Walras's law of markets as special case of the general Triangle Theorem: a laconic proof

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Abstract

From the set of the first three structural axioms follows the - economic - triangle theorem. It asserts that the product of the three key ratios, which characterize the firm, the market outcome, and the income distribution, is always equal to unity. The theorem contains only unit-free variables, is testable in principle, and involves no behavioral assumptions. The differentiated triangle theorem applies to an arbitrary number of firms. Therefrom Walras's Law can be derived without recourse to demand and supply functions or the notion of equilibrium.

JEL C00, D40, D50

Keywords new framework of concepts; structure-centric; axiom set; market clearing; commodity market; labor market; full employment; wage rate flexibility

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> But the "postulates" of the classical political economy, while restricted and scanty enough, were not as hypothetical or "assumed" as was supposed by the economists who formulated them. The psychology of the "economic man," faulty and unsatisfactory as it was, in the one characteristic essential to the economist above all others was not nearly as remote from reality as his creators supposed. In fact, it may almost be said that the ”economic man" was an actual Englishman of the commercial world, the description of whose behavior was correctly obtained by inductive inference from observation, but marred and distorted by faulty deductions from an inaccurate introspective, speculative psychology, in an attempt to obtain a rational explanation of the motivation of his behavior. (Viner, 1917, p. 248)

Standard economics rests on a set of behavioral axioms (Arrow and Hahn, 1991, p. v). It has been argued elsewhere that subjective-behavioral thinking leads, for deeper methodological reasons, to inconclusive filibustering about the agents' economic conduct and therefore has to be replaced by something fundamentally different (2013). The main point is, as Viner sensed already, that the axiomatic method, which is indispensable, is inapplicable to human behavior.

> If economic theory can be criticized, it is not for its abstraction, but for its bad abstraction. (Benetti and Cartelier, 1997, p. 217)

The correct abstraction therefore starts from the objective givens of the monetary economy. Section 1 of the present paper provides the formal point of departure. Therefrom the economic triangle theorem is derived. In Section 2 profit is defined and the zero profit conditions for the economy as a whole and the individual firms are established. These conditions in combination with the triangle theorem yield Walras's Law in Section 3. The Law holds for an arbitrary number of commodity markets. In Section 4 the conditions for full employment in the the labor market are complemented.

1 You can't think without it

> When you get it right, it is obvious that it is right – at least if you have any experience – because usually what happens is that more comes out than goes in. (Feynman, 1992, p. 171)

1.1 Axioms

The formal foundations of theoretical economics must be nonbehavioral and epitomize the interdependence of real and nominal variables that constitutes the monetary economy.

The first three structural axioms relate to income, production, and expenditure in a period of arbitrary length. The period length is conveniently assumed to be the calendar year. Simplicity demands that we have for the beginning one world economy, one firm, and one product. Axiomatization is about ascertaining the *minimum* number of premises. Three suffice for the beginning.

Total income of the household sector Y in period t is the sum of wage income, i.e. the product of wage rate W and working hours L, and distributed profit, i.e. the product of dividend D and the number of shares N.

$$Y = WL + DN \quad |t \tag{1}$$

If DN is set to zero then total income consists only of wage income.

Output of the business sector O is the product of productivity R and working hours.

$$O = RL \quad |t \tag{2}$$

The productivity R depends on the underlying production process. The 2nd axiom should therefore not be misinterpreted as a linear production function.

Consumption expenditures C of the household sector is the product of price P and quantity bought X.

$$C = PX \quad |t \tag{3}$$

The axioms represent the pure consumption economy, that is, no investment expenditures, no foreign trade, and no government.

The economic meaning is rather obvious for the set of structural axioms. What has to be emphasized is that total income in (1) is the sum of wage income and *distributed profit* and not of wage income and profit. Profit and distributed profit look similar but are entirely different economic phenomena.

1.2 Definitions

Definitions are supplemented by connecting variables on the right-hand side of the identity sign that have already been introduced by the axioms. With (4) wage income Y_W and distributed profit Y_D is defined:

$$Y_W \equiv WL \qquad Y_D \equiv DN \quad |t. \tag{4}$$

Definitions add no new content to the set of axioms but determine the logical context of concepts. New variables are introduced with new axioms.

We define the sales ratio as:

$$\rho_X \equiv \frac{X}{O} \quad |t. \tag{5}$$

A sales ratio $\rho_X = 1$ indicates that the quantity sold X and the quantity produced O are equal or, in other words, that the product market is cleared.

We define the expenditure ratio as:

$$\rho_E \equiv \frac{C}{Y} \quad |t. \tag{6}$$

An expenditure ratio $\rho_E = 1$ indicates that consumption expenditure C are equal to total income Y, in other words, that the household sector's budget is balanced.

We define the factor cost ratio as:

$$\rho_F \equiv \frac{W}{PR} \quad |t. \tag{7}$$

The factor cost ratio ρ_F summarizes the internal conditions of the firm. A value of $\rho_F < 1$ signifies that the real wage is lower than the productivity or, in other words, that unit wage costs are lower than the price, or in still other words, that the value of output exceeds the value of input.

We finally define the distributed profit ratio as:

$$\rho_D \equiv \frac{Y_D}{Y_W} \quad |t. \tag{8}$$

1.3 The triangle theorem

Axioms and definitions coalesce into a single equation that formally integrates the three constituents of the pure consumption economy: the firm ρ_F, the commodity market ω, and the income distribution ρ_D.

$$\rho_F\,\omega\,(1+\rho_D) = 1 \quad \text{with} \quad \omega \equiv \frac{\rho_E}{\rho_X} \quad |t \tag{9}$$

The triangle theorem asserts that the product of the three key ratios which characterize the firm, the market outcome, and the distribution is always equal to unity. In analogy to the geometric triangle, the third ratio/angle can be calculated exactly when two ratios/angles are known. When all ratios for the pure consumption economy are measured, eq. (9) will turn out to be true.

The differentiated triangle theorem applies to more than one firm. It is derived in the Appendix and reproduced here as (10). The differentiated equation looks a bit more sophisticated but is composed of the same basic constituents as (9):

$$\left(\rho_{FA}\frac{\rho_{EA}}{\rho_{XA}}+\rho_{FB}\frac{\rho_{EB}}{\rho_{XB}}+\rho_{FC}\frac{\rho_{EC}}{\rho_{XC}}\right)(1+\rho_D)=1 \quad |t. \tag{10}$$

Walras's Law is implicit in this equation and is now made explicit.

2 Profit

The business sector's financial profit in period t is defined with (11) as the difference between the sales revenues – for the economy as a whole identical with consumption expenditure C – and costs – here identical with wage income Y_W:[1]

$$Q_{fi} \equiv C - Y_W \quad |t. \tag{11}$$

Because of (3) and (4) this is identical with:

$$Q_{fi} \equiv PX - WL \quad |t. \tag{12}$$

This form is well-known from the theory of the firm. Due to the fact that the business sector is composed of a single firm microeconomics and macroeconomics coincide at the formal point of departure. This common core is the essential feature of a general theory.

2.1 Total zero profit

From (11) in combination with (4) and (6) follows for the differentiated financial profits of the three firms, respectively:

$$\begin{aligned}
Q_{fiA} &\equiv \rho_{EA}Y - W_A L_A \\
Q_{fiB} &\equiv \rho_{EB}Y - W_B L_B \\
\underline{Q_{fiC}} &\underline{\equiv \rho_{EC}Y - W_C L_C}
\end{aligned} \tag{13}$$

$$Q_{fi} \equiv (\rho_{EA}+\rho_{EB}+\rho_{EC})Y - Y_W$$

$$\text{with} \quad Y_W \equiv W_A L_A + W_B L_B + W_C L_C \quad |t.$$

Financial profit of the business sector as a whole is given as difference of total consumption expenditures and total wage income. This simplifies to:

[1] Nonfinancial profit is treated at length in (2012).

$$
\begin{aligned}
&Q_{fi} \equiv (\rho_{EA} + \rho_{EB} + \rho_{EC} - 1)\, Y \\
&\text{if} \quad Y_D = 0 \quad \Rightarrow \quad Y = Y_W \quad |t.
\end{aligned} \tag{14}
$$

The zero profit condition for the business sector as a whole then reads:

$$
\begin{gathered}
\rho_{EA} + \rho_{EB} + \rho_{EC} - 1 = 0 \\
\text{if} \quad Y_D = 0 \quad |t.
\end{gathered} \tag{15}
$$

General Equilibrium Theory assumes that distributed profit Y_D is equal to profit Q_{fi} (Patinkin, 2008, p. 1), (Buiter, 1980, p. 3) which is obviously a limiting case. It is a characteristic of the real world that retained profit as difference between profit and distributed profit is *never* zero. This point has been dealt with elsewhere (2013, Sec. 3) hence we put it here out of sight with the condition $Y_D = 0$. Eq. (15) is the balanced budget condition for an arbitrary number of firms.

2.2 Individual zero profit

From (12) in combination with (5) follows for the differentiated financial profits of firm A:

$$
Q_{fiA} \equiv P_A \rho_{XA} R_A L_A - W_A L_A \quad |t. \tag{16}
$$

Applying (7) this finally reduces to:

$$
\begin{aligned}
Q_{fiA} &\equiv \rho_{XA} P_A R_A L_A \left(1 - \frac{W_A}{\rho_{XA} P_A R_A}\right) \\
&\equiv \alpha_A \left(1 - \frac{\rho_{FA}}{\rho_{XA}}\right) \quad |t.
\end{aligned} \tag{17}
$$

The zero profit conditions for firm A then read

$$
\rho_{XA} = 1 \, \wedge \, \rho_{FA} = 1 \quad |t \tag{18}
$$

and analogous for all other firms. The conditions imply: if the market is cleared and the factor cost ratio is unity then the profit of the respective firm is zero.

In the general case without market clearing the sum of (17) over all firms gives a zero profit for the business sector as a whole:

$$0 = \alpha_A \left(1 - \underbrace{\frac{\rho_{FA}}{\rho_{XA}}}_{\neq 1} \right) + \alpha_B \left(1 - \underbrace{\frac{\rho_{FB}}{\rho_{XB}}}_{\neq 1} \right) + \alpha_C \left(1 - \underbrace{\frac{\rho_{FC}}{\rho_{XC}}}_{\neq 1} \right) \quad |t. \tag{19}$$

Since $\rho_X \neq 1$ signifies a difference between the quantity produced O and sold X the equation determines how the positive and negative excess demands of all firms are related. In the following the analysis is restricted to the case of market clearing, more precisely, the conditions of (18) apply.

3 Walras's Law

Walras's Law in the narrower version (Patinkin, 2008, p. 3) states that if n-1 markets are in equilibrium then the nth market is in equilibrium too. Equilibrium is a behavioral concept that presupposes demand and supply functions. It is therefore inapplicable in the structural-axiomatic context. Strictly speaking, the structural axiomatic approach produces an analogon to Walras's original law. The hypothetical intersections of fictional functions are replaced by objective conditions.

From the differentiated triangle theorem (10) follows:

$$\begin{gathered} \rho_{FC} \frac{\rho_{EC}}{\rho_{XC}} = 1 - \rho_{FBA} \frac{\rho_{EA}}{\rho_{XA}} - \rho_{FB} \frac{\rho_{EB}}{\rho_{XB}} \\ \text{if} \quad \rho_D = 0 \quad |t. \end{gathered} \tag{20}$$

Under the conditions of zero profit (18) in firm A and B follows:

$$\begin{gathered} \rho_{FC} \frac{\rho_{EC}}{\rho_{XC}} = 1 - \rho_{EA} - \rho_{EB} \\ \text{if} \quad \rho_D = 0,\, \rho_{XA} = 1,\, \rho_{FA} = 1,\, \rho_{XB} = 1,\, \rho_{FB} = 1 \quad |t. \end{gathered} \tag{21}$$

With the overall zero profit condition (15) inserted for ρ_{EC} this yields:

$$\begin{gathered} \frac{\rho_{XC}}{\rho_{FC}} = 1 \\ \text{if} \\ \rho_D = 0,\, \rho_{XA} = 1,\, \rho_{FA} = 1,\, \rho_{XB} = 1,\, \rho_{FB} = 1,\, \rho_{EC} = 1 - \rho_{EA} - \rho_{EB}. \end{gathered} \tag{22}$$

From (17) and (18) we know that a zero profit of firm C demands market clearing and a factor cost ratio of unity. Thus, if the markets of firm A and B are cleared, the market of firm C is also cleared, i.e. $\rho_{XC} = 1$ if $\rho_{FC} = 1$.

From a factor cost ratio of unity and an equal wage rate W for all firms follow the respective market clearing prices as:

$$P_A = \frac{W}{R_A} \quad P_B = \frac{W}{R_B} \quad P_C = \frac{W}{R_C} \quad |t. \tag{23}$$

The market clearing prices are equal to the respective unit wage costs. With an equal wage rate for all firms relative prices are solely determined by the productivities. Eqs. (23) corresponds to the vector of equilibrium prices.

From the derivation it is clear that Walras's Law in the structural axiomatic version relates to an arbitrary number of commodity markets. It does not relate to other types of markets. It would therefore be misleading to interpret the market B as money market and the market C as labor market. From (22) does not follow that the labor market is cleared if the commodity and the money market is cleared. As a matter of fact, it follows nothing definite about the labor market. The clearing of all commodity markets is compatible with any level of total employment.

4 Towards full employment

Let us return for a moment to the elementary case of a single firm. From the triangle theorem (9) follows the price as dependent variable:

$$P = \frac{\rho_E}{\rho_X} \frac{W}{R} \quad \text{if} \quad \rho_D = 0 \quad |t. \tag{24}$$

From this the market clearing price follows:

$$P = \frac{W}{R} \quad \text{if} \quad \rho_D = 0, \rho_X = 1, \rho_E = 1 \quad |t. \tag{25}$$

The market clearing price is equal to unit wage costs if the expenditure ratio is unity and the distributed profit ratio is zero. In the case of budget balancing the profit per unit is therefore zero. All changes of the wage rate and the productivity affect the market clearing price in the period under consideration. From (25) follows:

$$\frac{W}{P} = R \quad \text{if} \quad \rho_D = 0, \rho_X = 1, \rho_E = 1 \quad |t. \tag{26}$$

The real wage is equal to the productivity. This implies that the real wage is not separately determined in the labor market. The usual determination by means of demand and supply schedules for labor and the implicit optimization calculus of employees and employers is therefore redundant. Under the given conditions there is neither a relation between employment and real wage nor between employment and profit. The real wage is determined by the axiom set and the conditions $Y_D = 0$, $\rho_X = 1$, $\rho_E = 1$. This in turn implies that a fall of the wage rate can never affect the real wage but only the market clearing price. Neither wage flexibility nor stickiness has any effect on the real wage. By consequence, full employment cannot be achieved by a fall of the real wage. This alleged cure, however, is as popular among marginalists as bloodletting was among barber-surgeons.

What is needed, therefore, is an additional assumption about how the firms behave. Since we have no production function the optimization calculus is inapplicable. It would be illegitimate to introduce a production function for the sole purpose to make the *a priori* unconvincing profit maximization assumption applicable. It is assumed instead that the firm hires employees at the going wage rate until the labor market is cleared, that is, until there is no more labor supply at the going wage rate or, in still other words, until the unemployment rate is zero. Since profit is zero at any level of employment the firm can be indifferent between full employment or unemployment. In sum our behavioral assumption boils down to the assertion that the firm seeks to grow whenever possible.

Because of Walras's Law, which is supposed to hold as a limiting case, the business sector as a whole can be sure, at least in principle, that the commodity markets clear at any level of employment and that additional outputs can be sold (for details about the distribution of labor input between firms see 2011, Sec. 2). Vice versa, from Walras's Law does not logically follow that the labor market is cleared if all commodity markets are cleared.

Since the real wage (26) depends on the production conditions and can never be too high or too low the responsibility for full employment rests squarely with the business sector.

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Appendix

When the axioms (1) to (3) are differentiated we have in strict formal analogy for period t

$$Y = \underbrace{W_A L_A + W_B L_B + W_C L_C}_{Y_W} + \underbrace{D_A N_A + D_B N_B + D_C N_C}_{Y_D}$$
$$Y = Y_W\left(1 + \rho_D\right) \tag{27}$$

The differentiated output is given by

$$\begin{aligned} O_A &= R_A L_A \\ O_B &= R_B L_B \\ O_C &= R_C L_C \end{aligned} \tag{28}$$

The partitioning of the consumption expenditures is given by

$$\begin{aligned} C_A &= P_A X_A \\ C_B &= P_B X_B \\ C_C &= P_C X_C \end{aligned} \tag{29}$$

With the appropriately adapted definitions (27) boils down to the differentiated triangle theorem

$$\left(\rho_{FA}\,\frac{\rho_{EA}}{\rho_{XA}}+\rho_{FB}\,\frac{\rho_{EB}}{\rho_{XB}}+\rho_{FC}\,\frac{\rho_{EC}}{\rho_{XC}}\right)(1+\rho_{D})=1 \quad |t. \tag{30}$$