Approximate Nash equilibrium under the single crossing conditions

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Abstract

We consider strategic games where strategy sets are linearly ordered while the preferences of the players are described by binary relations. All restrictions imposed on the preferences are satisfied in the case of epsilon-optimization of a bounded-above utility function. A Nash equilibrium exists and can be reached from any strategy profile after a finite number of best response improvements if the single crossing conditions hold w.r.t. pairs [one player's strategy, a profile of other players' strategies], and the preference relations are transitive. If, additionally, there are just two players, every best response improvement path reaches a Nash equilibrium after a finite number of steps. If each player is only affected by a linear combination of the strategies of others, the single crossing conditions hold w.r.t. pairs [one player's strategy, an aggregate of the strategies of others], and the preference relations are interval orders, then a Nash equilibrium exists and can be reached from any strategy profile with a finite best response path.

Key words: strong acyclicity; single crossing; Cournot tatonnement; Nash equilibrium; aggregative game

JEL Classification Number: C 72.

1 Introduction

When the existence of the maximum of a function to be maximized cannot be guaranteed, a usual practice is to switch to ε-optimization. However, the practice is not costless: we forfeit the ability to apply necessary conditions. Similarly, faced with possible non-existence of the best responses in a strategic game, we may assume that the players only ε-optimize, but then we may find the usual sufficient conditions for the existence of a Nash equilibrium inapplicable.

As an example relevant to this paper, let us consider games of strategic complementarity (Topkis, 1979; Vives, 1990; Milgrom and Roberts, 1990). Suppose each utility function is supermodular and bounded above in own choice, but may be discontinuous, hence need not attain a maximum. Even if the increasing differences condition holds, the ε-best response correspondence need not be ascending; it is only *weakly* ascending under these strong assumptions. The set of ε-best responses to a particular profile of strategies of other players need not be a lattice and need not be chain-complete, hence the existence of an increasing selection cannot be derived even from Veinott (1989, Theorem 3.2), so there is nothing to apply Tarski's fixed point theorem to.

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More sophisticated, in particular, ordinal techniques developed later (Milgrom and Shannon, 1994; Shannon, 1995; Athey, 2001; Quah, 2007; Quah and Strulovici, 2009; Reny, 2011) do not help in this situation. To the best of my knowledge, the previous literature contains no existence result for ε-Nash equilibria in games of strategic complementarity (to say nothing of strategic substitutability) where the existence of the best responses is not guaranteed.

Although this paper has been motivated by the problems with ε-Nash equilibrium, we actually work in a much broader context. Namely, we consider agents whose preferences are described by binary relations, on which any restrictions are imposed only when necessary. The only assumption made throughout is *strong acyclicity*, which ensures the existence of undominated alternatives. In the case of ε-optimization of a function bounded above, this assumption holds. We also assume that the strategy sets are ordered; in all the theorems, the order is linear.

Besides the existence of equilibria, we also consider adaptive ("best response") dynamics. Actually, we consider two different scenarios that coincide in the "standard" case of preferences described by utility functions. In the case of ε-optimization, the difference is whether to demand that the new, ε-optimal, strategy should be a noticeable (more than ε) improvement over the current strategy or not.

Theorem 2 shows that a Nash equilibrium exists and can be reached from any strategy profile after a finite number of best response improvements if the strategy sets are chains, the single crossing conditions hold w.r.t. pairs <one player's strategy, a profile of other players' strategies>, and the preference relations are transitive. If there are just two players, *every* best response improvement path reaches a Nash equilibrium after a finite number of steps (Theorem 3). [Theorems 2 and 3, respectively, from Kukushkin et al. (2005) are a bit stronger, but restricted to the "standard" case of a finite game with preferences described by utility functions.]

Theorem 1 establishes the existence of an increasing selection from the best response correspondence when both available choices and parameters form chains, and the preference relation is an interval order. That theorem is applicable to *aggregative games* (Novshek, 1985; Kukushkin, 1994, 2004, 2005; Dubey et al., 2006; Jensen, 2010). For instance, in Theorem 4, each player is only affected by a linear combination of the strategies of others with a symmetrical matrix of coefficients, the single crossing conditions hold w.r.t. pairs <one player's strategy, an aggregate of the strategies of others>, and the preference relations are interval orders. Then a Nash equilibrium exists and can be reached from any strategy profile with a finite best response path (possibly, with insignificant improvements along the way).

Section 2 introduces basic notions concerning preferences and choice. In Section 3, the single crossing conditions and Theorem 1 are formulated. In Section 4, our principal model is introduced: a strategic game where the preferences of each player are defined by a family of binary relations on own strategies, with the choices of others as parameters. Section 5 contains applications of Theorem 1 to aggregative games. Most proofs are deferred to Section 6; a few concluding remarks are in Section 7.

2 Preferences and choice

We start with, more or less, standard notions related to individual choice. Let the preferences of an agent over alternatives from a set X be described by a binary relation $\succ$. For every $Y \subseteq X$, we denote

$$M(Y, \succ) := \{x \in Y \mid \nexists\, y \in Y\, [y \succ x]\}, \tag{1}$$

the set of "optimal," or rather acceptable, choices from Y.

Remark. Throughout this paper, we are only interested in the choice (by each agent) from a single set. However, considering the (potential) choices by the same agent from subsets helps to clarify the relationships between various assumptions in the theorems to follow.

A binary relation $\succ$ is *strongly acyclic* if there exists no infinite sequence $\langle x^k \rangle_{k \in \mathbb{N}}$ such that $x^{k+1} \succ x^k$ for each k.

Proposition 1. *Let $\succ$ be a binary relation on a set X. Then $\succ$ has the property that $M(Y, \succ) \neq \emptyset$ whenever $X \supseteq Y \neq \emptyset$ if and only if it is strongly acyclic.*

A straightforward proof is omitted.

A binary relation $\succ$ has the *NM-property* on a subset $Y \subseteq X$ if

$$\forall x \in Y \setminus M(Y, \succ)\, \exists y \in M(Y, \succ)\, [y \succ x]. \tag{2}$$

Proposition 2. *Let $\succ$ be a binary relation on a set X. Then $\succ$ has the NM-property on every nonempty subset $Y \subseteq X$ if and only if it is strongly acyclic and transitive.*

A straightforward proof is omitted.

A binary relation $\succ$ has the *strong NM-property* on a subset $Y \subseteq X$ if

$$\forall \{x^0, \ldots, x^m\} \subseteq Y \setminus M(Y, \succ)\, \exists y \in M(Y, \succ)\, \forall k \in \{0, \ldots, m\}\, [y \succ x^k]. \tag{3}$$

An irreflexive relation $\succ$ is called an *interval order* if it satisfies the condition

$$\forall x, y, a, b \in X\, \big[\, [y \succ x \;\&\; a \succ b] \Rightarrow [y \succ b \text{ or } a \succ x]\, \big]. \tag{4}$$

Equivalently, $\succ$ is an interval order if and only if there are a chain L and two mappings $u^+, u^- : X \to L$ such that, for all $x, y \in X$,

$$u^+(x) \geq u^-(x); \quad y \succ x \iff u^-(y) > u^+(x).$$

Proposition 3. *Let $\succ$ be a binary relation on a set X. Then $\succ$ has the strong NM-property on every nonempty subset $Y \subseteq X$ if and only if it is a strongly acyclic interval order.*

A routine proof is given for completeness.

Proof. To prove the sufficiency, we assume $\{x^0, \dots, x^m\} \subseteq Y \setminus M(Y, \succ)$. When $m = 0$, we just invoke Proposition 2. Then we argue by induction. For $m > 0$, the induction hypothesis implies the existence of $y' \in M(Y, \succ)$ such that $y' \succ x^k$ for each $k = 0, \dots, m-1$; we also have $y'' \in M(Y, \succ)$ such that $y'' \succ x^m$. For each $k = 0, \dots, m-1$, we apply (4) to x^k, y', y'', x^m, obtaining that either $y' \succ x^m$ or $y'' \succ x^k$ for each $k = 0, \dots, m-1$. In either case, we are home.

Conversely, if (4) does not hold, we have $M(\{x, y, a, b\}, \succ) = \{y, a\}$, hence (3) does not hold for $Y = \{x, y, a, b\}$ and $\{x, b\} \subseteq Y \setminus M(Y, \succ)$. □

As an example, let $u \colon X \to \mathbb{R}$ be bounded above and $\varepsilon > 0$; let the preference relation be

$$y \succ x \rightleftharpoons u(y) > u(x) + \varepsilon. \tag{5}$$

It is easy to see that $\succ$ is a strongly acyclic interval order (actually, a *semiorder*). $M(X, \succ)$ consists of all ε-maxima of u on X.

A binary relation $\succ$ has the *revealed preference property* on a subset $Y \subseteq X$ if

$$\forall x, y \in Y \, [x \notin M(Y, \succ) \ni y \Rightarrow y \succ x]. \tag{6}$$

An *ordering* is an irreflexive, transitive, and *negatively transitive* ($z \not\succ y \not\succ x \Rightarrow z \not\succ x$) binary relation. Equivalently, $\succ$ is an ordering if and only if there are a chain L and a mapping $u \colon X \to L$ such that

$$y \succ x \iff u(y) > u(x) \tag{7}$$

holds for all $x, y \in X$.

Proposition 4. *Let $\succ$ be a binary relation on a set X. Then $\succ$ has both the revealed preference property and $M(Y, \succ) \neq \emptyset$ on every nonempty subset $Y \subseteq X$ if and only if it is a strongly acyclic ordering.*

A straightforward proof is omitted.

Most often, the preferences in game theory are described with a *utility function* $u \colon X \to \mathbb{R}$ satisfying (7) for all $x, y \in X$. Actually, real values as such are not necessary to derive the existence of a Nash equilibrium from Tarski's fixed point theorem and to obtain the usual monotone comparative statics results under strategic complementarity. What *is* needed for the standard techniques to work is just the revealed preference property (6) for $Y = X$.

Technically speaking, this paper is about how, and to what extent, (6) could be replaced with weaker properties (2) or (3).

3 Parametric preferences and single crossing conditions

To the end of the paper, the preferences are described by a family $\langle \succ^s \rangle_{s \in S}$ of binary relations, rather than a single relation, parameter s reflecting the choices of other agents. We define the *best response correspondence*:

$$\mathcal{R}(s) := M(X, \succ^s). \tag{8}$$

Henceforth, we always assume alternatives and parameters to be *partially ordered sets* (*posets*). A parametric family $\langle \succ^s \rangle_{s \in S}$ on X has the *single crossing* property if these conditions hold:

$$\forall x, y \in X \, \forall s, s' \in S \left[[s' > s \,\&\, y \succ^s x \,\&\, y > x] \Rightarrow y \succ^{s'} x \right]; \tag{9a}$$

$$\forall x, y \in X \, \forall s, s' \in S \left[[s' > s \,\&\, y \succ^{s'} x \,\&\, y < x] \Rightarrow y \succ^s x \right]. \tag{9b}$$

This definition is equivalent to Milgrom and Shannon's (1994) if every $\succ^s$ is an ordering represented by a numeric function.

For a family of preference relations defined by ε-optimization (5) with a parameter s in the function, both conditions (9) hold if $u(x, s)$ satisfies Topkis's (1979) *increasing differences* condition:

$$\forall x, y \in X \, \forall s, s' \in S \left[[s' \geq s \,\&\, y \geq x] \Rightarrow u(y, s') - u(x, s') \geq u(y, s) - u(x, s) \right]. \tag{10}$$

When X and S are chains, the condition is equivalent to the *supermodularity* of u (as a function on the lattice $X \times S$).

Given a parametric family $\langle \succ^s \rangle_{s \in S}$ on X, an *increasing selection* from $\mathcal{R}$ is a mapping $r \colon S \to X$ such that $r(s) \in \mathcal{R}(s)$ for every $s \in S$ and $r(s'') \geq r(s')$ whenever $s', s'' \in S$ and $s'' \geq s'$.

Theorem 1. *Let X and S be chains such that both $\min S$ and $\max S$ exist. Let a parametric family $\langle \succ^s \rangle_{s \in S}$ of strongly acyclic relations on X satisfy single crossing conditions (9). Let every $\succ^s$ ($s \in S$) have the strong NM-property on X. Then there exists an increasing selection r from $\mathcal{R}$ on S such that $r(S)$ is finite.*

The proof is deferred to Section 6.1.

Corollary. *Let X and S be chains such that both $\min S$ and $\max S$ exist. Let a parametric family $\langle \succ^s \rangle_{s \in S}$ of strongly acyclic interval orders on X satisfy single crossing conditions (9). Then there exists an increasing selection from $\mathcal{R}$ on S such that $r(S)$ is finite.*

4 Strategic games and Cournot tâtonnement

We define a *strategic game* in a way that is not quite standard. There is a finite set N of players and a set X_i of strategies for each $i \in N$. We denote $X_N := \prod_{i \in N} X_i$ and $X_{-i} := \prod_{j \neq i} X_j$. Each player i's *preferences* are described by a parametric family of binary relations $\succ_i^{x_{-i}}$ ($x_{-i} \in X_{-i}$) on X_i. Then we have the *best response correspondence* $\mathcal{R}_i$ for each player $i \in N$, defined by (8) with $S := X_{-i}$, $X := X_i$, and $\succ_i^{x_{-i}}$ as $\succ^s$.

Remark. The idea that the preferences of players in a *non-cooperative* game should be only about their own strategies (with the choices of others as parameters) was championed by Olga Bondareva (1979).

With every strategic game, a number of improvement relations on X_N are associated ($i \in N$, $y_N, x_N \in X_N$):

$$y_N \rhd_i^{\mathrm{Ind}} x_N \rightleftharpoons [y_{-i} = x_{-i} \,\&\, y_i \succ_i^{x_{-i}} x_i]; \tag{11a}$$

$$y_N \rhd^{\mathrm{Ind}} x_N \rightleftharpoons \exists i \in N \, [y_N \rhd_i^{\mathrm{Ind}} x_N] \tag{11b}$$

(*individual improvement* relation);

$$y_N \triangleright_i^{\mathrm{BR}} x_N \rightleftharpoons [y_N \triangleright_i^{\mathrm{Ind}} x_N \;\&\; y_i \in \mathcal{R}_i(x_{-i})]; \tag{12a}$$

$$y_N \triangleright^{\mathrm{BR}} x_N \rightleftharpoons \exists i \in N\, [y_N \triangleright_i^{\mathrm{BR}} x_N] \tag{12b}$$

(*best response improvement* relation);

$$y_N \triangleright_i^{[\mathrm{BR}]} x_N \rightleftharpoons [y_{-i} = x_{-i} \;\&\; x_i \notin \mathcal{R}_i(x_{-i}) \ni y_i]; \tag{13a}$$

$$y_N \triangleright^{[\mathrm{BR}]} x_N \rightleftharpoons \exists i \in N\, [y_N \triangleright_i^{[\mathrm{BR}]} x_N] \tag{13b}$$

(*best response "quasi-improvement"* relation).

If $y_N \triangleright_i^{\mathrm{BR}} x_N$, then both $y_N \triangleright_i^{\mathrm{Ind}} x_N$ and $y_N \triangleright_i^{[\mathrm{BR}]} x_N$. The relation $y_N \triangleright_i^{\mathrm{Ind}} x_N$ implies $y_N \triangleright_i^{\mathrm{BR}} x_N$ if $\succ_i^{x_{-i}}$ is "two-levelled," i.e., $z_i \succ_i^{x_{-i}} y_i \succ_i^{x_{-i}} x_i$ is impossible for any $x_{-i} \in X_{-i}$; $y_N \triangleright_i^{[\mathrm{BR}]} x_N$ implies $y_N \triangleright_i^{\mathrm{BR}} x_N$ if $\succ_i^{x_{-i}}$ has the revealed preference property (6) on X_i.

In the case of ε-optimization (5), the difference between $\triangleright_i^{\mathrm{BR}}$ and $\triangleright_i^{[\mathrm{BR}]}$ is whether to demand that the dominating strategy should, besides being ε-optimal, be a significant (more than ε) improvement over the "inferior" strategy, or not. If the preferences are even less similar to those described by a utility function (e.g., a Pareto combination of several utility functions), a quasi-improvement may be no improvement at all, hence associated dynamics may be of very limited interest.

By definition, a Nash equilibrium is a maximizer of $\triangleright^{\mathrm{Ind}}$, i.e., an $x_N \in X_N$ such that $y_N \triangleright^{\mathrm{Ind}} x_N$ is impossible for any $y_N \in X_N$. When each player's preferences are represented, in the sense of (7), with a utility function $u_i(x_N)$, this definition is equivalent to the standard one. It may be worthwhile to note that the question of, say, (in)efficiency of equilibria makes no sense in our framework. Assuming that the preferences are described by (5), our definition transforms into that of an ε-Nash equilibrium (w.r.t. the utility functions).

Every Nash equilibrium is a maximizer of $\triangleright^{\mathrm{BR}}$ and $\triangleright^{[\mathrm{BR}]}$; the converse need not be true. However, if $x_N^0 \in X_N$ is a maximizer of $\triangleright^{[\mathrm{BR}]}$ and $\mathcal{R}_i(x_{-i}^0) \neq \emptyset$ for each $i \in N$, then x_N^0 is a Nash equilibrium. Similarly, $x_N^0 \in X_N$ is a Nash equilibrium if it is a maximizer of $\triangleright^{\mathrm{BR}}$ and each $\succ_i^{x_{-i}^0}$ has the NM-property.

A (finite or infinite) sequence of strategy profiles $\langle x_N^k \rangle_{k\in\mathbb{N}}$ is a *best response* (*quasi*)-*improvement path* if $x_N^{k+1} \triangleright^{\mathrm{BR}} x_N^k$ ($x_N^{k+1} \triangleright^{[\mathrm{BR}]} x_N^k$) whenever $k \geq 0$ and x_N^{k+1} is defined. A game Γ has the *finite best response improvement property* (*FBRP*) if it admits no infinite best response improvement path. Γ has the *weak FBRP* if, for every $x_N^0 \in X_N$, there is a finite best response improvement path $x_N^0, \ldots, x_N^m$ such that x_N^m is a Nash equilibrium. The (*weak*) *finite best response quasi-improvement property*, (*weak*) *F[BR]P*, is defined similarly.

The notion of a *restricted FBRP*, a property intermediate between FBRP and weak FBRP, was defined in Kukushkin (2004, Section 6, p. 103). Here we employ a similar version of F[BR]P. Let, for every $i \in N$ and $x_{-i} \in X_{-i}$, a nonempty subset $\mathcal{R}_i^*(x_{-i}) \subseteq \mathcal{R}_i(x_{-i})$ be given. We define the corresponding *admissible best response quasi-improvement* relation by:

$$y_N \triangleright_i^{[\mathrm{BR}]^*} x_N \rightleftharpoons [y_{-i} = x_{-i} \;\&\; x_i \notin \mathcal{R}_i(x_{-i}) \;\&\; y_i \in \mathcal{R}_i^*(x_{-i})]; \tag{14a}$$

$$y_N \triangleright^{[\mathrm{BR}]^*} x_N \rightleftharpoons \exists i \in N\, [y_N \triangleright_i^{[\mathrm{BR}]^*} x_N]. \tag{14b}$$

An *admissible best response quasi-improvement path* is a (finite or infinite) sequence of strategy profiles $\langle x_N^k \rangle_{k\in\mathbb{N}}$ such that $x_N^{k+1} \triangleright^{[\mathrm{BR}]^*} x_N^k$ whenever $k \geq 0$ and x_N^{k+1} is defined.

A game Γ has a *restricted finite best response quasi-improvement property* (*restricted F[BR]P*) if there is a collection of admissible best response correspondences $\mathcal{R}_i^*$ such that Γ admits no infinite admissible best response improvement path. As noted above, whether the "quasi-improvement-related" dynamic properties deserve much interest by themselves depends on the degree of rationality of the preferences. Nonetheless, even the weakest of those properties, the weak F[BR]P, implies the existence of a Nash equilibrium in any case.

In all the theorems to follow, we consider games where strategy sets X_i are posets, actually, chains. Then both $X_N := \prod_{i\in N} X_i$ and all $X_{-i} := \prod_{j\neq i} X_j$ are posets too with the Cartesian product of the orders on components.

Theorem 2. *Let each X_i in a game Γ be a chain containing its maximum and minimum. Let the parametric family of preference relations of each player satisfy both conditions* (9) *and every $\succ_i^{x_{-i}}$ be strongly acyclic and transitive. Then Γ has the weak FBRP, i.e., a Nash equilibrium can be reached from any strategy profile after a finite number of best response improvements* (12).

The proof is deferred to Section 6.2.

Theorem 3. *Let each X_i in a two player game Γ be a chain containing its maximum and minimum. Let the parametric family of preference relations of each player satisfy both conditions* (9)*, and every $\succ_i^{x_{-i}}$ be strongly acyclic and have the NM-property. Then Γ has the FBRP, i.e.,* every *best response improvement path reaches a Nash equilibrium after a finite number of steps.*

The proof is deferred to Section 6.3.

There may be no FBRP if $n > 2$ (Kukushkin et al., 2005, Example 4), and no weak FBRP if X_i are not chains (Kukushkin et al., 2005, Example 2), even if the preferences are described with utility functions and all X_i are finite.

Interestingly, there are no restrictions on the chains X_i apart from the existence of their maxima and minima; those assumptions, however, are essential.

Example 1. Let $N := \{1, 2\}$, $X_1 := X_2 :=]0, 1]$ (with the natural order); let preferences of the players be defined by (5) with utility functions $u_1(x_1, x_2) := \min\{2x_1 - x_2, (x_2 - 2x_1)/x_2\}$ and $u_2(x_1, x_2) := \min\{2x_2 - x_1, (x_1 - 2x_2)/x_1\}$, and $0 < \varepsilon < 1$. All assumptions of Theorem 3 are satisfied except for the existence of $\min X_i$; single crossing conditions (9) hold because both utility functions are supermodular. There is no (ε-)Nash equilibrium: $x_2 \leq (1 + \varepsilon)x_1/2$ whenever $x_2 \in \mathcal{R}_2(x_1)$, while $x_1 \leq (1 + \varepsilon)x_2/2$ whenever $x_1 \in \mathcal{R}_1(x_2)$; therefore, there should hold $x_1 \leq (1 + \varepsilon)^2 x_1/4 < x_1$ at any equilibrium.

5 Aggregative games

An *aggregative game* is a strategic game where each X_i is a subset of $\mathbb{R}$ and there are mappings $\sigma_i \colon X_{-i} \to \mathbb{R}$ (*aggregation rules*) such that every preference relation $\succ_i^{x_{-i}}$ only depends on $\sigma_i(x_{-i})$. For each $i \in N$, we denote $S_i := \sigma_i(X_{-i}) \subset \mathbb{R}$ in this case, and use notation $\succ_i^{s_i}$ instead of $\succ_i^{x_{-i}}$ and

$\mathcal{R}_i(s_i) := M(X_i, \succ^{s_i})$. Assuming that each player's preferences satisfy single crossing conditions (9), we can apply Theorem 1.

If each σ_i is increasing in each x_j, then Theorem 2 is applicable, so aggregation becomes redundant in a sense. Otherwise, certain restrictions should be imposed on the aggregation rules; exactly what is needed remains unclear, but quite a number of sufficient conditions have been established for the standard case of a game with utility functions where the best responses exist everywhere (Novshek, 1985; Kukushkin, 1994, 2004, 2005; Dubey et al., 2006; Jensen, 2010).

Without addressing the general problem, we just present a couple of appropriate collections of aggregation rules, which seem most interesting from the viewpoint of possible applications.

Theorem 4. *Let Γ be an aggregative game where each strategy set contains its maximum and minimum, $\sigma_i(x_{-i}) = \sum_{j\neq i} a_{ij}x_j$ with $a_{ij} = a_{ji} \in \mathbb{R}$ whenever $j \neq i$, every $\succ_i^{s_i}$ is strongly acyclic and has the strong NM-property. Let the parametric family of preference relations of each player satisfy both conditions (9). Then Γ has a restricted F[BR]P.*

The proof is deferred to Section 6.4.

Corollary. *Let Γ be a strategic game with a strategy set $X_i \subset \mathbb{R}$ for each $i \in N$ and utility functions of the form $u_i(x_N) = U_i(x_i, \sum_{j\neq i} a_{ij}x_j)$, where $a_{ij} = a_{ji} \in \mathbb{R}$ whenever $j \neq i$. Let each X_i contain its maximum and minimum, each $U_i(\cdot, s_i)$ be bounded above, and the increasing differences condition (10) be satisfied by each U_i. Then Γ possesses an ε-Nash equilibrium for every $\varepsilon > 0$.*

If the preferences of the players are described just by utility functions, then $\triangleright^{\mathrm{BR}}$ and $\triangleright^{[\mathrm{BR}]}$ are equivalent, hence F[BR]P and FBRP become the same thing. However, the FBRP cannot be asserted in Theorem 4, even for a finite game with such nice preferences.

Example 2. Let $N := \{1, 2, 3\}$, $X_1 := \{0, 1, 2, 3, 4\}$, $X_2 := \{0, 1, 2, 3, 4, 5\}$, $X_3 := \{0, 1\}$; let the preferences of the players be defined by the utility functions $u_i(x_N) = U_i(x_i, -\sum_{j\neq i} x_j)$. Clearly, we have an aggregative game as in Theorem 4 with $a_{ij} = -1$, hence $S_1 = \{-6, -5, \ldots, 0\}$, $S_2 = \{-5, -4, \ldots, 0\}$, $S_3 = \{-9, -8, \ldots, 0\}$. Let the utilities be:

$$U_1: \begin{bmatrix} 0 & 2 & 2 & 2 & 2 & 2 & 4 \\ 1 & 3 & 3 & 3 & 3 & 3 & 3 \\ 1 & 3 & 3 & 3 & 3 & 3 & 3 \\ 1 & 3 & 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{bmatrix} \quad U_2: \begin{bmatrix} 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad U_3: \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

where own choice, x_i, is on the ordinates axis, and s_i (= minus the sum of the partners' choices), on the abscissae axis. Conditions (9), even (10), are easy to check. By Theorem 4, the game has a restricted F[BR]P; actually, even a restricted FBRP. However, it does not have the FBRP since there is a best response improvement cycle:

$$\begin{array}{ccccccc} (3,0,0) & \xrightarrow{1} & (4,0,0) & \xrightarrow{3} & (4,0,1) & \xrightarrow{1} & (1,0,1) \\ \uparrow{\scriptstyle 2} & & & & & & \downarrow{\scriptstyle 2} \\ (3,5,0) & \xleftarrow{1} & (0,5,0) & \xleftarrow{3} & (0,5,1) & \xleftarrow{1} & (1,5,1) \end{array}.$$

Remark. If we retained the same sets X_i and utilities U_i, but redefined σ_i, setting $a_{ij} := 1$ for all $i, j \in N$, $j \neq i$, then there would be no best response improvement cycle (Kukushkin, 2004, Theorem 1).

Theorem 5. *Let Γ be an aggregative game where each strategy set contains its maximum and minimum, $\sigma_i(x_{-i}) = \max_{j \in I(i)} x_j$ with $j \in I(i) \iff i \in I(j)$, every $\succ_i^{s_i}$ is strongly acyclic and has the strong NM-property. Let the parametric family of preference relations of each player satisfy both conditions (9). Then Γ has a restricted F[BR]P.*

Theorem 6. *Let Γ be an aggregative game where each strategy set contains its maximum and minimum, $\sigma_i(x_{-i}) = -\max_{j \in I(i)} x_j$ with $j \in I(i) \iff i \in I(j)$, every $\succ_i^{s_i}$ is strongly acyclic and has the strong NM-property. Let the parametric family of preference relations of each player satisfy both conditions (9). Then Γ has a restricted F[BR]P.*

Both proofs are deferred to Section 6.5.

Corollary. *Let Γ be a strategic game with a strategy set $X_i \subset \mathbb{R}$ for each $i \in N$ and utility functions of the form $u_i(x_N) = U_i(x_i, -\max_{j \in I(i)} x_j)$, where $j \in I(i) \iff i \in I(j)$. Let each X_i contain its maximum and minimum, each $U_i(\cdot, s_i)$ be bounded above, and the increasing differences condition (10) be satisfied by each U_i. Then Γ possesses an ε-Nash equilibrium for every $\varepsilon > 0$.*

Theorem 7. *Let Γ be an aggregative game where each strategy set contains its maximum and minimum, $\sigma_i(x_{-i}) = \min_{j \in I(i)} x_j$ with $j \in I(i) \iff i \in I(j)$, every $\succ_i^{s_i}$ is strongly acyclic and has the strong NM-property. Let the parametric family of preference relations of each player satisfy both conditions (9). Then Γ has a restricted F[BR]P.*

Theorem 8. *Let Γ be an aggregative game where each strategy set contains its maximum and minimum, $\sigma_i(x_{-i}) = -\min_{j \in I(i)} x_j$ with $j \in I(i) \iff i \in I(j)$, every $\succ_i^{s_i}$ is strongly acyclic and has the strong NM-property. Let the parametric family of preference relations of each player satisfy both conditions (9). Then Γ has a restricted F[BR]P.*

If a game satisfies the assumptions of Theorem 7 or 8, then it satisfies the assumptions of Theorem 5 or 6 after the order on each strategy set is reversed (each X_i is replaced with $-X_i$).

Naturally, Theorem 8 admits a corollary virtually identical to that of Theorem 6. Similar statements related to Theorems 5 and 7 immediately follow from Theorem 2.

6 Proofs

6.1 Proof of Theorem 1

We call a subset $S' \subseteq S$ an *interval* if $s \in S'$ whenever $s' < s < s''$ and $s', s'' \in S'$. The intersection of any number of intervals is an interval too.

Lemma 6.1.1. *Let a parametric family $\langle \succ^s \rangle_{s \in S}$ of binary relations on a chain X satisfy both conditions (9). Let every $\succ^s$ have the NM-property on X. Then the set $\{s \in S \mid x \in \mathcal{R}(s)\}$, for every $x \in X$, is an interval.*

Proof. Suppose the contrary: $s' < s < s''$ and $x \in \mathcal{R}(s') \cap \mathcal{R}(s'')$, but $x \notin \mathcal{R}(s)$. By (2), we can pick $x^* \in \mathcal{R}(s)$ such that $x^* \succ^s x$. If $x^* > x$, we have $x^* \succ^{s''} x$ by (9a), contradicting the assumed $x \in \mathcal{R}(s'')$. If $x^* < x$, we have $x^* \succ^{s'} x$ by (9b) with the same contradiction. $\square$

The key role is played by the following recursive definition of sequences $x^k \in X$, $s^k \in S$, and $S^k \subseteq S$ ($k \in \mathbb{N}$) such that:

$$s^k \in S^k; \tag{15a}$$

$$S^k \text{ is an interval;} \tag{15b}$$

$$\forall s \in S^k \left[x^k \in \mathcal{R}(s)\right]; \tag{15c}$$

$$\forall m < k \left[S^k \cap S^m = \emptyset\right]; \tag{15d}$$

$$\forall m < k \left[[s^k < s^m \Rightarrow x^k < x^m] \;\&\; [s^k > s^m \Rightarrow x^k > x^m]\right]; \tag{15e}$$

$$\forall m < k \left[x^k \succ^{s^k} x^m \;\text{ or }\; x^m \in \mathcal{R}(s^k)\right]; \tag{15f}$$

$$\forall s \in S \left[[x^k \in \mathcal{R}(s) \;\&\; s \notin S^k] \Rightarrow \exists m < k \left(s \in S^m \;\text{ or }\; s < s^m < s^k \;\text{ or }\; s^k < s^m < s\right)\right]. \tag{15g}$$

We start with an arbitrary $s^0 \in S$, pick $x^0 \in \mathcal{R}(s^0)$, and set $S^0 := \{s \in S \mid x^0 \in \mathcal{R}(s)\}$. Now (15a), (15c), and (15g) for $k = 0$ immediately follow from the definitions; (15b), from Lemma 6.1.1; (15d), (15e), and (15f) hold vacuously.

Let $k \in \mathbb{N} \setminus \{0\}$, and let x^m, s^m, S^m satisfying (15) have been defined for all $m < k$. We define $\bar{S}^k := \bigcup_{m<k} S^m$. For every $s \in \bar{S}^k$, there is a unique, by (15d), $\mu(s) < k$ such that $s \in S^{\mu(s)}$. By (15c), $r(s) := x^{\mu(s)}$ is a selection from $\mathcal{R}$ on $\bar{S}^k$. The conditions (15b) and (15e) imply that r is increasing. If $\bar{S}^k = S$, then we already have an increasing selection, so we stop the process.

Otherwise, we pick $s^k \in S \setminus \bar{S}^k$ arbitrarily and denote $K^- := \{m < k \mid s^m < s^k\}$, $K^+ := \{m < k \mid s^m > s^k\}$ $K^* := \{m < k \mid x^m \notin \mathcal{R}(s^k)\}$, $m^- := \operatorname{argmax}_{m \in K^-} s^m$, $m^+ := \operatorname{argmin}_{m \in K^+} s^m$, and $I := \{s \in S \mid s^{m^-} < s < s^{m^+}\}$. If one of $K^\pm$ is empty (both cannot be), the respective $m^\pm$ is left undefined, in which case $I := \{s \in S \mid s^{m^-} < s\}$ or $I := \{s \in S \mid s < s^{m^+}\}$.

By the strong NM-property of $\succ^{s^k}$, we can pick $x^k \in \mathcal{R}(s^k)$ such that $x^k \succ^{s^k} x^m$ for each $m \in K^*$, hence (15f) holds. Finally, we define $S^k := \{s \in S \setminus \bar{S}^k \mid x^k \in \mathcal{R}(s)\} \cap I$. Now the conditions (15a), (15c), and (15d) immediately follow from the definitions; (15b) and (15g), from Lemma 6.1.1.

Checking (15e) needs a bit more effort. If we assume that $x^{m^-} \in \mathcal{R}(s^k)$, then the condition (15g) for m^- and s^k implies the existence of $m < m^-$ such that $s^{m^-} < s^m < s^k$, contradicting the definition of m^-; therefore, $x^k \succ^{s^k} x^{m^-}$ by (15f). If $x^k < x^{m^-}$ then $x^k \succ^{s^{m^-}} x^{m^-}$ by (9b), contradicting (15c) for m^-. Therefore, $x^k > x^{m^-} \geq x^m$ for all $m \in K^-$. A dual argument shows that $x^k < x^{m^+} \leq x^m$ for all $m \in K^+$. Thus, (15e) holds.

To summarize, either we obtain an increasing selection on some step, or our sequences are defined [and satisfy (15)] for all $k \in \mathbb{N}$.

Lemma 6.1.2. *If conditions (15) hold for all $k \in \mathbb{N}$, then there exists an increasing sequence $\langle k_h \rangle_{h \in \mathbb{N}}$ such that s^{k_h} is either monotone increasing or monotone decreasing in h, and $x^{k_{h+1}} \succ^{s^{k_{h+1}}} x^{k_h}$ for each $h \in \mathbb{N}$.*

Proof. We denote $\mathbb{N}^{\downarrow}$, respectively, $\mathbb{N}^{\uparrow}$, the set of $k \in \mathbb{N}$ such that $s^m < s^k$, or $s^m > s^k$, holds for an infinite number of $m \in \mathbb{N}$. Clearly, $\mathbb{N} = \mathbb{N}^{\downarrow} \cup \mathbb{N}^{\uparrow}$; without restricting generality, $\mathbb{N}^{\downarrow} \neq \emptyset$. We consider two alternatives.

Let there exist $\min\{s^k \mid k \in \mathbb{N}^{\downarrow}\} = s^*$; then the set $\{m \in \mathbb{N} \mid s^m < s^k\}$ is finite for every $s^k < s^*$, hence the set $\{m \in \mathbb{N} \mid s^k < s^m < s^*\}$ is infinite. We define $k_0 := \min\{k \in \mathbb{N} \mid s^k < s^*\}$, and then recursively define k_{h+1} as the least $k \in \mathbb{N}$ for which $s^{k_h} < s^k < s^*$. The minimality of k_h ensures that $k_{h+1} > k_h$. Whenever $s^{k_h} < s^m < s^{k_{h+1}}$, we have $m > k_{h+1}$ by the same minimality; therefore, $x^{k_h} \notin \mathcal{R}(s^{k_{h+1}})$ by (15g), hence $x^{k_{h+1}} \succ^{s^{k_{h+1}}} x^{k_h}$ by (15f).

Let $\min\{s^k \mid k \in \mathbb{N}^{\downarrow}\}$ not exist; then the set $\{m \in \mathbb{N}^{\downarrow} \mid s^m < s^k\}$ is nonempty (actually, infinite) for every $k \in \mathbb{N}^{\downarrow}$. We set $k_0 := \min \mathbb{N}^{\downarrow}$, and then recursively define k_{h+1} as the least $k \in \mathbb{N}^{\downarrow}$ for which $s^k < s^{k_h}$. The minimality of k_h again ensures that $k_{h+1} > k_h$. Whenever $s^{k_{h+1}} < s^m < s^{k_h}$, we have $m \in \mathbb{N}^{\downarrow}$, hence $m > k_{h+1}$; therefore, $x^{k_h} \notin \mathcal{R}(s^{k_{h+1}})$ by (15g), hence $x^{k_{h+1}} \succ^{s^{k_{h+1}}} x^{k_h}$ by (15f). $\square$

The final step of the proof consists in showing that the existence of a sequence described in Lemma 6.1.2 contradicts the strong acyclicity assumption. If s^{k_h} is increasing, the relations $x^{k_{h+1}} \succ^{s^{k_{h+1}}} x^{k_h}$ "translate," by (9a), to $x^{k_{h+1}} \succ^{\max S} x^{k_h}$ for each $h \in \mathbb{N}$. If s^{k_h} is decreasing, we obtain $x^{k_{h+1}} \succ^{\min S} x^{k_h}$ for each $h \in \mathbb{N}$ by (9b).

6.2 Proof of Theorem 2

We define

$$X^{\uparrow} := \{x_N \in X_N \mid \forall i \in N \, \forall y_N \in X_N \, [y_N \triangleright_i^{\mathrm{BR}} x_N \Rightarrow y_i > x_i]\}. \tag{16}$$

Lemma 6.2.1. *If $x_N \in X^{\uparrow}$ and $y_N \triangleright^{\mathrm{BR}} x_N$, then $y_N \in X^{\uparrow}$ too.*

Proof. Let $y_N \triangleright_i^{\mathrm{BR}} x_N$; then $y_i > x_i$ since $x_N \in X^{\uparrow}$. Suppose, to the contrary, that there are $z_N \in X_N$ and $j \in N$ such that $z_N \triangleright_j^{\mathrm{BR}} y_N$ and $y_j > z_j$. Since $y_i \in \mathcal{R}_i(y_{-i})$, we have $j \neq i$, hence $z_{-j} = y_{-j} > x_{-j}$, hence $z_j \succ_j^{x_{-j}} x_j$ by (9b), hence $x_j \notin \mathcal{R}_j(x_{-j})$. Now we have $z_j \notin \mathcal{R}_j(x_{-j})$ because we would have $(z_j, x_{-j}) \triangleright_j^{\mathrm{BR}} x_N$ and $z_j < x_j[= y_j]$ otherwise, contradicting the assumption $x_N \in X^{\uparrow}$. Since $\succ_j^{x_{-j}}$ has NM-property, there is $z'_N \in X_N$ such that $z'_N \triangleright_j^{\mathrm{BR}} (z_j, x_{-j})$. Since $\succ_j^{x_{-j}}$ is transitive, we have $z'_N \triangleright_j^{\mathrm{BR}} x_N$ as well. Therefore, $z'_j > x_j$ because $x_N \in X^{\uparrow}$, hence $(z'_j, y_{-j}) \triangleright_j^{\mathrm{BR}} z_N$ by (9a), hence $z_j \notin \mathcal{R}_j(y_{-j})$, contradicting our assumption $z_N \triangleright_j^{\mathrm{BR}} y_N$. $\square$

If $x_N^0 \in X^{\uparrow}$, but is not an equilibrium, we pick an arbitrary $x_N^1 \in X_N$ such that $x_N^1 \triangleright^{\mathrm{BR}} x_N^0$; then $x_N^1 \in X^{\uparrow}$ by Lemma 6.2.1. Iterating this operation, we obtain a best response improvement path $\langle x_N^k \rangle_k$ such that $x_N^k \in X^{\uparrow}$ whenever x_N^k is defined. Besides, $x_i^{k+1} > x_i^k$ whenever $x_N^{k+1} \triangleright_i^{\mathrm{BR}} x_N^k$; by (9a), we have $x_i^{k+1} \succ_i^{\max X_{-i}} x_i^k$ for all such k. If the path is infinite, then we will have an infinite number of improvements for, at least, one i (actually, two), contradicting the assumed strong acyclicity. Therefore, it must stop at some stage, and that is only possible at an equilibrium.

If $x_N^0 \notin X^{\uparrow}$, we pick $i \in N$ and $x_N^1 \in X_N$ such that $x_N^1 \triangleright_i^{\mathrm{BR}} x_N^0$ and $x_i^1 < x_i^0$; if $x_N^1 \notin X^{\uparrow}$, we behave similarly. Iterating this operation as long as $x_N^k \notin X^{\uparrow}$, we obtain a best response improvement path $\langle x_N^k \rangle_k$ such that $x_i^{k+1} < x_i^k$ whenever $x_N^{k+1} \triangleright_i^{\mathrm{BR}} x_N^k$. The path cannot be infinite for the same (or rather dual) reason as in the previous paragraph. Once $x_N^k \in X^{\uparrow}$, we already know that an infinite best response improvement path is impossible.

6.3 Proof of Theorem 3

Without restricting generality, we may assume $N = \{1, 2\}$. Suppose to the contrary that $\langle x_N^k \rangle_{k \in \mathbb{N}}$ is an infinite best response improvement path. Since we could start the path anyplace, we may assume that, for all $k \in \mathbb{N}$,

$$x_1^{2k} \notin \mathcal{R}_1(x_2^{2k}) \ni x_1^{2k+1} = x_1^{2k+2}; \quad \mathcal{R}_2(x_1^{2k}) \ni x_2^{2k} = x_2^{2k+1} \notin \mathcal{R}_2(x_1^{2k+1}).$$

Again without restricting generality, we may assume $x_1^1 > x_1^0$. Now if we suppose that $x_2^2 < x_2^0$, then the relation $x_2^2 \succ_2^{x_1^1} x_2^0$ and condition (9b) would imply $x_2^2 \succ_2^{x_1^0} x_2^0$, contradicting our assumption $x_2^0 \in \mathcal{R}_2(x_1^0)$. A straightforward inductive argument shows that $x_2^{2k+2} > x_2^{2k+1}$ and $x_1^{2k+1} > x_1^{2k}$ for all $k \in \mathbb{N}$. Now the relation $x_2^{2k+2} \succ_2^{x_1^{2k+1}} x_2^{2k}$ and condition (9a) imply $x_2^{2k+2} \succ_2^{\max X_1} x_2^{2k}$ for all $k \in \mathbb{N}$, which fact contradicts the strong acyclicity of $\succ_2^{\max X_1}$.

6.4 Proof of Theorem 4

Since each X_i contains its maximum and minimum, the same holds for each S_i. Applying Theorem 1, we obtain an increasing selection $r_i \colon S_i \to X_i$ from the best responses such that $r_i(S_i)$ is finite for each $i \in N$. Now we define an *admissible best response quasi-improvement* relation by (14) with $\mathcal{R}_i^*(x_{-i}) := \{r_i(\sigma_i(x_{-i}))\}$.

We have to show the impossibility of an infinite admissible best response quasi-improvement path. Quite a number of auxiliary constructions are needed for that; they generally follow Dubey et al. (2006), who, in their turn, used a trick developed by Huang (2002) for different purposes. The finiteness of each $r_i(S_i)$ simplifies something; the absence of upper hemicontinuity demands more subtlety, and here we follow Kukushkin (2005).

For each $i \in N$, we denote $r_i(S_i) = \{x_i^1, \ldots, x_i^{m_i}\}$, assuming $x_i^{h+1} > x_i^h$ for all relevant h. For every $x_i \in X_i$, we define $\eta_i(x_i) \in \{0, 1, \ldots, m_i\}$ as the minimal h such that $x_i^{h+1} > x_i$; if no such h exists, we set $\eta_i(x_i) := m_i$. Then we pick a list of $s_i^h \in [\min S_i, \max S_i]$, $h = 1, \ldots, m_i - 1$, such that $r_i(s_i) < x_i^{h+1}$ whenever $s_i < s_i^h$ and $r_i(s_i) > x_i^h$ whenever $s_i > s_i^h$. (If S_i is an interval in $\mathbb{R}$, then each s_i^h is determined uniquely.) For technical convenience, we add to the list $s_i^0 := \min S_i$ and $s_i^{m_i} := \max S_i$, and denote $\Delta_i^h := s_i^h - s_i^{h-1}$ $[> 0]$ for $h = 1, \ldots, m_i$.

For every $x_N \in X_N$, we define a set $N^0(x_N) := \{i \in N \mid x_i \in r_i(S_i)\}$ and a function

$$P(x_N) := \sum_{i \in N} \Big[-x_i \cdot s_i^{\eta_i(x_i)} + \sum_{j \in N \colon j \neq i} \frac{1}{2} a_{ij} \cdot x_i \cdot x_j + \sum_{h=1}^{\eta_i(x_i)} x_i^h \cdot \Delta_i^h \Big]. \tag{17}$$

For each $i \in N$, we define a binary relation $\triangleright_i$ on $r_i(S_i)$ by setting (for each $h \in \{1, \ldots, m_i - 1\}$ such that $s_i^h \in S_i$) $x_i^{h+1} \triangleright_i x_i^h$ if $r_i(s_i^h) = x_i^{h+1}$, and $x_i^h \triangleright_i x_i^{h+1}$ if $r_i(s_i^h) = x_i^h$.

Lemma 6.4.1. *Let $i \in N$, $z_i, y_i, x_i \in r_i(S_i)$, and $z_i \triangleright_i y_i \triangleright_i x_i$. Then either $z_i > y_i > x_i$ or $z_i < y_i < x_i$.*

Immediately follows from the definitions.

Then we extend $\triangleright_i$ to the whole X_i, setting $y_i \triangleright_i x_i$ whenever $x_i \notin r_i(S_i) \ni y_i$, and define $\triangleright\!\!\triangleright_i$ as the transitive closure of $\triangleright_i$ on X_i.

Lemma 6.4.2. *Each relation $\triangleright\!\!\triangleright_i$ is irreflexive and transitive.*

Immediately follows from Lemma 6.4.1.

Finally, we define the potential:

$$y_N \ggcurly x_N \iff \big[N^0(y_N) \supset N^0(x_N) \;\text{ or }\; [N^0(y_N) = N^0(x_N) \;\&\; P(y_N) > P(x_N)] \;\text{ or }\; \big(N^0(y_N) = N^0(x_N) \;\&\; P(y_N) = P(x_N) \;\&\; \forall i \in N\,[y_i = x_i \;\text{ or }\; y_i \triangleright\!\!\triangleright_i x_i] \;\&\; \exists i \in N\,[y_i \triangleright\!\!\triangleright_i x_i]\big)\big]. \tag{18}$$

Lemma 6.4.3. *The relation $\ggcurly$ is irreflexive and transitive.*

Immediately follows from the definition.

Lemma 6.4.4. *If $y_N \triangleright_i^{[\mathrm{BR}]^*} x_N$, then $y_N \ggcurly x_N$.*

Proof. By definition, $y_i = r_i(\sigma_i(x_{-i}))$ and $y_{-i} = x_{-i}$, hence $i \in N^0(y_N)$. If $x_i \notin r_i(S_i)$, then we have $N^0(x_N) \subset N^0(y_N)$ since $y_j = x_j$ for all $j \neq i$; therefore, $y_N \ggcurly x_N$ by the first lexicographic component in (18). Otherwise, we have $N^0(x_N) = N^0(y_N)$; let us compare $P(y_N)$ and $P(x_N)$.

Let $y_i = x_i^{h''}$ and $x_i = x_i^{h'}$; we denote $\bar{s}_i = \sigma_i(x_{-i})$. Since $y_i = r_i(\bar{s}_i)$, we have $s_i^{h''-1} \le \bar{s}_i \le s_i^{h''}$. Since $\sum_{j\in N:\, j\neq i} a_{ij} \cdot x_j = \bar{s}_i$, we have

$$P(y_N) = y_i \cdot (\bar{s}_i - s_i^{h''}) + \sum_{h=1}^{h''} x_i^h \cdot \Delta_i^h + C(x_{-i}) = y_i \cdot (\bar{s}_i - s_i^{h''-1}) + \sum_{h=1}^{h''-1} x_i^h \cdot \Delta_i^h + C(x_{-i}). \tag{19}$$

Let us consider two alternatives.

A. Let $y_i > x_i$, i.e., $h'' > h'$. Similarly to (19), we have

$$P(x_N) = x_i\cdot(\bar{s}_i - s_i^{h'}) + \sum_{h=1}^{h'} x_i^h\cdot\Delta_i^h + C(x_{-i}) = x_i\cdot(\bar{s}_i - s_i^{h''-1}) + x_i\cdot(s_i^{h''-1} - s_i^{h'}) + \sum_{h=1}^{h'} x_i^h\cdot\Delta_i^h + C(x_{-i}). \tag{20}$$

Note that $C(x_{-i})$ is indeed the same.

Subtracting (20) from (19), we obtain

$$P(y_N) - P(x_N) = (y_i - x_i)\cdot(\bar{s}_i - s_i^{h''-1}) + \sum_{h=h'+1}^{h''-1} (x_i^h - x_i)\cdot\Delta_i^h. \tag{21}$$

The first term is non-negative; the second is the sum of strictly positive numbers. Thus, we have $P(y_N) \ge P(x_N)$, and an equality is only possible if $\bar{s}_i = s_i^{h''-1}$ and $h'' = h' + 1$ (so the sum is empty). The last situation means that $y_i \triangleright_i x_i$, hence $y_i \triangleright\!\!\triangleright_i x_i$ as well. In other words, $y_N \ggcurly x_N$ in any case.

B. Let $y_i < x_i$, i.e., $h'' < h'$. We ignore what follows the second equality sign in (19), and replace (20) with

$$P(x_N) = x_i \cdot (\bar{s}_i - s_i^{h'}) + \sum_{h=1}^{h'} x_i^h \cdot \Delta_i^h + C(x_{-i}) = x_i \cdot (\bar{s}_i - s_i^{h''}) + x_i \cdot (s_i^{h''} - s_i^{h'}) + \sum_{h=1}^{h'} x_i^h \cdot \Delta_i^h + C(x_{-i}). \quad (22)$$

Subtracting (22) from (19), we obtain

$$P(y_N) - P(x_N) = (x_i - y_i) \cdot (s_i^{h''} - \bar{s}_i) + \sum_{h=h''+1}^{h'} (x_i - x_i^h) \cdot \Delta_i^h. \quad (23)$$

Again, $P(y_N) \geq P(x_N)$, and an equality is only possible if $\bar{s}_i = s_i^{h''}$ and $h' = h'' + 1$, which means that $y_i \triangleright_i x_i$, hence $y_i \ggcurly_i x_i$ as well. In other words, $y_N \ggcurly x_N$ again. □

Finally, let $\langle x_N^k \rangle_{k=0,1,\ldots}$ be an admissible best response quasi-improvement path, i.e., whenever x_N^{k+1} is defined, there holds $x_N^{k+1} \triangleright_i^{[\mathrm{BR}]^*} x_N^k$ for some (unique) $i \in N$. By Lemma 6.4.4, we have $x_N^{k+1} \ggcurly x_N^k$. We set $N^* := \{i \in N \mid \exists k\, [x_i^{k+1} = r_i(x_{-i}^k)]\}$. If $i \in N \setminus N^*$, then x_i^k is the same for all k. Thus, our path moves upwards (in the sense of $\ggcurly$) in a finite set $\prod_{i \in N^*} r_i(S_i)$, hence it cannot be infinite.

6.5 Proof of Theorems 5 and 6

The two proofs are so similar that we do not have to distinguish almost to the very end.

Exactly as in the proof of Theorem 4, we apply Theorem 1, obtaining an increasing selection from the best responses $r_i \colon S_i \to X_i$ such that $r_i(S_i)$ is finite for each $i \in N$. Then we again define an *admissible best response quasi-improvement* relation by (14) with $\mathcal{R}_i^*(x_{-i}) := \{r_i(\sigma_i(x_{-i}))\}$.

Denoting $X_i^0 := r_i(S_i)$ and $\mathcal{X} := \bigcup_{i \in N} X_i^0 \subset \mathbb{R}$, we define strictly increasing mappings $\rho \colon \mathcal{X} \to \mathbb{N}$ by $\rho(x) := \#\{y \in \mathcal{X} \mid y < x\}$ (rank function) and $\varphi \colon \mathcal{X} \to \mathbb{R}$ by $\varphi(x) := n^{\rho(x)}$, where $n := \#N$.

Lemma 6.5.1. *Let $I \subset N$, $y_I, x_I \in X_I^0$, and $\max_{i \in I} y_i > \max_{i \in I} x_i$. Then $\sum_{i \in I} \varphi(y_i) > \sum_{i \in I} \varphi(x_i)$.*

Proof. Let $\max_{i \in I} x_i = \mu$. Then $\sum_{i \in I} \varphi(y_i) \geq n^{\mu+1}$, while $\sum_{i \in I} \varphi(x_i) \leq \#I \cdot n^{\mu} < n^{\mu+1}$. □

Supposing, to the contrary, that $\langle x_N^k \rangle_{k \in \mathbb{N}}$ is an infinite admissible best response quasi-improvement path, we denote $N^* := \{i \in N \mid \exists k \in \mathbb{N}\, [x_i^{k+1} = r_i(x_i^k)]\}$ and consider two alternatives.

A. Let $N^* = N$. We pick $\bar{k} \in \mathbb{N}$ such that $x_i^k \in X_i^0$ whenever $k \geq \bar{k}$. Clearly, $\langle x_N^k \rangle_{k \geq \bar{k}}$ is an infinite admissible best response quasi-improvement path in a subgame where each player is restricted to strategies from X_i^0. On the other hand, Lemma 6.5.1 implies that the subgame can be perceived as generated by the aggregation rules $\sigma_i^*(x_{-i}) := \sum_{j \in I(i)} x_j$ in the case of Theorem 5, or $\sigma_i^*(x_{-i}) := \sum_{j \in I(i)} (-x_j)$ in the case of Theorem 6. Therefore, it is covered by Theorem 4 in either case. The contradiction proves both theorems.

B. Let $N^* \subset N$. For each $i \in N \setminus N^*$, we have $x_i^k = x_i^0$ for all k. Therefore, we may consider a reduced game with the set of active players N^*, and each $i \in N \setminus N^*$ always choosing x_i^0. The game satisfies all assumptions of our theorem and $\langle x_N^k \rangle_{k \in \mathbb{N}}$ remains an infinite admissible best response quasi-improvement path; besides, Alternative A holds. Now the argument of the previous paragraph applies.

7 Concluding remarks

7.1. If S in Theorem 1 does not contain either minimum or maximum, then an increasing selection still exists, but $r(S)$ need not be finite, hence the proof of Theorem 4 is no longer valid. Moreover, Example 1 shows that a Nash equilibrium may fail to exist in this case. (A two-person game with scalar strategies is aggregative by our definition.)

7.2. If every $\mathcal{R}_i(x_{-i})$ is a singleton, then a restricted FBRP (F[BR]P) is equivalent to the FBRP (F[BR]P). If, additionally, every $\succ_i^{x_{-i}}$ has the NM-property, then the FBRP and F[BR]P are equivalent. If, additionally, $\#N = 2$, then the FBRP is equivalent to the weak FBRP.

7.3. It is instructive to compare our Theorems 2 and 3 with Theorems 2 and 3, respectively, from Kukushkin et al. (2005). The assertive parts are the same, whereas the assumptions are incomparable: we do not require X_i's to be finite, nor the preferences to be orderings, here; on the other hand, non-scalar strategies were allowed there (to a certain extent).

7.4. A closer look at the proofs of Theorems 2 and 3 shows that both remain valid if the maxima and minima exist in all X_i but one. It remains unclear whether the transitivity assumption in Theorem 2 could be weakened to the NM-property.

7.5. It remains unclear whether the assumption that both X and S are chains can be dropped or weakened in Theorem 1. From the game-theoretic viewpoint, however, the question does not seem pressing. The existence of an ε-Nash equilibrium in a game with increasing best responses does not require increasing selections (Theorem 2). If the best responses are, say, decreasing, then, indeed, all existence results in the literature need increasing selections, but they also need the assumption that each player is only affected by a scalar aggregate of the partners/rivals' choices. In principle, Theorem 1 of Jensen (2010) *could* be applicable to a game with non-scalar strategies, so an extension of our Theorem 1 to non-scalar X *could* be useful; however, no interesting example of such a game has emerged so far.

On the other hand, an extension of Theorem 1 to non-scalar S would allow us to add monotone comparative statics of equilibria to Theorems 4–8. Such an extension looks quite plausible (as long as X remains scalar), but there is no clear-cut theorem as yet.

7.6. Theorems 5 and 6 can be extended, with only minor changes in the proofs, to "lexicographic aggregation" such as the Leximax or Leximin orderings. Aggregation rules σ_i should then be mappings from X_{-i} to chains "longer" than $\mathbb{R}$.

7.7. The fact that we had to assume each strategy set in each theorem to be a chain is extremely irritating. Unfortunately, I have no idea at the moment whether and how the assumption could be dispensed with. On the other hand, a conjecture that (6) is indispensable when dealing with multi-dimensional strategies seems premature: there is no counterexample, nor even a hint wherefrom such an example could emerge.

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