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Abstract

In the paper, we are encouraged to investigate the effect of game structure imposed on the minimum-time needed to economic maturity in a dynamic macroeconomic model. Indeed, we have established a basic framework for the comparative study of the cooperative stochastic differential game and dynamic sequential game of economic maturity. Moreover, in a simple stochastic growth model, closed-form solution of the minimum-time needed to economic maturity has been derived with the explicit condition, under which it is confirmed that cooperation between the representative household and the self-interested politician will definitely lead us to much faster economic maturity than that of sequential action, supplied, too. Finally, our model supports the comparative study of the minimum-time needed to economic maturity under different political-institution constraints.

Keywords: Economic maturity; Minimum-time objective; Political economy; Sequential equilibrium; Cooperative stochastic differential game.

JEL classification: C70; D72; O11.

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1. Introduction

It is widely noted that institutional difference is one of the major differences between the developing economies and the developed economies. Usually, different institutional arrangements will form different incentive structures, induce different economic behaviors of the individuals, different fiscal policies of the government, and hence different speeds and levels of economic maturity. That is, different game structures lead to different institutional arrangements (e.g., North, 1990; Hurwicz, 1996; Williamson, 2000; Amable, 2003), hence producing different economic performances (see, North, 1994; Acemoglu et al., 2005a, 2005b). The major goal of the current exploration is to construct a basic framework to comparatively study the minimum-time needed to economic maturity under different game structures, i.e., dynamic sequential game and cooperative stochastic differential game. Indeed, we have derived closed-form solution of the minimum-time needed to economic maturity in a simple model of endogenous economic growth (e.g., Barro, 1990; Rebelo, 1991; Turnovsky, 2000; Aghion, 2004; Wälde, 2011; Dai, 2012), where competitive assumption is employed for the firm, endogenous savings rate is determined by the representative household and the goal of the self-interested politician is to choose a tax policy such that the utility from tax revenue, which can be viewed as the rent, is maximized. Leong and Huang (2010) confirm that uncertainty will produce more realistic solution than that of the deterministic case (see, Kaitala and Pohjola, 1990). We also consider a stochastic environment as in Merton (1975), i.e., the source of uncertainty is the population size. In addition to that, the explicit condition, under which cooperation between the representative household and the self-interested politician will lead to much faster economic maturity than that of sequential action, has been supplied for the first time. And in fact, the explicit condition is determined by the relevant model parameters, such as discount factor, technology parameter, depreciation factor, variance term, and the natural rate of population growth.

The current investigation focuses on the issue of economic maturity for any underdeveloped economy. We argue that the state of economic maturity can be interpreted as a Golden Age (e.g., Phelps, 1961) or a turnpike (see, McKenzie, 1963a; Dai, 2012) of the economy and the formal definition of the minimum-time needed to economic maturity has been stated using mathematical language. Indeed, we pay more attention to economic development rather than purely economic growth (see, Solow, 2003; Aghion, 2004). Moreover, it's easy to notice that our paper is a natural extension of the seminal and interesting paper of Kurz (1965), where optimal paths of capital accumulation under the minimum-time objective are thoroughly investigated. It is, nevertheless, worthwhile emphasizing that the minimum-time needed to economic maturity is endogenously determined in our model. And this would be regarded as an advantage of the optimal stopping theory used here.

Some seminal papers (see, Judd, 1985; Chamley, 1986; Phelan and Stacchetti, 2001, and among others) study dynamic optimal Ramsey taxation under the crucial assumption that taxes are set by benevolent governments. Nevertheless, in practice and also in line with the public choice theory (e.g., Buchanan and Tullock, 1962; Barro, 1973; Ferejohn, 1986), the politician's preferences may diverge from those of his constituents and that he may pursue his self-interest. Indeed, some existing literatures study the dynamic taxation under the assumption that taxes are decided by a self-interested politician. For example, Acemoglu et al. (2008, 2010, 2011) consider the case where the self-interested politicians have the power to set taxes and meanwhile the citizens can discipline politicians using elections or other means. Moreover, Acemoglu et al. (2008, 2010, 2011) analyze the political economy distortions by supplying that the politician has the power to allocate some of the tax revenue to himself as rents or government consumption, and also a formal politician utility, which is usually different from that of the individual or citizen, is supplied. Yared (2010) characterizes optimal tax policies in the presence of rent-seeking politicians whose utilities increase in rents, which are defined as excessive public spending with no social value, and also highlights how the incentives of rent-seeking politicians affect optimal policy prescriptions. As you can see below, we also suppose a self-interested politician in our model. And we further consider three types of self-interested politician, i.e., strongly self-interested politician, semi-strongly self-interested politician, and weakly self-interested politician, in order to sufficiently reflect different political institutions in reality. That is, we have provided a general framework for the study of the economic effect of the minimum-time needed to economic maturity with respect to different political institutions. Noting that North (1994) has emphasized that political and economic institutions are the underlying determinants of economic performance, hence our model provides us with a useful framework in which we can explicitly explore in which way and to what extent political institutions affect economic performance in a specific growth model.

Starting with time inconsistency being introduced by the seminal paper of Kydland and Prescott (1977), latter papers, such as Chari and Kehoe (1990, 1993), argue that fiscal-policy problems should be better studied as a dynamic game between the government and the households. For instance, in a repeated-game framework, Chari and Kehoe (1990) focus on sustainable plans characterized by symmetric perfect Bayesian equilibria. Similar to the sustainable equilibrium defined and analyzed by Chari and Kehoe (1990), Phelan and Stacchetti (2001) provide a formal definition of a sequential equilibrium for the dynamic policy game between the government and the households, and also develop a strategic dynamic programming method. Acemoglu et al. (2008, 2010, 2011) study dynamic taxation policy in the context of a dynamic game between a self-interested government and citizens, and characterize the best sub-game perfect equilibrium of this game from the viewpoint of the citizens. Yared (2010) considers an infinitely repeated game between citizens and rent-seeking politicians with double-sided lack of commitment in which reputation mechanism sustains efficient equilibrium policies. Also, Farhi and Werning (2008) study efficient nonlinear taxation in a dynamic game with political economy constraints and without commitment, it is revealed that reputational mechanism induces a trigger-strategy equilibrium, where a deviation would be followed by the worst possible continuation equilibrium. In our study, it is however illustrated that under certain conditions the unique sub-game perfect equilibrium may result in dynamic inefficiency when compared to the cooperative equilibrium, and also sub-game consistency, which is a much stronger concept than that of time consistency (see, Fischer, 1980; Klein et al., 2008) in some sense, has been demonstrated to be met for the current model by heavily employing the technique developed by Yeung and Petrosyan (2006). In other words, we employ backward induction principle to ensure time consistency in the dynamic sequential game while using sub-game consistency to ensure time consistency in the cooperative stochastic differential game. To sum up, the present model has supplied a useful framework for the comparative study of the dynamic sequential game emphasized by Phelan and Stacchetti (2001), Acemoglu et al. (2008, 2010, 2011) and the cooperative differential game studied by Kaitala and Pohjola (1990), Yeung and

Petrosyan (2006) and Leong and Huang (2010) in a stochastic growth model under political-economy constraint. And hence our study would be regarded as a natural extension of existing literatures.

The current paper proceeds as follows. Section 2 thoroughly introduces some basic definitions and the computation algorithm of the sequential-equilibrium minimum-time needed to economic maturity. Sections 3 and 4 derive the sequential-equilibrium and cooperative-equilibrium minimum-time needed to economic maturity, respectively, in a very general framework. In section 5, we discuss some specific examples with closed-form solutions derived by applying the general model established in sections 3 and 4. There is a brief concluding section.

2. Computation Algorithm of Sequential Equilibrium

2.1 Minimum-Time Needed to Economic Maturity

Suppose that we are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the optimization problem facing the economic agent is expressed as follows,

$$\max_{c(t)\geq 0} \mathbb{E}_s \left[\int_s^{\tau^*} e^{-\rho(t-s)} u\big(c(t)\big) dt + U^{\tau^*} \chi_{\{\tau^* < \infty\}} \right] \tag{1}$$

subject to the corresponding law of motion of capital accumulation with c denoting per capita consumption and with U^{τ^*} given by,

$$U^{\tau^*} \chi_{\{\tau^* < \infty\}} := \sup_{\tau \in \mathcal{T}} \mathbb{E}_s \left[e^{-\rho(\tau-s)} u\big(c(\tau)\big) \chi_{\{\tau < \infty\}} \right] \tag{2}$$

subject to the law of motion of capital accumulation and $\mathcal{T} := \{\mathcal{F} -\text{stopping times}\}$. Hence, we give,

Definition 1 (Minimum-Time Needed to Economic Maturity). The optimal stooping time τ^* determined by (2) defines a minimum-time needed to economic maturity in the sense of Radner Preference.

About the definition of Radner Preference, one may refer to the classical paper of Radner (1961). And one can easily notice that the specification in (2) efficiently captures the Ratchet effect emphasized by traditional consumption theory and hence we would also call it the "peak preference" with the purpose of reflecting the psychological effect in consumption. As is well-known, when discussing efficient capital accumulation, efficiency is usually defined with reference to the final state (see, Radner, 1961; Kurz, 1965; Dai, 2012) or the terminal stock (see, McKenzie, 1963b, 1976). In this paper, the terminal stock, equivalent to efficient capital accumulation in some sense, is endogenously determined as well as the minimum-time needed to economic maturity, which is an optimal stopping time that maximizes the final-state objective function of the economic agent, i.e., choosing a minimum time so as to maximize the discounted utility function, which, to some extent, resembles Kurz's (1965) specification, that is, minimizing the time needed to reach the state of economic maturity.

2.2 Types of Self-Interested Politician

Now, we introduce three types of self-interested politician according to the above specification. Firstly, we give the respective preference of the representative household and the self-interested politician as follows,

$$\max_{c^H(t)\geq 0} \mathbb{E}_s\left[\int_s^\tau e^{-\rho(t-s)}u^H\big(c^H(t)\big)dt + e^{-\rho(\tau-s)}u^H\big(c^H(\tau)\big)\chi_{\{\tau<\infty\}}\right] \tag{3}$$

and,

$$\max_{c^P(t)\geq 0} \mathbb{E}_s\left[\int_s^\tau e^{-\rho(t-s)}u^P\big(c^P(t)\big)dt + e^{-\rho(\tau-s)}u^P\big(c^P(\tau)\big)\chi_{\{\tau<\infty\}}\right] \tag{4}$$

for any $\tau \in \mathcal{T}$. Then, we give the following definitions,

Definition 2 (Self-Interested Politician). We call the politician the self-interested politician when he consumes the tax revenue as rent. That is, the politician is not benevolent in the usual sense.

Definition 3 (Strongly Self-Interested Politician). We call the politician the strongly self-interested politician when he is self-interested and also the minimum-time needed to economic maturity τ^* is determined by,

$$U^{P,\tau^*}\chi_{\{\tau^*<\infty\}} := \sup_{\tau\in\mathcal{T}} \mathbb{E}_s\left[e^{-\rho(\tau-s)}u^P\big(c^P(\tau)\big)\chi_{\{\tau<\infty\}}\right]$$

subject to the law of motion of capital accumulation.

Remark 2.1 In this case, the preference of the representative household is given by (3) while the preference of the politician given by,

$$\mathbb{E}_s\left[\int_s^\tau e^{-\rho(t-s)}u^P\big(c^P(t)\big)dt + U^{P,\tau}\chi_{\{\tau<\infty\}}\right]$$

Definition 4 (Semi-Strongly Self-Interested Politician). We call the politician the semi-strongly self-interested politician when he is self-interested and also the minimum-time needed to economic maturity $\tau^* := \tau^{P*} \wedge \tau^{H*}$ is determined by,

$$U^{P,\tau^{P*}}\chi_{\{\tau^{P*}<\infty\}} := \sup_{\tau\in\mathcal{T}} \mathbb{E}_s\left[e^{-\rho(\tau-s)}u^P\big(c^P(\tau)\big)\chi_{\{\tau<\infty\}}\right]$$

and,

$$U^{H,\tau^{H*}}\chi_{\{\tau^{H*}<\infty\}} := \sup_{\tau\in\mathcal{T}} \mathbb{E}_s\left[e^{-\rho(\tau-s)}u^H\left(c^H(\tau)\right)\chi_{\{\tau<\infty\}}\right]$$

subject to the corresponding law of motion of capital accumulation, respectively.

Remark 2.2 In this case, the preferences of the politician and the representative household are respectively given as follows,

$$\mathbb{E}_s\left[\int_s^\tau e^{-\rho(t-s)}u^P\left(c^P(t)\right)dt + U^{P,\tau}\chi_{\{\tau<\infty\}}\right]$$

and,

$$\mathbb{E}_s\left[\int_s^\tau e^{-\rho(t-s)}u^H\left(c^H(t)\right)dt + U^{H,\tau}\chi_{\{\tau<\infty\}}\right]$$

with $\tau \in \mathcal{T}$.

Definition 5 (Weakly Self-Interested Politician). We call the politician the weakly self-interested politician when he is self-interested and also the minimum-time needed to economic maturity τ^* is determined by,

$$U^{H,\tau^*}\chi_{\{\tau^*<\infty\}} := \sup_{\tau\in\mathcal{T}} \mathbb{E}_s\left[e^{-\rho(\tau-s)}u^H\left(c^H(\tau)\right)\chi_{\{\tau<\infty\}}\right]$$

subject to the law of motion of capital accumulation.

Remark 2.3 In this case, the preferences of the politician and the representative household are respectively given as follows,

$$\mathbb{E}_s\left[\int_s^\tau e^{-\rho(t-s)}u^P\left(c^P(t)\right)dt + e^{-\rho(\tau-s)}u^P\left(c^P(\tau)\right)\chi_{\{\tau<\infty\}}\right]$$

and,

$$\mathbb{E}_s\left[\int_s^\tau e^{-\rho(t-s)}u^H\left(c^H(t)\right)dt + U^{H,\tau}\chi_{\{\tau<\infty\}}\right]$$

with $\tau \in \mathcal{T}$.

2.3 Computation Algorithm

We introduce the following computation algorithm by employing the well-known backward induction principle,

Case 1. There is a strongly self-interested politician in the economy.

The economic agents will act sequentially and the order of action reads as follows,

i) The politician determines the minimum-time needed to economic maturity based upon any given taxation policy and any given consumption strategy of the representative household.

ii) Based on i), the politician chooses the taxation policy to maximize his welfare given any possible consumption strategy of the representative household.

iii) Based upon i) and ii), the representative household determines his optimal consumption.

And hence the corresponding computation algorithm is given by,

Computation Algorithm I.

Step 1. The representative household chooses his\her optimal consumption strategy given the taxation policy of the politician and the time horizon of the program.

Step 2. The self-interested politician chooses the taxation policy to maximize his welfare\utility given the optimal consumption strategy of the representative household derived in Step 1 and any possible time horizon of the program.

Step 3. Based upon the results derived in Steps 1 and 2, the minimum-time needed to economic maturity is established by the strongly self-interested politician.

Case 2. There is a semi-strongly self-interested politician in the economy.

Now, the order of action reads as follows,

i) The politician determines the minimum-time needed to economic maturity τ^{P*} based upon any given taxation policy of himself and any given consumption strategy of the representative household.

ii) The politician chooses the taxation policy to maximize his welfare based on i) and given any possible consumption strategy of the representative household.

iii) The representative household determines the minimum-time needed to economic maturity τ^{H*} based upon the taxation policy derived in ii).

iv) The representative household chooses his optimal consumption strategy based upon τ^{H*} and the taxation policy given by ii).

It follows from the well-known backward induction principle that the computation algorithm is given as follows,

Computation Algorithm II.

Step 1. The representative household chooses his\her optimal consumption strategy given the taxation policy of the politician and the time horizon of the program.

Step 2. Based upon Step 1, the minimum-time needed to economic maturity τ^{H*} is determined given any taxation policy of the politician.

Step 3. Based upon the result derived in Step 1, the taxation policy is determined by the semi-strongly self-interested politician.

Step 4. The minimum-time needed to economic maturity τ^{P*} is determined by the politician based upon Steps 1 and 3.

Step 5. Given the minimum-time needed to economic maturity τ^{H*} and τ^{P*} derived in Steps 2 and 4, respectively, we define the unique minimum-time needed to economic maturity as $\tau^* := \tau^{P*} \wedge \tau^{H*}$.

Case 3. There is a weakly self-interested politician in the economy.

Noting that the economic agents will act sequentially, then the order of action reads as follows,

i) The minimum-time needed to economic maturity is derived by the representative household for any given taxation policy of the politician and any given consumption strategy of the representative household.

ii) Based on i), the taxation policy is determined by the politician to maximize his welfare for any given consumption strategy of the representative household.

iii) Based upon i) and ii), the optimal consumption strategy is determined by the representative household.

So, by applying the backward induction principle, we get the following computation algorithm,

Computation Algorithm III.

Step 1. The representative household chooses his\her optimal consumption strategy based upon any given taxation policy of the politician and any given time horizon of the program.

Step 2. Provided the result derived in Step 1, the taxation policy is chosen by the self-interested politician to maximize his welfare\utility for any possible time horizon of the economy.

Step 3. The minimum-time needed to economic maturity is established by the representative household based upon the results given in Steps 1 and 2.

Therefore, we have stated all the computation algorithms for the sequential-equilibrium minimum-time needed to economic maturity for the above three cases corresponding to three types of self-interested politician.

3. Sequential-Equilibrium Economic Maturity

3.1 Basic Environment

We consider the following neoclassical production function,

$$Y(t) = F\big(K(t), L(t)\big) \tag{5}$$

which is a strictly concave function, and it also exhibits constant returns to scale effect with K denoting the aggregate capital stock and L representing the labor force or population size. Thus, the following law of motion of capital accumulation is derived,

$$\dot{K}(t) := \frac{dK}{dt} = \big(1-\tau_K(t)\big)\Big(F_K\big(K(t), L(t)\big) - \delta\Big)K(t) + \big(1-\tau_W(t)\big)F_L\big(K(t), L(t)\big)L(t) - \big(1+\tau_C(t)\big)C(t) \tag{6}$$

where δ, a given constant, denotes the depreciation factor, $F_K\big(K(t), L(t)\big) := \frac{\partial F}{\partial K}\big(K(t), L(t)\big)$, $F_L\big(K(t), L(t)\big) := \frac{\partial F}{\partial L}\big(K(t), L(t)\big)$, $C(t)$ stands for aggregate consumption, and $\tau_K(t)$, $\tau_W(t)$ and $\tau_C(t)$ represent capital-income tax rate, labor-income tax rate, and consumption tax rate, respectively, at period t.

Now, suppose that $(B(t), s \le t \le T)$ stands for a standard Brownian motion defined on the following filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{s \le t \le T}, \mathbb{P})$ with $\mathbb{F} := \{\mathcal{F}_t\}_{s \le t \le T}$ the $\mathbb{P}-$augmented filtration generated by $(B(t), s \le t \le T)$ with $\mathcal{F} := \mathcal{F}_T$ for $\forall T > 0$, that is, the underlying stochastic basis satisfies the well-known usual conditions. Then, based upon the given probability space and also in line with Merton (1975), we define,

$$dL(t) = nL(t)dt + \sigma L(t)dB(t) \tag{7}$$

subject to $B(s) = 0$ and $\sigma \in \mathbb{R}_0 := \mathbb{R} - \{0\}$, a constant. Thus, combining (6) with (7) and applying Itô's Rule leads us to,

$$\begin{aligned} dk(t) = \Big\{\Big[\big(1-\tau_K(t)\big)\Big(F_K\big(K(t), L(t)\big) - \delta\Big) + \sigma^2 - n\Big]k(t) + \big(1-\tau_W(t)\big)F_L\big(K(t), L(t)\big) \\ - \big(1+\tau_C(t)\big)c(t)\Big\}dt - \sigma k(t)dB(t) \end{aligned} \tag{8}$$

with initial value $k(s) := k_0 > 0$ and $k(t) := \frac{K(t)}{L(t)}$, $c(t) := \frac{C(t)}{L(t)}$ denoting the capital-labor ratio and per capita consumption, respectively, at time t.

So, based upon (8), we give the following differential operator for the new process $\zeta(t)$ $:= (t - s, k(t))$,

$$\mathcal{A}\phi(\zeta_0) := \frac{\partial\phi}{\partial s}(\zeta_0)$$

$$+\{[(1-\tau_K(0))(F_K(K(0),L(0))-\delta)+\sigma^2-n]k_0+(1-\tau_W(0))F_L(K(0),L(0))$$

$$-(1+\tau_C(0))c(0)\}\frac{\partial\phi}{\partial k_0}(\zeta_0)+\frac{1}{2}\sigma^2 k_0^2\frac{\partial^2\phi}{\partial k_0^2}(\zeta_0) \tag{9}$$

for $\zeta_0 := (0, k_0)$ and $\forall\phi \in C_0^2(\mathbb{R}\times\mathbb{R}_+)$.

Noting that both continuity and differentiability are neighborhood properties, we hence fix a domain D in $\mathbb{R}\times\mathbb{R}_+$ and the probability law of $\zeta(t)$ starting at $\zeta_0 := (0, k_0)$ for $t = s$ is (with slight abuse of notation) also denoted by $\mathbb{P}^{\zeta_0}$. And define,

$$\tau_D := \inf\{t > s;\ \zeta(t) \notin D\}$$

Definition 6 (Regular Boundary). Let ∂D denote the boundary of the domain D, a point $\zeta_0 \in \partial D$ is called regular for D (w.r.t. $\zeta(t)$) if $\mathbb{P}^{\zeta_0}(\tau_D = s) = 1$.

This definition implies that a.a. paths of $\zeta(t)$ starting from ζ_0 leave D immediately.

Assumption 1. *Without loss of generality, D is chosen such that τ_D is sufficiently large a.s. $\mathbb{P}^{\zeta_0}$.*

3.2 Sequential-Equilibrium Minimum-Time Needed to Economic Maturity

Case 1. Suppose that there is a strongly self-interested politician in the economy.

Applying Computation Algorithm I, we then obtain,

Problem 1. It is assumed that the economy consists of $L(t)$ identical individuals, each of whom possesses perfect foresight in period t. Thus, the optimization problem facing the representative household is expressed as follows,

$$\max_{c(t)\geq 0} J^c(\zeta_0) := \mathbb{E}_s\left[\int_s^{\tau^*} e^{-\rho(t-s)}u^H(c(t))dt + e^{-\rho(\tau^*-s)}u^H(c(\tau^*))\chi_{\{\tau^*<\infty\}}\right]$$

subject to (8) with $\mathbb{E}_s$ denoting the expectation operator depending on $\mathcal{F}_s$ and $u^H(\cdot)$ the strictly increasing, strictly concave instantaneous utility function of per capita consumption with the well-known Inada conditions satisfied.

So, we get,

Theorem 1 (Necessity). *Define,*

$$\psi^H(\zeta_0) = \sup\{J^c(\zeta_0); c = c(\zeta) Markov\ control\}$$

Suppose that $\psi^H \in C^2(D) \cap C(\overline{D})$ satisfies,

$$\mathbb{E}_s\left[\int_s^{\tau^*} |\mathcal{A}\psi^H(\zeta(t))|dt + |\psi^H(\zeta(\tau^*))|\right] < \infty$$

for all bounded stopping times $\tau^* \leq \tau_D$. *Moreover, suppose that an optimal Markov control* c^* *exists and that* ∂D *is regular in the sense of Definition 6 for* $\zeta(t)$. *Then,*

$$\sup_{c\geq 0}\{e^{\rho s} u^H(c) + \mathcal{A}\psi^H(\zeta_0)\} = 0$$

for all $\zeta_0 \in D$, *and,*

$$\psi^H(\zeta_0) = e^{\rho s} u^H(c)$$

for all $\zeta_0 \in \partial D$. *In other words, the optimal consumption* c^* *meets,*

$$e^{\rho s} u^H(c^*) + \mathcal{A}J^{c^*}(\zeta_0) = 0$$

for all $\zeta_0 \in D$.

Proof. This is a direct application of the Theorem of Hamilton-Jacobi-Bellman (HJB) Equation (see, Øksendal, 2003). ■

Theorem 2 (Sufficiency). *Let* φ^H *be a function in* $C^2(D) \cap C(\overline{D})$ *such that for all* $c \geq 0$,

$$e^{\rho s} u^H(c) + \mathcal{A}\varphi^H(\zeta_0) \leq 0$$

for all $\zeta_0 \in D$ *with boundary values,*

$$\lim_{t\to\tau_D} \varphi^H(\zeta(t)) = e^{-\rho(\tau_D - s)} u^H\big(c(\tau_D)\big)\chi_{\{\tau_D<\infty\}}$$

a.s. $\mathbb{P}^{\zeta_0}$, *and such that,*

$$\{\varphi^{H-}(\zeta(\tau^*));\ \tau^* stopping\ time\ with\ \tau^* \leq \tau_D\}$$

is uniformly $\mathbb{P}^{\zeta_0}$ *−integrable for all Markov controls* c *and all* $\zeta_0 \in D$. *Then,*

$$\varphi^H(\zeta_0) \geq J^c(\zeta_0)$$

for all Markov controls c *and all* $\zeta_0 \in D$. *Moreover, if for each* $\zeta_0 \in D$ *we have found* $c^*(\zeta_0)$ *such that,*

$$e^{\rho s} u^H\big(c^*(\zeta_0)\big) + \mathcal{A}\varphi^H(\zeta_0) = 0$$

and,

$$\left\{\varphi^{H,c^*}(\zeta(\tau^*));\ \tau^* stopping\ time\ with\ \tau^* \leq \tau_D\right\}$$

is uniformly $\mathbb{P}^{\zeta_0}$ *−integrable for all* $\zeta_0 \in D$. *Then,* $c^* = c^*(\zeta)$ *is a Markov control such that,*

$$\varphi^H(\zeta_0) = J^{c^*}(\zeta_0)$$

and hence if c^* *is admissible then* c^* *must be an optimal control and* $\varphi^H(\zeta_0) = \psi^H(\zeta_0)$*, which appears in Theorem 1.*

Proof. A canonical application of the Verification Theorem of HJB Equation (see, Øksendal, 2003) shows the desired assertion. ■

Some papers such as Karatzas and Wang (2000), Jeanblanc et al. (2004), and also the textbook of Øksendal and Sulem (2005) study utility maximization with discretionary stopping. Instead of deriving the optimal stopping time and the optimal controls simultaneously, Theorems 1 and 2 establish optimal consumption for any given stopping time based upon our Computation Algorithm defined in Section 2. In other words, the dynamic sequential game structure between the representative household and the self-interested politician will naturally make the corresponding computation of the optimal controls much easier. And this would be regarded as a byproduct of the dynamic sequential game discussed here. Moreover, it would be interesting to notice that optimal controls indeed interact with each other when the economic agents are faced with various types of decisions, i.e., optimal stopping time and optimal consumption appear in Theorems 1 and 2 can be regarded as totally different control variables in some sense. Thus, in contrast to the traditional consumption theory, Theorems 1 and 2 show us that optimal consumption will endogenously affect the underlying minimum-time needed to economic maturity on the one hand, and on the other hand, the minimum-time needed to economic maturity will in turn constraint the choice of optimal consumption behavior as a stochastic boundary condition in the corresponding optimization problem facing the representative household. And this would be interpreted as the new characteristic of Theorems 1 and 2 when compared to existing papers focusing on optimal consumption theory.

Thus, in what follows, we substitute c^* into (9) and we will use $\mathcal{A}^{c^*}\phi(\zeta_0)$ instead of $\mathcal{A}\phi(\zeta_0)$ for all $\zeta_0 \in D$. And also (8) would be rewritten as follows,

$$dk(t) = \Big\{\big[\big(1-\tau_K(t)\big)\big(F_K\big(K(t),L(t)\big)-\delta\big)+\sigma^2-n\big]k(t)+\big(1-\tau_W(t)\big)F_L\big(K(t),L(t)\big)$$
$$-\big(1+\tau_C(t)\big)c^*(t)\Big\}dt-\sigma k(t)dB(t) \tag{10}$$

Now, the optimization problem facing the self-interested politician can be expressed as follows,

Problem 2. Here, we particularly consider the taxation-revenue consumption per capita for the politician. That is, the self-interested politician faces the following optimization problem,

$$\max_{\tau_K(t),\tau_W(t),\tau_C(t)} J^{(\tau_K,\tau_W,\tau_C)}(\zeta_0)$$

$$:= \mathbb{E}_s\left[\int_s^{\tau^*} e^{-\rho(t-s)} u^P\left(\tau_K(t)\big(F_K\big(K(t),L(t)\big)-\delta\big)k(t), \tau_W(t)F_L\big(K(t),L(t)\big), \tau_C(t)c^*(t)\right)dt \right.$$

$$\left. + U^{P,\tau^*}(\zeta_0)\chi_{\{\tau^*<\infty\}}\right]$$

subject to (10) with $\mathbb{E}_s$ denoting the expectation operator depending on $\mathcal{F}_s$ and $u^P(\cdot,\cdot,\cdot)$ the smooth and increasing instantaneous utility function. Indeed, the specification of u^P can efficiently reflect the type of politician in the sense of preference, i.e., risk-aversion politician, risk-neutral politician, and risk-preference politician.

So, quite similar to Theorems 1 and 2, we obtain,

Theorem 3 (Necessity). *Define,*

$$\psi^P(\zeta_0) = \sup\{J^{(\tau_K,\tau_W,\tau_C)}(\zeta_0); \tau_K = \tau_K(\zeta), \tau_W = \tau_W(\zeta), \tau_C = \tau_C(\zeta) Markov\ controls\}$$

Suppose that $\psi^P \in C^2(D) \cap C(\overline{D})$ *satisfies,*

$$\mathbb{E}_s\left[\int_s^{\tau^*} \left|\mathcal{A}^{c^*}\psi^P(\zeta(t))\right|dt + \left|U^{P,\tau^*}(\zeta_0)\chi_{\{\tau^*<\infty\}}\right|\right] < \infty$$

for all bounded stopping times $\tau^* \le \tau_D$. *Moreover, suppose that an optimal Markov control* $(\tau_K^*, \tau_W^*, \tau_C^*)$ *exists and that* ∂D *is regular in the sense of Definition 6 for* $\zeta(t)$. *Then,*

$$\sup_{\tau_K,\tau_W,\tau_C}\left\{e^{\rho s}u^P(\tau_K(F_K(K,L)-\delta)k_0, \tau_W F_L(K,L), \tau_C c^*) + \mathcal{A}^{c^*}\psi^P(\zeta_0)\right\} = 0$$

for all $\zeta_0 \in D$, *and,*

$$\psi^P(\zeta_0) = U^{P,\tau^*}(\zeta_0) \tag{11}$$

for all $\zeta_0 \in \partial D$. *In other words, the optimal control* $(\tau_K^*, \tau_W^*, \tau_C^*)$ *fulfills,*

$$e^{\rho s}u^P(\tau_K^*(F_K(K,L)-\delta)k_0, \tau_W^* F_L(K,L), \tau_C^* c^*) + \mathcal{A}^{c^*}J^{(\tau_K^*,\tau_W^*,\tau_C^*)}(\zeta_0) = 0$$

for all $\zeta_0 \in D$.

And also,

Theorem 4 (Sufficiency). *Let* φ^P *be a function in* $C^2(D) \cap C(\overline{D})$ *such that for all* (τ_K, τ_W, τ_C),

$$e^{\rho s}u^P(\tau_K(F_K(K,L)-\delta)k_0, \tau_W F_L(K,L), \tau_C c^*) + \mathcal{A}^{c^*}\varphi^P(\zeta_0) \le 0$$

for all $\zeta_0 \in D$ *with boundary values,*

$$\lim_{t \to \tau_D} \varphi^P(\zeta(t)) = U^{P,\tau_D}(\zeta_0)\chi_{\{\tau_D<\infty\}}$$

a.s. $\mathbb{P}^{\zeta_0}$*, and such that,*

$$\{\varphi^{P-}(\zeta(\tau^*));\ \tau^* stopping\ time\ with\ \tau^* \le \tau_D\}$$

is uniformly $\mathbb{P}^{\zeta_0}$ *−integrable for all Markov controls* (τ_K, τ_W, τ_C) *and all* $\zeta_0 \in D$*. Then,*

$$\varphi^P(\zeta_0) \ge J^{(\tau_K,\tau_W,\tau_C)}(\zeta_0)$$

for all Markov controls (τ_K, τ_W, τ_C) *and all* $\zeta_0 \in D$*. Moreover, if for each* $\zeta_0 \in D$ *we have found that* $\left(\tau_K^*(\zeta_0), \tau_W^*(\zeta_0), \tau_C^*(\zeta_0)\right)$ *such that,*

$$e^{\rho s} u^P\left(\tau_K^*(\zeta_0)(F_K(K,L)-\delta)k_0, \tau_W^*(\zeta_0)F_L(K,L), \tau_C^*(\zeta_0)c^*(\zeta_0)\right) + \mathcal{A}^{c^*}\varphi^P(\zeta_0) = 0$$

and,

$$\left\{\varphi^{P,\left(\tau_K^*,\tau_W^*,\tau_C^*\right)}(\zeta(\tau^*));\ \tau^* stopping\ time\ with\ \tau^* \le \tau_D\right\}$$

is uniformly $\mathbb{P}^{\zeta_0}$ *−integrable for all* $\zeta_0 \in D$*. Then,* $(\tau_K^*, \tau_W^*, \tau_C^*) = \left(\tau_K^*(\zeta), \tau_W^*(\zeta), \tau_C^*(\zeta)\right)$ *is a Markov control such that,*

$$\varphi^P(\zeta_0) = J^{\left(\tau_K^*,\tau_W^*,\tau_C^*\right)}(\zeta_0)$$

and hence if $(\tau_K^*, \tau_W^*, \tau_C^*)$ *is admissible then* $(\tau_K^*, \tau_W^*, \tau_C^*)$ *must be an optimal control and* $\varphi^P(\zeta_0) = \psi^P(\zeta_0)$*, which appears in Theorem 3.*

Many existing literatures (e.g., Chamley, 1986; Jones et al., 1993; Phelan and Stacchetti, 2001; Kocherlakota, 2005; Acemoglu et al., 2011, and among others) focusing on taxation theory build up discrete-time models with exogenously prescribed time horizon. And some seminal papers (see, Chamley, 1986; Jones et al., 1993; Acemoglu et al., 2011) would heavily depend on the existence of the long-run steady state of the economy while Theorems 3 and 4 holding along the whole path of capital accumulation with the tax rates exhibiting Markov properties. And also the time horizon of the planning problem is endogenously determined in our model. That is to say, Theorems 3 and 4 show us formulas characterizing the taxation rates under political-economy constraint and also for very general preference functions, technology functions and endogenous time horizon. Generally, economic intuition will lead us to investigate how tax rates would affect the equilibrium minimum-time needed to economic maturity. In other words, we are usually inclined to focus on the policy effect of the government imposed on the equilibrium minimum-time needed to economic maturity. The results presented in Theorems 3 and 4, however, show us the inverse effect, i.e., the equilibrium minimum-time needed to economic maturity, characterized via a stochastic stopping time, as a stochastic boundary condition will also affect the equilibrium (in the sense of sub-game perfect) choice of tax rates of the self-interested politician. And hence such kind of interaction would shed some new insights into taxation theory from the viewpoint of economic development.

Hence, in what follows, we use the characteristic operator $\mathcal{A}^{c^*,(\tau_K^*,\tau_W^*,\tau_C^*)}\phi(\zeta_0)$ instead of $\mathcal{A}^{c^*}\phi(\zeta_0)$ for all $\zeta_0 \in D$. Also, inserting $(\tau_K^*, \tau_W^*, \tau_C^*)$ into (10) produces,

$$dk(t) = \{[(1-\tau_K^*(t))(F_K(K(t),L(t)) - \delta) + \sigma^2 - n]k(t) + (1-\tau_W^*(t))F_L(K(t),L(t)) - (1+\tau_C^*(t))c^*(t)\}dt - \sigma k(t)dB(t) \tag{12}$$

Thus, we can give,

Problem 3. Let $\mathcal{T}$ denote the set of all $\mathcal{F}-$stopping times $\tau \le \tau_D$. Consider the following problem facing the self-interested politician,

$$U^{P,\tau^*}(\zeta_0)\chi_{\{\tau^*<\infty\}} := \sup_{\tau\in\mathcal{T}} \mathbb{E}_s\left[e^{-\rho(\tau-s)}u^P\left(\tau_K^*(\tau)(F_K(K(\tau),L(\tau)) - \delta)k(\tau), \tau_W^*(\tau)F_L(K(\tau),L(\tau)), \tau_C^*(\tau)c^*(\tau)\right)\chi_{\{\tau<\infty\}}\right] := \sup_{\tau\in\mathcal{T}}\mathbb{E}_s\left[e^{\rho s}\tilde{u}^P(\zeta(\tau))\chi_{\{\tau<\infty\}}\right]$$

subject to (12).

It follows from Problem 3 that we have extended the concept of self-interested politician widely used by Acemoglu et al. (2008, 2010, 2011) and Yared (2010), and among others. Since the major issue of the present exploration is to compute the minimum-time needed to economic maturity for underdeveloped economies, the strongly self-interested politician rather than the representative household will determine the optimal stopping time. That is, the corresponding minimum-time needed to economic maturity only takes into account the utility or welfare of the self-interested politician. Indeed, this specification reflects certain type of political institutional arrangement of planning economies in reality. In addition, it is easily seen that the specification in Problem 3 is totally different from that in Dai (2012), where there is a benevolent government in the underlying economy. And it is insisted that such kind of difference indeed reflects different institutional arrangements in reality. For example, in many planning economies, it is the politician's or the government's interests instead of the households' interests that will determine the long-run economic development policy, i.e., the minimum-time needed to economic maturity. Obviously, such kind of institutional arrangement will induce an incentive structure among the economic agents leading to very poor economic performance, especially in the long run. Undoubtedly, both Dai (2012) and Problem 3 just consider special or extreme cases. Nevertheless, what's the corresponding lesson? For underdeveloped economies, in order to promote long-run and sustainable economic development, good institutions such as democratic institutions and market-economy institutions in Western world should be established first with the purpose of endogenously producing efficient incentive structure in the economy. In other words, the corresponding political and economic institutions should play a quite positive role in increasing the encompassing interests (see, Olson, 2000) between the politician and the household.

We then obtain by solving Problem 3,

Theorem 5 (Sequential-Equilibrium Minimum-Time Needed to Economic Maturity: Existence).

a) Suppose that we can find a function $\phi^P : \overline{D} \to \mathbb{R}$ *such that,*

(i) $\phi^P \in C^1(D) \cap C(\overline{D})$

(ii) $\phi^P \geq e^{\rho s}\tilde{u}^P$ *on* D *and* $\lim_{t \to \tau_D} \phi^P(\zeta(t)) = e^{\rho s}\tilde{u}^P(\zeta(\tau_D))\chi_{\{\tau_D < \infty\}}$ *a.s.* $\mathbb{P}^{\zeta_0}$

Define $G := \{\zeta_0 \in D;\ \phi^P(\zeta_0) > e^{\rho s}\tilde{u}^P(\zeta_0)\}$ *and suppose* $\zeta(t)$ *spends 0 time on* ∂G *a.s.* $\mathbb{P}^{\zeta_0}$*, i.e.,*

(iii) $\mathbb{E}_{\zeta_0}\left[\int_s^{\tau_D} \chi_{\partial G}(\zeta(t))dt\right] = 0$ *for all* $\zeta_0 \in D$*, and suppose that,*

(iv) ∂G *is a Lipschitz surface.*

Moreover, suppose the following conditions:

(v) $\phi^P \in C^2(D \setminus \partial G)$ *and the second order derivatives of* ϕ^P *are locally bounded near* ∂G

(vi) $\mathcal{A}^{c^*,(\tau_K^*,\tau_W^*,\tau_C^*)}\phi^P \leq 0$ *on* $D \setminus \partial G$

Then, $\phi^P(\zeta_0) \geq U^{P,\tau^*}(\zeta_0)$ *for all* $\zeta_0 \in D$

b) Suppose, in addition to the above conditions, that,

(vii) $\mathcal{A}^{c^*,(\tau_K^*,\tau_W^*,\tau_C^*)}\phi^P = 0$ *on* G

(viii) $\tau_G := \inf\{t > s;\ \zeta(t) \notin G\} < \infty$ *a.s.* $\mathbb{P}^{\zeta_0}$ *for* $\zeta_0 \in D$*, and*

(ix) The family $\{\phi^{P-}(\zeta(\tau));\ \tau \leq \tau_G, \tau \in \mathcal{T}\}$ *is uniformly integrable w.r.t.* $\mathbb{P}^{\zeta_0}$ *for all* $\zeta_0 \in D$

Then, $\phi^P(\zeta_0) = U^{P,\tau^*}(\zeta_0) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_s[e^{\rho s}\tilde{u}^P(\zeta(\tau))]$ *for all* $\zeta_0 \in D$

and,

$$\tau^* = \tau_G$$

is an optimal stopping time for this problem.

Proof. A direct application of the Variational Inequalities for optimal stopping (see, Øksendal, 2003) produces the required assertion. ∎

While one may notice certain similarity of the present approach to those literatures studying endogenous lifetime or endogenous longevity in growth models (see, Chakraborty, 2004; de la Croix and Ponthiere, 2010, and among others), there exist obvious differences especially when referring to economic intuitions and economic implications behind the formal models. For example, existing studies mainly focus on OLG models and health-investment behaviors while the current exploration emphasizing issues of macroeconomic development, i.e., formal characterization of economic maturity for underdeveloped economies and the corresponding characteristics of their optimal paths of capital accumulation. Furthermore, it is easily seen that the maximum sustainable capital-labor ratio corresponding to the state of economic maturity as well as the minimum time needed to economic maturity is endogenously determined by using stochastic optimal stopping theory that is widely applied in mathematical finance (see, Øksendal and Sulem (2005) and references therein). As is well known, in Kurz's (1965) study, the targets or the maximum sustainable level of terminal path capital-labor ratios are exogenously specified, and the corresponding minimum time problem is expressed as: for any given initial capital-labor ratios, to chose strategies so that the prescribed targets can be reached as soon as possible. The major innovation of the present approach, therefore, is that we endogenously determine the terminal path, the minimum time and also take the economic-welfare considerations of the strongly self-interested politician into account in solving the minimum-time problem. Last but not least, Theorem 5 indeed provides us with a general and complete characterization of the minimum-time needed to economic maturity when compared to the corresponding result in Dai (2012). And most importantly, this kind of generalization will sufficiently capture the economic effects of preferences and technologies on the minimum-time needed to economic maturity, which hence implies that Theorem 5 would be of independent interest.

Corollary 1. I*n principle, the sequential-equilibrium minimum-time needed to economic maturity* τ^* *can be further computed by the following equality,*

$$\psi^P(\zeta_0) = \phi^P(\zeta_0)$$

Proof. Combining (11) with Theorem 5 produces the desired result. ■

It is particularly worth emphasizing that Corollary 1 as well as the corollaries in Section 4 is one major innovation of the model because these corollaries provide simple conditions under which the corresponding minimum-time needed to economic maturity can be explicitly computed as is shown in Section 5. In addition to that, one may easily notice that the equilibrium minimum-time needed to economic maturity in Corollary 1 clearly reflects the reasonable combination of the optimal stopping theory and the stochastic dynamic programming method.

Case 2. Suppose that there is a semi-strongly self-interested politician in the economy.

Case 3. Suppose that there is a weakly self-interested politician in the economy.

Noting that the discussions corresponding to Cases 2 and 3 are quite similar to that of Case 1, we hence take them omitted and leave them to the interested readers.

4. Cooperative-Equilibrium Economic Maturity

In the present section, we will introduce a new approach to economic maturity, i.e., cooperative-equilibrium economic maturity. Kaitala and Pohjola (1990), and Leong and Huang (2010) study the differential cooperative game between the firm and the government in deterministic and stochastic environments, respectively. However, we will investigate the differential cooperative game between the representative household and the self-interested politician with the time horizon endogenously determined. As a result, our following theorems are new relative to those of Kaitala and Pohjola (1990), and Leong and Huang (2010). Additionally, the following results will be much more complicated owing to the general preference and technology functions we employed here.

We will first introduce Markov feedback Nash equilibrium solution, and then cooperative equilibrium which fulfills the following requirements: individual rationality, group rationality, sub-game consistency and also Pareto-optimality. Moreover, we derive the payoff distribution procedure (PDP) of the cooperative game based upon the sub-game consistent imputation and provided that the players agree to act according to all agreed upon Pareto-optimal principle, for example, Nash bargaining solution and Shapley value. In particular, we give,

Assumption 2. *Here, and throughout the current paper, it is assumed that payoffs\utilities are transferable across players, i.e., the representative household and the self-interested politician, and over time.*

Case 1. Suppose that there is a strongly self-interested politician in the economy.

Theorem 6 (Markov Feedback Nash Equilibrium Solution). *We denote by $\Gamma(k_0, \hat{\tau} - s)$ the differential game between the representative household and the self-interested politician, and hence a set of feedback strategies $\left[\hat{c}^{(s)}(t,k); \hat{\tau}_K^{(s)}(t,k), \hat{\tau}_W^{(s)}(t,k), \hat{\tau}_C^{(s)}(t,k)\right]$ provides a Nash equilibrium solution to the game $\Gamma(k_0, \hat{\tau} - s)$, if there exist continuously differentiable functions, $V^{(s)i}(t,k): [s, \hat{\tau}] \times \mathbb{R}_+ \to \mathbb{R}$, $i \in \{H, P\}$, satisfying the following Fleming-Bellman-Isaacs partial differential equations,*

$$\begin{aligned} -V_t^{(s)H}(t,k) - \frac{1}{2}\sigma^2 k^2 V_{kk}^{(s)H}(t,k) &= \max_{c \geq 0}\Big(e^{-\rho(t-s)} u^H(c) \\ &+ V_k^{(s)H}(t,k)\{[(1-\hat{\tau}_K)(F_K(K,L)-\delta)+\sigma^2-n]k + (1-\hat{\tau}_W)F_L(K,L) - (1+\hat{\tau}_C)c\}\Big) \end{aligned}$$

and,

$$\begin{aligned} -V_t^{(s)P}(t,k) - \frac{1}{2}\sigma^2 k^2 V_{kk}^{(s)P}(t,k) &= \max_{\tau_K, \tau_W, \tau_C}\Big(e^{-\rho(t-s)} u^P(\tau_K(F_K(K,L)-\delta)k, \tau_W F_L(K,L), \tau_C \hat{c}) \\ &+ V_k^{(s)P}(t,k)\{[(1-\tau_K)(F_K(K,L)-\delta)+\sigma^2-n]k + (1-\tau_W)F_L(K,L) - (1+\tau_C)\hat{c}\}\Big) \end{aligned}$$

with the following boundary conditions,

$$V^{(s)H}\big(\hat{\tau}, k(\hat{\tau})\big) = e^{-\rho(\hat{\tau}-s)} u^H\big(c(\hat{\tau})\big)$$

$$V^{(s)P}\big(\hat{\tau}, k(\hat{\tau})\big) = U^{P,\hat{\tau}}(\zeta_0) \tag{13}$$

As argued by Fischer (1980), the problem of dynamic inconsistency can arise if the policy maker's utility function differs from that of the representative household. That is, there will be no dynamic inconsistency if the politician and the representative household face exactly the same optimization problem except for the variables they control. Noting that the Markov feedback Nash equilibrium $\left[\hat{c}^{(s)}(t,k); \hat{\tau}_K^{(s)}(t,k), \hat{\tau}_W^{(s)}(t,k), \hat{\tau}_C^{(s)}(t,k)\right]$ given in Theorem 6 is Markovian in the sense that they are functions of current time t and current state $k = k(t)$, and thus independent of past values of state. This implies that the optimal solutions do not depend on the choice of the starting time of the optimal path, and accordingly the problem of dynamic inconsistency disappears even though the self-interested politician and the representative household face totally different optimization problems in Theorem 6. Moreover, besides the Feedback-Nash equilibrium solution established in Theorem 6, many literatures such as Pohjola (1983), and Başar et al. (1985) also studied Feedback-Stackelberg solution (see, Simaan and Cruz, 1973) in a differential game model of capitalism (e.g., Lancaster, 1973; Hoel, 1978). It is therefore asserted that Theorem 6 can also be extended to derive the corresponding Feedback-Stackelberg solution and one, if motivated, may also investigate the difference and similarity between the two kinds of solution in the present framework.

Now, inserting the feedback strategies derived in Theorem 6 into (8) gives rise to,

$$\begin{aligned} dk(t) = \Big\{\Big[\Big(1 - \hat{\tau}_K^{(s)}\big(t,k(t)\big)\Big)\big(F_K\big(K(t),L(t)\big) - \delta\big) + \sigma^2 - n\Big]k(t) + \Big(1 - \hat{\tau}_W^{(s)}\big(t,k(t)\big)\Big)F_L\big(K(t),L(t)\big) \\ - \Big(1 + \hat{\tau}_C^{(s)}\big(t,k(t)\big)\Big)\hat{c}^{(s)}\big(t,k(t)\big)\Big\}dt \\ - \sigma k(t)dB(t) \end{aligned} \tag{14}$$

Provided the Markov feedback Nash equilibrium $\left[\hat{c}^{(s)}(t,k); \hat{\tau}_K^{(s)}(t,k), \hat{\tau}_W^{(s)}(t,k), \hat{\tau}_C^{(s)}(t,k)\right]$ given in Theorem 6, then the corresponding stopping time $\hat{\tau}$ given in Theorem 6 is a solution to the following problem,

Problem 4. Similar to Problem 3, let $\mathcal{T}$ denote the set of all $\mathcal{F}$ −stopping times $\tau \le \tau_D$. Then the optimal stopping problem facing the strongly self-interested politician reads as follows,

$$\begin{aligned} &U^{P,\hat{\tau}}(\zeta_0)\chi_{\{\hat{\tau}<\infty\}} \\ &\qquad := \sup_{\tau\in\mathcal{T}} \mathbb{E}_s\Big[e^{-\rho(\tau-s)}u^P\Big(\hat{\tau}_K^{(s)}\big(\tau,k(\tau)\big)\big(F_K\big(K(\tau),L(\tau)\big) \\ &\qquad - \delta\big)k(\tau), \hat{\tau}_W^{(s)}\big(\tau,k(\tau)\big)F_L\big(K(\tau),L(\tau)\big), \hat{\tau}_C^{(s)}\big(\tau,k(\tau)\big)\hat{c}^{(s)}\big(\tau,k(\tau)\big)\Big)\chi_{\{\tau<\infty\}}\Big] \\ &\qquad := \sup_{\tau\in\mathcal{T}}\mathbb{E}_s\big[e^{\rho s}\tilde{u}^P(\zeta(\tau))\chi_{\{\tau<\infty\}}\big] \end{aligned}$$

subject to (14).

Solving Problem 4 establishes the following theorem, which is quite similar to Theorem 5.

Theorem 7 (Markov-Equilibrium Minimum-Time Needed to Economic Maturity: Existence).

a) Suppose that we can find a function $\phi^P: \overline{D} \to \mathbb{R}$ *such that,*

(i) $\phi^P \in C^1(D) \cap C(\overline{D})$

(ii) $\phi^P \geq e^{\rho s}\tilde{u}^P$ *on* D *and* $\lim_{t\to\tau_D} \phi^P(\zeta(t)) = e^{\rho s}\tilde{u}^P(\zeta(\tau_D))\chi_{\{\tau_D<\infty\}}$ *a.s.* $\mathbb{P}^{\zeta_0}$

Define $G := \{\zeta_0 \in D;\ \phi^P(\zeta_0) > e^{\rho s}\tilde{u}^P(\zeta_0)\}$ *and suppose* $\zeta(t)$ *spends 0 time on* ∂G *a.s.* $\mathbb{P}^{\zeta_0}$*, i.e.,*

(iii) $\mathbb{E}_{\zeta_0}\left[\int_s^{\tau_D} \chi_{\partial G}(\zeta(t))dt\right] = 0$ *for all* $\zeta_0 \in D$*, and suppose that,*

(iv) ∂G *is a Lipschitz surface.*

Moreover, suppose the following conditions:

(v) $\phi^P \in C^2(D \setminus \partial G)$ *and the second order derivatives of* ϕ^P *are locally bounded near* ∂G

(vi) $\mathcal{A}^{\hat{c},(\hat{\tau}_K,\hat{\tau}_W,\hat{\tau}_C)}\phi^P \leq 0$ *on* $D \setminus \partial G$

Then, $\phi^P(\zeta_0) \geq U^{P,\hat{\tau}}(\zeta_0)$ *for all* $\zeta_0 \in D$

b) Suppose, in addition to the above conditions, that,

(vii) $\mathcal{A}^{\hat{c},(\hat{\tau}_K,\hat{\tau}_W,\hat{\tau}_C)}\phi^P = 0$ *on* G

(viii) $\tau_G := \inf\{t > s;\ \zeta(t) \notin G\} < \infty$ *a.s.* $\mathbb{P}^{\zeta_0}$ *for* $\zeta_0 \in D$*, and*

(ix) The family $\{\phi^{P-}(\zeta(\tau));\ \tau \leq \tau_G, \tau \in \mathcal{T}\}$ *is uniformly integrable w.r.t.* $\mathbb{P}^{\zeta_0}$ *for all* $\zeta_0 \in D$

Then, $\phi^P(\zeta_0) = U^{P,\hat{\tau}}(\zeta_0) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_s[e^{\rho s}\tilde{u}^P(\zeta(\tau))]$ *for all* $\zeta_0 \in D$

and,

$$\hat{\tau} = \tau_G$$

is an optimal stopping time for this problem.

Corollary 2. I*n principle, the Markov-equilibrium minimum-time needed to economic maturity* $\hat{\tau}$ *can be further computed by the following equality,*

$$V^{(s)P}\left(\hat{\tau}, k(\hat{\tau})\right) = \phi^P(\zeta_0)$$

Proof. Combining (13) with Theorem 7 produces the desired result. ∎

Generally, the set of sub-game perfect sequential equilibrium is a subset of that of Nash equilibrium. For example, in many interesting games, there exist multiple Nash equilibra while the uniqueness of the sub-game perfect Nash equilibrium can be ensured. Therefore, the Markov-equilibrium minimum time in Theorem 7 may rightly coincide with the sequential equilibrium minimum time in Theorem 5 on the one hand, while on the other hand, the Markov equilibrium minimum time may be also a relatively new concept under certain specifications of preference and technology in the model.

Now, we focus on the following cooperative stochastic differential game. First, we introduce,

Problem 5. Based upon Assumption 2, and suppose that the representative household and the self-interested politician agree to maximize the sum of their expected payoffs, i.e.,

$$\max_{c,\tau_K,\tau_W,\tau_C} \mathbb{E}_s \left\{ \int_s^{\tau^{**}} e^{-\rho(t-s)} \Big[u^H(c(t)) + u^P\Big(\tau_K(t)\big(F_K\big(K(t),L(t)\big)-\delta\big)k(t), \tau_W(t)F_L\big(K(t),L(t)\big), \tau_C(t)c(t)\Big)\Big] dt + U^{Cooperation\ ,\tau^{**}}(\zeta_0)\chi_{\{\tau^{**}<\infty\}} \right\}$$

subject to (8).

In particular, both $U^{Cooperation\ ,\tau^{**}}(\zeta_0)$ and τ^{**} are determined by the following problem,

Problem 6. When there is cooperation between the representative household and the strongly self-interested politician, then the minimum-time needed to economic maturity τ^{**} is determined by solving the following problem,

$$\begin{aligned} &U^{Cooperation\ ,\tau^{**}}(\zeta_0)\chi_{\{\tau^{**}<\infty\}} \\ &:= \sup_{\tau\in\mathcal{T}} \mathbb{E}_s \left\{ e^{-\rho(\tau-s)} \Big[u^H\Big(c^{**(s)}\big(\tau,k(\tau)\big)\Big) \right. \\ &+ u^P\Big(\tau_K^{**(s)}\big(\tau,k(\tau)\big)\big(F_K\big(K(\tau),L(\tau)\big) \\ &\left. -\delta\big)k(\tau), \tau_W^{**(s)}\big(\tau,k(\tau)\big)F_L\big(K(\tau),L(\tau)\big), \tau_C^{**(s)}\big(\tau,k(\tau)\big)c^{**(s)}\big(\tau,k(\tau)\big)\Big)\Big]\chi_{\{\tau<\infty\}} \right\} \\ &:= \sup_{\tau\in\mathcal{T}} \mathbb{E}_s\big[e^{\rho s}\tilde{u}^{Cooperation}\ \ (\zeta(\tau))\chi_{\{\tau<\infty\}}\big] \end{aligned}$$

subject to,

$$k(t) = \Big\{\Big[\Big(1-\tau_K^{**(s)}\big(t,k(t)\big)\Big)\Big(F_K\big(K(t),L(t)\big)-\delta\Big)+\sigma^2-n\Big]k(t)+\Big(1-\tau_W^{**(s)}\big(t,k(t)\big)\Big)F_L\big(K(t),L(t)\big)$$
$$-\Big(1+\tau_C^{**(s)}\big(t,k(t)\big)\Big)c^{**(s)}\big(t,k(t)\big)\Big\}dt$$
$$-\sigma k(t)dB(t) \tag{15}$$

with $\tau_K^{**(s)}\big(t,k(t)\big)$, $\tau_W^{**(s)}\big(t,k(t)\big)$, $\tau_C^{**(s)}\big(t,k(t)\big)$ and $c^{**(s)}\big(t,k(t)\big)$ determined by Problem 5.

By solving Problem 5, we derive,

Theorem 8. *We denote by $\Gamma^C(k_0,\tau^{**}-s)$ the cooperative differential game between the representative household and the strongly self-interested politician, and consequently a set of Markov feedback strategies* $\Big[c^{**(s)}\big(t,k(t)\big);\tau_K^{**(s)}\big(t,k(t)\big),\tau_W^{**(s)}\big(t,k(t)\big),\tau_C^{**(s)}\big(t,k(t)\big)\Big]$ *provides a cooperative equilibrium solution to the cooperative game $\Gamma^C(k_0,\tau^{**}-s)$, if there exist continuously differentiable functions $W^{(s)}(t,k)\colon[s,\tau^{**}]\times\mathbb{R}_+\to\mathbb{R}$, $i\in\{H,P\}$, satisfying the following Fleming-Bellman-Isaacs partial differential equation,*

$$-W_t^{(s)}(t,k)-\frac{1}{2}\sigma^2k^2W_{kk}^{(s)}(t,k)$$
$$=\max_{c,\tau_K,\tau_W,\tau_C}\Big(e^{-\rho(t-s)}[u^H(c)+u^P(\tau_K(F_K(K,L)-\delta)k,\tau_WF_L(K,L),\tau_Cc)]$$
$$+W_k^{(s)}(t,k)\{[(1-\tau_K)(F_K(K,L)-\delta)+\sigma^2-n]k+(1-\tau_W)F_L(K,L)-(1+\tau_C)c\}\Big)$$

with the following boundary condition,

$$W^{(s)}\big(\tau^{**},k(\tau^{**})\big)=U^{Cooperation\ ,\tau^{**}}(\zeta_0) \tag{16}$$

It will be shown below that the boundary condition in (16) is of great importance in identifying the cooperative equilibrium minimum-time needed to economic maturity. As you may notice, Theorem 8 is established relying on Assumption 2, i.e., payoffs\utilities are transferable across players and over time. Nonetheless, technically, Theorem 8 can also be extended to study the case of nontransferable utilities\payoffs (see, Yeung and Petrosyan, 2006) across players and over time. For example, here we may consider the weighted social welfare function (see, Harsanyi, 1955, and among others) regarding the representative household and the self-interested politician. And one may further interpret such kind of specification from the following viewpoints: first, the choice of the social welfare function will to some extent reflect the social institution or social structure of the underlying economy (see, for example, Akerlof (1997) and references therein), for instance, the representative household and the self-interested politician share asymmetric social status, and therefore asymmetric bargaining power in the game of resource allocation; second, here we specifically employ the methodology that utility is comparable among the economic individuals (e.g., Harsanyi, 1955; Sen, 1970; Kalai, 1977, and among others). To sum up, Theorem 8 has provided us with a useful starting point in this direction for future exploration.

In order to make sure that $\tau_K^{**(s)}(t,k(t))$, $\tau_W^{**(s)}(t,k(t))$, $\tau_C^{**(s)}(t,k(t))$ and $c^{**(s)}(t,k(t))$ derived in Theorem 8 indeed provides us with a cooperative equilibrium solution, we need to introduce the following definitions and theorems,

Definition 7 (Group Rationality). If it is confirmed that $W^{(s)}(t,k(t)) > \sum_{i\in\{H,P\}} V^{(s)i}(t,k(t))$ along the trajectory $\{k(t)\}_{t=s}^{\tau^{**}}$ that is given by (15), then we claim that the optimal solution $\left[c^{**(s)}(t,k(t)); \tau_K^{**(s)}(t,k(t)), \tau_W^{**(s)}(t,k(t)), \tau_C^{**(s)}(t,k(t))\right]$ satisfies group rationality.

Chang and Malliaris (1987), by using the Reflection Principle, demonstrated the existence and uniqueness of the solution to the classic Solow equation under continuous time uncertainty for the class of strictly concave production functions which are continuously differentiable on the non-negative real numbers. This class contains all CES functions with elasticity of substitution less than unity. Hence, we directly give,

Assumption 3. *Suppose that the solution to the SDE given in (15) exists and it can be expressed as follows,*

$$k^{**}(t) = k_0 + \int_s^t \Big\{\Big[\Big(1-\tau_K^{**(s)}\big(\lambda, k^{**}(\lambda)\big)\Big)\big(F_K\big(K(\lambda),L(\lambda)\big)-\delta\big)+\sigma^2-n\Big]k^{**}(\lambda)$$
$$+\Big(1-\tau_W^{**(s)}\big(\lambda,k^{**}(\lambda)\big)\Big)F_L\big(K(\lambda),L(\lambda)\big)-\Big(1+\tau_C^{**(s)}\big(\lambda,k^{**}(\lambda)\big)\Big)c^{**(s)}\big(\lambda,k^{**}(\lambda)\big)\Big\}d\lambda$$
$$-\int_s^t \sigma k^{**}(\lambda)dB(\lambda) \tag{17}$$

We let Ξ_t^{**} denote the set of reliable values of $k^{**}(t)$ at time t generated by (17). The term k_t^{**} is also used to represent an element in the set Ξ_t^{**}. Let $\eta(t_0) := [\eta_H(t_0), \eta_P(t_0)]$ denote the instantaneous payoff of the cooperative game $\Gamma^C(s,k_s^{**})$ at time $t_0 \in [s,\tau^{**}]$ with $k_s^{**} \in \Xi_s^{**}$. In particular, along the cooperative trajectory $\{k^{**}(t)\}_{t=s}^{\tau^{**}}$ we put,

$$\xi^{(s)i}\big(t_0,k_{t_0}^{**}\big) := \mathbb{E}_{t_0}\left[\int_{t_0}^{\tau^{**}} e^{-\rho(\lambda-t_0)}\eta_i(\lambda)\,d\lambda\middle| k(t_0)=k_{t_0}^{**}\right]$$

$$\xi^{(s)i}(t,k_t^{**}) := \mathbb{E}_t\left[\int_t^{\tau^{**}} e^{-\rho(\lambda-t)}\eta_i(\lambda)\,d\lambda\middle| k(t)=k_t^{**}\right]$$

for $i\in\{H,P\}$, $k_{t_0}^{**}\in\Xi_{t_0}^{**}$, $k_t^{**}\in\Xi_t^{**}$ and $t\geq t_0\geq s$.

Thus, based upon an agreed-upon optimality principle such as Nash bargaining solution or

Shapley value introduced below, the vectors $\xi^{(s)}(t_0, k_{t_0}^{**}) := [\xi^{(s)H}(t_0, k_{t_0}^{**}), \xi^{(s)P}(t_0, k_{t_0}^{**})]$ for $t_0 \geq s$, are valid imputations if the following conditions are satisfied,

Definition 8 (Valid Imputation). The vector $\xi^{(s)}(t_0, k_{t_0}^{**})$ is a valid imputation of the differential cooperative game $\Gamma^C(t_0, k_{t_0}^{**})$, for $t_0 \in [s, \tau^{**}]$ and $k_{t_0}^{**} \in \Xi_{t_0}^{**}$, if it satisfies,

(i) $\xi^{(s)}(t_0, k_{t_0}^{**}) := [\xi^{(s)H}(t_0, k_{t_0}^{**}), \xi^{(s)P}(t_0, k_{t_0}^{**})]$ is a Pareto optimal imputation vector;

(ii) Individual rationality requirement, i.e., $\xi^{(s)i}(t_0, k_{t_0}^{**}) \geq V^{(s)i}(t_0, k_{t_0}^{**})$, for $i \in \{H, P\}$.

Moreover, we define,

$$\gamma^{(s)i}(t_0; t_0, k_{t_0}^{**}) := \mathbb{E}_{t_0}\left[\int_{t_0}^{\tau^{**}} e^{-\rho(\lambda - t_0)} \eta_i(\lambda)\, d\lambda \,\middle|\, k(t_0) = k_{t_0}^{**}\right] = \xi^{(s)i}(t_0, k_{t_0}^{**})$$

and,

$$\gamma^{(s)i}(t_0; t, k_t^{**}) := \mathbb{E}_t\left[\int_t^{\tau^{**}} e^{-\rho(\lambda - t_0)} \eta_i(\lambda)\, d\lambda \,\middle|\, k(t) = k_t^{**}\right]$$

for $i \in \{H, P\}$ and $t \geq t_0 \geq s$. Noting that,

$$\begin{aligned} \gamma^{(s)i}(t_0; t, k_t^{**}) &:= e^{-\rho(t - t_0)} \mathbb{E}_t\left[\int_t^{\tau^{**}} e^{-\rho(\lambda - t)} \eta_i(\lambda)\, d\lambda \,\middle|\, k(t) = k_t^{**}\right] = e^{-\rho(t - t_0)} \xi^{(s)i}(t, k_t^{**}) \\ &= e^{-\rho(t - t_0)} \gamma^{(s)i}(t; t, k_t^{**}) \end{aligned} \tag{18}$$

for $i \in \{H, P\}$ and $k_t^{**} \in \Xi_t^{**}$, we now give,

Definition 9 (Sub-Game Consistency). The condition in (18) guarantees sub-game consistency of the solution imputation throughout the game interval in the sense that the extension of the solution policy to a situation with a later starting time and any feasible state brought about by prior optimal behaviors would remain optimal.

Indeed, Definition 9 is directly brought from Yeung and Petrosyan (2006). Furthermore, we can get the PDP as follows,

Theorem 9 (Sub-Game Consistent Solution). *An instantaneous payment at time* $t_0 \in [s, \tau^{**}]$ *equaling,*

$$\eta_i(t_0) = -\xi_{t_0}^{(s)i}\left(t_0, k_{t_0}^{**}\right) - \frac{1}{2}\sigma^2\left(k_{t_0}^{**}\right)^2 \xi_{k_{t_0}^{**}k_{t_0}^{**}}^{(s)i}\left(t_0, k_{t_0}^{**}\right)$$
$$- \xi_{k_{t_0}^{**}}^{(s)i}\left(t_0, k_{t_0}^{**}\right)\left\{\left[\left(1 - \tau_K^{**(s)}\left(t_0, k_{t_0}^{**}\right)\right)\left(F_K\left(K(t_0), L(t_0)\right) - \delta\right) + \sigma^2 - n\right] k_{t_0}^{**}\right.$$
$$\left. + \left(1 - \tau_W^{**(s)}\left(t_0, k_{t_0}^{**}\right)\right) F_L\left(K(t_0), L(t_0)\right) - \left(1 + \tau_C^{**(s)}\left(t_0, k_{t_0}^{**}\right)\right) c^{**(s)}\left(t_0, k_{t_0}^{**}\right)\right\}$$

for $i \in \{H, P\}$ *and* $k_{t_0}^{**} \in \Xi_{t_0}^{**}$, *and this yields a sub-game consistent solution or the PDP of the cooperative game* $\Gamma^C\left(t_0, k_{t_0}^{**}\right)$.

Proof. It is quite similar to the proof of Theorem 5.8.3 of Yeung and Petrosyan (2006), so we take it omitted. ■

As noted above, one may consider sub-game consistent solutions under specific optimality principles. For example, one may use,

Definition 10 (Nash Bargaining Solution\Shapley Value). In the cooperative game $\Gamma^C(s, k_0)$, at time s an imputation,

$$\xi^{(s)i}(s, k_0) = V^{(s)i}(s, k_0) + \tfrac{1}{2}\left[W^{(s)}(s, k_0) - \textstyle\sum_{j \in \{H,P\}} V^{(s)j}(s, k_0)\right].$$

is assigned to player i, for $i \in \{H, P\}$; and at time $t_0 \in [s, \tau^{**}]$, an imputation,

$$\xi^{(s)i}\left(t_0, k_{t_0}^{**}\right) = V^{(s)i}\left(t_0, k_{t_0}^{**}\right) + \tfrac{1}{2}\left[W^{(s)}\left(t_0, k_{t_0}^{**}\right) - \textstyle\sum_{j \in \{H,P\}} V^{(s)j}\left(t_0, k_{t_0}^{**}\right)\right].$$

is assigned to player i, for $i \in \{H, P\}$ and $k_{t_0}^{**} \in \Xi_{t_0}^{**}$.

Here, it is especially worth emphasizing that Nash bargaining solution and Shapley value coincide with each other in the present two-player game (see, Yeung and Petrosyan, 2006) while they generally showing us different cooperative mechanisms when there are over two players in the game.

Theorem 10 (Sub-Game Consistency of the Nash Bargaining Solution\Shapley Value). *It is confirmed that the Nash bargaining solution\Shapley value* $\xi^{(s)i}\left(t_0, k_{t_0}^{**}\right)$ *given in Definition 10 is a sub-game consistent imputation for the present cooperative game* $\Gamma^C(s, k_0)$ *for* $\tau^{**} \geq t_0 \geq s$.

Proof. Noting that the equilibrium feedback strategies or the stochastic controls in Theorems 6 and 8 are Markovian in the sense that they depend on current state and current time, one can readily observe by comparing the corresponding stochastic Bellman equations in Theorems 6 and 8 for different values of $t_0 \in [s, \hat{\tau}]$ and $t_0 \in [s, \tau^{**}]$, respectively, that,

$$\begin{pmatrix} \hat{c}^{(t_0)}(t,\hat{k}_t) \\ \hat{\tau}_K^{(t_0)}(t,\hat{k}_t) \\ \hat{\tau}_W^{(t_0)}(t,\hat{k}_t) \\ \hat{\tau}_C^{(t_0)}(t,\hat{k}_t) \end{pmatrix} = \begin{pmatrix} \hat{c}^{(s)}(t,\hat{k}_t) \\ \hat{\tau}_K^{(s)}(t,\hat{k}_t) \\ \hat{\tau}_W^{(s)}(t,\hat{k}_t) \\ \hat{\tau}_C^{(s)}(t,\hat{k}_t) \end{pmatrix}$$

for $\hat{\tau} \geq t \geq t_0 \geq s$ and $\hat{k}_t = \hat{k}(t)$ the corresponding optimal trajectory of capital-labor ratio determined by (14) at time t, and also,

$$\begin{pmatrix} c^{**(t_0)}(t,k_t^{**}) \\ \tau_K^{**(t_0)}(t,k_t^{**}) \\ \tau_W^{**(t_0)}(t,k_t^{**}) \\ \tau_C^{**(t_0)}(t,k_t^{**}) \end{pmatrix} = \begin{pmatrix} c^{**(s)}(t,k_t^{**}) \\ \tau_K^{**(s)}(t,k_t^{**}) \\ \tau_W^{**(s)}(t,k_t^{**}) \\ \tau_C^{**(s)}(t,k_t^{**}) \end{pmatrix}$$

for $\tau^{**} \geq t \geq t_0 \geq s$ and $k_t^{**} = k^{**}(t)$ the corresponding optimal trajectory of capital-labor ratio determined by (17). Moreover, along the optimal trajectory $\{\hat{k}_t\}_{t=s}^{\hat{\tau}}$, one can obtain,

$$\begin{aligned} V^{(s)H}(t_0,\hat{k}_{t_0}) &:= \mathbb{E}_s\left[\left.\int_{t_0}^{\hat{\tau}} e^{-\rho(\lambda-s)} u^H\left(\hat{c}^{(s)}(\lambda,\hat{k}_\lambda)\right)d\lambda + e^{-\rho(\hat{\tau}-s)} u^H\left(\hat{c}^{(s)}(\hat{\tau},\hat{k}_{\hat{\tau}})\right)\right| \hat{k}(t_0)=\hat{k}_{t_0}\right] \\ &= \mathbb{E}_s\left[\left.\int_{t_0}^{\hat{\tau}} e^{-\rho(\lambda-t_0)} u^H\left(\hat{c}^{(t_0)}(\lambda,\hat{k}_\lambda)\right)d\lambda + e^{-\rho(\hat{\tau}-t_0)} u^H\left(\hat{c}^{(t_0)}(\hat{\tau},\hat{k}_{\hat{\tau}})\right)\right| \hat{k}(t_0)=\hat{k}_{t_0}\right] e^{-\rho(t_0-s)} \\ &= \mathbb{E}_{t_0}\left[\left.\int_{t_0}^{\hat{\tau}} e^{-\rho(\lambda-t_0)} u^H\left(\hat{c}^{(t_0)}(\lambda,\hat{k}_\lambda)\right)d\lambda + e^{-\rho(\hat{\tau}-t_0)} u^H\left(\hat{c}^{(t_0)}(\hat{\tau},\hat{k}_{\hat{\tau}})\right)\right| \hat{k}(t_0)=\hat{k}_{t_0}\right] e^{-\rho(t_0-s)} \\ &:= e^{-\rho(t_0-s)}\, V^{(t_0)H}(t_0,\hat{k}_{t_0}) \end{aligned}$$

where $V^{(s)H}(t_0,\hat{k}_{t_0})$ measures the expected present value of the representative household's payoff in the time interval $[t_0,\hat{\tau}]$ when $\hat{k}(t_0)=\hat{k}_{t_0}$ and when the game starts from time $s \leq t_0$. For the self-interested politician,

$$V^{(s)P}(t_0,\hat{k}_{t_0}) := \mathbb{E}_s\left[\int_{t_0}^{\hat{\tau}} e^{-\rho(\lambda-s)}u^P\left(\hat{\tau}_K^{(s)}(\lambda,\hat{k}_\lambda)\big(F_K(K(\lambda),L(\lambda))\right.\right.$$

$$\left.\left.-\delta\big)\hat{k}_\lambda,\hat{\tau}_W^{(s)}(\lambda,\hat{k}_\lambda)F_L(K(\lambda),L(\lambda)),\hat{\tau}_C^{(s)}(\lambda,\hat{k}_\lambda)\hat{c}^{(s)}(\lambda,\hat{k}_\lambda)\right)d\lambda\,\middle|\,\hat{k}(t_0)=\hat{k}_{t_0}\right]$$

$$=\mathbb{E}_s\left[\int_{t_0}^{\hat{\tau}} e^{-\rho(\lambda-t_0)}u^P\left(\hat{\tau}_K^{(t_0)}(\lambda,\hat{k}_\lambda)\big(F_K(K(\lambda),L(\lambda))\right.\right.$$

$$\left.\left.-\delta\big)\hat{k}_\lambda,\hat{\tau}_W^{(t_0)}(\lambda,\hat{k}_\lambda)F_L(K(\lambda),L(\lambda)),\hat{\tau}_C^{(t_0)}(\lambda,\hat{k}_\lambda)\hat{c}^{(t_0)}(\lambda,\hat{k}_\lambda)\right)d\lambda\,\middle|\,\hat{k}(t_0)=\hat{k}_{t_0}\right]e^{-\rho(t_0-s)}$$

$$=\mathbb{E}_{t_0}\left[\int_{t_0}^{\hat{\tau}} e^{-\rho(\lambda-t_0)}u^P\left(\hat{\tau}_K^{(t_0)}(\lambda,\hat{k}_\lambda)\big(F_K(K(\lambda),L(\lambda))\right.\right.$$

$$\left.\left.-\delta\big)\hat{k}_\lambda,\hat{\tau}_W^{(t_0)}(\lambda,\hat{k}_\lambda)F_L(K(\lambda),L(\lambda)),\hat{\tau}_C^{(t_0)}(\lambda,\hat{k}_\lambda)\hat{c}^{(t_0)}(\lambda,\hat{k}_\lambda)\right)d\lambda\,\middle|\,\hat{k}(t_0)=\hat{k}_{t_0}\right]e^{-\rho(t_0-s)}$$

$$:= e^{-\rho(t_0-s)}\,V^{(t_0)P}(t_0,\hat{k}_{t_0})$$

where $V^{(s)P}(t_0,\hat{k}_{t_0})$ measures the expected present value of the strongly self-interested politician's payoff in the time interval $[t_0,\hat{\tau})$ when $\hat{k}(t_0)=\hat{k}_{t_0}$ and when the game starts from time $s\le t_0$. Similarly, for the cooperative game, we obtain,

$$W^{(s)}(t_0,k_{t_0}^{**}) = e^{-\rho(t_0-s)}W^{(t_0)}(t_0,k_{t_0}^{**})$$

where $W^{(s)}(t_0,k_{t_0}^{**})$ measures the expected present value of the cooperative payoff in the time interval $[t_0,\tau^{**})$ when $k^{**}(t_0)=k_{t_0}^{**}$ and when the game starts from time $s\le t_0$.

Now, we can obtain the Nash bargaining solution\Shapley value along the cooperative optimal trajectory $\{k_{t_0}^{**}\}_{t_0=s}^{\tau^{**}}$ as follows,

$$\xi^{(s)i}(t_0,k_{t_0}^{**}) = V^{(s)i}(t_0,k_{t_0}^{**})+\frac{1}{2}\left[W^{(s)}(t_0,k_{t_0}^{**})-\sum_{j\in\{H,P\}}V^{(s)j}(t_0,k_{t_0}^{**})\right] = e^{-\rho(t_0-s)}\left\{V^{(t_0)i}(t_0,k_{t_0}^{**})+\frac{1}{2}\left[W^{(t_0)}(t_0,k_{t_0}^{**})-\sum_{j\in\{H,P\}}V^{(t_0)j}(t_0,k_{t_0}^{**})\right]\right\} = e^{-\rho(t_0-s)}\xi^{(t_0)i}(t_0,k_{t_0}^{**}).$$

for $i\in\{H,P\}$, $s\le t_0\le\tau^{**}$and $k_{t_0}^{**}\in\Xi_{t_0}^{**}$. And this proof is complete. ■

As noted by Yeung and Petrosyan (2006) that though one of the most commonly used allocation principles is the Shapley value, however, equal imputation of cooperative gains may not be totally agreeable to the players when players are asymmetric in their sizes of noncooperative payoffs. For example, in the current context, the noncooperative payoffs of the representative household and the self-interested politician may be asymmetric in reality owing to unequal social status. So, to overcome this, we also consider the following allocation principle in which the players' shares of the gain from cooperation are proportional to the relative sizes of their expected noncooperative payoffs. To be exact, the corresponding imputation scheme satisfies,

Definition 11 (Proportional Distribution). In the present cooperative game $\Gamma^C(k_0, \tau^{**} - s)$, an imputation,

$$\xi^{(s)i}(s, k_0) = \frac{V^{(s)i}(s, k_0)}{\sum_{j \in \{H,P\}} V^{(s)j}(s, k_0)} W^{(s)}(s, k_0)$$

should be assigned to player i, for $i \in \{H, P\}$; and in the sub-game $\Gamma^C\left(k_{t_0}^{**}, \tau^{**} - t_0\right)$ for $t_0 \in [s, \tau^{**}]$, an imputation,

$$\xi^{(s)i}\left(t_0, k_{t_0}^{**}\right) = \frac{V^{(s)i}\left(t_0, k_{t_0}^{**}\right)}{\sum_{j \in \{H,P\}} V^{(s)j}\left(t_0, k_{t_0}^{**}\right)} W^{(s)}\left(t_0, k_{t_0}^{**}\right)$$

is assigned to player i, for $i \in \{H, P\}$ and $k_{t_0}^{**} \in \Xi_{t_0}^{**}$.

Theorem 11 (Sub-Game Consistency of the Proportional Distribution). *The proportional-distribution imputation* $\xi^{(s)i}\left(t_0, k_{t_0}^{**}\right)$ *given in Definition 11 provides us with a sub-game consistent imputation for the cooperative game* $\Gamma^C(k_0, \tau^{**} - s)$ *for* $i \in \{H, P\}$, $s \leq t_0 \leq \tau^{**}$ *and* $k_{t_0}^{**} \in \Xi_{t_0}^{**}$.

Proof. The proof is quite similar to that of Theorem 10, so we take it omitted. ■

Up to the present step, we have discussed the relevant issues, i.e., group rationality, individual rationality, Pareto-optimal principle, and sub-game consistency of the above cooperative stochastic differential game between the representative household and the strongly self-interested politician. Now, we are in the position to derive the cooperative-equilibrium minimum-time needed to economic maturity.

By solving Problem 6 and also employing similar arguments as in Theorem 5, we get,

Theorem 12 (Cooperative-Equilibrium Minimum-Time Needed to Economic Maturity: Existence).

a) Suppose that we can find a function $\phi^C: \overline{D} \to \mathbb{R}$ *such that,*

(i) $\phi^C \in C^1(D) \cap C(\overline{D})$

(ii) $\phi^C \geq e^{\rho s}\tilde{u}^{Cooperation}$ *on* D *and* $\lim_{t\to\tau_D}\phi^C(\zeta(t)) = e^{\rho s}\tilde{u}^{Cooperation}(\zeta(\tau_D))\chi_{\{\tau_D<\infty\}}$ *a.s.* $\mathbb{P}^{\zeta_0}$

Define $G := \{\zeta_0 \in D;\ \phi^C(\zeta_0) > e^{\rho s}\tilde{u}^{Cooperation}(\zeta_0)\}$ *and suppose* $\zeta(t)$ *spends 0 time on* ∂G *a.s.* $\mathbb{P}^{\zeta_0}$*, i.e.,*

(iii) $\mathbb{E}_{\zeta_0}\left[\int_s^{\tau_D}\chi_{\partial G}(\zeta(t))dt\right] = 0$ *for all* $\zeta_0 \in D$*, and suppose that,*

(iv) ∂G *is a Lipschitz surface.*

Moreover, suppose the following conditions:

(v) $\phi^C \in C^2(D \setminus \partial G)$ *and the second order derivatives of* ϕ^C *are locally bounded near* ∂G

(vi) $\mathcal{A}^{c^{**},(\tau_K^{**},\tau_W^{**},\tau_C^{**})}\phi^C \leq 0$ *on* $D \setminus \partial G$

Then, $\phi^C(\zeta_0) \geq U^{Cooperation,\tau^{**}}(\zeta_0)$ *for all* $\zeta_0 \in D$

b) Suppose, in addition to the above conditions, that,

(vii) $\mathcal{A}^{c^{**},(\tau_K^{**},\tau_W^{**},\tau_C^{**})}\phi^C = 0$ *on* G

(viii) $\tau_G := \inf\{t > s;\ \zeta(t) \notin G\} < \infty$ *a.s.* $\mathbb{P}^{\zeta_0}$ *for* $\zeta_0 \in D$*, and*

(ix) *The family* $\{\phi^{C-}(\zeta(\tau));\ \tau \leq \tau_G, \tau \in \mathcal{T}\}$ *is uniformly integrable w.r.t.* $\mathbb{P}^{\zeta_0}$ *for all* $\zeta_0 \in D$

Then, $\phi^C(\zeta_0) = U^{Cooperation,\tau^{**}}(\zeta_0) = \sup_{\tau\in\mathcal{T}}\mathbb{E}_s\left[e^{\rho s}\tilde{u}^{Cooperation}(\zeta(\tau))\right]$ *for all* $\zeta_0 \in D$

and,

$$\tau^{**} = \tau_G$$

is an optimal stopping time for this problem.

In addition to that, we have,

Corollary 3. I*n principle, the cooperative-equilibrium minimum-time needed to economic maturity* τ^{**} *can be further computed by the following equality,*

$$W^{(s)}\left(\tau^{**}, k(\tau^{**})\right) = \phi^C(\zeta_0)$$

Proof. Combining (16) with Theorem 12 produces the desired result. ■

Case 2. Suppose that there is a semi-strongly self-interested politician in the economy.

Case 3. Suppose that there is a weakly self-interested politician in the economy.

Noting that the discussions corresponding to Cases 2 and 3 are quite similar to that of Case 1, we hence take them omitted and leave them to the interested readers. Furthermore, it would be interesting to comparatively study the cooperative equilibrium minimum-time needed to economic maturity corresponding to different cases, i.e., different political institutional arrangements. That is, the framework presented here makes it possible to evaluate the economic efficiency of political institutions from the perspective of economic development. As emphasized by North (1994) that economic and political institutions are the underlying determinants of economic performance and also argued by Acemoglu et al. (2005b) that institutions are the fundamental cause of economic growth, the paper has built up a baseline framework for us to explore the role institutions play in economic maturity, especially from the viewpoint of time dimension. And hence our results would be seen as a supplement to those of North (1994) and Acemoglu et al. (2005b).

Now, provided the sequential-equilibrium minimum-time needed to economic maturity τ^*, Markov-equilibrium minimum-time needed to economic maturity $\hat{\tau}$, and the cooperative-equilibrium minimum-time needed to economic maturity τ^{**}, given in Theorems 5, 7 and 12, respectively, we can then investigate the following issue: which approach will lead us to much faster economic maturity? Lancaster (1973) and Kaitala and Pohjola (1990) argued that cooperation between the government and the firm will be more beneficial compared to the dynamic inefficiency of capitalism. Moreover, Leong and Huang (2010) demonstrates that cooperation is always Pareto optimal compared to the non-cooperative Markovian Nash equilibrium although the cooperative solution is indeterminate. Apart from these papers, the present model defines the concept of dynamic inefficiency of capitalism in the sense of the minimum-time needed to economic maturity. In other words, if the cooperation between the self-interested politician and the representative household will lead to much faster speed of economic maturity, then there exists dynamic inefficiency of capitalism in the underlying economy. Furthermore, if we interpret different game structures as different institutional arrangements (e.g., North, 1990; Hurwicz, 1996; Williamson, 2000; Amable, 2003), then we provide a basic framework to analyze different speeds of economic maturity corresponding to different institutional arrangements. This indeed shows new approach and also new perspective for those studies focused on underdeveloped economies.

Finally, it is particularly worth emphasizing that the equilibrium minimum-times needed to economic maturity derived in the above theorems strictly depend on the initial value of the underlying economic system. This has to some extent reflected the well-known path-dependence effect analyzed and emphasized by North (1990). In other words, we argue that, besides in the process of institutional changes, path-dependence effect also plays a crucial role in economic development for those underdeveloped economies. What is more, as you can see in the following section, one can even proceed to the comparative static analysis of the equilibrium minimum-time needed to economic maturity with respect to the initial capital stock of the abstract economy. This of course will show us very rich and also interesting economic intuition and economic implication of the mathematical model. And it, therefore, would be regarded as an advantage of the framework established in the paper.

5. Examples: Closed-Form Solutions

In this section, we will take the following case for example,

Case 1. Suppose that there is a strongly self-interested politician in the economy.

In order to make things easier and also derive closed-form solutions, we adopt the following production technology instead of that in (5),

$$Y(t) = AK(t)$$

with $A > 0$, an exogenously given constant. Also, we shall consider a simple form of (6), i.e.,

$$\dot{K}(t) = \big(1 - \tau_K(t)\big)(A - \delta)K(t) - \big(1 - \theta(t)\big)AK(t)$$

where θ stands for savings rate of the representative household. So, (8) becomes,

$$dk(t) = \big[\big(1 - \tau_K(t)\big)(A - \delta) + \sigma^2 - n - \big(1 - \theta(t)\big)A\big]k(t)dt - \sigma k(t)dB(t) \tag{19}$$

5.1 Risk-Aversion Politician

It is assumed that there is a risk-aversion politician in the economy. And both the self-interested politician and the representative household exhibit log preference. In particular, here we without loss of generality put $t \geq 0$ instead of $t \geq s$ which is used in the previous sections. So, the optimization problem facing the representative household reads as follows,

$$\max_{0<\theta(t)<1} \mathbb{E}\left[\int_0^{\tau^*} e^{-\rho(s+t)} \ln\Big(\big(1 - \theta(t)\big)Ak(t)\Big)\,dt + e^{-\rho(s+\tau^*)} \ln\Big(\big(1 - \theta(\tau^*)\big)Ak(\tau^*)\Big)\chi_{\{\tau^*<\infty\}}\right] \tag{20}$$

subject to (19). Applying Computation Algorithm I shows that,

Proposition 1. *Provided the optimal stopping time τ^* and the taxation policy of the strongly self-interested politician, we can get the optimal savings rate as $\theta^*(t) \equiv \theta^* = 1 - \frac{\rho}{A}$ by solving the problem in (20).*

Proof. The proof is quite easy and hence we take it omitted. ■

The optimization problem facing the self-interested politician is expressed as follows,

$$\max_{0<\tau_K(t)<1} \mathbb{E}\left[\int_0^{\tau^*} e^{-\rho(s+t)} \ln\big(\tau_K(t)(A - \delta)k(t)\big)\,dt + U^{P,\tau^*}\chi_{\{\tau^*<\infty\}}\right] \tag{21}$$

subject to (19) and Proposition 1.

Proposition 2. *Conditional on Computation Algorithm I and Proposition 1, we get by solving the problem in (21) the sub-game perfect Nash equilibrium capital-income tax rate as $\tau_K^*(t) \equiv \tau_K^* = \frac{\rho}{A-\delta}$ and also the* *following boundary condition,*

$$J^P\big(\tau^*, k(\tau^*)\big) := e^{-\rho(s+\tau^*)}\left\{\left[\frac{1}{\rho}\ln\rho + \frac{1}{\rho^2}(A-\delta-n+\sigma^2-\rho) - \frac{1}{\rho} - \frac{\sigma^2}{2\rho^2}\right] + \frac{1}{\rho}\ln k(\tau^*)\right\} = U^{P,\tau^*}\chi_{\{\tau^*<\infty\}}.$$

Proof. Based on Proposition 1, we have the following Bellman-Isaacs-Fleming equation,

$$\begin{aligned} &-J_t^P\big(t,k(t)\big) - \frac{1}{2}\sigma^2 k^2(t) J_{kk}^P\big(t,k(t)\big) \\ &= \max_{0<\tau_K(t)<1}\Big\{e^{-\rho(s+t)}\ln\big(\tau_K(t)(A-\delta)k(t)\big) \\ &+ J_k^P\big(t,k(t)\big)k(t)\big[\big(1-\tau_K(t)\big)(A-\delta)+\sigma^2-n-\rho\big]\Big\} \end{aligned} \tag{22}$$

Performing the maximization gives,

$$J_k^P\big(t,k(t)\big)k(t)(A-\delta) = e^{-\rho(s+t)}\frac{1}{\tau_K(t)} \tag{23}$$

If we put,

$$J^P\big(t,k(t)\big) = e^{-\rho(s+t)}\big[C_1 + C_2\ln\big(k(t)\big)\big] \tag{24}$$

which combines with (23) implies that,

$$1 = (A-\delta)\tau_K(t)C_2 \tag{25}$$

Inserting (24) and (25) into (22) yields,

$$\rho\big[C_1 + C_2\ln\big(k(t)\big)\big] + \frac{1}{2}\sigma^2 C_2 = \ln\big(k(t)\big) - \ln C_2 + C_2(A-\delta-n+\sigma^2-\rho) - 1$$

which shows that $C_2 = \frac{1}{\rho}$ and,

$$C_1 = \frac{1}{\rho}\ln\rho + \frac{1}{\rho^2}(A-\delta-n+\sigma^2-\rho) - \frac{1}{\rho} - \frac{\sigma^2}{2\rho^2}$$

which gives the desired result and hence the proof is complete. ■

Now, applying Propositions 1 and 2 reveals that (19) can be rewritten as follows,

$$dk(t) = (A-\delta-n+\sigma^2-2\rho)k(t)dt - \sigma k(t)dB(t) \tag{19'}$$

And so the corresponding optimal stopping problem can be written as follows,

$$U^{P,\tau^*}\chi_{\{\tau^*<\infty\}} := \sup_{\tau\in\mathcal{T}}\mathbb{E}_{(s,k)}\big[e^{-\rho(s+\tau)}\ln\big(\rho k(\tau)\big)\chi_{\{\tau<\infty\}}\big]$$

subject to (19’). The generator in (9) can be written as,

$$\mathcal{A}\phi(s,k) = \frac{\partial\phi}{\partial s} + (A-\delta-n+\sigma^2-2\rho)k\frac{\partial\phi}{\partial k} + \frac{1}{2}\sigma^2 k^2 \frac{\partial^2\phi}{\partial k^2}$$

If we try a function ϕ of the form,

$$\phi(s,k) = e^{-\rho s}k^{\mu}$$

for some constant $\mu \in \mathbb{R}$. We thus get,

$$\mathcal{A}\phi(s,k) = e^{-\rho s}k^{\mu}\left[-\rho + (A-\delta-n+\sigma^2-2\rho)\mu + \frac{1}{2}\sigma^2\mu(\mu-1)\right] := e^{-\rho s}k^{\mu}h(\mu)$$

Solving equation $h(\mu) = 0$ gives the unique positive root,

$$\mu = \frac{-[2(A-\delta-n-2\rho)+\sigma^2] + \sqrt{[2(A-\delta-n-2\rho)+\sigma^2]^2 + 8\sigma^2\rho}}{2\sigma^2} \tag{26}$$

With this value of μ we put,

$$\phi(s,k) = \begin{cases} e^{-\rho s}Ck^{\mu}, & (s,k) \in G \\ e^{-\rho s}\ln(\rho k), & (s,k) \notin G \end{cases} \tag{27}$$

for some constant C, to be determined. If we let $g(s,k) := e^{-\rho s}\ln(\rho k)$, we have,

$$\mathcal{A}g(s,k) = e^{-\rho s}\left[-\rho\ln(\rho k) + \left(A-\delta-n-2\rho+\frac{1}{2}\sigma^2\right)\right] > 0 \Leftrightarrow k < \frac{1}{\rho}\exp\left[\frac{1}{\rho}\left(A-\delta-n-2\rho+\frac{1}{2}\sigma^2\right)\right]$$

Therefore, we put,

$$\Sigma := \left\{(s,k);\ k < \frac{1}{\rho}\exp\left[\frac{1}{\rho}\left(A-\delta-n-2\rho+\frac{1}{2}\sigma^2\right)\right]\right\}$$

Thus, we guess that the continuation region G has the form,

$$G := \{(s,k);\ 0 < k < k^*\} \tag{28}$$

for some k^* such that $\Sigma \subseteq G$, i.e.,

$$k^* \geq \frac{1}{\rho}\exp\left[\frac{1}{\rho}\left(A-\delta-n-2\rho+\frac{1}{2}\sigma^2\right)\right]$$

Hence, by (28) we can rewrite (27) as follows,

$$\phi(s,k) = \begin{cases} e^{-\rho s}Ck^{\mu}, & 0 < k < k^* \\ e^{-\rho s}\ln(\rho k), & k \geq k^* \end{cases}$$

We without loss of generality guess that the value function ϕ is C^1 at $k = k^*$ and this will naturally lead to the following smooth-fit conditions,

$$C(k^*)^{\mu} = \ln(\rho k^*) \quad \text{(continuity at } k = k^*\text{)}$$

$$C\mu(k^*)^{\mu-1} = (k^*)^{-1} \quad \text{(differentiability at } k = k^*)$$

from which we thus derive,

$$k^* = \frac{1}{\rho}\exp\left(\frac{1}{\mu}\right) \,\&\, C = \mu^{-1}\left[\frac{1}{\rho}\exp\left(\frac{1}{\mu}\right)\right]^{-\mu} \tag{29}$$

Proposition 3. *Under the above constructions and certain parameter constraints, we obtain the sequential-equilibrium minimum-time needed to economic maturity denoted by* $\tau^* = \tau_G := \inf\{t > 0; k(t) = k^*\}$. *In other words,* $g^*(s,k) := e^{-\rho s}\mu^{-1}(k^*)^{-\mu}k^{\mu} = U^{P,\tau^*}$ *is a supermeanvalued majorant of* $g(s,k)$ *with* k^* *and* μ *given by (29) and (26), respectively.*

Proof. See the proof of Theorem 1 of Dai (2012). ■

Corollary 4. *There is a closed-form solution for the sequential-equilibrium minimum-time needed to economic maturity* τ^*, *and indeed,*

$$\tau^* = \frac{1}{\rho}\ln\left(\mu(k^*)^{\mu}k^{-\mu}\left\{\left[\frac{1}{\rho}\ln\rho + \frac{1}{\rho^2}(A-\delta-n+\sigma^2-\rho) - \frac{1}{\rho} - \frac{\sigma^2}{2\rho^2}\right] + \frac{1}{\rho}\ln k^*\right\}\right)$$

where k^* *and* μ *are given by (29) and (26), respectively, and* k *denotes the initial condition.*

Proof. Combining the boundary condition in Proposition 2 with Proposition 3 easily confirms the required assertion. ■

In what follows, we will derive the closed-form solution corresponding to cooperative economic maturity. Before doing this, we establish,

Proposition 4. *There exists a Markov feedback Nash equilibrium solution denoted by* $\{\hat{\theta}, \hat{\tau}_K\} = \left\{1 - \frac{\rho}{A}, \frac{\rho}{A-\delta}\right\}$, *and the corresponding value functions are given by,*

$$V^H(t,k(t)) = V^P(t,k(t)) = e^{-\rho(s+t)}\left\{\left[\tfrac{1}{\rho}\ln\rho + \tfrac{1}{\rho^2}(A-\delta-n+\sigma^2-\rho) - \tfrac{1}{\rho} - \tfrac{\sigma^2}{2\rho^2}\right] + \tfrac{1}{\rho}\ln k(t)\right\}.$$

Proof. This proof is quite similar to those of Propositions 1 and 2, and hence we omit it and leave it to the interested reader. ■

Now, if the representative household and the strongly self-interested politician can cooperate with each other, then the corresponding optimization problem amounts to,

$$\max_{\substack{0<\tau_K(t)<1\\0<\theta(t)<1}} \mathbb{E}\left\{\int_0^{\tau^{**}} e^{-\rho(s+t)}\left[\ln\big(\tau_K(t)(A-\delta)k(t)\big) + \ln\Big(\big(1-\theta(t)\big)Ak(t)\Big)\right]dt\right.$$

$$\left. + U^{Cooperation\ ,\tau^{**}}\chi_{\{\tau^{**}<\infty\}}\right\} \tag{30}$$

subject to (19). By solving the problem in (30), one can establish,

Proposition 5. *There is a cooperative solution denoted by* $\{\theta^{**}, \tau_K^{**}\} = \left\{1 - \frac{\rho}{2A}, \frac{\rho}{2(A-\delta)}\right\}$, *and the corresponding value function is,*

$$W(t, k(t)) = e^{-\rho(s+t)}\left\{\left[-\frac{2}{\rho}\ln\frac{2}{\rho} + \frac{2}{\rho^2}(A - \delta - n + \sigma^2) - \frac{2}{\rho} - \frac{\sigma^2}{\rho^2}\right] + \frac{2}{\rho}\ln k(t)\right\}.$$

Proof. This proof is quite similar to the above propositions, so we take it omitted. ∎

Proposition 6. *Provided the cooperative solution in Proposition 5, it is shown that group rationality, individual rationality and sub-game consistency are all fulfilled when we employ Nash bargaining solution \ Shapley value as the imputation scheme.*

Proof. Based upon Theorem 10, Propositions 4 and 5, the required assertions are easily demonstrated, and we therefore leave the details to the interested reader. ∎

Now, we are in the position to consider the following optimal stopping problem,

$$U^{Cooperation\ ,\tau^{**}}\chi_{\{\tau^{**}<\infty\}} := \sup_{\tau\in\mathcal{T}} \mathbb{E}_{(s,k)}\left[e^{-\rho(s+\tau)}\ln\left(\frac{\rho}{2}k(\tau)\right)^2 \chi_{\{\tau<\infty\}}\right]$$

subject to,

$$dk(t) = (A - \delta - n + \sigma^2 - \rho)k(t)dt - \sigma k(t)dB(t)$$

So, the generator in (9) can be rewritten as follows,

$$\mathcal{A}\phi(s,k) = \frac{\partial\phi}{\partial s} + (A - \delta - n + \sigma^2 - \rho)k\frac{\partial\phi}{\partial k} + \frac{1}{2}\sigma^2 k^2 \frac{\partial^2\phi}{\partial k^2}$$

If we try,

$$\phi(s,k) = e^{-\rho s}k^{\epsilon}$$

for some constant $\epsilon \in \mathbb{R}$. We thus get,

$$\mathcal{A}\phi(s,k) = e^{-\rho s}k^{\epsilon}\left[-\rho + (A - \delta - n + \sigma^2 - \rho)\epsilon + \frac{1}{2}\sigma^2\epsilon(\epsilon - 1)\right] := e^{-\rho s}k^{\epsilon}h(\epsilon)$$

Solving equation $h(\epsilon) = 0$ produces,

$$\epsilon = \frac{-[2(A - \delta - n - \rho) + \sigma^2] + \sqrt{[2(A - \delta - n - \rho) + \sigma^2]^2 + 8\sigma^2\rho}}{2\sigma^2} \tag{31}$$

With this value of ϵ we put,

$$\phi(s,k) = \begin{cases} e^{-\rho s} C k^{\epsilon}, & (s,k) \in G \\ e^{-\rho s} \ln\left(\frac{\rho}{2}k\right)^2, & (s,k) \notin G \end{cases} \tag{32}$$

for some constant C remains to be determined. If we let $g(s,k) := e^{-\rho s}\ln\left(\frac{\rho}{2}k\right)^2$, we obtain,

$$\mathcal{A}g(s,k) = e^{-\rho s}\left[-\rho\ln\left(\frac{\rho}{2}k\right)^2 + 2(A-\delta-n-\rho)+\sigma^2\right] > 0 \Leftrightarrow k < \frac{2}{\rho}\exp\left\{\frac{1}{2\rho}\left[2(A-\delta-n-\rho)+\sigma^2\right]\right\}$$

So, we put,

$$\Sigma := \left\{(s,k);\ k < \frac{2}{\rho}\exp\left(\frac{1}{2\rho}\left[2(A-\delta-n-\rho)+\sigma^2\right]\right)\right\}$$

Thus, we guess that the continuation region G has the form,

$$G := \{(s,k);\ 0 < k < k^{**}\} \tag{33}$$

for some k^{**} such that $\Sigma \subseteq G$, i.e.,

$$k^{**} \geq \frac{2}{\rho}\exp\left\{\frac{1}{2\rho}\left[2(A-\delta-n-\rho)+\sigma^2\right]\right\}$$

Hence, by (33) we can rewrite (32) as follows,

$$\phi(s,k) = \begin{cases} e^{-\rho s} C k^{\epsilon}, & 0 < k < k^{**} \\ e^{-\rho s} \ln\left(\frac{\rho}{2}k\right)^2, & k \geq k^{**} \end{cases}$$

And hence we have the following smooth-fit conditions,

$$C(k^{**})^{\epsilon} = \ln\left(\frac{\rho}{2}k^{**}\right)^2 \quad (\text{continuity at } k = k^{**})$$

$$C\epsilon(k^{**})^{\epsilon-1} = 2(k^{**})^{-1} \quad (\text{differentiability at } k = k^{**})$$

from which we thus obtain,

$$k^{**} = \frac{2}{\rho}\exp\left(\frac{1}{\epsilon}\right) \ \&\ C = 2\epsilon^{-1}\left[\frac{2}{\rho}\exp\left(\frac{1}{\epsilon}\right)\right]^{-\epsilon} \tag{34}$$

Proposition 7. *Under the above constructions and certain parameter constraints, then there exists a cooperative-equilibrium minimum-time needed to economic maturity denoted by* $\tau^{**} = \tau_G := \inf\{t > 0; k(t) = k^{**}\}$. *In other words,* $g^{**}(s,k) := 2\, e^{-\rho s}\epsilon^{-1}(k^{**})^{-\epsilon}k^{\epsilon} = U^{Cooperation\ ,\tau^{**}}$ *is a supermeanvalued majorant of* $g(s,k)$ *with* k^{**} *and* ϵ *given by (34) and (31), respectively.*

Proof. See the proof of Theorem 1 of Dai (2012). ■

Similar to Corollary 4, we establish,

Corollary 5. *There is a closed-form solution for the cooperative-equilibrium minimum-time needed to economic maturity* τ^{**}, *and in fact,*

$$\tau^{**} = \frac{1}{\rho}\ln\left(\frac{1}{2}\epsilon(k^{**})^{\epsilon}k^{-\epsilon}\left\{\left[-\frac{2}{\rho}\ln\frac{2}{\rho}+\frac{2}{\rho^2}(A-\delta-n+\sigma^2)-\frac{2}{\rho}-\frac{\sigma^2}{\rho^2}\right]+\frac{2}{\rho}\ln k^{**}\right\}\right)$$

where k^{**} *and* ϵ *are given by (34) and (31), respectively, and* k *denotes the initial condition.*

Proof. Combining Proposition 5 with Proposition 7 easily confirms the required result. ■

Corollary 6. *Cooperation between the representative household and the strongly self-interested politician will lead us to much faster economic maturity than that of sequential action when,*

$$\epsilon(k^{**})^{\epsilon}k^{-\epsilon}\left\{\left[-\frac{1}{\rho}\ln\frac{2}{\rho}+\frac{1}{\rho^2}(A-\delta-n+\sigma^2)-\frac{1}{\rho}-\frac{\sigma^2}{2\rho^2}\right]+\frac{1}{\rho}\ln k^{**}\right\}$$

$$< \mu(k^{*})^{\mu}k^{-\mu}\left\{\left[\frac{1}{\rho}\ln\rho+\frac{1}{\rho^2}(A-\delta-n+\sigma^2-\rho)-\frac{1}{\rho}-\frac{\sigma^2}{2\rho^2}\right]+\frac{1}{\rho}\ln k^{*}\right\}$$

in which k^{**}, ϵ, k^{*} *and* μ *are given by (34), (31), (29) and (26), respectively. Otherwise, decentralized sequential action will do a better job than that of differential cooperation in the sense of the minimum-time needed to economic maturity.*

Proof. It follows from Corollaries 4 and 5 that we have the required result. ■

5.2 Risk-Neutral and Risk-Preference Politician

One can still suppose that the representative household exhibits log preference while the criterion of the risk-neutral self-interested politician expressed as follows,

$$\mathbb{E}\left[\int_0^{\tau^*} e^{-\rho(s+t)}\big(\tau_K(t)(A-\delta)k(t)\big)\,dt + U^{P,\tau^*}\chi_{\{\tau^*<\infty\}}\right]$$

And also, the objective of the corresponding optimal stopping problem is given by,

$$\mathbb{E}_{(s,k)}\left[e^{-\rho(s+\tau)}\big(\tau_K^*(\tau)(A-\delta)k(\tau)\big)\chi_{\{\tau<\infty\}}\right]$$

Similarly, for the risk-preference self-interested politician, we have the following criterions for the politician,

$$\mathbb{E}\left[\int_0^{\tau^*} e^{-\rho(s+t)}\frac{\big(\tau_K(t)(A-\delta)k(t)\big)^{\varpi}}{\varpi}dt + U^{P,\tau^*}\chi_{\{\tau^*<\infty\}}\right]$$

and,

$$\mathbb{E}_{(s,k)}\left[e^{-\rho(s+\tau)}\frac{\left(\tau_K^*(\tau)(A-\delta)k(\tau)\right)^{\varpi}}{\varpi}\chi_{\{\tau<\infty\}}\right]$$

in which $\varpi > 1$, some given constant.

Noting that the following discussion is quite similar to that appears in Section 5.1, thus we plan to omit it and leave it to the interested reader. Undoubtedly, closed-form solutions can be derived, too. Moreover, one can comparatively study the minimum-time needed to economic maturity corresponding to different types of politician, and accordingly different types of political institution. For example, one specific type of political institution will induce much higher level of economic maturity while much slower speed of economic maturity when compared with other types of political institution.

6. Concluding Remarks

Dai (2012), by employing optimal stopping theory, discussed efficient capital accumulation with reference to the final state or terminal stocks. And Dai derived closed-form solution by using AK production technology. Nevertheless, the present exploration indeed extends Dai's results from the following directions: first, we have provided very general conditions under which the minimum-time needed to economic maturity can be computed corresponding to a wide range of preferences and technologies; second, in the present study, the role of game structure or institutional arrangement has been sufficiently emphasized in determining the minimum-time needed to economic maturity; last but not least, we study the minimum-time needed to economic maturity for underdeveloped economies and especially under political-economy constraint, i.e., the self-interested politician indeed maximizes the corresponding utility from the rent. Dai mainly demonstrated the strong convergence of capital accumulation to the efficient capital stock while the current paper focusing on the explicit computation and complete characterization of the minimum-time needed to economic maturity for those underdeveloped economies and also under political-economy constraint. What is more, Dai, in a given institutional arrangement and for given preference and technology, provides the condition under which the efficient state is achievable in the sense of uniform topology while the present exploration constructing a general framework in which one can comparatively evaluate the economic efficiency of different institutional arrangements from the viewpoint of the efficient speed (i.e., based on welfare maximization) of economic development. In other words, Dai strictly follows the neoclassical framework while the current paper is indeed in line with new institutional economics. In particular, we have to some extent modeled the underlying idea of Coase (1988) that we need a baseline framework to comparatively and sufficiently evaluate the economic efficiency of different institutional arrangements in order to make a wise choice during the corresponding institutional changes in reality.

Although optimal stopping theory has been widely used in mathematical finance, no literatures except for Dai (2012) notice that this mathematical technique would be very helpful in endogenously determining the minimum-time needed to economic maturity in macroeconomics or development economics. Indeed, the results stated and proved in Sections 3 and 4 are new to the best of our knowledge. In other words, these theorems should be of independent interest in macroeconomics although the major techniques are brought from stochastic analysis (see, Øksendal, 2003) and cooperative stochastic differential game (see, Yeung and Petrosyan, 2006). For example, our specification will naturally lead to the explicit computation of the minimum-time needed to economic maturity, as is shown in Section 5. We have provided a general framework by which one can establish the minimum-time needed to economic maturity with respect to different game structures (or institutional arrangements) between the representative household and the self-interested politician. Moreover, our mathematical results show us, for the first time, in which way and to what extent preference, technology, economic and political institutions affect the minimum-time needed to economic maturity in a stochastic growth model. And this would be regarded as one innovation of the paper when compared to Kurz (1965), Phelan and Stacchetti (2001), Acemoglu et al. (2008, 2010, 2011), Kaitala and Pohjola (1990), and Leong and Huang (2010).

It is plausible to argue that in an underdeveloped economy such as China (see, Song et al., 2011), the government and the households are motivated to choose appropriate fiscal policies and investment strategies, respectively, such that the economy reaches its maturity state as soon as possible. Our study has formally modeled the state of economic maturity in a stochastic growth model. Moreover, we by employing the optimal stopping theory widely used in mathematical finance give a formal definition of the concept of minimum-time needed to economic maturity. And it would be regarded as an advantage of the stopping theory that the maximal and sustainable capital stock per capita as well as the minimum-time is endogenously determined. Indeed, the major goal of the paper is to investigate the effect of game structure on the minimum-time needed to economic maturity. That is, if we interpret different game structures as different institutional arrangements, then this study provides a basic framework for the comparative study of economic maturity under different institutional arrangements. In a simple model of endogenous growth, the closed-form solution of the minimum-time needed to economic maturity has been derived with the explicit condition, under which cooperation between the representative household and the self-interested politician will induce much faster economic maturity than that of decentralized sequential action, supplied, too. That is, we have shown an example where individual rationality results in dynamic inefficiency under certain institutional arrangement. Nonetheless, it would be noticed that our model can also produce the corresponding condition under which dynamic sequential game structure corresponding to capitalism in some sense will induce much faster economic maturity than that of cooperative stochastic differential game structure in a stochastic growth model.

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