0, \bar{n}]$, with $\mu([0, \bar{n}]) \leq \bar{m}$, via the map $\lambda \mapsto \mu_\lambda$, where μ_λ is the unique Borel measure on $[0, \bar{n}]$ that satisfies $\mu_\lambda([0, z]) = \lambda(z)$ for all $z \in [0, \bar{n}]$. Now we give Λ the topology that is carried over through this map from the narrow topology on the space of Borel measures on $[0, \bar{n}]$, so that Λ becomes a compact metrizable space.

We will address the existence of an optimal income tax via a game played by the government and the individuals. Modeling income taxes as above is technically

[10] Without this assumption, we would need to add an auctioneer to the game used to show existence of equilibrium.

convenient since it implies that the government's choice set is compact. This would not be the case would one focus on income taxes that are continuous or on general incentive-feasible mechanisms (the latter being considered in Golosov, Kocherlakota, and Tsyvinski (2003)).[11] However, allowing for discontinuous income taxes implies that the individuals' and the government's payoff functions are discontinuous. Nevertheless, despite of such discontinuities, continuous security will hold and will allow to prove existence of an optimal income tax.

To unify several optimal taxation problems, we consider the possibility that the choice of the income tax by the government is restricted. To this end, let $\hat{C}$ be the closure of $\{u \in C(M \times L) : u = \tilde{u}_t \text{ for some } t \in \hat{T}\}$, $C = \hat{C} \times N$ the set of players' attributes, $c : \hat{T} \to C$ the attribute function, defined by $c(t) = (\tilde{u}_t, n_t)$ for all $t \in \hat{T}$, and $K = \{\gamma \in M^1_+(C \times M \times L) : \gamma_C = \hat{\nu} \circ c^{-1}\}$ the set of distributions over attributes and actions. The set K is regarded as being endowed with the narrow topology. Let $\Theta\colon K \rightrightarrows \Lambda$ be a correspondence with the interpretation that, given $\gamma \in K$, the government may choose income taxes only from $\Theta(\gamma)$.

Given $\gamma \in K$, the distribution of the map $(u, n, m, l) \mapsto nl\colon C \times M \times L \to [0, \bar{n}]$ is denoted by $\hat{\gamma}$. Thus $\hat{\gamma}$ is the distribution of outputs determined by γ, or, in other words, the pre-tax income distribution given by γ. We write $\hat{K} \subseteq M^1_+([0, \bar{n}])$ for the image of K under the map $\gamma \mapsto \hat{\gamma}$. As no confusion can arise, the symbol $\hat{\gamma}$ will also be used to denote generic elements of $\hat{K}$. We give $\hat{K}$ the subspace topology defined from the narrow topology of $M^1_+([0, \bar{n}])$, so that the map $\gamma \mapsto \hat{\gamma}$ becomes continuous.

Below we will consider an optimal tax problem where the government cannot commit to an income tax announced to the individuals before they make their decisions (see Example 8). In this case, an optimal income tax should have the property that after the individuals have made their choices, the government has no incentive to change the tax. This is in contrast with the commitment case, considered in Example 4, where no such requirement is made. Despite this difference, we can treat these two cases, as well as other optimal tax problems, in an unified way by introducing an auxiliary utility function $v : \Lambda \times K \to \mathbb{R}$ for the government and specifying it in accordance with the different cases we want to capture.

We are now ready to state the following assumptions.

(T5) Θ is well-behaved and takes convex values.

(T6) v is usc, $v(\cdot, \gamma)$ is quasi-concave for all $\gamma \in K$, $v(\lambda, \cdot)$ is continuous for each continuous $\lambda \in \Lambda$, and $v(\lambda, \gamma) \geq 0$ for all $(\lambda, \gamma) \in \Lambda \times K$.

[11] Another reason for focusing on taxes instead of general incentive-feasible mechanisms has to do with decentralization; see, for instance, Section 4.3 in Kocherlakota (2010).

(T7) For all $\gamma \in K$ and $\varepsilon > 0$, there exists an open neighborhood O of γ and a continuous $\psi : O \to \Lambda$ such that for all $\gamma' \in O$,

(i) $\psi(\gamma') \in \Theta(\gamma')$,

(ii) $\int \psi(\gamma')(z) \mathrm{d}\hat{\gamma}'(z) = \int z \mathrm{d}\hat{\gamma}'(z)$,

(iii) $v(\psi(\gamma'), \gamma') > v(\lambda, \gamma) - \varepsilon$ for all $\lambda \in \Theta(\gamma)$ such that $\int \lambda(z) \mathrm{d}\hat{\gamma}(z) = \int z \mathrm{d}\hat{\gamma}(z)$.

Specifying the correspondence Θ and the function v, we can obtain several particular cases.

Example 4. Let $\Theta(\gamma) = \Lambda$ and $v(\lambda, \gamma) = 0$ for all $(\lambda, \gamma) \in \Lambda \times K$. This is the case considered in Mirrlees (1971). It is clear that (T5) and (T6) are satisfied in this example. As for (T7), simply let $O = K$ and $\psi(\gamma') = \lambda_0$ for all $\gamma' \in O$, where λ_0 is the identity, i.e. $\lambda_0(z) = z$ for all $z \in N$.

Example 5. Let $\Theta(\gamma) = \{\lambda : \lambda(\bar{n} - \xi) \geq \zeta\}$ for all $\gamma \in K$, where $\xi > 0$ is a small number, and $0 < \zeta < \bar{n} - \xi$ a high number. This case can be interpreted as one where the government commits to income taxes that give high work incentives for highly skilled individuals. Alternatively, this case can be regarded as arising because, if taxed at a high tax rate, high skill individuals will choose to evade taxation. Clearly Θ takes convex values, and by Lemma 7 in Section 4.11, Θ is well-behaved. Thus (T5) holds. Let $v(\lambda, \gamma) = 0$ for all $(\lambda, \gamma) \in \Lambda \times K$. Then (T6) also holds. As in the previous example, letting $O = K$ and $\psi(\gamma') = \lambda_0$ for all $\gamma' \in O$ shows that (T7) holds, too.

Example 6. In this example we consider the case where the government ceases to function as total output approaches zero. This is modeled by specifying Θ as follows. First, only the 0% income tax λ_0 is allowed if total output is zero, with the interpretation that in this case the government no longer exists and thus, in particular, cannot redistribute income. Second, for total output larger than zero, the income taxes the government can implement are those with a distance to the 0% income tax not exceeding a number which depends continuously on $\hat{\gamma}$, i.e., on the distribution of outputs.

Specifically, let $\hat{\gamma}_0 \in \hat{K}$ be Dirac measure at $0 \in [0, \bar{n}]$. Set $\Theta(\gamma) = \{\lambda_0\}$ if $\hat{\gamma} = \hat{\gamma}_0$, and for some continuous function $\hat{\gamma} \mapsto \varepsilon(\hat{\gamma}) \colon \hat{K} \to \mathbb{R}$, with $\varepsilon(\hat{\gamma}_0) = 0$ and $\varepsilon(\hat{\gamma}) > 0$ for $\hat{\gamma} \neq \hat{\gamma}_0$, set $\Theta(\gamma) = C_{\varepsilon(\hat{\gamma})}(\lambda_0)$ for $\gamma \in K$ with $\hat{\gamma} \neq \hat{\gamma}_0$, writing $C_{\varepsilon(\hat{\gamma})}(\lambda_0)$ for the closed ball of radius $\varepsilon(\hat{\gamma})$ around λ_0 for the metric on Λ induced by Huntingdon's metric on the space of Borel measures on $[0, \bar{n}]$.[12] Further, let $v(\lambda, \gamma) = 0$ for all $(\lambda, \gamma) \in \Lambda \times K$.

[12]Recall that Huntingdon's metric on the space $M_+([0, \bar{n}])$ of Borel measures on $[0, \bar{n}]$ is the metric ρ defined by setting $\rho(\mu, \mu') = \sup\{\left|\int h \mathrm{d}\mu - \int h \mathrm{d}\mu'\right| : h \in L\}$ for all $\mu, \mu' \in M_+([0, \bar{n}])$, where $L = \{h \in C([0, \bar{n}]) \colon \|h\|_\infty \leq 1$ and h is 1-Lipschitz$\}$. Recall also that Huntingdon's metric induces the narrow topology (see Fremlin (2003, 437L and 437Y(i))).

As in the previous two examples, (T6) and (T7) hold. Clearly Θ is well-behaved, as both the maps $\gamma \mapsto \hat{\gamma}$ and $\hat{\gamma} \mapsto \varepsilon(\hat{\gamma})$ are continuous. Also, $C_{\varepsilon(\hat{\gamma})}(\lambda_0)$ is convex for all $\gamma \in K$, by the definition of Huntingdon's metric. Thus (T5) holds.

Two additional examples will be given below.

An *economy* is a list $E = \langle(\hat{T}, \hat{\Sigma}, \hat{\nu}), M, L, N, \hat{n}, \Lambda, \Theta, v, \langle \tilde{u}_t, n_t\rangle_{t\in\hat{T}}\rangle$. An *equilibrium* for an economy E consists of an income tax λ^* and a pair $g^* = (m^*, l^*)$, where $m^* : \hat{T} \to M$ and $l^* : \hat{T} \to L$ are measurable functions, such that:

(a) λ^* solves $\max_\lambda v(\lambda, \hat{\nu} \circ (c, g^*)^{-1})$ subject to the conditions $\lambda \in \Theta(\hat{\nu} \circ (c, g^*)^{-1})$ and $\int_{\hat{T}} \lambda(n_t l^*(t))\mathrm{d}\hat{\nu}(t) = \int_{\hat{T}} n_t l^*(t)\mathrm{d}\hat{\nu}(t)$.[13]

(b) For almost all $t \in \hat{T}$, $g^*(t)$ solves $\max_{(m,l)\in M\times L} \tilde{u}_t(m, l)$ subject to $m \leq \lambda^*(n_t l)$.

Conditions (b) and (a) together imply that g^* is a competitive equilibrium allocation. Indeed, by the monotonicity assumption in (T2), we must have $m^*(t) = \lambda^*(n_t l^*(t))$ for almost all $t \in \hat{T}$. Hence $\int_{\hat{T}} m^*(t)\mathrm{d}\hat{\nu}(t) = \int_{\hat{T}} \lambda^*(n_t l^*(t))\mathrm{d}\hat{\nu}(t) = \int_{\hat{T}} n_t l^*(t)\mathrm{d}\hat{\nu}(t)$ and thus market clearing holds.

Note that when an individual's effort equals l, then she provides a quantity of labor equal to $n_t l$, and thus $\lambda^*(n_t l)$ is the maximum amount of consumption she can consume. In an equilibrium, individual t's effort equals $l^*(t)$ and the amount of tax she pays equals $n_t l^*(t) - \lambda^*(n_t l^*(t))$, which is the difference between the pre-tax income $n_t l^*(t)$ and the after-tax income $\lambda^*(n_t l^*(t))$. Thus, condition (a) means, in particular, that the government has a balanced budget.

In choosing an optimal income tax, the government is constrained by an implementability condition: the allocation g that results from the choice of a given income tax λ must be such that (λ, g) is an equilibrium of the economy. Thus, writing $S(E)$ for the set of equilibria of E, the *government's optimization problem* is

$$\max_{(\lambda,g)\in S(E)} \int_{\hat{T}} \tilde{u}_t(g(t))\mathrm{d}\hat{\nu}(t).$$

An income tax λ^* is an *optimal income tax* if there exists a $g^* : \hat{T} \to M \times L$ such that (λ^*, g^*) is a solution of the government's optimization problem.

We can now consider a fourth example, obtained by specifying Θ so as to capture the basic idea of the credible income taxation problem in Farhi, Sleet, Werning, and Yeltekin (2011).

Example 7. As in Golosov, Kocherlakota, and Tsyvinski (2003), assume, in addition to (T1) and (T2), that $\tilde{u}_t(m, l) = \pi(m) + \eta(l)$ for all $t \in \hat{T}$ and $(m, l) \in M \times L$, where

[13] Abusing notation, we sometimes write m to denote a function from $\hat{T}$ to M, instead a generic element of M; similarly for l and L. The meaning should be clear from the context.

$\pi : M \to \mathbb{R}$ and $\eta : L \to \mathbb{R}$ are functions with π strictly concave and $\pi(0) = \eta(0) = 0$. Let

$$\Theta(\gamma) = \begin{cases} \tilde{\Theta}(\gamma) & \text{if } \pi\left(\int z \mathrm{d}\hat{\gamma}(z)\right) + \int \eta(l) \mathrm{d}\gamma(u, n, m, l) > 0, \\ \Lambda & \text{otherwise,} \end{cases}$$

$\gamma \in K$, where

$$\tilde{\Theta}(\gamma) = \left\{ \lambda' \in \Lambda : \int \pi(\lambda'(z)) \mathrm{d}\hat{\gamma}(z) + \delta \int \eta(l) d\gamma(u, n, m, l) \geq (1 - \delta)\pi\left(\int z \mathrm{d}\hat{\gamma}(z)\right) \right\}.$$

Further, let $v(\lambda, \gamma) = 0$ for all $(\lambda, \gamma) \in \Lambda \times K$. In Lemma 13 in Section 4.14 it is shown that, for this specifications of Θ and v, assumptions (T5)-(T7) are satisfied and for an equilibrium (λ^*, g^*) of E the following must hold, writing $g^*(t) = (m^*(t), l^*(t))$:

$$(1) \qquad \int_{\hat{T}} \big(\pi(\lambda^*(n_t l^*(t))) + \eta(l^*(t))\big) \mathrm{d}\hat{\nu}(t) \\ \geq (1 - \delta) \left(\pi \left(\int_{\hat{T}} \lambda^*(n_t l^*(t)) \mathrm{d}\hat{\nu}(t) \right) + \int_{\hat{T}} \eta(l^*(t)) \mathrm{d}\hat{\nu}(t) \right).$$

This condition is analogous to the credibility condition of Farhi, Sleet, Werning, and Yeltekin (2011) and implies that the government credibly commits to λ^* in the following sense. Suppose that in each one of infinitely many periods $k \in \mathbb{N}$, the government and the individuals simultaneously choose an income tax λ_k and consumption/effort pairs $(m_k(t), l_k(t))$, respectively, knowing the entire history of previous taxes and joint distributions over attributes and actions.[14] In addition, assume that for each individual and the government the utility in the repeated interaction is the discounted sum of the period-wise utilities, with discount factor $\delta \in (0, 1)$, where the period-wise utility function of the government is given by $\int_{\hat{T}} \big(\pi(\lambda_k(n_t l_k(t))) + \eta(l_k(t))\big) \mathrm{d}\hat{\nu}(t)$ for all $k \in \mathbb{N}$, λ_k being the income tax and l_k the effort allocation in period k.[15] In this setting, the

[14]Note that we are restricting the government to choose income taxes that do not depend on the tax paid previously by individuals, which is something that the government observes. This is an important restriction. In fact, if the outcome in the first period is fully revealing (i.e. individuals with different skills have different income levels) then the first-best could be achieved from period 2 onwards. Alternative assumptions to rule out this case include: (1) the government is legally obliged to tax only individual's income, or (2) each individual lives only for one period and so pays taxes only once. Assumption (2) means that individuals are short-lived and the government is long-lived; this poses no difficulties within a repeated-game framework and is, in fact, standard (see, for instance, Mailath and Samuelson (2006, Section 2.7) or Sabourian (1990)). Finally, we note that Assumption (2) is similar to the overlapping generations assumption in Farhi, Sleet, Werning, and Yeltekin (2011).

[15]Note that $\int_{\hat{T}} (\pi(\lambda_k(n_t l_k(t))) + \eta(l_k(t))) \mathrm{d}\hat{\nu}(t) = \int_{N \times L} (\pi(\lambda_k(nl)) + \eta(l)) \mathrm{d}\hat{\nu} \circ (\hat{n}, l_k)^{-1}(n, l)$ for all $k \in \mathbb{N}$, so all the government needs to know is the joint distribution of skills and efforts.

above credibility condition states that the stationary outcome with (λ^*, g^*) in every period is a subgame perfect equilibrium of the repeated game just described.[16]

At the core of the above example is the inability of the government to commit to an income tax. In fact, the right-hand side of condition (1) describes the best short-run deviation of the government, which, by the fact that individuals' choices are made simultaneously, consists in choosing an income tax that gives all individuals the same after-tax income.

In a final example, we consider another non-commitment problem. Specifically, it is supposed that, after individuals have made effort decisions but before they carry out consumption, the government may revise a tax announced earlier.

Example 8. Let Θ be as in Example 6, and set $v(\lambda, \gamma) = \int u(\lambda(nl), l) \mathrm{d}\gamma(u, n, m, l)$ for all $(\lambda, \gamma) \in \Lambda \times K$. Then, as in Example 6, (T5) holds. If, in addition to what is supposed in (T1) and (T2), $\tilde{u}_t(\cdot, l)$ is concave for all $t \in \hat{T}$ and $l \in L$, then by Lemma 14 in Section 4.14, (T6) and (T7) hold, too. Now under the assumption stated in the previous paragraph, an equilibrium implementing an optimal income tax must have the property that the income tax maximizes aggregate utility subject to the feasibility constraints set by Θ and the given total output. In view of (a) of the equilibrium definition, this is guaranteed by specifying the government's auxiliary utility function v as above. (Recall in this regard that the specification of Θ in Example 6 means, in particular, that the feasibility sets $\Theta(\gamma)$ of the government do not depend on individuals' consumption, but only on the distribution of outputs. While in the previous examples an equilibrium can be easily constructed (there is an equilibrium with a 0% income tax in Examples 4-6, and an equilibrium with a 100% income tax in Examples 4 and 7), this is not the case in this example. In particular, there cannot be an equilibrium with a 0% income tax given the specification of v, and there cannot be an equilibrium with a 100% income tax given the specification of Θ.

Here is our theorem on existence of an optimal income tax.

Theorem 10. *If the economy $E = \langle (\hat{T}, \hat{\Sigma}, \hat{\nu}), M, L, N, \hat{n}, \Lambda, \Theta, v, \langle \tilde{u}_t, n_t \rangle_{t \in \hat{T}} \rangle$ satisfies (T1)-(T7), then there exists an optimal income tax.*

(See Section 4.13 for the proof.) The proof of this theorem requires, in particular, to show that the choice set $S(E)$ of the government is non-empty. We will do this by using Theorem 5 to establish existence of a Nash equilibrium in the following game.

Let $E = \langle (\hat{T}, \hat{\Sigma}, \hat{\nu}), M, L, N, \hat{n}, \Lambda, \Theta, v, \langle \tilde{u}_t, n_t \rangle_{t \in \hat{T}} \rangle$ be an economy satisfying (T1)-(T7). The government will be denoted by player $\bar{t}$, where $\bar{t} \notin \hat{T}$. Let $\bar{T} = \{\bar{t}\}$ and set

[16]Condition (1) is also necessary for a stationary outcome to be subgame perfect (see Chari and Kehoe (1990)) since the payoff of the government in the worst subgame perfect equilibrium from its point of view is $\pi(0) + \eta(0) = 0$.

$T = \bar{T} \cup \hat{T}$, $\Sigma = \hat{\Sigma} \cup \{B \cup \bar{T} : B \in \hat{\Sigma}\}$ and, for all $B \in \Sigma$, $\nu(B) = \hat{\nu}(B \cap \hat{T}) + \chi_B(\bar{t})$, where χ_B is the characteristic function of B. Since $(\hat{T}, \hat{\Sigma}, \hat{\nu})$ is a super-atomless complete probability space, it follows that Assumptions (A1), (A9) and (S2) are satisfied.

Concerning players' action spaces, let $X_{\bar{t}} = \Lambda$ and $X_t = M \times L$ for all $t \in \hat{T}$. As said above, we may identify Λ with the set of Borel measures μ on $[0, \bar{n}]$ satisfying $\mu([0, \bar{n}]) \leq \bar{m}$. Let X_0 be the vector space of all finite signed Borel measures on $[0, \bar{n}]$, endowed with the narrow topology. Then X_0 is a Souslin locally convex space. Let $X_1 = \mathbb{R}^2$, and let X be obtained from X_0 and X_1 in the sense of Remark 1(a). We may then identify Λ and $M \times L$ with subsets of X, so that (A3) and (A4) are satisfied. Moreover, we may view strategy profiles as functions from T to X.

The attribute space C and the attribute function c are defined as above. Thus, it follows from (T1) and (T3) that (A10) is satisfied.

Given the above specification, we obtain the following regarding the externality map $\tilde{e}$. Given $f \in S_G$ (i.e. a measurable $f \colon T \to X$ with $f(\bar{t}) \in \Lambda$ and $f(t) \in M \times L$ for almost all $t \in \hat{T}$), let $\hat{f} = f|_{\hat{T}}$ and note that $\tilde{e}(f) = (f(\bar{t}), \hat{\nu} \circ (c, \hat{f})^{-1})$. Thus, with K as defined above, we have $\tilde{E}_G = \Lambda \times K$. In view of this, given $y \in \tilde{E}_G$, we will often write (λ, γ) for y.

Regarding players' payoff functions, define $u_{\bar{t}} \colon \Lambda \times \tilde{E}_G \to \mathbb{R}$ by setting

$$u_{\bar{t}}(\lambda', \lambda, \gamma) = \begin{cases} v(\lambda', \gamma) & \text{if } \int (z - \lambda'(z)) \, \mathrm{d}\hat{\gamma}(z) = 0, \\ -1 & \text{otherwise,} \end{cases}$$

for all $\lambda' \in \Lambda$ and $(\lambda, \gamma) \in \tilde{E}_G$. For $t \in \hat{T}$, define $u_t \colon M \times L \times \tilde{E}_G \to \mathbb{R}$ by setting $u_t(m, l, \lambda, \gamma) = \tilde{u}_t(m, l)$ for all $(m, l) \in M \times L$ and $(\lambda, \gamma) \in \tilde{E}_G$.

Finally, we specify the constraint correspondences. Define $A_{\bar{t}} \colon \tilde{E}_G \rightrightarrows \Lambda$ by setting $A_{\bar{t}}(y) = \Theta(\gamma)$ for all $y = (\lambda, \gamma) \in \tilde{E}_G$. For $t \in \hat{T}$, define $A_t \colon \tilde{E}_G \rightrightarrows M \times L$ by setting $A_t(y) = \{(m, l) \in M \times L : m \leq \lambda(n_t l)\}$ for $y = (\lambda, \gamma) \in \tilde{E}_G$. Since the three sets $\Theta(\gamma)$, $\{\lambda' \in \Lambda : \int (z - \lambda'(z)) \, \mathrm{d}\hat{\gamma}(z) = 0\}$, and $\{\lambda' \in \Lambda : v(\lambda', \gamma) \geq \alpha\}$, $\alpha \in \mathbb{R}$, are convex, it follows that (A11) is satisfied.

Let $G = ((T, \Sigma, \nu), X, \langle X_t, u_t, A_t \rangle_{t \in T}, \tilde{e})$ be the game just defined. It is clear that every equilibrium of the economy E is a Nash equilibrium of G, and that by (T6) and (T7), every Nash equilibrium of G is an equilibrium of E.

We note that the payoff function $u_{\bar{t}}$ in the game G need not be usc, so that the existence result of Balder (2002, Theorem 2.2.1) cannot be applied to conclude the existence of an equilibrium of G. E.g., suppose $\hat{\nu} \circ \hat{n}^{-1}$ is uniform on $[0, \bar{n}]$, $\Theta(\gamma) = \Lambda$ for all $\gamma \in K$, and $v(\lambda, \gamma) = 0$ for all $(\lambda, \gamma) \in \Lambda \times K$. Define $\lambda' \in \Lambda$ by setting

$$\lambda'(z) = \begin{cases} z \text{ if } z < \bar{n}/2, \\ \bar{n} \text{ if } z \geq \bar{n}/2. \end{cases}$$

Let $\hat{f} : \hat{T} \to M \times L$ and $\hat{f}_k : \hat{T} \to M \times L$, $k \in \mathbb{N}\backslash\{0\}$, be given by $\hat{f}(t) = \hat{f}_k(t) = (n_t, 1)$ if $n_t < \bar{n}/2$, and by $\hat{f}_k(t) = (\bar{n}/2 - n_t/k, \bar{n}/(2n_t) - 1/k)$ and $\hat{f}(t) = (\bar{n}/2, \bar{n}/(2n_t))$ otherwise. Fix any $\lambda \in \Lambda$. Set $y_k = (\lambda, \hat{\nu} \circ (c, \hat{f}_k)^{-1})$ and $y = (\lambda, \hat{\nu} \circ (c, \hat{f})^{-1})$. Then $(\lambda', y_k) \to (\lambda', y)$. Also, $\int (\lambda'(nl) - nl)\, \mathrm{d}\hat{\nu} \circ (c, \hat{f}_k)^{-1}(u, n, m, l) = 0$ for all k, and $\int (\lambda'(nl) - nl)\, \mathrm{d}\hat{\nu} \circ (c, \hat{f})^{-1}(u, n, m, l) = \bar{n}/4 > 0$. Thus $u_{\bar{t}}(\lambda', y_k) = 0$ for all k, but $u_{\bar{t}}(\lambda', y) = -1$, showing that $u_{\bar{t}}$ is not usc. (In fact, $u_{\bar{t}}$ is not even weakly usc.)

As announced, we will establish existence of an equilibrium of the economy E by showing that the game G has an equilibrium. To this end, we will show that G is continuously secure:

Theorem 11. *The game G satisfies* CS'*.*

(The proof may be found in Section 4.12.) In view of this theorem and of what was noted above, the game G satisfies the assumptions of Theorem 5. Thus:

Theorem 12. *The game G has an equilibrium.*

4 Appendix: Proofs

4.1 Notation

Let Z be a topological space. For a sequence $\langle z_n \rangle$ in Z, $\mathrm{LS}\, z_n$ denotes the set of its cluster points. For a sequence $\langle A_n \rangle$ of subsets of Z, $\mathrm{KLS}\, A_n$ denotes the Kuratowski limes superior, i.e., the set of those points z in Z such that for every neighborhood U of z there are infinitely many n with $A_n \cap U \neq \emptyset$.

The fact in the following lemma is more or less well-known.

Lemma 1. *Let K be a compact subset of a Hausdorff locally convex vector space X. If $\langle F_n \rangle$ is a sequence of subsets of K, then $\overline{\mathrm{co}}\, \mathrm{KLS}\, \mathrm{co} F_n \subseteq \overline{\mathrm{co}}\, \mathrm{KLS}\, F_n$.*

Proof. Suppose $x \notin \overline{\mathrm{co}}\, \mathrm{KLS}\, F_n$. Then by the separation theorem (see e.g. Aliprantis and Border (2006, Theorem 5.79, p. 207)) there is an open half-space $H \subseteq X$ (i.e., a set of the form $\{y \in X : p(y) < r\}$ where p is a continuous linear functional on X, and r a real number) such that $\overline{\mathrm{co}}\, \mathrm{KLS}\, F_n \subseteq H$ and $x \notin \bar{H}$, where $\bar{H}$ is the closure of H. Since K is compact, so is $K \backslash H$. Hence since $F_n \subseteq K$ for each n, and since a compact space is countably compact, for large n we must have $F_n \cap (K \setminus H) = \emptyset$, i.e., $F_n \subseteq H$. Hence also $\mathrm{co} F_n \subseteq H$ for large n, so $\mathrm{KLS}\, \mathrm{co} F_n \subseteq \bar{H}$, and thus $\overline{\mathrm{co}}\, \mathrm{KLS}\, \mathrm{co} F_n \subseteq \bar{H}$. Thus $x \notin \overline{\mathrm{co}}\, \mathrm{KLS}\, F_n$ implies $x \notin \overline{\mathrm{co}}\, \mathrm{KLS}\, \mathrm{co} F_n$. $\square$

4.2 Young measures

This section contains, in particular, the material needed for the fixed point part of the proof of our results on existence of Nash equilibrium. The idea to perform the fixed point argument in a space of Young measures is taken from Balder (2002).

We start by stating some definitions. Let us fix a complete measure space (T, Σ, ν) with $0 < \nu(T) < \infty$ and a completely regular Souslin space X.

Recall that a Young measure from T to X is just a function $g\colon T \to M^1_+(X)$ which is measurable for the narrow topology of $M^1_+(X)$. Recall that this property is equivalent to the property that the map $t \mapsto g(t)(B)$ is measurable for each $B \in \mathcal{B}(X)$, and also to the property that, for each bounded continuous $p\colon X \to \mathbb{R}$, the map $t \mapsto \int p \mathrm{d}g(t)$ is measurable. The first equivalence shows in particular that if $f\colon T \to X$ is a measurable map, then the map $t \mapsto \delta_{f(t)}$ is a Young measure, where $\delta_{f(t)}$ denotes Dirac measure at $f(t)$.

Let $\mathcal{R}$ denote the set of all Young measures from T to X, endowed with the narrow topology for Young measures. Recall that this topology is defined to be the coarsest topology on $\mathcal{R}$ such that for each $q \in \mathcal{G}$ the functional

$$g \mapsto \int_T \int_X q(t,x)\mathrm{d}g(t)\mathrm{d}\nu(t)\colon \mathcal{R} \to \mathbb{R}$$

is continuous, where $\mathcal{G}$ is the set of all measurable functions $q\colon T \times X \to \mathbb{R}$ such that $q(t,\cdot)$ is continuous for each $t \in T$ and such that, for some integrable $\theta_q\colon T \to \mathbb{R}_+$, $\sup\{|q(x)|\colon x \in X\} \leq \theta_q(t)$ for each $t \in T$. It should be noted that, in general, the narrow topology for Young measures is not a Hausdorff topology.

In the sequel, if $\kappa\colon T \rightrightarrows X$ is a correspondence, then $\mathcal{R}_\kappa$ denotes the subset of $\mathcal{R}$ defined by setting

$$\mathcal{R}_\kappa = \{g \in \mathcal{R}\colon \operatorname{supp} g(t) \subseteq \kappa(t) \text{ for almost all } t \in T\}.$$

With (T, Σ, ν) and X as before, we next collect some facts that are fundamental. In the following theorem, the compactness part is a result due to Balder (1989), which provides the mathematical basis for the fixed point part of the proof of our equilibrium existence result.

Theorem 13. *Let $\kappa\colon T \rightrightarrows X$ be a correspondence with measurable graph such that $\kappa(t)$ is non-empty and compact for all $t \in T$. Then the subset $\mathcal{R}_\kappa$ of $\mathcal{R}$ is non-empty, closed, compact, and sequentially compact.*

Proof. As noted above, if $f\colon T \to X$ is measurable, then the map $t \mapsto$ "Dirac measure at $f(t)$" belongs to $\mathcal{R}$. Thus non-emptiness of $\mathcal{R}_\kappa$ is implied by the von Neumann-Aumann-Sainte Beuve measurable selection theorem (see Castaing and Valadier (1977, Theorem III.22)). From Balder (1989, Theorem 2.2 and Remark 2.4) it follows that $\mathcal{R}_\kappa$ is both relatively compact and relatively sequentially compact. Hence, given that $\mathcal{R}_\kappa$ is closed, it is both compact and sequentially compact. That $\mathcal{R}_\kappa$ is indeed closed may be seen as follows. Recall that a Souslin space is separable and that on any such space, if it is completely regular, there is metric that gives a topology weaker than the original one but such that the Borel sets for both topologies are the same.[17] Choose such a metric, say ρ, on the space X under consideration. In particular, then, X is separable for ρ. Define $q_\rho\colon T \times X \to \mathbb{R}$ by setting

$$q_\rho(t,x) = \min\{1, \rho(x,\kappa(t))\}, \;\; (t,x) \in T \times X.$$

Using Castaing and Valadier (1977, Theorems III.22, III.9, and Lemma III.14) it may be seen that q belongs to the set $\mathcal{G}$ defined above. Now as κ takes closed values, an element $g \in \mathcal{R}$ belongs to $\mathcal{R}_\kappa$ if and only if $\int_T \int_X q_\rho(t,x)\mathrm{d}g(t)\mathrm{d}\nu(t) = 0$. Thus, by the definition of the narrow topology for Young measures, $\mathcal{R}_\kappa$ is closed in $\mathcal{R}$. □

Corollary 1. *Let $\kappa\colon T \rightrightarrows X$ be as in Theorem 13, and S_κ the set of all measurable a.e. selections of κ. Then the set $\left\{\frac{1}{\nu(T)}\nu \circ f^{-1} : f \in S_\kappa\right\}$ of distributions of the members of S_κ is a relatively compact subset of $M^1_+(X)$ for the narrow topology.*

Proof. For each $g \in \mathcal{R}_\kappa$, define $\mu_g \in M^1_+(X)$ by setting $\mu_g(B) = \frac{1}{\nu(T)} \int_T g(t)(B)\mathrm{d}\nu(t)$ for every $B \in \mathcal{B}(X)$. (That μ_g is indeed countably additive may be easily seen with the help of the monotone convergence theorem). Note that, for any $f \in S_\kappa$, setting $g(t) = \delta_{f(t)}$ for $t \in T$, where $\delta_{f(t)}$ is Dirac measure at $f(t)$, defines an element g of $\mathcal{R}_\kappa$ for which $\mu_g = \frac{1}{\nu(T)}\nu \circ f^{-1}$. Now the map $g \mapsto \mu_g\colon \mathcal{R}_\kappa \to M^1_+(X)$ is continuous for the narrow topology of $M^1_+(X)$ (see Balder (2002, proof of Theorem 3.1.1)), and thus the assertion follows from Theorem 13. □

The following version of Corollary 1 is appealed to in the proof of Theorem 5 given below in Section 4.7.

Lemma 2. *Let C be another completely regular Souslin spaces, $c\colon T \to C$ a measurable function, and $\kappa\colon T \rightrightarrows X$ a correspondence as in Theorem 13. Then the set*

$$\left\{\tfrac{1}{\nu(T)}\nu \circ (f,c)^{-1} : f \textit{ is a measurable a.e. selection of } \kappa\right\}$$

is a relatively compact subset of $M^1_+(X \times C)$ for the narrow topology.

[17] Recall that a Souslin space X is Hausdorff by definition, and therefore, if it is completely regular, the set of continuous functions on X separates the points of X. But on a Souslin space X, any set of continuous functions that separates the points of X contains a countable subset with the same property (Castaing and Valadier (1977, III.31)). This yields the assertion concerning the metric. Now the assertion concerning the Borel sets follows from Schwartz (1973, p. 101 Corollary 2).

Proof. Note first that the facts that c is measurable and C is a Souslin space imply that c has a measurable graph (see Castaing and Valadier (1977, p. 74)). Thus the correspondence $\kappa_1 \colon T \rightrightarrows C$, where $\kappa_1(t) = \{c(t)\}$ for each $t \in T$, has a measurable graph. Define $\kappa_2 \colon T \rightrightarrows X \times C$ by setting $\kappa_2(t) = \kappa(t) \times \kappa_1(t)$ for each $t \in T$. By the fact that $\mathcal{B}(X \times C) \supseteq \mathcal{B}(X) \otimes \mathcal{B}(C)$, it is elementary to check that the properties of κ and κ_1 to have a measurable graph imply that κ_2 has a measurable graph. Finally note that as both X and C are completely regular Souslin spaces, so is $X \times C$. The lemma is now easily seen to follow from Corollary 1, with X there replaced by $X \times C$, and κ there replaced by κ_2. (Note that if h is any measurable a.e. selection of κ_2, we may modify h on a null set, if necessary, so that the C-component of h takes value $c(t)$ for each $t \in T$.) $\square$

The final result stated in this section is just a translation of Theorem 4.15 in Balder (2000) into our notation.

Theorem 14. *If $g_n \to g$ in $\mathcal{R}$, then* $\operatorname{supp} g(t) \subseteq \operatorname{KLS} \operatorname{supp} g_n(t)$ *for almost all $t \in T$.*

4.3 A purification result

We need the following purification result, which will be proved as a consequence of a result in Podczeck (2009). As in the previous section, (T, Σ, ν) is a complete measure space with $0 < \nu(T) < \infty$, and X a completely regular Souslin space.

Theorem 15. *Let $\kappa \colon T \rightrightarrows X$ be as in Theorem* 13. *Writing Γ_κ for the graph of κ, let $\mathcal{C}$ be a countable set of functions $q \colon \Gamma_\kappa \to \mathbb{R}$ such that* (i) *q is measurable for the subspace σ-algebra of Γ_κ defined from $\Sigma \otimes \mathcal{B}(X)$,* (ii) *$q(t, \cdot)$ is continuous on $\kappa(t)$ for each $t \in T$,* (iii) *there is an integrable $\theta_q \colon T \to \mathbb{R}_+$ such that* $\sup\{|q(t,x)| \colon x \in \kappa(t)\} \leq \theta_q(t)$ *for almost all $t \in T$. Suppose (T, Σ, ν) is super-atomless. Then given any $g \in \mathcal{R}_\kappa$, there is a measurable $f \colon T \to X$ such that*

(1) *$f(t) \in \operatorname{supp} g(t)$ for almost all $t \in T$;*

(2) *$\int_T \int_X q(t,x) \mathrm{d}g(t)(x) \mathrm{d}\nu(t) = \int_T q(t, f(t)) \mathrm{d}\nu(t)$ for all $q \in \mathcal{C}$;*

(3) *$\int_T g(t)(B) \mathrm{d}\nu(t) = \nu(f^{-1}(B))$ for all $B \in \mathcal{B}(X)$.*

Proof. By Podczeck (2009), the theorem is true in the special case where X is a compact metric space and the maps q are defined on all of $T \times X$, with (i) assumed to hold with $T \times X$ in place of Γ_κ, and (ii) and (iii) with X in place of $\kappa(t)$. We will show that the situation of the present theorem can be reduced to this case.

As in the proof of Theorem 13, recall that a Souslin space is separable and that on any such space there is metric that gives a topology weaker than the original one but such that the Borel sets for both topologies are the same.

This fact implies that we may view X as a subset of a compact metric space K such that the inclusion map $i\colon X \to K$ is continuous and such that $\mathcal{B}(X)$ coincides with the subspace σ-algebra defined from $\mathcal{B}(K)$ (use Engelking (1989, Theorem 3.5.2), which is a compactification result, and for the assertion on the Borel σ-algebras, use Castaing and Valadier (1977, Lemma III.20) in addition). In particular, for each $t \in T$, since $\kappa(t)$ is a compact subset of X, $\kappa(t)$ is also compact as a subset of K, and the topologies of X and K give the same subspace topology on $\kappa(t)$. Furthermore, a map $f\colon T \to X$ is measurable for $\mathcal{B}(X)$ if and only if it is measurable for $\mathcal{B}(K)$ when viewed as a map into K.

Fix any $g \in \mathcal{R}_\kappa$. As the inclusion $i\colon X \to K$ is continuous, we may define a map $g_1\colon T \to M_+^1(K)$ by setting $g_1(t) = g(t) \circ i^{-1}$ for each $t \in T$ (i.e., $g_1(t)$ is the image measure of $g(t)$ under i). In particular, for any continuous map $p\colon K \to \mathbb{R}$, we have $\int_K p \mathrm{d}g_1(t) = \int_X p \circ i \mathrm{d}g(t)$ for every $t \in T$, and therefore the map $t \mapsto \int_K p \mathrm{d}g_1(t)$ is measurable, by the fact that g is a Young measure. Moreover, because $\operatorname{supp} g(t) \subseteq \kappa(t)$ for almost $t \in T$ by definition of $\mathcal{R}_\kappa$, we must have $\operatorname{supp} g_1(t) = \operatorname{supp} g(t)$ for almost all $t \in T$, by what was noted in the previous paragraph about the sets $\kappa(t)$.

We assert the following.

Claim: For each $q \in \mathcal{C}$ there is a $q'\colon T \times K \to \mathbb{R}$ such that (a) $q'|_{\Gamma_\kappa} = q$, (b) q' is measurable, (c) $q'(t, \cdot)$ is continuous for each $t \in T$, (d) $\sup\{|q'(t,x)|\colon x \in K\} \le \theta_q(t)$ for almost all $t \in T$.

Assuming the claim has been established, the theorem can be proved as follows. By Podczeck (2009, Corollary and Lemma 2), the fact that $t \mapsto \int_K p \mathrm{d}g_1(t)$ is measurable for each continuous $p\colon K \to \mathbb{R}$ implies that there is a measurable $f\colon T \to K$ such that (1)-(3) of the theorem hold with g_1 substituted for g, $\mathcal{B}(K)$ for $\mathcal{B}(X)$, and with each $q \in \mathcal{C}$ replaced by an element q' associated with q according to the claim. Now since $\operatorname{supp} g_1(t) = \operatorname{supp} g(t)$ for almost all $t \in T$, (1) of the theorem must also hold with f and g, i.e., $f(t) \in \operatorname{supp} g(t)$ for almost all $t \in T$, and therefore, in view of (a) of the claim, (2) must actually hold with f, g, and the given $\mathcal{C}$, because $\operatorname{supp} g(t) \subseteq \kappa(t)$ for almost all $t \in T$, and because $g_1(t) = g(t) \circ i^{-1}$ for each $t \in T$. Note that since $f(t) \in \kappa(t)$ for almost all $t \in T$, we have $f(t) \in X$ for almost all $t \in T$. Changing f on a null set of T, if necessary, we may assume that f takes all of its values in X. Now, because for every $B \in \mathcal{B}(X)$ there is a $B' \in \mathcal{B}(K)$ with $B = B' \cap X = i^{-1}(B')$, f must be measurable for $\mathcal{B}(X)$, and the fact that (3) of the theorem holds with f, g_1, and $\mathcal{B}(K)$ implies that (3) of the theorem holds with f, g, and $\mathcal{B}(X)$ as well.

Thus it remains to establish the above claim. Take any $q \in \mathcal{C}$ and note first that continuity of $q(t, \cdot)$ on $\kappa(t)$ as subspace of X, which holds for each $t \in T$ by hypothesis, implies continuity on $\kappa(t)$ as a subspace of K, by what was noted above. Define

$q_1\colon \Gamma_\kappa \to \mathbb{R}$ by setting

$$q_1(t,x) = 3/2 + (1/2)\arctan q(t,x),\ \ (t,x) \in \Gamma_\kappa.$$

Then q_1 is measurable for the subspace σ-algebra on Γ_κ defined from $\Sigma \otimes \mathcal{B}(X)$, because q is. Also, $q_1(t,\cdot)$ is continuous on $\kappa(t)$ for each $t \in T$, and q_1 takes all of its values in $(1,2)$.

Let ρ denote the metric of K. As $\kappa(t)$ is compact and therefore closed in K for each $t \in T$, and as q_1 takes values in $(1,2)$, we can define a function $q_2\colon T \times K \to \mathbb{R}$, also taking values in $(1,2)$, by setting $q_2(t,x) = q_1(t,x)$ if $x \in \kappa(t)$ and

$$q_2(t,x) = \inf\left\{\frac{q_1(t,u)\rho(x,u)}{\rho(x,\kappa(t))} : u \in \kappa(t)\right\}$$

otherwise. Note that for each $t \in T$, $q_2(t,\cdot)$ is a continuous extension of $q_1(t,\cdot)$ to K (see, e.g., Mandelkern (1990)). We claim that q_2 is measurable. As $q_2(t,\cdot)$ is continuous for each $t \in T$, to establish this claim it suffices by Castaing and Valadier (1977, Lemma III.14) to show that $q(\cdot,x)$ is measurable for each $x \in K$.

To this end, we appeal to Castaing and Valadier (1977, Theorem III.22) to choose a countable set $\{h_i\colon i \in I\}$ of measurable functions $h_i\colon T \to X$ such that $\{h_i(t)\colon i \in I\}$ is a dense subset of $\kappa(t)$ for each $t \in T$. By the measurability property of q_1 mentioned before, measurability of the h_i's implies in particular that the maps $t \mapsto q_1(t,h_i(t))$ are measurable. By what was said in the third paragraph of this proof, each h_i is measurable also when viewed as map into K, and for each $t \in T$, $\{h_i(t)\colon i \in I\}$ is dense in $\kappa(t)$ also for the topology of K. Taking some $x \in K$ now as given, it follows that for each $i \in I$ the map $t \mapsto \rho(x,h_i(t))$ is measurable, and therefore the map $t \mapsto \rho(x,\kappa(t))$ must be measurable as well. Thus if we set $T_1 = \{t \in T\colon \rho(x,\kappa(t)) > 0\}$ and define $e_i\colon T \to \mathbb{R}$, $i \in I$, by setting

$$e_i(t) = \begin{cases} \dfrac{q_1(t,h_i(t))\rho(x,h_i(t))}{\rho(x,\kappa(t))} & \text{if } t \in T_1, \\ q_1(t,x) & \text{if } t \in T \setminus T_1, \end{cases}$$

then e_i is measurable (recall that the sets $\kappa(t)$ are closed in K, so e_i is indeed defined on $T \setminus T_1$). Now because $\{h_i(t)\colon i \in I\}$ is dense in $\kappa(t)$, and because the function $u \mapsto q_1(t,u)\rho(x,u)/\rho(x,\kappa(t))$ is continuous on $\kappa(t)$ for each $t \in T_1$, we must have $q_2(t,x) = \inf\{e_i(t)\colon i \in I\}$ for each $t \in T$. As I is countable, it follows that $q_2(\cdot,x)$ is measurable.

Recalling that q_2 takes values in $(1,2)$, define $q_3\colon T \times K \to \mathbb{R}$ by setting

$$q_3(t,x) = \tan(2q_2(t,x) - 3),\ \ (t,x) \in T \times K.$$

Then, because q_2 is measurable, so is q_3, and as $q_2(t,\cdot)$ is continuous for each $t \in T$, so is $q_3(t,\cdot)$ for each $t \in T$. Thus (b) and (c) of the claim above hold for q_3. By construction, for each $t \in T$ we have $q_3(t,x) = q(t,x)$ if $x \in \kappa(t)$, i.e., (c) of the claim holds for q_3. As for (d), consider any $t \in T$, and note that the choice of $q_2(t,\cdot)$ implies that for any $x \in K \backslash \kappa(t)$,

$$\inf\{q_1(t,y)\colon y \in \kappa(t)\} \leq q_2(t,x) \leq \sup\{q_1(t,y)\colon y \in \kappa(t)\},$$

and therefore, by choice of $q_3(t,\cdot)$, since $q_3(t,x) = q(t,x)$ for all $y \in \kappa(t)$,

$$\inf\{q(t,y)\colon y \in \kappa(t)\} \leq q_3(t,x) \leq \sup\{q(t,y)\colon y \in \kappa(t)\},$$

from which it follows that $\sup\{|q_3(t,x)|\colon x \in K\} = \sup\{|q(t,x)|\colon x \in \kappa(t)\} \leq \theta_q(t)$. Thus (d) of the claim holds for q_3. This completes the proof. □

4.4 Proof of Theorems 1 and 2

Let us fix a game $G = ((T,\Sigma,\nu), X, \langle X_t, u_t, A_t\rangle_{t\in T}, e)$ such that (A1)-(A4) hold. Thus X is now a locally convex space in addition to being Souslin.

Note that assumptions (S1) and (S2), which are involved in Theorems 1 and 2, respectively, are incompatible. However, the following condition contains (S1) and (S2) as special cases.

(S3) (i) The subspace measure on $\bar{T}$ defined from ν is separable; (ii) the set $\hat{\mathcal{C}}$ is finite or the subspace measure on $\hat{T}$ defined from ν is super-atomless.

Assumption (S3) is an auxiliary assumption, introduced just to get in a position to prove our existence results with (S1) and (S2) in a unified way. It is not a generalization of (S1) and (S2) at a deeper level. It is assumed in the sequel that (S3) is satisfied.

As before, $\mathcal{R}$ denotes the set of all Young measures from T to X, endowed with the narrow topology for Young measures. By $\mathcal{R}_G$ we denote the subset of $\mathcal{R}$ defined as

$$\mathcal{R}_G = \{g \in \mathcal{R}\colon \operatorname{supp} g(t) \subseteq X_t \text{ for almost all } t \in T\}.$$

Now by Balder (2002, Corollary 4.2.1), (A4)(i) and (A4)(iii) together imply that there is a map $h_1\colon \mathcal{R}_G \to \bar{S}_G$ such that for each $g \in \mathcal{R}_G$, $h_1(g)(t)$ is the barycenter of $g(t)$ for almost all $t \in \bar{T}$, i.e., the unique element $x \in X_t$ for which $p(x) = \int_{X_t} p\mathrm{d}g(t)$ for every continuous linear functional p on X. Define $h_2\colon \mathcal{R}_G \to \mathbb{R}^{\hat{\mathcal{C}}}$ by setting $h_2(g) = \left\langle \int_{\hat{T}} \int_{X_t} q(t,x)\mathrm{d}g(t)(x)\mathrm{d}\nu(t)\right\rangle_{q\in\hat{\mathcal{C}}}$ for each $g \in \mathcal{R}_G$, where $\hat{\mathcal{C}}$ is the countable set involved in the externality map e. Finally define $h\colon \mathcal{R}_G \to \bar{S}_G \times \mathbb{R}^{\hat{\mathcal{C}}}$ by setting $h(g) = (h_1(g), h_2(g))$ for $g \in \mathcal{R}_G$.

Recall from Section 2.2 that $\mathbb{R}^{\hat{\mathcal{C}}}$ is endowed with the product topology, $\bar{S}_G$ with the feeble topology, $\bar{S}_G \times \mathbb{R}^{\hat{\mathcal{C}}}$ with the corresponding product topology, and the set $E_G \equiv e(S_G) \subseteq \bar{S}_G \times \mathbb{R}^{\hat{\mathcal{C}}}$ with the subspace topology.

Lemma 3. *The following hold for the map h.*

(a) *h takes values in E_G.*

(b) *h is continuous as map from $\mathcal{R}_G$ to E_G*

(c) *Given any $g \in \mathcal{R}_G$, there is an $f \in S_G$, with $e(f) = h(g)$, such that for almost all $t \in \hat{T}$, $f(t) \in \operatorname{supp} g(t)$, and for almost all $t \in \bar{T}$, $f(t) \in \overline{\mathrm{co}} \operatorname{supp} g(t)$.*

Proof. We first show that (c) holds. Consider any $g \in \mathcal{R}_G$. If the set $\hat{\mathcal{C}}$ is finite, then by Lyapunov's theorem for Young measures (see Balder (2002, Theorem, 4.2.3)), assumption (A2) implies that there is a measurable function $\hat{f} \colon \hat{T} \to X$ such that $h_2(g) = \big\langle \int_{\hat{T}} q(t, \hat{f}(t)) \mathrm{d}\nu(t) \big\rangle_{q \in \hat{\mathcal{C}}}$ and such that $\hat{f}(t) \in \operatorname{supp} g(t)$ for almost all $t \in \hat{T}$. If $\hat{\mathcal{C}}$ is countably infinite, then by (S3), the subspace measure on $\hat{T}$ defined from ν is super-atomless, and we get an $\hat{f} \colon \hat{T} \to X$ with the same properties by Theorem 15 (with T there replaced by $\hat{T}$, and $\mathcal{C}$ by $\hat{\mathcal{C}}$), noting that by (A4), X_t is compact for each $t \in \hat{T}$ and the graph of the correspondence $t \mapsto X_t \colon \hat{T} \to X$ is measurable.

Define $f \colon T \to X$ by setting $f(t) = h_1(g)(t)$ for $t \in \bar{T}$, and $f(t) = \hat{f}(t)$ for $t \in \hat{T}$. Thus $f|_{\bar{T}} = h_1(g) \in \bar{S}_G$. In particular, $f|_{\bar{T}}$ is measurable and we have $(f|_{\bar{T}})(t) \in X_t$ for almost all $t \in \bar{T}$. As $g \in \mathcal{R}_G$ means $\operatorname{supp} g(t) \subseteq X_t$ for almost all $t \in T$, the fact that $\hat{f}(t) \in \operatorname{supp} g(t)$ for almost all $t \in \hat{T}$ implies that we have $f(t) \in X_t$ also for almost all $t \in \hat{T}$. Thus $f(t) \in X_t$ for almost all $t \in T$, and because $\bar{T}$ and $\hat{T}$ are measurable subsets of T, it follows that f is measurable. That is, $f \in S_G$. In particular, $e(f)$ is defined. Recalling that the definition of e says $e(f) = (f|_{\bar{T}}, \big\langle \int_{\hat{T}} q(t, \hat{f}(t)) \mathrm{d}\nu(t) \big\rangle_{q \in \hat{\mathcal{C}}})$, we may conclude that $e(f) = (h_1(g), h_2(g)) = h(g)$. Now by construction, we have $f(t) \in \operatorname{supp} g(t)$ for almost all $t \in \hat{T}$. Also, because $f(t) = h_1(g)(t)$ for $t \in \bar{T}$, and because by the definition of h_1, $h_1(g)(t)$ is the barycenter of $g(t)$ for almost all $t \in \bar{T}$, we must have $f(t) \in \overline{\mathrm{co}} \operatorname{supp} g(t)$ for almost all $t \in \bar{T}$, by the definition of barycenter and by the separation theorem. Thus (c) holds.

Now as $E_G = e(S_G)$ by definition, (a) is directly implied by (c). As for (b), by Balder (2002, Theorem 4.2.2) it follows that the map h_1 continuous, and by the argument in Step 1 in the proof of Theorem 2.2.1 in Balder (2002), it follows that for each $q \in \hat{\mathcal{C}}$ the map $g \mapsto \int_{\hat{T}} \int_{X_t} q(t, x) \mathrm{d}g(t) \mathrm{d}\nu(t) \colon \mathcal{R}_G \to \mathbb{R}$ is continuous. Thus, by choice of the topologies of $\bar{S}_G \times \mathbb{R}^{\hat{\mathcal{C}}}$ and E_G, (b) follows. □

Lemma 4. *Given any sequence $\langle f_n \rangle$ in S_G, there is an $f \in S_G$ and a subsequence $\langle f_k \rangle$ of $\langle f_n \rangle$ such that $e(f_k) \to e(f)$ and such that for almost all $t \in \hat{T}$, $f(t) \in \operatorname{LS} f_n(t)$, and for almost all $t \in \bar{T}$, $f(t) \in \overline{\mathrm{co}} \operatorname{LS} f_n(t)$.*

Proof. For each n let $g_n \in \mathcal{R}_G$ be defined by setting $g_n(t) = \delta_{f_n(t)}$ for $t \in T$, where $\delta_{f_n(t)}$ denotes Dirac measure at $f_n(t)$. Then by Theorem 13 there are a $g \in \mathcal{R}_G$ and a subsequence $\langle g_k \rangle$ of the sequence $\langle g_n \rangle$ such that $g_k \to g$ in $\mathcal{R}$. By Theorem 14, then, $\operatorname{supp} g(t) \subseteq \operatorname{KLS} \operatorname{supp} g_k(t) \subseteq \operatorname{KLS} \operatorname{supp} g_n(t)$ for almost all $t \in T$, and by Lemma 3(b), $h(g_k) \to h(g)$. Choose $f \in S_G$ such that f corresponds to g according to Lemma 3(c), and let $\langle f_k \rangle$ be the subsequence of $\langle f_n \rangle$ corresponding to $\langle g_k \rangle$. Observing that $h(g_n) = e(f_n)$ for each n, and that $\operatorname{KLS} \operatorname{supp} g_n(t) = \operatorname{LS} f_n(t)$ for each $t \in T$ by definition of the g_n's, we may see that f and $\langle f_k \rangle$ are as required. □

Lemma 5. *The set E_G is compact and pseudo-metrizable.*

Proof. Assumption (S3)(i) implies that $\bar{S}_G$ is pseudo-metrizable (see Balder (2002, Remark 4.3.1)), and because $\hat{\mathcal{C}}$ is countable, $\mathbb{R}^{\hat{\mathcal{C}}}$ is metrizable. Thus $E_G \subseteq \bar{S}_G \times \mathbb{R}^{\hat{\mathcal{C}}}$ is pseudo-metrizable. As $E_G = e(S_G)$, Lemma 4 implies that E_G is sequentially compact. As E_G is pseudo-metrizable, this means E_G is compact. □

In the proof of Theorem 3 below, it will be convenient to also have the following consequence of Lemma 4 to hand.

Lemma 6. *Let $\langle f_n \rangle$ be a sequence in S_G with $e(f_n) \to y$ for some $y \in E_G$. Then there is an $f \in S_G$ such that $e(f) = y$ and such that for almost all $t \in \hat{T}$, $f(t) \in \operatorname{LS} f_n(t)$, and for almost all $t \in \bar{T}$, $f(t) \in \overline{\operatorname{co}} \operatorname{LS} f_n(t)$.*

Proof. Let ρ be a pseudo-metric for E_G. Note that if y, $y' \in E_G$ are both limits of the same sequence $\langle y_k \rangle$ in E_G, then $\rho(y, y') = 0$. In view of Lemma 4, it suffices therefore to show that if $y \in E_G$ and $f_1 \in S_G$ are such that $\rho(e(f_1), y) = 0$, then there is an $f_2 \in S_G$ with $e(f_2) = y$ such that $f_2(t) = f_1(t)$ for almost all $t \in T$.

Let any such elements y and f_1 be given. Pick any $f' \in S_G$ with $e(f') = y$. Define $f_2 \in S_G$ by setting $f_2(t) = f_1(t)$ for each $t \in \hat{T}$, and $f_2(t) = f'(t)$ for each $t \in \bar{T}$. Then for the maps $\hat{e}$ and $\bar{e}$ from the definition of e, we have $e(f_2) = (\bar{e}(f'), \hat{e}(f_1))$. Now, because $\rho(e(f_1), e(f')) = 0$, we must have $\hat{e}(f_1) = \hat{e}(f')$, and it follows that $e(f_2) = e(f') = y$. Suppose there is a non-negligible $F \subseteq \bar{T}$ such that $f_1(t) \neq f_2(t)$ for all $t \in F$. By (A3) and what was noted in footnote 17, there is a countable set D of continuous linear functions on X which separates the points of X. Since the maps f_1 and f_2 are measurable and disagree on a non-negligible subset of $\bar{T}$, the properties of D imply that there are a $p \in D$ and a measurable non-negligible $G \subseteq \bar{T}$ such that $p(f_1(t)) \neq p(f_2(t))$ for all $t \in G$. But this means that we can construct an element $q \in \bar{\mathcal{G}}$ for which $\int_{\bar{T}} q(t, f_1(t)) \mathrm{d}\nu(t) \neq \int_{\bar{T}} q(t, f_2(t)) \mathrm{d}\nu(t)$, where $\bar{\mathcal{G}}$ is the set of functions from the definition of the feeble topology on $\bar{S}_G$. Since $h \mapsto \int_{\bar{T}} q(t, h(t)) \mathrm{d}\nu(t) \colon \bar{S}_G \to \mathbb{R}$ is continuous for the feeble topology, the composition of this map with the projection of E_G onto $\bar{S}_G$ is continuous as well. As the projections of $e(f_1)$ and $e(f_2)$ onto $\bar{S}_G$

are just $f_1|_{\bar{T}}$ and $f_2|_{\bar{T}}$ respectively, it follows that $\rho(e(f_1), e(f_2)) > 0$, contradicting the fact that $e(f_2) = y$ and $\rho(e(f_1), y) = 0$. Thus $f_2(t) = f_1(t)$ must hold for almost all $t \in T$. $\square$

We are now ready for the proof of Theorems 1 and 2. Since condition (S3), which was introduced at the beginning of this section, is implied by (S1) as well as by (S2), both these theorems are special cases of the following auxiliary result (noting that (A2), which is not explicitly assumed in Theorem 2, is implied by (S2)).

Theorem 16. *Let $G = ((T, \Sigma, \nu), X, \langle X_t, u_t, A_t \rangle_{t \in T}, e)$ be a game satisfying (A1)-(A4), (S3), and CS. Then G has a Nash equilibrium.*

Proof. Note first that according to Lemma 5, E_G is compact and pseudo-metrizable. Let ρ denote a corresponding pseudo-metric.

Suppose by way of contradiction that the game G has no equilibrium. Then by CS and compactness of E_G, there is $\mathcal{X} : T \times E_G \rightrightarrows X$ and a finite family $\langle U_j, \varphi_j, \alpha_j \rangle_{j \in J}$ where $\langle U_j \rangle_{j \in J}$ is a covering of E_G by open subsets, and φ_j, α_j correspond to $\mathcal{X}$ and U_j according to CS for each $j \in J$.

Recall that Lebesgue's covering theorem holds in compact pseudo-metric spaces. We can therefore find an $\varepsilon > 0$ so that each closed ρ-ball in E_G of radius 2ε is included in some member of the open covering $\langle U_j \rangle_{j \in J}$ of E_G. Let $\langle B_i \rangle_{i \in I}$ be a finite covering of E_G by closed ρ-balls of radius ε. Then the choice of ε implies that whenever $\langle B_i \rangle_{i \in I'}$ is subfamily of $\langle B_i \rangle_{i \in I}$ with $\bigcap_{i \in I'} B_i \neq \emptyset$, there is a $j \in J$ such that $\bigcup_{i \in I'} B_i \subseteq U_j$.

Fix any $i \in I$. Define a Caratheodory correspondence $\varphi^i : T \times B_i \rightrightarrows X$ as follows. Let $H = \{j \in J : B_i \subseteq U_j\}$. By the previous paragraph, H is non-empty. Using the fact that the functions α_j are measurable, we can find a finite partition $\langle T_k \rangle_{k \in K}$ of T into measurable sets and a corresponding family $\langle j_k \rangle_{k \in K}$ of elements of H such that if $t \in T_k$ then $\alpha_{j_k}(t) \geq \alpha_j(t)$ for each $j \in H$. Now define $\varphi^i : T \times B_i \rightrightarrows X$ by setting $\varphi^i(t, y) = \varphi_{j_k}(t, y)$ for $(t, y) \in T_k \times B_i$, $k \in K$. It is clear that $\varphi^i(t, \cdot)$ is well-behaved for each $t \in T$. Furthermore, for every $y \in B_i$, we have

$$\operatorname{graph}(\varphi^i(\cdot, y)) = \bigcup_{k \in K} \left(\operatorname{graph}(\varphi_{j_k}(\cdot, y)) \cap (T_k \times X)\right),$$

showing that the graph of $\varphi^i(\cdot, y)$ is measurable, because K is finite. Thus φ^i is a Caratheodory correspondence. By (a) in the definition of CS, $\varphi^i(t, y) \subseteq \mathcal{X}(t, y)$ for each $(t, y) \in T \times B_i$. Also, given any $y \in B_i$, (c) in the definition of CS implies that, for almost all $t \in T$, if $x \in \varphi^i(t, y)$ then $u_t(x, y) \geq \alpha_j(t)$ for each j with $B_i \subseteq U_j$.

Do this construction for each $i \in I$. For each $y \in E_G$, set $I^y = \{i \in I : y \in B_i\}$. Let $\varphi : T \times E_G \rightrightarrows X$ be the correspondence defined by setting

$$\varphi(t, y) = \bigcup_{i \in I^y} \varphi^i(t, y), \ (t, y) \in T \times E_G.$$

Then, because each φ^i is a Caratheodory correspondence, so is φ. (Indeed, first it is clear that φ takes non-empty values, and as I is finite, φ takes closed values. To see that $\varphi(t,\cdot)$ is uhc for each $t \in T$, fix $t \in T$ and $y \in E_G$, and consider an open $O \subseteq X$ such that $\varphi(t,y) \equiv \bigcup_{i \in I^y} \varphi^i(t,y) \subseteq O$. As each $\varphi^i(t,\cdot)$ is uhc and I^y is finite, there is a neighborhood V of y in E_G such that $\bigcup_{i \in I^y} \varphi^i(t,y') \subseteq O$ for each $y' \in V$. As $I \backslash I^y$ is finite and all the B_i's are closed, setting $V' = V \backslash (\bigcup_{i \in I \backslash I^y} B_i)$, V' is still a neighborhood of y, but such that $I^{y'} \subseteq I^y$ for all $y' \in V'$, implying that $\varphi(t,y') \subseteq O$ for all $y' \in V'$. Finally, for any $y \in E_G$, it is clear that since each $\varphi^i(\cdot,y)$ has a measurable graph, so does $\varphi(\cdot,y)$, again since I is finite, and since the graph of $\varphi(\cdot,y)$ is the union over I^y of the graphs of the correspondences $\varphi^i(\cdot,y)$.)

We claim that there are a $y^* \in E_G$ and a measurable $f^*\colon T \to X$ such that $y^* = e(f^*)$ and for almost all $t \in T$, $f^*(t) \in \varphi(t,y^*)$ if $t \in \hat{T}$, and $f^*(t) \in \overline{\mathrm{co}}\,\varphi(t,y^*)$ if $t \in \bar{T}$. Assuming for the time being that this has been established, let us see how to finish the proof.

Note first that by definition of φ, we have $f^*(t) \in \bigcup_{i \in I^{y^*}} \varphi^i(t,y^*)$ for almost all $t \in \hat{T}$, and $f^*(t) \in \overline{\mathrm{co}} \bigcup_{i \in I^{y^*}} \varphi^i(t,y^*)$ for almost all $t \in \bar{T}$. By (b) in CS, if $t \in \bar{T}$ then $\varphi^i(t,y^*)$ is convex or included in a finite-dimensional subspace of X, in addition to being compact, so we actually have $f^*(t) \in \mathrm{co} \bigcup_{i \in I^{y^*}} \varphi^i(t,y^*)$ for almost all $t \in \bar{T}$ (by the general fact that the convex hull of the union of finitely many such subsets of a Hausdorff topological vector space is closed; e.g., combine Lemma 5.29 and Corollary 5.33 in Aliprantis and Border (2006)). Now by definition of the sets I^y, applied to I^{y^*}, we have $y^* \in \bigcap_{i \in I^{y^*}} B_i$. According to what was noted in the last sentence of the third paragraph of this proof, this means there is a $j^* \in J$ such that $B_i \subseteq U_{j^*}$ for all $i \in I^{y^*}$. But from the fourth paragraph, if $y^* \in B_i \subseteq U_{j^*}$ then, for almost all $t \in T$, $x \in \varphi^i(t,y^*)$ implies $u_t(x,y^*) \geq \alpha_{j^*}(t)$. As $x \in \varphi^i(t,y^*)$ also implies $x \in \mathcal{X}(t,y^*)$, we may conclude that $f^*(t) \in \{x \in \mathcal{X}(t,y^*)\colon u_t(x,y^*) \geq \alpha_{j^*}(t)\}$ for almost all $t \in \hat{T}$, and that $f^*(t) \in \mathrm{co}\{x \in \mathcal{X}(t,y^*)\colon u_t(x,y^*) \geq \alpha_{j^*}(t)\}$ for almost all $t \in \bar{T}$. However, by (d) in the definition of CS, this is impossible because $e(f^*) = y^* \in U_{j^*}$, and this contradiction establishes the theorem.

Thus it remains to be shown that the above claim is correct. To this end, consider the correspondence $\varphi_1\colon E_G \rightrightarrows \mathcal{R}_G$ defined by setting

$$\varphi_1(y) = \{g \in \mathcal{R}_G\colon \mathrm{supp}\, g(t) \subseteq \varphi(t,y) \text{ for almost all } t \in T\},\ y \in E_G.$$

Then by Theorem 13, the fact that φ is a Caratheodory correspondence implies that φ_1 takes non-empty closed values. The fact that φ is a Caratheodory correspondence means, in particular, that $\varphi(t,\cdot)$ is well-behaved for each $t \in T$, which implies that if $y \in E_G$ is a limit of a sequence $\langle y_n \rangle$ in E_G, then $\mathrm{KLS}\, \varphi(t,y_n) \subseteq \varphi(t,y)$ for each $t \in T$ (because $\varphi(t,\cdot)$ takes its values in the compact, hence regular space X_t). Therefore, by Theorem 14, φ_1 has a sequentially closed graph in $E_G \times \mathcal{R}_G$. As E_G is pseudo-metrizable by Lemma 5, and $\mathcal{R}_G$ is sequentially compact by Theorem 13, it follows that φ_1 is uhc as may readily be seen.

Let $\varphi_2 \colon \mathcal{R}_G \rightrightarrows \mathcal{R}_G$ be the composition $\varphi_2 = \varphi_1 \circ h$, where h is the map from Lemma 3. Then φ_2 is uhc, because φ_1 is and because by Lemma 3, h is continuous. Also, φ_2 takes non-empty closed values. These properties of φ_2 guarantee that φ_2 has a fixed point, g^* say (see the discussion in steps 2 and 6 in the proof of Theorem 4.1.2 in Balder (2002)). Choose an element $f^* \in S_G$ which corresponds to g^* according to Lemma 3(c), and set $y^* = e(f^*)$. Then $g^* \in \varphi_1(y^*)$, and by the definition of φ_1 it follows that f^* and y^* are as required in the claim above. □

4.5 Proof of Theorem 3:

Let $G = ((T, \Sigma, \nu), X, \langle X_t, u_t, A_t\rangle_{t\in T}, e)$ be a game satisfying (A1)-(A4) and (S1) or (S2). If (A5)-(A8) hold in addition, then G satisfies CS.

Proof. Let $\varphi \colon T \times E_G \rightrightarrows X$ denote the best reply correspondence of the game. Thus $\varphi(t, y) = \{x \in A_t(y) \colon u_t(x, y) = w_t(y)\}$ for each $(t, y) \in T \times E_G$. We will show that CS holds with certain restrictions of φ, and with $\mathcal{X}$ chosen to be $(t, y) \mapsto A_t(y)$.

Recall first that a compact subset of a Souslin space is metrizable (for the subspace topology); see Schwartz (1973, p. 96, Theorem 3, and p. 106, Corollary 2). Thus (A3) and (A4)(i) imply that the sets X_t are metrizable. Using this fact, it is easily seen that upper semi-continuity of the payoff functions and (A6)(i) imply the following:

$$(*)\quad \begin{array}{l} \text{If } y_n \to y \text{ in } E_G,\ x_n \in A_t(y_n) \text{ for each } n, \text{ and } \underline{\lim}\, u_t(x_n, y_n) \geq w_t(y), \\ \qquad \text{then } x \in \mathrm{LS}\, x_n \text{ implies } x \in A_t(y) \text{ and } u_t(x, y) = \overline{\lim}\, u_t(x_n, y_n) = w_t(y). \end{array}$$

From this fact together with (A4)(i) and the hypothesis that the functions w_t are lsc it follows that for any $t \in T$, $\varphi(t, \cdot)$ is well-behaved and w_t is actually continuous. Also, by (A5) and (A6)(ii), for any $y \in E_G$, the map $t \mapsto w_t(y)$ is measurable and $\varphi(\cdot, y)$ has a measurable graph (see Castaing and Valadier (1977, Lemma III.39 and remarks in the sequel)[18]). In particular, φ is a Caratheodory correspondence.

Now fix any $y \in E_G$ and assume there is no $f \in S_G$ such that f is an equilibrium strategy with $e(f) = y$. Note that by Lemma 5, E_G is compact and pseudo-metrizable. Choose a corresponding pseudo-metric ρ. For $n \in \mathbb{N}\backslash\{0\}$, write $B_{1/n}(y)$ for the open ρ-ball around y of radius $1/n$. Also, for each $n \in \mathbb{N}\backslash\{0\}$ and each $y' \in B_{1/n}(y)$, let

$$\varphi^n(t, y') = \{x \in A_t(y') \colon u_t(x, y') \geq w_t(y) - 1/n\}.$$

[18] We mention that this reference involves a measurable selection theorem.

We claim that there is an integer $n_1 > 0$ such that (d) in the definition of CS holds, with $(t, y) \mapsto A_t(y)$ substituted for $\mathcal{X}$, and with $U = B_{1/n_1}(y)$, $\alpha\colon T \to [-\infty, +\infty]$ given by $\alpha(t) = w_t(y) - 1/n_1$, and with $T' \subseteq T$ being measurable with $\nu(T') > 2^{-n_1}$. Indeed, otherwise there would be a sequence $\langle f_n \rangle$ of strategies, with $e(f_n) \to y$, such that for each n there is a $T_n \subseteq T$ with $\nu(T_n) \leq 2^{-n}$ such that $f_n(t) \in \varphi^n(t, e(f_n))$ for almost all $t \in \hat{T} \backslash T_n$, and $f_n(t) \in \mathrm{co}\varphi^n(t, e(f_n))$ for almost all $t \in \bar{T} \backslash T_n$. Now by Lemma 6 there is a strategy f, with $e(f) = y$, such that $f(t) \in \mathrm{LS}\, f_n(t)$ for almost all $t \in \hat{T}$, and $f(t) \in \overline{\mathrm{co}}\, \mathrm{LS}\, f_n(t)$ for almost all $t \in \bar{T}$.[19] Noting that the sequence $\big\langle \bigcup_{n \geq m} T_n \big\rangle_{m \in \mathbb{N}}$ of sets is decreasing with $\nu\big(\bigcup_{n \geq m} T_n\big) \to 0$, it follows that for almost all $t \in \hat{T}$, $f(t) \in \mathrm{KLS}\, \varphi^n(t, e(f_n))$, and for almost all $t \in \bar{T}$, $f(t) \in \overline{\mathrm{co}}\, \mathrm{KLS}\, \mathrm{co}\varphi^n(t, e(f_n))$. By $(*)$ above, we have $\mathrm{KLS}\, \varphi^n(t, e(f_n)) \subseteq \varphi(t, y)$ for all $t \in T$. In particular, by Lemma 1, $\overline{\mathrm{co}}\, \mathrm{KLS}\, \mathrm{co}\varphi^n(t, e(f_n)) \subseteq \overline{\mathrm{co}}\, \varphi(t, y)$. By (A8) and the fact that φ, being a Caratheodory correspondence, takes closed values, we have $\overline{\mathrm{co}}\, \varphi(t, y) = \varphi(t, y)$ for $t \in \bar{T}$. It follows that $f(t) \in \varphi(t, y)$ for almost all $t \in T$. As $e(f) = y$, this means f is an equilibrium strategy, and we get a contradiction to the assumption made about y.

Choose and fix an integer n_1 according to the previous paragraph. We next claim that there is an integer $n_2 > 0$ and a $T^{n_2} \subseteq T$ such that $\nu(T^{n_2}) > 1 - 2^{-n_1}$ and such that for each $t \in T^{n_2}$, if $y' \in B_{1/n_2}(y)$ and $x \in \varphi(t, y')$ then $u_t(x, y') > w_t(y) - 1/n_1$. To see this, for each $n \in \mathbb{N} \backslash \{0\}$ let

$$T^n = \{t \in T \colon \inf_{y' \in B_{1/n}(y)} w_t(y') \geq w_t(y) - 1/n_1\}.$$

As was noted above, the map w_t is continuous for each $t \in T$, and the map $t \mapsto w_t(y')$ is measurable for each $y' \in E_G$. Also, as E_G, being compact and pseudo-metrizable, is separable, $B_{1/n}(y)$ contains a countable dense subset. Combining these facts, it follows that the map $t \mapsto \inf_{y' \in B_{1/n}(y)} w_t(y')$ is measurable, and hence that the set T^n is measurable. Now as each w_t is continuous, we have $T^n \uparrow T$ as $n \to \infty$, and it follows that $\nu(T^n) > 1 - 2^{-n_1}$ for n large enough. Thus, since $x \in \varphi(t, y')$ means $u_t(x, y') = w_t(y')$, n_2 and T^{n_2} with the desired properties do exist.

Choose such n_2 and T^{n_2}, and set $n_3 = \max\{n_1, n_2\}$. Let φ^y be the restriction of φ to $T \times B_{1/n_3}(y)$. Then, since φ is a Caratheodory correspondence, so is φ^y. Modify the function α of the penultimate paragraph on $T \backslash T^{n_2}$, if necessary, so as to get

$$\alpha(t) = \begin{cases} w_t(y) - 1/n_1 & \text{if } t \in T^{n_2} \\ -\infty & \text{otherwise.} \end{cases}$$

Then, with $(t, y) \mapsto A_t(y)$ in place of $\mathcal{X}$, (d) of CS still holds with $U = B_{1/n_3}(y)$, as $1/n_3 \leq 1/n_1$ and $\nu(T \backslash T^{n_2}) < 2^{-n_1}$. Also, by construction, (a) and (c) of CS hold for $U = B_{1/n_3}(y)$, φ^y, and α. Finally, by (A8), (b) of CS holds for φ^y, too. □

[19] Recall that (S3), which is assumed for Lemma 6, is implied by both (S1) and (S2).

4.6 Proof of Theorem 4:

Let $G = ((T, \Sigma, \nu), X, \langle X_t, u_t, A_t \rangle_{t \in T}, e)$ be a game satisfying (A1)-(A4) and (S1) or (S2). If (A5)-(A8) hold in addition, with "usc" in (A7) replaced by "weakly usc," then G is continuously secure.

Proof. With the hypothesis that a game be weakly usc, it still follows that $(*)$ in the proof of Theorem 3 holds, and therefore that proof still does the job (again with φ being the best reply correspondence of the game). □

4.7 Proof of Theorem 5:

Let $G = ((T, \Sigma, \nu), X, \langle X_t, u_t, A_t \rangle_{t \in T}, \tilde{e})$ be a game satisfying (A1), (A3), (A4), (A9), (A10), (S2), and CS'. Then G has a Nash equilibrium.

Proof. Recall that any Hausdorff locally convex space is completely regular. Hence by (A3) and (A10), $X \times C$ is a completely regular Souslin space. By what was noted in footnote 17, there is a countable family $\langle p_i \rangle_{i \in I}$ of continuous bounded functions on $X \times C$ which separates the points of $X \times C$. We may assume that the family $\langle p_i \rangle_{i \in I}$ is stable with respect to multiplication. Then by Schwartz (1973, p. 388, Corollary 1), the family of maps $\gamma \mapsto \int p_i \mathrm{d}\gamma \colon M^1_+(X \times C) \to \mathbb{R}$, $i \in I$, separates the points of $M^1_+(X \times C)$. For each $j \in J$ and $i \in I$, define a map $q_{ij} : \Gamma_G \cap (\hat{T} \times X) \to \mathbb{R}$ by setting

$$q_{ij}(t, x) = \begin{cases} \frac{1}{\nu(T_j)} p_i(x, c(t)) & \text{if } t \in T_j \\ 0 & \text{if } t \in \hat{T} \setminus T_j \end{cases}$$

(where the sets T_j are the subsets of $\hat{T}$ from the definition of the externality map $\tilde{e}$).

Let $\hat{\mathcal{C}} = \{q_{ij} \colon i \in I, j \in J\}$, and with this choice of $\hat{\mathcal{C}}$ let $e \colon S_G \to \bar{S}_G \times \mathbb{R}^{\hat{\mathcal{C}}}$ be the externality mapping of the general model as developed in Section 2.2. Note that $\hat{\mathcal{C}}$ satisfies the requirements in that model; in particular, $\hat{\mathcal{C}}$ is countable. As in Section 2.2, let $E_G = e(S_G)$, endowed with the same topology as there.

We claim that there is a homeomorphism $h \colon \tilde{E}_G \to E_G$ such that $e = h \circ \tilde{e}$. Given such an h, we may identify $\tilde{E}_G$ with E_G via h. In particular, we may view the constraint correspondences A_t as being defined on E_G, and the payoff functions u_t as being defined on the respective sets $X_t \times E_G$. Moreover, under this identification, CS' is equivalent to CS. The theorem under proof is therefore implied by Theorem 2.

To establish the claim, define $h' \colon \prod_{t \in \bar{T}} X_t \times \left(M^1_+(X \times C)\right)^J \to \prod_{t \in \bar{T}} X_t \times \mathbb{R}^{\hat{\mathcal{C}}}$ by setting

$$h'\big(z, \langle \gamma_j \rangle_{j \in J}\big) = \left(z, \left\langle \left\langle \int_{X \times C} p_i \mathrm{d}\gamma_j \right\rangle_{i \in I} \right\rangle_{j \in J}\right), \; z \in \prod_{t \in \bar{T}} X_t \,, \; \langle \gamma_j \rangle_{j \in J} \in \left(M^1_+(X \times C)\right)^J.$$

Recall from Section 2.6 that we are viewing each X_t as being endowed with the subspace topology defined from X, $M^1_+(X \times C)$ as being endowed with the narrow topology, and all products involving these spaces as being endowed with the product topology. Thus h' is continuous. Note also that by choice of the family $\langle p_i \rangle_{i\in I}$, h' is an injection. Let h be the restriction of h' to $\tilde{E}_G$.

For each $f \in S_G$ and each $j \in J$, set $\gamma_j(f) = (1/\nu(T_j))(\nu|_{T_j}) \circ (f|_{T_j}, c|_{T_j})^{-1}$. Note that for any $f \in S_G$, and each $i \in I$ and $j \in J$, we have

$$\int_{X\times C} p_i \mathrm{d}\gamma_j(f) = \int_{T_j} \tfrac{1}{\nu(T_j)} p_i(f(t), c(t))\mathrm{d}\nu(t) = \int_{\hat{T}} q_{ij}(t, f(t))\mathrm{d}\nu(t).$$

Using this fact, it follows that $e = h \circ \tilde{e}$, by the definitions of the three maps involved. In particular, as $E_G = e(S_G)$ by definition of E_G, h is a surjection from $\tilde{E}_G$ onto E_G.

Now by (A4), $\prod_{t\in\bar{T}} X_t$ is compact, and by Lemma 2 in Section 4.2, (A3), (A4), and (A10) imply that for each $j \in J$ there is a compact set $K_j \subseteq M^1_+(X \times C)$ such that $\{(1/\nu(T_j))(\nu|_{T_j}) \circ (f|_{T_j}, c|_{T_j})^{-1} \colon f \in S_G\} \subseteq K_j$. Consequently there is a compact $K \subseteq \prod_{t\in\bar{T}} X_t \times \left(M^1_+(X\times C)\right)^J$ such that $\tilde{E}_G \subseteq K$. Compactness of K and the fact that h' is a continuous injection mean that the restriction of h' to K is a homeomorphism from K onto $h'(K)$. As $\tilde{E}_G \subseteq K$ and h is the restriction of h' to $\tilde{E}_G$, it follows that h is a homeomorphism from $\tilde{E}_G$ onto $h(\tilde{E}_G)$ (recall that the topology of $\tilde{E}_G$ is the subspace topology defined from $\prod_{t\in\bar{T}} X_t \times \left(M^1_+(X \times C)\right)^J$). By what was stated in Remark 3, (A9) means that the feeble topology on $\bar{S}_G \equiv \prod_{t\in\bar{T}} X_t$, which is involved in the definition of E_G, is the same as the product topology of $\prod_{t\in\bar{T}} X_t$. By the fact that $h(\tilde{E}_G) = E_G$, we may now conclude that h is a homeomorphism from $\tilde{E}_G$ to E_G. This completes the proof. $\square$

4.8 Proof of Theorem 6

Let $G = ((T, \Sigma, \nu), X, \langle X_t, u_t, A_t\rangle_{t\in T}, \tilde{e})$ be a game satisfying (A1), (A3), (A4), (A9)-(A11), (S2), GPS and BRC. Then G also satisfies CS', and consequently, by Theorem 5, G has a Nash equilibrium.

Proof. Recall first from the proof of Theorem 5 that, for some choice of an externality map e as defined in Section 2.2, $\tilde{E}_G \equiv \tilde{e}(S_G)$ may be homeomorphically identified with $E_G \equiv e(S_G)$. Consequently, Lemmas 6 and 5 continue to hold with $\tilde{e}$ in place of e, and $\tilde{E}_G$ in place of E_G. In particular, $\tilde{E}_G$ is pseudo-metrizable, therefore first-countable.

Let $y \in \tilde{E}_G$ be such that there is no equilibrium strategy f of G with $\tilde{e}(f) = y$. We claim that there is an $\varepsilon > 0$ and a neighborhood V of y such that if $f \in S_G$ satisfies $\tilde{e}(f) \in V$ and $f(t) \in A_t(\tilde{e}(f))$ for almost all $t \in T$, there is $T' \subseteq T$ with $\nu(T') > \varepsilon$ such that $u_t(f(t), \tilde{e}(f)) < w_t(y) - \varepsilon$ for all $t \in T'$.

Indeed, otherwise there is a sequence $\langle f_k \rangle$ in S_G with $\tilde{e}(f_k) \to y$ such that for each k, $f_k(t) \in A_t(\tilde{e}(f_k))$ for almost all $t \in T$, and for some $T_k \subseteq T$ with $\nu(T \backslash T_k) < 2^{-k}$, $u_t(f_k(t), \tilde{e}(f_k)) \geq w_t(y) - 2^{-k}$ for all $t \in T_k$. Now the sequence $\big\langle \bigcup_{n \geq m} T_n \big\rangle_{m \in \mathbb{N}}$ of sets is decreasing with $\nu\big(\bigcup_{n \geq m} T_n\big) \to 0$, and thus we must have $\underline{\lim}_n\, u_t(f_n(t), \tilde{e}(f_n)) \geq w_t(y)$ for almost every $t \in T$. Lemma 6 gives an $f \in S_G$ such that $f(t) \in \mathrm{LS}\, f_n(t)$ for almost all $t \in \hat{T}$ and such that $\tilde{e}(f) = y$. Note that $\tilde{e}(f_k) \to \tilde{e}(f)$ implies $f(t) = \lim_k f_k(t)$ for $t \in \bar{T}$; see the paragraph following the statement of Assumption (A9). Thus we must have $f(t) \in \mathrm{LS}\, f_n(t)$ for almost all $t \in T$. By BRC, it follows that f is an equilibrium of G, and as $\tilde{e}(f) = y$ we thus get a contradiction to the assumption made about y.

Fix a neighborhood V of y and a number $\varepsilon > 0$ as just established. Relative to this ε, let U, φ, and α be chosen according to GPS, and let the correspondence $\mathcal{X}$ required in the definition of CS' be given by $\mathcal{X}(t, y') = A_t(y')$ for all $(t, y') \in T \times \tilde{E}_G$. Then (U, φ, α) satisfies (1)-(3) in CS'. By (A11) and the choice of $\mathcal{X}$, (4) in CS' is equivalent to the following statement: Whenever f is a strategy with $\tilde{e}(f) \in U$ and $f(t) \in A_t(\tilde{e}(f))$ for almost all $t \in T$, then there is a non-negligible set $T' \subseteq T$ such that $u_t(f(t), \tilde{e}(f)) < \alpha(t)$ for all $t \in T'$. Thus, shrinking the set U, if necessary, so that $U \subseteq V$, (4) in CS' must hold because of 4 in GPS. □

4.9 Proof of Theorem 7:

Let $G = ((T, \Sigma, \nu), X, \langle X_t, u_t, A_t \rangle_{t \in T}, \tilde{e})$ be a game satisfying (A1), (A3)-(A6), (A9)-(A12), (S2), and GBRS$^\bullet$. Then G also satisfies CS', and consequently, by Theorem 5, G has a Nash equilibrium.

Proof. Consider any $y \in \tilde{E}_G$ and suppose there is no $f \in S_G$ with $\tilde{e}(f) = y$ such that f is an equilibrium of G. Arguing similarly as in the proof of Theorem 6, but with $\underline{w}_y^\bullet$ in place of w_t, and GBRS$^\bullet$ in place of BRC, it follows that there is an $\varepsilon > 0$ and a neighborhood V of y such that if $f \in S_G$ satisfies $\tilde{e}(f) \in V$ and $f(t) \in A_t(\tilde{e}(f))$ for almost all $t \in T$, there is $T' \subseteq T$ with $\nu(T') > \varepsilon$ such that $u_t(f(t), \tilde{e}(f)) < \underline{w}_y^\bullet(t) - \varepsilon$ for all $t \in T'$.

Recall that $\underline{w}_y^\bullet = \sup\{\alpha^\bullet \colon \alpha \in \mathcal{Q}_y\} \subseteq L^0(\nu)$ and note that the set $\{\alpha^\bullet \colon \alpha \in \mathcal{Q}_y\}$ is upwards directed. Therefore, because $L^0(\nu)$ has the countable sup-property (see Fremlin (2001, 241Y(d))), there is a sequence $\langle \alpha_n \rangle$ in $\mathcal{Q}_y$ such that $\alpha_n(t) \to \underline{w}_y^\bullet(t)$ for almost all $t \in T$. Using Egoroff's theorem we may find an integer n_0 and a set $T_0 \subseteq T$ with $\nu(T_0) > 1 - \varepsilon$ such that $\alpha_{n_0}(t) > \underline{w}_y^\bullet(t) - \varepsilon$ for almost all $t \in T_0$, where ε is the number from the previous paragraph. By (A11) and the definition of the set $\mathcal{Q}_y$, we may conclude that CS' is satisfied, with $\mathcal{X}$ given by $\mathcal{X}(t, y) = A_t(y)$ for all $(t, y) \in T \times \tilde{E}_G$ (cf. the last paragraph of the proof of Theorem 6). □

4.10 Proof of Theorem 8:

Let $G = ((T,\Sigma,\nu), X, \langle X_t, u_t\rangle_{t\in T}, \tilde{e})$ be a game satisfying (A1), (A3), (A4), (A9), (A10), (A11') and (A13). Then for each $y \in \tilde{E}_G$, $\underline{w}^\bullet_y(t) = \underline{w}_t(y)$ for almost all $t \in T$.

Proof. Fix any $y \in \tilde{E}_G$. It is straightforward to check that $\underline{w}^\bullet_y(t) \le \underline{w}_t(y)$ for almost all $t \in T$. To see that $\underline{w}^\bullet_y(t) \ge \underline{w}_t(y)$, note first that (A13)(iii) implies that the map $t \mapsto \inf\{u_t(x,y)\colon x \in \hat{X}\}\colon T \to \mathbb{R}$ is a measurable. In view of this and the definition of $\underline{w}^\bullet_y$, it suffices to show that the map $t \mapsto \underline{w}_t(y)\colon T \to \mathbb{R}$ is measurable and that for each integer $n > 0$ there is a neighborhood U of y, a Caratheodory correspondence $\varphi\colon T \times U \rightrightarrows \hat{X}$, and a measurable set $T' \subseteq T$ with $\nu(T\backslash T') < 1/n$ such that for each $(t,y') \in T' \times U$, $u_t(x,y) \ge \underline{w}_t(y) - 1/n$ whenever $x \in \varphi(t,y')$, and such that $\varphi(t,y')$ is convex or included in a finite-dimensional subspace of X for all $t \in \bar{T}$.

Fix any $n \in \mathbb{N}$. Set $\mathcal{U} = \{u_t\colon t \in \hat{T}\}$ and let $\mathcal{U}$ be endowed with the subspace topology defined from $B(\hat{X} \times \tilde{E}_G)$. By (A13)(iv) there is a countable partition $\langle E_i\rangle_{i\in I}$ of $\mathcal{U}$ into Borel sets E_i, each with diameter less than $1/(3n)$. For each $i \in I$, let $F_i = \{t \in \hat{T}\colon u_t \in E_i\}$ and pick a point $t_i \in F_i$. Note that by (A13)(iii), F_i is a measurable subset of $\hat{T}$ for each $i \in I$. By definition of the functions $\underline{w}_t$, for each $i \in I$ there is a neighborhood U_i of y and a well-behaved correspondence $\varphi_i\colon U_i \rightrightarrows \hat{X}$ such that $u_{t_i}(x,y') > w_{t_i}(y) - 1/(3n)$ whenever $y' \in U_i$ and $x \in \varphi_i(y')$.

There is a finite $J \subseteq I$ such that $\nu(\hat{T}\backslash\bigcup_{i\in J} F_i) < 1/(2n)$. Furthermore, since $\bar{T}$ is countable, there is a finite $\tilde{T} \subseteq \bar{T}$ such that $\nu(\bar{T} \setminus \tilde{T}) < 1/(2n)$. By definition of the functions $\underline{w}_t$ for t belonging to $\bar{T}$, for each $t \in \tilde{T}$ there is a neighborhood U_t of y and a well-behaved correspondence $\varphi_t\colon U_t \rightrightarrows X_t$ such that $u_t(x,y') > \underline{w}_t(y) - 1/n$ whenever $y' \in U_t$ and $x \in \varphi_t(y')$ and such that φ_t takes convex values or is included in a finite-dimensional subspace of X.

Set $U = \left(\bigcap_{i\in J} U_i\right) \cap \left(\bigcap_{t\in\tilde{T}} U_t\right)$. Then U is a neighborhood of γ. Define a correspondence $\varphi\colon T \times U \rightrightarrows X$ by setting

$$\varphi(t,y') = \begin{cases} \varphi_i(y') & \text{if } t \in F_i \text{ and } i \in J, \\ \varphi_t(y') & \text{if } t \in \tilde{T}, \\ X_t & \text{if } t \notin \tilde{T} \cup \bigcup_{i\in J} F_i. \end{cases}$$

Clearly φ is a Caratheodory correspondence. Also, $\varphi(t,y')$ is convex or included in a finite-dimensional subspace of X for all $t \in \bar{T}$ and $\nu\big(T\backslash\big(\tilde{T} \cup \bigcup_{i\in J} F_i\big)\big) < 1/n$.

Note that for any $t, t' \in \hat{T}$ we have

$$(*) \qquad \|\underline{w}_t - \underline{w}_{t'}\|_\infty \le \|u_t - u_{t'}\|_\infty.$$

Using this fact, we may see that for each $t \in \bigcup_{i\in J} F_i$ we have $u_t(x,y') \ge \underline{w}_t(y) - 1/n$ whenever $y' \in U$ and $x \in \varphi(t,y')$. As noted above, the same holds for all $t \in \tilde{T}$.

Using $(*)$ it may also be seen that the map $t \mapsto \underline{w}_t(y) : T \to \mathbb{R}$ is measurable. Indeed, for each $u \in \mathcal{U}$ pick a point t_u in $\hat{T}$ so that $u = u_{t_u}$. Define $v\colon \mathcal{U} \to \mathbb{R}$ by setting $v(u) = \underline{w}_{t_u}(y)$ for every $u \in \mathcal{U}$. Clearly, as $\underline{w}_t(y) = \underline{w}_{t'}(y)$ whenever $u_t = u_{t'}$, the function $t \mapsto \underline{w}_t(y) : \hat{T} \to \mathbb{R}$ can be written as the composition of the function $t \mapsto u_t$ with the function v. Now from $(*)$, $|v(u) - v(u')| \leq \|u - u\|_\infty$ for any u, $u' \in \mathcal{U}$. That is, v is continuous, and by (A13)(iii) we may conclude that $t \mapsto \underline{w}_t(y) : \hat{T} \to \mathbb{R}$ is measurable. It now follows from (A9) that $t \mapsto \underline{w}_t(y)$ is measurable as a function on the entire set T. $\square$

4.11 Lemmas for the proof of Theorems 10 and 11

For the following lemma recall that Λ is endowed with the topology obtained from the narrow topology of $M_+([0, \bar{n}])$ by identifying an element $\lambda \in \Lambda$ with the Borel measure μ_λ on $[0, \bar{n}]$ defined by $\mu_\lambda([0, z]) = \lambda(z)$ for all $z \in [0, \bar{n}]$.

Lemma 7. *Suppose $\lambda_k \to \lambda$ in Λ and $z_k \to z$ in $[0, \bar{n}]$. Then $\lambda(z) \geq \overline{\lim}_k \lambda_k(z_k)$. If $z > 0$ and z is a point of continuity of λ, then $\lambda(z) = \lim_k \lambda_k(z_k)$.*

Proof. As $\lambda_k \to \lambda$ implies $\lambda_k(\bar{n}) \to \lambda(\bar{n})$, it is clear that the first assertion holds in case $\lambda = \bar{n}$. Suppose $z < \bar{n}$ and pick any $z^* \in (z, \bar{n}]$. Then

$$\overline{\lim}_k \lambda_k(z_k) = \overline{\lim}_k \mu_{\lambda_k}([0, z_k]) \leq \overline{\lim}_k \mu_{\lambda_k}([0, z^*]) \leq \mu_\lambda([0, z^*]) = \lambda(z^*).$$

As $z^* > z$ was arbitrary, $\overline{\lim}_k \lambda_k(z_k) \leq \lambda(z)$ by right-continuity of λ.

Suppose $z > 0$ is a continuity point of λ. Then given $\varepsilon > 0$ there is a $z' < z$ with $\lambda(z') > \lambda(z) - \varepsilon$, and thus a $z^* < z$ such that $\mu_\lambda([0, z^*)) > \lambda(z) - \varepsilon$. Now

$$\underline{\lim}_k \lambda_k(z_k) = \underline{\lim}_k \mu_{\lambda_k}([0, z_k]) \geq \underline{\lim}_k \mu_{\lambda_k}([0, z^*)) \geq \mu_\lambda([0, z^*)) > \lambda(z) - \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, $\underline{\lim}_k \lambda_k(z_k) \geq \lambda(z)$. $\square$

For the next lemma, recall that if $\gamma \in K$ then $\hat{\gamma} \in \hat{K}$ denotes the distribution of the map $(u, n, m, l) \mapsto nl\colon C \times M \times N \to [0, \bar{n}]$, and that the map $\gamma \mapsto \hat{\gamma}$ is continuous.

Lemma 8. *Let $\lambda_k \to \lambda$ in Λ and $\gamma_k \to \gamma$ in K. Then (a) $\int u(\lambda(nl), l) \mathrm{d}\gamma(u, n, m, l) \geq \overline{\lim}_k \int u(\lambda_k(nl), l) \mathrm{d}\gamma_k(u, n, m, l)$ and (b) $\int \pi(\lambda(z)) \mathrm{d}\hat{\gamma}(z) \geq \overline{\lim}_k \int \pi(\lambda_k(z)) \mathrm{d}\hat{\gamma}_k(z)$ if $\pi\colon [0, \bar{m}] \to \mathbb{R}$ is continuous and non-decreasing.*

Proof. (b) Note that $\gamma_k \to \gamma$ implies $\hat{\gamma}_k \to \hat{\gamma}_k$. Let μ be Lebesgue measure on $[0, 1]$. By Skorokhod's Theorem we can select measurable maps h, $h_k\colon [0, 1] \to [0, \bar{n}]$, $k \in \mathbb{N}$, such that $\hat{\gamma} = \mu \circ h^{-1}$ and $\hat{\gamma}_k = \mu \circ h_k^{-1}$ for each k and such that $h_k(a) \to h(a)$ for almost all $a \in [0,1]$. Using Fatou's lemma, Lemma 7, and the properties of π, we get

$$\overline{\lim}_k \int \pi(\lambda_k(z))\mathrm{d}\hat{\gamma}_k(z) = \overline{\lim}_k \int \pi(\lambda(h_k(a)))\mathrm{d}\mu(a) \leq \int \overline{\lim}_k \, \pi(\lambda_k(h_k(a)))\mathrm{d}\mu(a)$$
$$\leq \int \pi(\lambda(h(a)))\mathrm{d}\mu(a) = \int \pi(\lambda(z))\mathrm{d}\hat{\gamma}(z).$$

This proves (b). Part (a) follows similarly, but this time with the maps h and h_k going to $\hat{C} \times N \times M \times N$, observing that if $(u_k, n_k, m_k, l_k) \to (u, n, m, l)$ in $\hat{C} \times N \times M \times N$ and $\lambda_k \to \lambda$ in Λ, then $\overline{\lim}_k \, u_k(\lambda_k(n_k l_k), l_k) \leq u(\lambda(nl), l)$, by Lemma 7 and because $(u, n, m, l) \in \hat{C} \times N \times M \times N$ implies that u is non-decreasing in m. □

Lemma 9. *For each $t \in T$, the correspondence A_t is well-behaved. Furthermore, for each $y \in \tilde{E}_G$, the correspondence $t \mapsto A_t(y)$ has a measurable graph.*

Proof. It follows from (T5) that $A_{\bar{t}}$ is well-behaved. Consider any $t \in \hat{T}$. As $\lambda(z) \in \mathbb{R}_+$ for all $z \in [0, \bar{n}]$ and $\lambda \in \Lambda$, we have $(0,0) \in A_t(y)$ for all $y \in \tilde{E}_G$. i.e., A_t takes non-empty values. Let $y = (\lambda, \gamma) \in \tilde{E}_G$, $(m, l) \in M \times L$, and suppose $\langle(\lambda_k, \gamma_k)\rangle$ and $\langle(m_k, l_k)\rangle$ are sequences in $\tilde{E}_G$ and $M \times L$, respectively, with $(\lambda_k, \gamma_k) \to (\lambda, \gamma)$, $(m_k, l_k) \to (m, l)$ and $(m_k, l_k) \in A_t(\lambda_k, \gamma_k)$ for all k. Then by Lemma 7, we have $m = \lim_k m_k \leq \overline{\lim}_k \, \lambda(n_t l_k) \leq \lambda(n_t l)$, so $(m, l) \in A_t(\lambda, \gamma)$. Thus A_t is closed and therefore well-behaved as $M \times L$ is compact.

Let $y = (\lambda, \gamma) \in \tilde{E}_G$. To show that $t \mapsto A_t(y)$ has a measurable graph, it suffices to show that Γ_A is measurable, where Γ_A is the graph of the restriction of $t \mapsto A_t(y)$ to $\hat{T}$. Define $p : \hat{T} \times M \times L \to \mathbb{R}$ by $p(t, m, l) = \lambda(n_t l) - m$ for all $(t, m, l) \in \hat{T} \times M \times L$. Then p is measurable and $\Gamma_A = p^{-1}(\mathbb{R}_+)$. Thus Γ_A is measurable. □

Lemma 10. *Given $y = (\lambda, \gamma) \in \tilde{E}_G$ and $\varepsilon > 0$, there are measurable maps $\beta \colon \hat{T} \to \mathbb{R}$ and $f \colon \hat{T} \to M \times L$ and a neighborhood W of λ in Λ such that:*

1. *$f(t) \in A_t(y')$ for all $t \in \hat{T}$ and all $y' = (\lambda', \gamma) \in \tilde{E}_G$ with $\lambda' \in W$.*
2. *$u_t(f(t), y') \geq \beta(t)$ for all $t \in \hat{T}$ and all $y' = (\lambda', \gamma) \in \tilde{E}_G$ with $\lambda' \in W$.*
3. *$\nu(\{t \in \hat{T} \colon \beta(t) \geq w_t(y) - \varepsilon\}) \geq 1 - \varepsilon$.*

Proof. Let $y = (\lambda, \gamma) \in \tilde{E}_G$ and $\varepsilon > 0$ be given. Note first that (T1) implies that if $h \colon \hat{T} \to M \times L$ is measurable, then so is the map $t \mapsto \tilde{u}_t(h(t)) \colon \hat{T} \to \mathbb{R}$. Moreover, (T1) also implies that the map $(t, x) \mapsto \tilde{u}_t(x) \colon \hat{T} \times M \times L \to \mathbb{R}$ is measurable. Thus, by Lemma 9, since $\tilde{u}_t$ is continuous for each $t \in \hat{T}$, the map $t \mapsto w_t(y)$ is measurable and there is a measurable $g = (m, l) \colon \hat{T} \to M \times L$ such that $m(t) \leq \lambda(n_t l(t))$ and $\tilde{u}_t(g(t)) = w_t(y)$ for all $t \in \hat{T}$ (see Castaing and Valadier (1977, Lemma III.39 and the remarks following its proof)). Let $D \subseteq N$ be the set of discontinuity points of λ, and note that D is countable as λ is non-decreasing. Let $D_1 = \{d_1, d_2, \dots\}$ be a countable dense subset of N such that $D \cap D_1 = \emptyset$. Let $T_1 = \{t \in \hat{T} \colon m(t) > 0, n_t > 0, n_t \notin D\}$ and note that T_1 is a measurable subset of $\hat{T}$. For $i, j \in \mathbb{N}\backslash\{0\}$, set

$$T_{ij} = \big\{t \in T_1 : d_i/n_t \leq 1,\ \tilde{u}_t\big(\tfrac{j-1}{j}m(t), d_i/n_t\big) > w_t(y) - \varepsilon,\ \tfrac{j-1}{j}m(t) < \lambda(d_i)\big\}.$$

Then T_{ij} is measurable, and by the definition of T_1 and the fact that λ is non-decreasing and right-continuous, we have $\bigcup_{i,j\in\mathbb{N}\backslash\{0\}} T_{ij} = T_1$. We can therefore find a measurable map $g_1 = (m_1, l_1) \colon T_1 \to M \times L$ such that for all $t \in T_1$, $\tilde{u}_t(g_1(t)) > w_t(y) - \varepsilon$, $m_1(t) < \lambda(n_t l_1(t))$, and $n_t l_1(t) \in D_1$. Now as each $d \in D_1$ is a continuity point of λ, and as D_1 is countable, Lemma 7 implies that there are a neighborhood W of λ and a measurable $T_2 \subseteq T_1$, with $\nu(T_1 \backslash T_2) < \varepsilon$, such that $m_1(t) < \lambda'(n_t l_1(t))$ for each $t \in T_2$ and $\lambda' \in W$. Now define $f \colon \hat{T} \to M \times L$ by setting $f(t) = g_1(t)$ if $t \in T_2$ and $f(t) = (0,0)$ otherwise, and $\beta \colon \hat{T} \to \mathbb{R}$ by setting $\beta(t) = \tilde{u}_t(f(t))$ for all $t \in \hat{T}$. Concerning $t \in \hat{T}\backslash T_1$, note that if $m(t) = 0$ then $w_t(y) = u_t(0,0)$ by (T2), and that $\{t \in \hat{T} \colon n_t \in D \text{ or } n_t = 0\}$ is a null set by (T4), as D is countable. □

Lemma 11. *For all $y \in \tilde{E}_G$, w_t is lsc at y for almost all $t \in \hat{T}$.*

Proof. Fix $y = (\lambda, \gamma) \in \tilde{E}_G$. Clearly, if $w_t(y) = \tilde{u}_t(0,0)$ then w_t is lsc at y because $(0,0) \in A_t(y')$ for every $y' \in \tilde{E}_G$. Now by the proof of Lemma 10, for almost all $t \in T$, if $w_t(y) > \tilde{u}_t(0,0)$ then, given $\varepsilon > 0$, there is a point $(m,l) \in M \times L$, with $m < n_t l$ and $\tilde{u}_t(m,l) > w_t(y) - \varepsilon$, such that $n_t l$ is a continuity point of λ. In view of Lemma 7, this shows that the assertion holds. □

Lemma 12. *Let $f \in S_G$ and $\langle f_k \rangle$ a sequence in S_G such that (a) $\tilde{e}(f_k) \to \tilde{e}(f)$ and, for almost all $t \in T$, (b) $f_k(t) \in A_t(\tilde{e}(f_k))$ for all $k \in \mathbb{N}$, (c) $f(t) \in \mathrm{LS}\, f_k(t)$, and (d) $\underline{\lim}_k\, u_t(f_k(t), \tilde{e}(f_k)) \geq w_t(\tilde{e}(f))$. Then f is an equilibrium of G.*

Proof. By Lemma 9, (b) and (c) imply that $f(t) \in A_t(\tilde{e}(f))$ for almost all $t \in T$. As $\tilde{u}_t$ is continuous for each $t \in \hat{T}$, (c) implies $\tilde{u}_t(f(t)) \in \mathrm{LS}\, \tilde{u}_t(f_k(t))$ for almost all $t \in \hat{T}$. Hence, by the definitions of u_t and w_t for $t \in \hat{T}$, (d) implies $u_t(f(t), \tilde{e}(f(t))) = w_t(\tilde{e}(f))$ for almost all $t \in \hat{T}$.

It remains to see that $f(\bar{t})$ is optimal for $\bar{t}$ in $A_{\bar{t}}(\tilde{e}(f))$. For this, set $\gamma = \hat{\nu} \circ (\hat{n}, \hat{f})^{-1}$, and $\gamma_k = \hat{\nu} \circ (\hat{n}, \hat{f}_k)^{-1}$ for $k \in \mathbb{N}$. Write $\bar{\lambda}_k = f_k(\bar{t})$, $k \in \mathbb{N}$, and $\bar{\lambda} = f(\bar{t})$.

Now (a) implies both $\lambda_k \to \lambda$ and $\gamma_k \to \gamma$. Thus $\int nl\mathrm{d}\gamma_k(n,m,l) \to \int nl\mathrm{d}\gamma(n,m,l)$ as well as $\int m\mathrm{d}\gamma_k(n,m,l) \to \int m\mathrm{d}\gamma(n,m,l)$. Moreover, from Lemma 8(b), with π being the identity on $[0,\bar{m}]$, we have $\int \bar{\lambda}(nl)\mathrm{d}\gamma(n,m,l) \geq \overline{\lim}_k \int \bar{\lambda}_k(nl)\mathrm{d}\gamma_k(n,m,l)$.

Next, note that for all k, we have $\int m\mathrm{d}\gamma_k(n,m,l) \leq \int \bar{\lambda}_k(nl)d\gamma_k(n,m,l)$ because $f_k(t) \in A_t(\tilde{e}(f_k))$ for almost all $t \in \hat{T}$. Moreover, $\int m\mathrm{d}\gamma(n,m,l) = \int \bar{\lambda}(nl)d\gamma(n,m,l)$

because for almost all $t \in \hat{T}$, both $f(t) \in A_t(\tilde{e}(f))$ and $u_t(f(t), \tilde{e}(f(t))) = w_t(\tilde{e}(f))$, and because the functions u_t are strictly increasing in m. Hence, by the last two facts stated in the previous paragraph, $\int \bar{\lambda}_k(nl)d\gamma_k(n,m,l) \to \int \bar{\lambda}(nl)\mathrm{d}\gamma(n,m,l)$.

Finally, note that by (T7) and the definition of $u_{\bar{t}}$, $w_{\bar{t}}(\tilde{e}(f)) \geq 0$, since v is non-negative by (T6). Thus $\underline{\lim}_k\, u_{\bar{t}}(\lambda_k, \tilde{e}(f_k)) \geq w_{\bar{t}}(\tilde{e}(f))$ implies that we must have $v(\bar{\lambda}_k, \gamma_k) = u_{\bar{t}}(\bar{\lambda}_k, \tilde{e}(f_k))$ as well as $\int \bar{\lambda}_k(nl)\mathrm{d}\gamma_k(n,m,l) = \int nl\mathrm{d}\gamma_k(n,m,l)$ for all sufficiently large k, again by definition of $u_{\bar{t}}$ and non-negativity of v. By the fact that $\int nl\mathrm{d}\gamma_k(n,m,l) \to \int nl\mathrm{d}\gamma(n,m,l)$ and the conclusion of the previous paragraph, it follows that $\int \bar{\lambda}(nl)\mathrm{d}\gamma(n,m,l) = \int nl\mathrm{d}\gamma(n,m,l)$ and hence that $u_{\bar{t}}(\bar{\lambda}, \tilde{e}(f)) = v(\bar{\lambda}, \gamma)$. As v is usc by (T6), we can conclude that

$$u_{\bar{t}}(\bar{\lambda}, \tilde{e}(f)) = v(\bar{\lambda}, \gamma) \geq \overline{\lim}_k\, v(\bar{\lambda}_k, \gamma_k) = \overline{\lim}_k\, u_{\bar{t}}(\bar{\lambda}_k, \tilde{e}(f_k)) \geq w_{\bar{t}}(\tilde{e}(f)),$$

so that $u_{\bar{t}}(\bar{\lambda}, \tilde{e}(f)) = w_{\bar{t}}(\tilde{e}(f))$. This completes the proof. □

4.12 Proof of Theorem 11:

The game G satisfies CS'.

Proof. Lemma 12 implies that G satisfies BRC. We claim that G also satisfies GPS. Once this claim is established, the result follows from Theorem 6.

Let $y = (\lambda, \gamma) \in \tilde{E}_G$ and $\varepsilon > 0$. Let W, f, and β be chosen according to Lemma 10, and let O and ψ be chosen according to Assumption (T7). Set $U = W \times O$. Define $\varphi\colon T \times U \rightrightarrows (M \times L) \cup \Lambda$ by setting $\varphi(t, y') = \{f(t)\}$ for $t \in \hat{T}$, and by setting $\varphi(\bar{t}, (\lambda', \gamma')) = \{\psi(\gamma')\}$, $y' = (\lambda', \gamma') \in U$. Define $\alpha\colon T \to \mathbb{R}$ by setting $\alpha(t) = \beta(t)$ for $t \in \hat{T}$, and by setting $\alpha(\bar{t}) = w_{\bar{t}}(y) - \varepsilon$. It is now readily seen that (U, φ, α) satisfies the requirements in GPS. □

4.13 Proof of Theorem 10:

If the economy $E = \langle (\hat{T}, \hat{\Sigma}, \hat{\nu}), M, L, N, \hat{n}, \Lambda, \Theta, v, \langle \tilde{u}_t, n_t \rangle_{t \in \hat{T}} \rangle$ satisfies (T1)-(T7), then there exists an optimal income tax.

Proof. Let

$$\tilde{S}(E) = \{\tilde{e}(f) : f \text{ is an equilibrium of } G\}$$

and let $F : \tilde{S}(E) \to \mathbb{R}$ be defined by setting $F(\lambda, \gamma) = \int u(m,l)d\gamma(u,n,m,l)$ for all $y = (\lambda, \gamma) \in \tilde{S}(E)$. Consider the problem $\max_{y \in \tilde{S}(E)} F(y)$. By change of variables, this problem has a solution if and only if the problem $\max_{f \in S(E)} \int_{\hat{T}} u_t(f(t))\mathrm{d}\nu(t)$ has a solution, where $S(E)$ is the set of all equilibria of G. As pointed out earlier, every equilibrium of the game G defines an equilibrium of the economy E, with the same utilities for the individuals in $\hat{T}$, and vice versa. Thus it suffices to show that the problem $\max_{y \in \tilde{S}(E)} F(y)$ has a solution.

To this end, note by Theorem 12, $\tilde{S}(E)$ is nonempty. Moreover, as a subset of the compact and metrizable space $\tilde{E}_G$, $\tilde{S}(E)$ is closed and therefore compact. To see this, let $y \in \tilde{E}_G$ and $\langle y_k \rangle$ a sequence in $\tilde{E}(G)$ such that $y_k \to y$. For each $k \in \mathbb{N}$, let f_k be an equilibrium of G such that $\tilde{e}(f_k) = y_k$. By Lemma 6, there is an $f \in S_G$ such that $\tilde{e}(f) = y$ and $f(t) \in \mathrm{LS}\, f_k(t)$ for almost all $t \in T$. Since w_t is lsc for almost all $t \in T$ (by Lemma 11 and (T7)), we must have

$$\underline{\lim}_k\, u_t(f_k(t), \tilde{e}(f_k)) = \underline{\lim}_k\, w_t(\tilde{e}(f_k)) \geq w_t(\tilde{e}(f))$$

for almost all $t \in T$. It now follows from Lemma 12 that f is an equilibrium of G, and thus $y \in \tilde{S}(E)$.

Finally, note that by (T1) the map $(u, n, m, l) \mapsto u(m, l) \colon \tilde{S}(E) \to \mathbb{R}$ is continuous. As $\tilde{S}(E)$ is compact, it follows first that the map F is continuous, and then that the problem $\max_{y \in \tilde{S}(E)} F(y)$ has a solution. This completes the proof. □

4.14 Lemmas for Examples 7 and 8

Recall for the proofs below that if $\gamma \in K$ then $\hat{\gamma} \in \hat{K}$ denotes the distribution of the map $(u, n, m, l) \mapsto nl \colon C \times M \times N \to [0, \bar{n}]$, and that the map $\gamma \mapsto \hat{\gamma}$ is continuous.

Lemma 13. *Let Θ and v be as in Example 7. Then (T5)-(T7) hold. Moreover, (1) holds for any equilibrium (λ^*, g^*) of $E = \langle (\hat{T}, \hat{\Sigma}, \hat{\nu}), M, L, N, \hat{n}, \Lambda, \Theta, \langle \tilde{u}_t, n_t \rangle_{t \in \hat{T}} \rangle$.*

Proof. (T6) holds trivially. As for (T5), note that as π is concave, Θ has convex values. To show that Θ is well-behaved, it suffices to show that Θ is closed. To see this, note first that the two functions π and η must be continuous. Because π is increasing in addition, Lemma 8 implies that the map $(\lambda, \gamma) \mapsto \int \pi(\lambda(z)) \mathrm{d}\hat{\gamma}(z) \colon \Lambda \times K \to \mathbb{R}$ is usc. Using these facts, in conjunction with the fact that Λ and K are endowed with the narrow topology, is easily seen that Θ has a closed graph.

Concerning (T7), fix $\gamma \in K$ and $\varepsilon > 0$. Let $O = K$ and define $\psi \colon O \to \Lambda$ by setting $\psi(\gamma') = \left(\int z \mathrm{d}\hat{\gamma}'(z)\right) \chi_N$ for all $\gamma' \in O$, where χ_N is the characteristic function of N. Then ψ is continuous and (T7)(ii) holds. Also, $\int \pi(\psi(\gamma')(z)) \mathrm{d}\hat{\gamma}'(z) = \pi\left(\int z \mathrm{d}\hat{\gamma}'(z)\right)$. Using this fact, it follows that (T7)(i) holds. Finally, it is clear that (T7)(iii) holds, as v constantly takes value 0.

As for the last part of the lemma, suppose (λ^*, g^*) is an equilibrium of E and set $\gamma = \hat{\nu} \circ (c, g^*)^{-1}$. Assume first that $\pi\left(\int z d\hat{\gamma}(z)\right) + \int \eta(l) d\gamma(u, n, m, l) > 0$. By (a) of the equilibrium definition, (1) is equivalent to $\int \pi(\lambda^*(z)) d\hat{\gamma}(z) + \delta \int \eta(l) d\gamma(u, n, m, n) \geq (1 - \delta)\pi\left(\int z d\hat{\gamma}(z)\right)$. Thus (1) holds, since $\lambda^* \in \Theta(\gamma)$ by definition of equilibrium.

Assume $\pi\left(\int z d\hat{\gamma}(z)\right) + \int \eta(l) d\gamma(u,n,m,l) \leq 0$. Note that (b) of the equilibrium definition implies $\pi(\lambda^*(n_t l^*(t))) + \eta(l^*(t)) \geq 0$ for almost all $t \in \hat{T}$, because $0 \leq \lambda^*(0)$ and $\pi(0) + \eta(0) = 0$. Thus, by concavity of π and by (a) of the equilibrium definition,

$$\begin{aligned} 0 \leq \int \pi(\lambda^*(z)) \mathrm{d}\hat{\gamma}(z) + \int \eta(l) \mathrm{d}\gamma(u,n,m,l) \\ \leq \pi\left(\int \lambda^*(z) \mathrm{d}\hat{\gamma}(z)\right) + \int \eta(l) \mathrm{d}\gamma(u,n,m,l) \\ = \pi\left(\int z \mathrm{d}\hat{\gamma}(z)\right) + \int \eta(l) \mathrm{d}\gamma(u,n,m,l) \leq 0. \end{aligned}$$

Now this shows that the first two sums must be zero, from which (1) follows. □

Lemma 14. *Let Θ and v be as in Example 8 and assume that $\tilde{u}_t(\cdot, l)$ is concave for all $t \in \hat{T}$ and $l \in L$. Then (T6) and(T7) hold.*

Proof. To see that (T6) holds, note first that by Lemma 8, v is usc. Now if $\lambda \in \Lambda$ is continuous, then the bounded map $(u,n,m,l) \mapsto u(\lambda(nl), l) \colon C \times M \times L \to \mathbb{R}$ is continuous, and therefore the map $\gamma \mapsto \int u(\lambda(nl), l) \mathrm{d}\gamma(u,n,m,l) \equiv v(\lambda, \gamma) \colon K \to \mathbb{R}$ is continuous (by definition of the narrow topology). By the assumption that $\tilde{u}_t(\cdot, l)$ is concave for all $t \in \hat{T}$ and $l \in L$, it follows that $v(\cdot, \gamma)$ is quasi-concave for all $\gamma \in K$. Finally, by (T2), $v(\lambda, \gamma) \geq 0$ for all $(\lambda, \gamma) \in \Lambda \times K$. Thus (T6) holds.

As for (T7), fix $\gamma \in K$ and $\varepsilon > 0$. Let ρ denote the metric on Λ induced by Huntingdon's metric on the space of of Borel measures on $[0, \bar{n}]$. Recall that λ_0 is the element of Λ satisfying $\lambda_0(z) = z$ for all $z \in [0, \bar{n}]$. Let $B = \{\lambda \in \Lambda \colon \int \lambda(z) \mathrm{d}\hat{\gamma}(z) = \int z \mathrm{d}\hat{\gamma}(z)\}$ and note that $\lambda_0 \in B$. Let $S = \{\lambda \in B \colon \rho(\lambda, \lambda_0) \leq \varepsilon(\hat{\gamma})\}$ and set $s = \sup\{v(\lambda, \gamma) \colon \lambda \in S\}$.

Suppose $\int z \mathrm{d}\hat{\gamma}(z) = 0$. Then $v(\lambda_0, \gamma) = s$. Since λ_0 is continuous, by (T6) there is a neighborhood O of γ such that $v(\lambda_0, \gamma') > v(\lambda_0, \gamma) - \varepsilon$ for all $\gamma' \in O$. Set $\psi(\gamma') = \lambda_0$ for all $\gamma' \in O$. Evidently ψ is as required in (T7).

Suppose $\int z \mathrm{d}\hat{\gamma}(z) > 0$ which implies $\varepsilon(\hat{\gamma}) > 0$. It suffices to find a $\lambda \in B$ with the following properties: (a) $v(\lambda, \gamma) > s - \varepsilon$, (b) $\rho(\lambda, \lambda_0) < \varepsilon(\hat{\gamma})$, and (c) λ is continuous.

Indeed, given such a λ, by (c) we have $\int \lambda(z) \mathrm{d}\hat{\gamma}'(z) > 0$ for all γ' in some neighborhood O of γ, and thus $\int \alpha(\gamma')\lambda(z) \mathrm{d}\hat{\gamma}'(z) = \int z \mathrm{d}\hat{\gamma}'(z)$ for some number $\alpha(\gamma') > 0$. Note that the map $\gamma' \mapsto \alpha(\gamma')$ is continuous with $\alpha(\gamma) = 1$. Consider a sequence $\langle \gamma_k \rangle$ in O with $\gamma_k \to \gamma$; in particular, $\alpha(\gamma_k) \to 1$. Now (c) implies that we have $u_k(\alpha(\gamma_k)\lambda(n_k l_k), l_k)) \to u(\lambda(nl), l)$ whenever $(u_k, n_k, m_k, l_k) \to (u, n, m, l)$. Using Billingsley (1968, Theorem 5.5, p. 34) and the definition of v, it follows that $v(\alpha(\gamma_k)\lambda, \gamma_k) \to v(\lambda, \gamma)$. In view of this and (a), shrinking the set O, if necessary, we have $v(\alpha(\gamma')\lambda, \gamma') > s - \varepsilon$ for all $\gamma' \in O$. Shrinking the set O another time, if necessary, (b) implies that we can arrange to also have $\rho(\alpha(\gamma')\lambda, \lambda_0) < \varepsilon(\hat{\gamma}')$ for each $\gamma' \in O$, using the fact that the maps $\gamma' \mapsto \hat{\gamma}'$ and $\hat{\gamma}' \mapsto \varepsilon(\hat{\gamma}')$ are continuous, together with continuity of a metric. Now define $\psi\colon O \to \Lambda$ by setting $\psi(\gamma') = \alpha(\gamma')\lambda$ for each $\gamma' \in O$. Then ψ is as required in (T7).

Now to find a λ as desired, choose some $\lambda_1 \in S$ such that (a) holds for λ_1. Note that the function $\alpha \mapsto v(\alpha\lambda_1 + (1-\alpha)\lambda_0, \gamma)\colon [0,1] \to \mathbb{R}$ is continuous, and that $\rho(\alpha\lambda_1 + (1-\alpha)\lambda_0, \lambda_0) < \varepsilon(\hat{\gamma})$ for each $0 < \alpha < 1$, by definition of Huntingdon's metric. Therefore, by the definition of λ_0, there is a $\lambda_2 \in S$ such that (b) holds in addition to (a). Being non-decreasing and right-continuous, λ_2 is usc, so there is a sequence $\langle\lambda_k\rangle$ of continuous elements of Λ such that $\lambda_k \geq \lambda_2$ for each k and $\lambda_k(z) \to \lambda_2(z)$ for each $z \in [0, \bar{n}]$. Now there is a sequence $\langle\alpha_k\rangle$ of real numbers, with $\alpha_k \to 1$, such that $\int \alpha_k\lambda_k(z)\mathrm{d}\hat{\gamma}(z) = \int z\mathrm{d}\hat{\gamma}(z)$ for each k. By Lebesgue's dominated convergence theorem, we must have $v(\alpha_k\lambda_k, \gamma) \to v(\lambda_2, \gamma)$, so λ_k satisfies (a) if k is large enough. Moreover, viewing λ_2 and each λ_k as Borel measures on $[0, \bar{n}]$, we have $\lambda_k \to \lambda_2$ in the narrow topology (Billingsley (1968, p. 18)), so also (b) is satisfied by λ_k if k is large enough. Thus, for sufficiently large k, λ_k is in B and satisfies all of (a)-(c) above. □

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