Rational bubbles and the spirit of capitalism

Zhou, Ge

College of Economics, Zhejiang University

February 2013

Rational Bubbles and The Spirit of Capitalism

Ge ZHOU
College of Economics,
Academy of Financial Research,
Zhejiang University,
Hangzhou, China
Email: flhszh@gmail.com

January, 2013

Abstract

This paper provides a simple infinite-horizon model of rational bubbles in a production economy. The bubble can arise because of the pursuit of status, which captures the main point of "the spirit of capitalism". I illustrate that the extent of "the spirit of capitalism" determines whether a bubble can exist or not. A stronger extent leads to a larger size of the bubble. Bubbles crowd out investment and stimulate consumption, so that retard growth. I also discuss a stochastic bubble that bursts with an exogenous probability. There are actually multiple equilibria with the stochastic bubbles. At each equilibrium, bubble collapses with a probability that depends on the initial economic condition. Moreover, I demonstrate that a tax policy on wealth can both eliminate bubbles and achieve the social optimal.

Keywords: Bubbles, Spirit of Capitalism, Multiple Equilibra, Wealth Tax

JEL Classification: E2, E44

1 Introduction

"The spirit of capitalism seems to be a driving force behind stock-market volatility and economic growth", Bakshi and Chen (1996) suggests, according to their analytical and empirical examinations. However, they does not discuss the phenomenon of bubbles, which is one culprit to induce huge volatility of stock market. This paper is motivated by exploring the linkage between "the spirit of capitalism" and bubbles, and, develops a theory on rational bubbles driven by the pursuit of status.

My theory is based on a simple infinite-horizon model of a production economy with "the spirit of capitalism". As Max Weber argued, "the spirit of capitalism" signifies that people acquire wealth not just for implied material rewards but also for the social status led to by the accumulation of wealth. Following the method of Kurz (1968) and Zou (1994), I model "the spirit of capitalism" by setting the wealth term directly into the utility function. The bubble in my model is in the pricing of an intrinsicly useless paper. Doubtlessly, we can look the useless paper as the zero-dividend asset, or, fiat money. From a more ambitious perspective, however, the bubble abstractly represents the term that exceeds the fundamental value in the pricing of any property.

The model sheds light on how people's pursuit of status can lead to rational bubbles. More concretely, in an infinite-horizon general equilibrium framework, bubbles can arise provided that the ratio of the marginal utility of wealth over the marginal utility of consumption is positive as time goes to infinity. The condition, technically, prevents the transversality condition (TVC) in ruling out bubbles. Intuitively, when the happiness of holding one more unit of wealth is not trivial relative to the utility of consuming one more unit of goods, individuals would like to hold the asset with a bubble in its price in order to enjoy the resulting status but not sell it out for the purpose of consumption. However, once the ratio of marginal utilities is zero, there is no incentive for the pursuit of status. Sooner or later, individuals will sell out the asset with bubble for material rewards. It is the behavior that expels bubbles.

The way to introduce rational bubbles by "the spirit of capitialism", technically, is similar to the prevailing method to model a credit-driven bubble, which is adopted in Kocherlakota (2009) and a series of papers by Miao and Wang.

Both methods focus on preventing the TVC in ruling out bubbles. The credit-driven bubble can exist because of the binding credit constraint. When credit is scarce, indicated by the binding credit constraint, individuals have incentives to hold the asset with bubble as collateral for more outside credit. The mechanism guarantees the existence of bubbles. More technique details are given in the appendix B.

By a parameterized model, this paper provides specific conditions to determine whether a bubble can exist or not. The conditions mainly depend on the extent of "the spirit of capitalism". When the enthusiasm for status seeking is strong enough, physical capital will be over-accumulated so that the bubbleless economy is dynamically inefficient. Under the situation, bubbles driven by the pursuit of status can arise and mitigate the over-accumulation of capitals. It is because that the incentive for holding an asset with bubble is not trivial relative to the incentive for consumption in an economy with heavy enough culture of status seeking. The requirement for dynamic inefficiency in the bubbleless economy is consistent with the necessary condition for the existence of bubbles in Tirole's overlapping generations (OLG) framework.

The impact of the bubble driven by status seeking is robust both in the neoclassical growth model and in the endogenous growth model. Bubbles stimulate consumption, crowd out investment and slow down economic growth. The result is similar to the findings that are based on Tirole's OLG framework, such as Saint-Paul (1992), Yanagawa and Grossman (1992), and King and Ferguson (1993). The infinite-horizon framework that I adopt, however, eliminates the concern of incomplete markets induced by the OLG framework and easily connects with a vast literature on asset pricing. However, the credit-driven bubble in the infinite-horizon framework has an opposite impact. The reason is simple. The bubble driven by status seeking, like Tirole's bubble, absorbs funds from capital market while the credit-driven bubble helps to provide funds for investment.

Besides the deterministic rational bubbles, my simple model can also analyze stochastic bubbles. A stochastic bubble that bursts with an exogenous constant probability can exist only if the probability of bursting is less than some upper limit, which measures the extent of dynamic inefficiency in the bubbleless economy. The more dynamically inefficient, the more possible for the stochastic bubble to arise. The finding is consistent with what suggested by Weil (1987), which is based on Tirole's OLG framework. As demonstrated by my explicit solution, given the capital stock, a higher probability that the bubble collapses will reduce the size of the bubble, lower consumption, and raise economic growth. We can explain the finding by following intuitions. Facing a higher likelihood of collapsing, individuals will lessen their bubbly assets and invest more into physical capital. The higher investment stimulates economic growth but sacrifices some consumption. Moreover, there are multiple stochastic bubbly equilibria. Each equilibrium has a distinct probability that bubble crashes.

This paper also explores the social optimal policy. The reason for the inefficiency of competitive economy is that "the spirit of capitalism" induces the overaccumulation of physical capital. Two tax policies that restrict capital accumulation can make the competitive economy back to the social optimal. One tax is imposed on investment return; the other on the holding wealth. The optimal tax rates are both equal to the ratio of marginal utility of wealth over the marginal utility of consumption. The former policy focuses on return from physical capital and still allows bubbles. While tax on wealth can also eliminate bubbles.

Putting wealth into the utility function seems to be similar to putting money into the utility function (MIU). Someones might think that the bubble in the pricing of intrinsically useless paper is just identical to the positive value of money in MIU models. The opinion however is not accurate. In typical MIU models, money enters the utility function independently of the physical capital stock. In my model, however, wealth term includes the bubble and the physical capital. It implies that the marginal utility of the bubble depends on the physical capital. Whereas money is neutral as argued by Sidrauski (1967), the bubble in my model has an impact on the real economy.

This paper is related to many current papers on rational bubbles. Introducing financial fricitons into Tirole's OLG framework, Farhi and Tirole (2012) analyzes the interaction of liquidity and bubbles, and, Martin and Ventura (2012) provides a stylized model of economic growth with bubbles. Kocherlakota (2009), and, Miao and Wang (2012) demonstrate that bubbles on collaterals can emerge when the credit is scarce enough. Unlike these papers, I offer a very simple model of rational bubbles driven by the pursuit of status and illustrate a robust result that is consistent with the finding by Tirole (1985). In addition, Kamihigashi (2008) also models a rational bubble by "the spirit of capitalism". Different from his paper, my model does not need any further restriction on the property of the preference function and my results are robust and independent of the production function.

The rest of the paper is organized as follows. Section 2 sets up an infinite-horizon model with "the spirit of capitalism" and provides the necessary condition for the existence of bubbles. Section 3 uses a parameterized function of preference to study the bubbleless equilibrium and the bubbly equilibrium both in a neoclassical growth model and in an endogenous growth model. Section 4 analyzes a stochastic bubbly economy where the bubble bursts with an exogenous probability. Section 5 discusses social optimal policies to eliminate bubbles. Section 6 gives a brief review of the mehodologies to model rational bubbles in a genenral equilibrium framework. Section 7 concludes.

2 The Existence of Bubbles

In this section, I present the necessary condition for the existence of rational bubbles in a standard infinite-horizon general equilibrium model with a finite number of agents. As Tirole (1980) and Blanchard and Watson (1982) argued, it is that the transversality condition (TVC) rules out bubbles. The introducing of "the spirit of capitalism" is necessary to make sure that the transversality condition still holds even at an equilibrium with bubbles. I also illustrate that a binding credit constraint technically has the same function.

We set up the general model as follows.

Time is continuous. An infinite number of identical individuals, who live forever, are continuously and evenly distributed in the area of [0,1]. Every individual can rent his physical capital to firms that are owned by all of the individuals, receives the lump-sum transfer of the firms' profit, Π, and a rental at the rate of r. In the model, the capital stock is denoted by k, and r is equal to the real interest rate. Each individual is also able to invest in financial assets. For convenience, I suppose that there is only one kind of zero-dividend asset in this economy. Based on the standard definition, the fundmental value of the asset should be zero. Therefore, once the price of the asset, which is denoted by

q, is positive, we say that an asset bubble exists. The total supply of the asset is normalized by 1. The amount of the financial asset held by the individual is denoted by s.

Each individual wishes to maximize the sum of time discounted utility values

$$\int_0^\infty e^{-\rho t} U(c, a) dt, \quad \rho > 0,$$

facing his budget constraint given by

$$\dot{a} = rk - c + \dot{q}s + \Pi,$$

where ρ is the rate of time preference, $U(c, a)$ is the utility function, which is continuous, differentiable, strictly increasing and concave in all of its arguments. Here, c is the amount of consumption, and $a \equiv qs + k$ is the amount of wealth, which is equal to the sum of values of asset and physical capital. Following the method of Hengfu Zou (1991) and Bakshi and Chen (1996), I set the wealth term directly into the utility function in order to model "the spirit of capitalism".

The Hamiltonian of the representative agent's optimal problem can be written as

$$\begin{aligned} \mathcal{H} &= U(c, a) + \lambda(a - qs - k) \\ &\quad + \mu(rk - c + \dot{q}s + \Pi). \end{aligned}$$

The first order conditions are given by the Euler equation

$$\frac{\dot{\mu}}{\mu} = \rho - \frac{U'_a}{\mu} - r, \tag{1}$$

where $\mu = U'_c$, and the non-arbitrage condition

$$\frac{\dot{q}}{q} = r, \tag{2}$$

which means the growth rate of bubble is equal to the real interest rate. The transversality conditions can be written as

$$\lim_{t \to \infty} e^{-\rho t} \mu k = 0, \tag{3}$$

$$\lim_{t\to\infty} e^{-\rho t}\mu q s = 0. \tag{4}$$

Appendix A proves that the forms of above transversality conditions are correct.

There are infinite number of homogeneous firms exist in this economy. Each of them wishes to maximize its current profit

$$\Pi \equiv f(k) - \delta k - rk,$$

where $\delta > 0$ is the depreciation rate of physical capital. From the first order condition, the rate of rental (also the real interest rate) is given by

$$r = f'(k) - \delta. \tag{5}$$

At equilibrium, the goods market clearing condition is given by

$$\dot{k} = f(k) - \delta k - c, \tag{6}$$

and the asset market clearing condition is

$$s = 1.$$

Thus, the transversality condition (4) can be rewritten as

$$\lim_{t\to\infty} e^{-\rho t}\mu q = 0. \tag{7}$$

If the innitial value of q is some positive number, then a bubble exists. From equation (2), we know that the bubble grows at the speed of r. To make sure that the transversality condition (7) will not be diviated, we need the growth rate of the product of μ and q is eventually less than ρ. We can find that the growth rate of μq is equal to $\rho - \frac{U'_a}{U'_c}$ by combining the Euler equation (1) with equation (2). Here, I use the fact that $\mu = U'_c$. Thus, as long as

$$\lim_{t\to\infty} \frac{U'_a}{U'_c} > 0, \tag{8}$$

the product of μ and q will eventually grow at a rate, which is less than ρ, and the transversality condition (7) will hold. Therefore, technically, the condition of (8) is necessary for the existence of bubbles.

We can understand above necessary condition (8) by below intuitions. Individuals, in an economy with "the spirit of capitalism", not only care about the expected consumption flows provided by their wealth, but also enjoy holding the wealth itself. When the condition (8) is satisfied, the happiness from holding one more unit of wealth is nontrivial relative to the happiness to consume one more good even at the end of the world. Therefore, even if a bubble cannot provides any material rewards, people still like to hold it for enjoying the increase of their wealth.

The condition also reveals the necessity of "the spirit of capitalism" for the existence of bubbles. If there is no "the spirit of capitalism" in the economy, then the marginal utility of wealth, U_a', is zero. The tranversality condition (7) will be divated if there is a bubble. Thus, the transversality condition rules out bubbles. The reason can be explained intuitively. In the case, people hold wealth is only for the purpose of the expected consumption flows. They do not feel happy by holding the wealth itself. Thus, no one likes to hold an asset whose price includes a bubble forever since no material rewards flow support the bubble. Sooner or later, they will sell out the asset just for the purpose of material rewards. The behavior expels bubbles.

Here, we need to highlight that any further restriction on the form of preference function to guarantee the existence of bubbles is not necessary at all. For example, we do not need the restriction of $\lim_{a\to\infty} U_a' > 0$ required in Kamihigashi (2008). When $\lim_{a\to\infty} U_a' = 0$, as long as the margnial utility of consumption, U_c', also converges to zero, the condition (8) may also hold. In the case, it is still possible for bubbles to arise. The following sections in this paper verify this point.

Moreover, I would like to stress that another method to introduce rational bubbles into an infinite-horizon general equilibrium framework by a binding credit constraint, which is used in Kocherlakota (2008, 2009) and Miao and Wang's a series of papers, technically, is simialr to the method I advocate in this paper. The binding credit constraint makes sure that the transversality condition is not violated when bubbles emerge. We can see the point clear by below intuitions. The credit constraint binds only when the credit is scarce. Since bubbles can work as collaterals, individuals have incentive to hold them for outside credit. Therefore, individuals might not sell out bubbles just for the purpose of consumption. We can see more technique details in the appendix B.

3 The Parameterized Model

This section uses a parameterized model to illustrate that rational bubbles can arise in an economy with "the spirit of capitalism" provided that the condition (8) is satisfied. I consider both a neoclassical growth model and an endogenous growth model and offer explicit conditions under which the bubbly equilibrium can exist. The dynamic analysis is also included.

For convenience, the parameterized utility function of the representive individual takes the form of

$$\log c + \eta \log \frac{a}{\bar{a}}, \tag{9}$$

where $\eta > 0$ measures the weight of "the spirit of capitalism", $\bar{a}$ denotes the average wealth level, and the ratio of one's own wealth to the average wealth dertermines the individual's wealth status. The specification is consistent with the main point of the capitalism spirit and also shares the same idea of "catching up with the Joneses" argued by Abel (1990). Moreover, the log-form utility function guarantees the uniqueness of the bubbleless steady state and dispels the concern about multiple steady states introduced by the wealth effect.[1] The adoption of utility function (9) simplifies our analysis. Here, I have to highlight that utility function (9) implies that

$$\lim_{a \to \infty} U'_a = 0.$$

It means that Kamihigashi's condition is violated. The specific form of the condition (8) currently is given by

$$\lim_{t \to \infty} \frac{\eta c}{a} > 0. \tag{10}$$

However, it is natural to ask why not consider the other similar utility func-

[1] Kurz (1968) demonstrates that it is possible for multiple steady states when wealth term enters the utility function directly.

tions, such as, *absolute wealth as status*,

$$\log c + \eta \log a, \tag{11}$$

or, *difference between one's own wealth and some social standard as status*,

$$\log c + \eta \log(a - \kappa \bar{a}), \tag{12}$$

where $\kappa > 0$ measures the degree of poverty-aversion. Both of them, like utility function (9), catch the main spirit of the capitalism and generate very similar aggregate dynamics in an competitive economy. I choose the form of (9) as the utility function only because that the wealth affects utility only by externality in the form. It makes the discussion on social optimal easier.

The following section explores the bubbleless equilibrium, bubbly equilibrium and their dynamics in a neoclassical growth model and an endogenous growth model respectively.

3.1 Neoclassical Growth Model

In the model, the production function, $f(k)$, takes the form of

$$Ak^{\alpha},$$

where A is the technology level, and $0 < \alpha < 1$. The real interest rate, r, is equal to

$$\alpha A k^{\alpha-1} - \delta,$$

which is a decreasing function of the physical capital, k. With the production of decreasing return to scale, our discussions focus on steady states at which all variables are constants. At the steady states, the transversality conditions are trivial.

The following proposition presents that there exist two steady states in the economy. One is the bubbleless steady state, where the value of bubble, q, is equal to zero. The other is the bubbly steady state, at which the value of bubble is some positive constant. I use a variable with an asterisk to denote its value at the bubbleless steady state and double asterisks, at the bubbly steady state.

Proposition 1 *(a) There always exists a unique bubbleless steady state, at which*

$$q^* = 0,$$

$$k^* = [\frac{\rho + (1+\eta)\delta}{(\eta+\alpha)A}]^{\frac{1}{\alpha-1}},$$

and

$$c^* = [\frac{\rho + (1+\eta)\delta}{\eta+\alpha} - \delta]k^*.$$

(b) With the parameters restriction of

$$\eta\delta(1-\alpha) > \rho\alpha, \tag{13}$$

there exists a unique bubbly steady state, where

$$k^{**} = (\frac{\delta}{\alpha A})^{\frac{1}{\alpha-1}},$$

$$c^{**} = \frac{\delta(1-\alpha)}{\alpha}k^{**},$$

and

$$q^{**} = [\frac{\eta\delta(1-\alpha)}{\rho\alpha} - 1]k^{**} > 0.$$

(c) Under the parameters restriction (13),

$$k^{**} < k^*,$$

and

$$c^{**} > c^*.$$

The parameters restriction (13) in above proposition illustrates that bubbles more possibly arise in an economy where people are more patient and more care about the status, i.e., the value of ρ is lower, and, the value of η is higher.] A lower value of ρ, or, a larger value of η, leads a larger size of an bubble. Above results are intuitive. Impatient people prefer less saving so that less funds flow into the financial market to blow bubbles. While rampant speculations usually acompany an entrenched culture of the capitalism.

The proposition also implies that the existence of bubble requires the dynamic inefficiency of the bubbleless economy. The extent of the dynamic inefficiency generally is measured by the difference between the growth rate and the real interest rate. The difference at the bubbleless steady state is given by

$$0 - r(k^*) = \frac{(1-\alpha)\eta\delta - \alpha\rho}{\eta + \alpha}.$$

Its value is positive under the restricition (13). It means that the bubbleless economy is dynamically inefficient. The requirement is same as the argument by Tirole (1985), which bases on an overlapping generation (OLG) framework. The dynamic inefficiency in both papers comes from the overaccumulation of physical capital. Bubbles would absorb redundant funds and mitigate the overaccumulation of capital. However, causes of capital's overaccumulation in the two papers are different. In this paper, people accumulate too much capital for status seeking; while, Tirole (1985)'s overlapping generation framework makes the physical capital also work as a store of value.

From the comparison of the bubbleless steady state and the bubbly steady state, it is easy to figure out bubble's impact on the real economy. Bubbles, as one type of wealth, can substitute the physical capital in the process of status seeking. Thus, they crow out investment. As a result of the bubbles' wealth effect, individuals also consume more. The implicaitons of bubbles are also consistent with those given by Tirole (1985).

Next, we consider the stability of the steady states and the local dynamics of the economic equilibrium. The following proposition summarizes the analysis on the stability of the bubbly steady state and the bubbleless steady state.

Proposition 2 *Under the restriction (13), both the bubbly steady state and and the bubbleless steady state are local saddle points. Moreover, the stable manifold of the bubbly steady state is one dimensional, while that of the bubbleless steady state is two dimensional.*

From above proposition, we can also figure out the local dynamics in the neighbourhood of the steady states. Around the bubbly steady state, given initial value of the state varialbe, physical capital, k_0, there is a unique pair of initial consumption and initial bubble, $\{c_0, q_0\}$, which makes sure the initial economy eventually converges to the bubbly steady state. However, in the neighbourhood of the bubbleless steady state, given the initial values of capital and bubble, k_0 and q_0, there always is a unique value of consumption, c_0, to guarantee the initial economy finally converges to the bubbleless steady state. In this sense, there exist a series of asymptotically bubbleless equilibra whose innital values of bubbles are positive. The result is consistent with the finding by Tirole (1985). Miao and Wang (2012) also suggests a similar local dynamics in a model of credit-driven bubbles.

3.2 Endogenous Growth Model

Following Romer (1986) and Xie (1991), I suppose that the production function in the endogenous growth model has some positive externality. The specific form of $f(k)$ is given by

$$Ak^{\alpha}\bar{k}^{1-\alpha}, \tag{14}$$

where $\bar{k}$ is the average capital stock, and $0 < \alpha < 1$. At equilibrium, $k = \bar{k}$, and the real interest rate is equal to $\alpha A - \delta$, which is a constant. The form of the production function that I adopt here is same as that in Yanagawa and Grossman (1992), which discusses bubbles in an OLG framework. The setup makes it easy to compare with the results of each other.

In the endogeous growth economy, our discussions focus on the balanced growth paths (BGP), at which all variables grow at some constant speeds. Now, we have to pay more attention on the condition (8) that prevents transversality condition (7) from ruling out bubbles. From the real resource constraint, we can obtain that

$$\frac{\dot{c}}{c} \leq \frac{\dot{k}}{k} \leq \frac{\dot{a}}{a}. \tag{15}$$

In order to make sure that the condition (10) is satisfied, we need that

$$\lim_{t\to\infty} \frac{\dot{c}}{c} \geq \lim_{t\to\infty} \frac{\dot{a}}{a}.$$

Therefore, at the balanced growth path with bubbles, the growth rate of consumption cannot be less than the growth rate of bubbles, or, the growth rate of physical capital, i.e.,

$$\frac{\dot{c}}{c} \geq \frac{\dot{q}}{q},$$

and

$$\frac{\dot{c}}{c} \geq \frac{\dot{k}}{k}.$$

Otherwise, bubbles will be ruled out by the transversality condition (7). Together with the constraint (15), we obtain that

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} \geq \frac{\dot{q}}{q}$$

at the balanced growth path with bubbles. We can see the point clearer in the appendix 3.

The following proposition provides three possible balanced growth paths in the endogenous growth economy. One is the bubbleless balanced growth path, at which the value of bubble is zero. The second is the quasi-bubbleless balanced growth path, at which the value of bubble is positive but eventually trivial relative to the values of real economic variables. The third is the bubbly balanced growth path.

Proposition 3 *(a) There always exists a bubbleless balanced growth path, at which*

$$q = 0,$$

$$c = \frac{(1-\alpha)A + \rho}{\eta + 1} k,$$

and

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = A - \delta - \frac{(1-\alpha)A + \rho}{\eta + 1}.$$

(b) With the parameters restriction

$$\eta(1-\alpha)A > \rho, \tag{16}$$

there exists a quasi-bubbleless balanced growth path, at which the value of bubble eventually is trivial relative to the values of real economic variables. The quasi-

bubbleless balanced growth path can be described by

$$\frac{\dot{q}}{q} = r = \alpha A - \delta,$$

$$c = \frac{(1-\alpha)A + \rho}{\eta + 1} k,$$

and

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = A - \delta - \frac{(1-\alpha)A + \rho}{\eta + 1} > r.$$

(c) Given the restriction (16), there exists a bubbly balanced growth path, at which

$$c = (1-\alpha)Ak,$$

$$q = [\frac{\eta(1-\alpha)A}{\rho} - 1]k,$$

and

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = \frac{\dot{q}}{q} = r = \alpha A - \delta.$$

Above proposition suggests that the economic environment has similar effects on bubbles in the endogenous growth model as in the neoclassical growth model. The condition for the existence of bubbles and the size of an bubble {both depend on the extents of how patient people are, and, how they care about their status}. The difference between the growth rate of bubbleless economy and the real interest rate, measuring the dynamic inefficiency, is given by

$$\frac{\eta(1-\alpha)A - \rho}{\eta + 1}.$$

Its value is positive under the parameters restriciton (16). Therefore, the existence of bubbles still requires the dynamic inefficiency of the bubbleless economy in the case of endogenous growth.

Moreover, the implicaitons of bubbles in the endogenous growth economy are also same as those in the neoclassical growth economy. We can clarify the point by comparing the bubbly balanced growth path and the bubbleless balanced growth path. The comparison is listed in the following table.

	Bubbly BGP	v.s.	Bubbleless BGP
Consumption	$(1-\alpha)Ak$	$>$	$\frac{(1-\alpha)A+\rho}{\eta+1}k$
Bubble	$[\frac{\eta(1-a)A}{\rho}-1]k$	$>$	0
Growth Rate	$\alpha A-\delta$	$<$	$A-\delta-\frac{(1-\alpha)A+\rho}{\eta+1}$
Saving Rate	α	$<$	$[A-\frac{(1-\alpha)A+\rho}{\eta+1}]/A$

Given the parameters restriction (16) and the same capital level, consumption in the bubbly economy is more than that in the bubbleless economy. Together with the fact that the investment is equal to $Ak-c$, investment in the bubbly economy must be lower than that in the bubbleless economy. Therefore, bubbles stimulate consumption and crowd out investment. It also explains why the bubbleless economy has a higher growth rate than the bubbly economy. The result is consistent with the findings by Saint-Paul (1992), Yanagawa and Grossman (1992), and King and Ferguson (1993), whose bubbles exist in Tirole's OLG framework.

There could be some doubt as to whether the individual's optimal problem is well-defined or not in the case when the real interest rate is not larger than the growth rate of the real economy, as presented in part (b) of Proposition 3. The concavity of the utility function makes sure that the value of the objective function will not diverge. Appendix A also proves that the solution determined by the first order conditions and tranversality conditions is the optimal choice for the individual.

Now, we consider the dynamics in the endogenous growth economy. For convenience to use a phase diagram, we can define that

$$\begin{aligned}\tilde{c} &\equiv ce^{-rt},\\ \tilde{k} &\equiv ke^{-rt},\\ \tilde{q} &\equiv qe^{-rt}.\end{aligned}$$

Here, $r\equiv\alpha A-\delta$ is the real interest rate. The economy can be described by the following equations system.

$$\tilde{q}\equiv q_0,$$

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{\eta \tilde{c}}{q_0 + \tilde{k}} - \rho,$$

$$\dot{\tilde{k}} = (1-\alpha)A\tilde{k} - \tilde{c}.$$

On the bubbly balanced growth path, $\tilde{c}$ and $\tilde{k}$ both eventually converge to some non-negative constants, which are increasing in the initial value of bubble, q_0. However, on the bubbleless balanced growth path and the quasi-bubbleless balanced growth path, both of {the discounted variables} will diverge.

When the initial vlaue of bubble, q_0, is equal to zero, the dynamics of the bubbleless economy is same as those of the classical endogenous growth models. Given the initial physical capital, k_0, an appropriate value of initial consumption, c_0, makes sure that the economy stays at the bubbleless balanced growth path forever.

Next, we consider the case of positive initial value of bubble, i.e., $q_0 > 0$. The phase diagram is given by the figure of endogenous growth economy. In the figure, the locus of $\frac{\dot{\tilde{c}}}{\tilde{c}} = 0$ and the locus of $\dot{\tilde{k}} = 0$ are both straight lines. Given the restriction (16), the slope of the locus of $\frac{\dot{\tilde{c}}}{\tilde{c}} = 0$ is less than that of the locus of $\dot{\tilde{k}} = 0$. The two locuses have an intersection point whose horizontal ordinate is denoted by $\tilde{k}_0^*(q_0)$. The intersection point represents the bubbly balance growth path. The value of $\tilde{k}_0^*(q_0)$ is given by

$$\frac{1}{\frac{\eta[(1-\alpha)A]}{\rho} - 1} q_0.$$

{In the space that is above the locus of $\frac{\dot{\tilde{c}}}{\tilde{c}} = 0$ and below the locus of $\dot{\tilde{k}} = 0$,} there exists at least one trajectory converging to the quasi-bubbleless balanced growth path.

By the phase diagram, it is easy to have following findings about the economic dynamcis. Given $q_0 > 0$, and $\tilde{k}_0 \equiv k_0$, if the initial value of capital, k_0, is equal to $\tilde{k}_0^*(q_0)$, then an appropriate value of initial cosumption, $c_0 = \tilde{c}_0$, can make sure the initial economy located at the intersection point of the two locuses. In the case, the economy will stay on the bubbly balance growth path forever; if $k_0 > \tilde{k}_0^*(q_0)$, then an appropriate value of initial consumption can guarantee the initial economy is on the trajectory converging to the quasi-bubbleless balanced growth path; if $k_0 < \tilde{k}_0^*(q_0)$, then no economic equilirium exists.

Above findings can also be reexpressed from the perspective of the initial value of bubble. The inverse function of $\tilde{k}_0^*(q_0)$ implies that a threshold value of bubble, $q_0^*(k_0)$, is equal to

$$[\frac{\eta}{\rho}(1-\alpha)A - 1]k_0.$$

Here, I use the fact that $\tilde{k}_0 \equiv k_0$. With the restriction (16), for any initial capital stock, $k_0 > 0$, there is a unique corresponding threshold value, $q_0^*(k_0) > 0$. If the initial value of bubble q_0 is equal to $q_0^*(k_0)$, the locus of $\frac{\dot{\tilde{c}}}{\tilde{c}} = 0$ and the locus of $\dot{\tilde{k}} = 0$ intersect at the point whose horizontal ordinate is equal to k_0. An appropriate initial consumption, c_0, can make sure that the economy is located at the intersection point. If the initial value of bubble q_0 is positive but less than the threshold value of $q_0^*(k_0)$, then the horizontal ordinate of the intersection point is less than k_0. Thus, an appropriate initial value of consumption will make sure that the initial economy is on the trajectory converging to the quasi-bubbleless balanced growth path. If the initial value of bubble q_0 is larger than the threshold value of $q_0^*(k_0)$, then the horizontal ordinate of the intersection point is larger than k_0. There is no economic equilibrium in this case.

The following proposition summarizes our analysis on the dynamics.

Proposition 4 *(a) If the initial value of bubble, q_0, is equal to zero, the economy stays on the bubbleless balanced growth path.*

(b) Under the parameters restriction (16), given any initial capital stock, $k_0 > 0$, there is a unique threshold value of bubble, $q_0^*(k_0) = [\frac{\eta}{\rho}(1-\alpha)A-1]k_0$. If the initial bubble, q_0, is positive but less than $q_0^*(k_0)$, then the economy converges to the quasi-bubbleless balanced growth path; if the initial bubble, q_0, is equal to $q_0^*(k_0)$, then the economy stays on the bubbly balanced growth path forever; if the initial bubble, q_0, is larger than $q_0^*(k_0)$, then no economic equilibrium exists.

4 Stochastic Bubbles

This section discusses a stochastic bubbly economy, where bubbles might burst with an exogenous constant probability. The utility function and the production function are given by (9) and (14), respectively. Following the methodology of Rebelo and Xie (1999), we can obtain an explicit solution of the stochastic bubbly economy.

Suppose that the bubble still exists at the current moment, i.e., $q > 0$. The process of the bubble can be described as follows,

$$dq = \left\langle \begin{array}{c} \varphi q dt, \text{ with probability } 1 - \varepsilon dt \\ -q, \text{ with probability } \varepsilon dt \end{array} \right. ,$$

where $\varphi > 0$, and $\varepsilon > 0$. If the bubble bursts, i.e., $q = 0$, then the price of the asset will always be zero as shown by the above process. This implies that bubble cannot be reborn.

I conjecture the process of average capital $\bar{k}$ as follows,

$$d\bar{k} = \varphi \bar{k} dt.$$

Suppose the process of asset volume held by the representative individual, is given by

$$ds = \iota dt,$$

where ι is a choice variable that measures the increment of the assets held by the representative individual, given its price level. Thus, the budget can be written as follows,

$$dk = (Ak^{\alpha}\bar{k}^{1-\alpha} - \delta k - c - q\iota)dt.$$

The Hamilton-Jacobi-Bellman equation is given by

$$\begin{aligned} 0 \quad = \quad & \max_{c,\iota}\{U(c, ps + k) + V_1(q, \bar{k}, k, s)\varphi q + V_2(q, \bar{k}, k, s)\varphi \bar{k} \\ & + V_3(q, \bar{k}, k, s)(Ak^{\alpha}\bar{k}^{1-\alpha} - \delta k - c - q\iota) \\ & + V_4(q, \bar{k}, k, s)\iota - (\varepsilon + \rho)V(q, \bar{k}, k, s) + \varepsilon V(0, \bar{k}, k, s)\}. \end{aligned}$$

I guess the form of value function to be

$$V(q,\bar{k},k,s) \equiv \chi + h\log(qs+k) + b\log\bar{k} + \psi\log(q+\bar{k}).$$

It is easy to obtain that

$$V_1(q,\bar{k},k,s) = \frac{hs}{qs+k} + \frac{\psi}{q+\bar{k}},$$

$$V_2(q,\bar{k},k,s) = \frac{b}{\bar{k}} + \frac{\psi}{q+\bar{k}},$$

$$V_3(q,\bar{k},k,s) = \frac{h}{qs+k},$$

$$V_4(q,\bar{k},k,s) = \frac{hq}{qs+k}.$$

Thus, the Hamilton-Jacobi-Bellman equation above can be rewritten as

$$\begin{aligned} 0 \;=\; & \max_{c,\iota}\{\log c + \eta\log(qs+k) - \eta\log(q+\bar{k}) \\ & + \frac{h\varphi qs}{qs+k} + b\varphi + \psi\varphi + \frac{h}{qs+k}(Ak^{\alpha}\bar{k}^{1-\alpha} - \delta k - c) \\ & -(\varepsilon+\rho)[\chi + h\log(qs+k) + b\log\bar{k} + \psi\log(q+\bar{k})] \\ & +\varepsilon[\chi + h\log k + (b+\psi)\log\bar{k}]\} \end{aligned} \tag{17}$$

The optimal condition for consumption is given by

$$c = \frac{qs+k}{h}. \tag{18}$$

The partial derivatives of equation (17) with respect to $\bar{k}$, s, q, and k, respectively, should all be zero. That is,

$$\frac{h(1-\alpha)}{qs+k}Ak^{\alpha}\bar{k}^{-\alpha} + \frac{\varepsilon\psi}{\bar{k}} = \frac{\eta}{q+\bar{k}} + \frac{\rho b}{\bar{k}} + \frac{\psi(\varepsilon+\rho)}{q+\bar{k}}, \tag{19}$$

$$\eta(qs+k) + h\varphi k = h(\varepsilon+\rho)(qs+k) + h(Ak^{\alpha}\bar{k}^{1-\alpha} - \delta k - c) \tag{20}$$

$$\begin{aligned} & \eta s(qs+k) + hs\varphi k - \frac{[\eta + \psi(\varepsilon+\rho)](qs+k)^2}{q+\bar{k}} \\ = \quad & hs(\varepsilon+\rho)(qs+k) + hs(Ak^{\alpha}\bar{k}^{1-\alpha} - \delta k - c) \end{aligned} \tag{21}$$

$$\begin{aligned} & \eta(qs+k) + h(\alpha Ak^{\alpha-1}\bar{k}^{1-\alpha} - \delta)(qs+k) + \frac{\varepsilon h}{k}(qs+k)^2 \\ = \quad & h(\varepsilon+\rho)(qs+k) + h\varphi qs + h(Ak^{\alpha}\bar{k}^{1-\alpha} - \delta k - c) \end{aligned} \tag{22}$$

From equation (20) and equation (21), we obtain that

$$\psi = -\frac{\eta}{\varepsilon+\rho}.$$

Thus, equation (19) can be rewritten as

$$h(1-\alpha)Ak^{\alpha}\bar{k}^{1-\alpha} = (\rho b + \frac{\varepsilon\eta}{\varepsilon+\rho})(qs+k). \tag{23}$$

Equation (22) minus equation (20) is

$$\varepsilon(ps+k) = [\varphi - (\alpha Ak^{\alpha-1}\bar{k}^{1-\alpha} - \delta)]k. \tag{24}$$

Together with equation (23), we obtain that

$$k^{\alpha-1}\bar{k}^{1-\alpha} = \frac{(\varphi+\delta)(\rho b + \frac{\varepsilon\eta}{\varepsilon+\rho})}{A[\varepsilon h(1-\alpha) + (\rho b + \frac{\varepsilon\eta}{\varepsilon+\rho})\alpha]}.$$

It means that the ratio of k over $\bar{k}$ is some constant. At equilibrium, $k = \bar{k}$. Therefore, we know that

$$(\varphi+\delta)(\rho b + \frac{\varepsilon\eta}{\varepsilon+\rho}) = A[\varepsilon h(1-\alpha) + (\rho b + \frac{\varepsilon\eta}{\varepsilon+\rho})\alpha] \tag{25}$$

and

$$k = \bar{k}$$

Substituting equation (18) into equation (20), we can get that

$$h\varphi k - h(Ak^{\alpha}\bar{k}^{1-\alpha} - \delta k) = [h(\varepsilon+\rho) - 1 - \eta](qs+k).$$

Substituing equation (24) into above equation, we obtain that

$$(\varphi+\delta)(\rho h-1-\eta)=A\{\alpha[h(\varepsilon+\rho)-1-\eta]-\varepsilon h\}. \tag{26}$$

From equation (23), we know that

$$\frac{(1-\alpha)A}{\rho b+\frac{\varepsilon\eta}{\varepsilon+\rho}}k=\frac{qs+k}{h}=c. \tag{27}$$

By the fact that $k=\bar{k}$ and equation (18), the budget constraint can be rewritten as

$$dk=(Ak-\delta k-\frac{(1-\alpha)A}{\rho b+\frac{\varepsilon\eta}{\varepsilon+\rho}}k-q\iota)dt.$$

Since at equilibrium

$$\iota\equiv 0,$$

we can get that

$$d\bar{k}=[A-\delta-\frac{(1-\alpha)A}{\rho b+\frac{\varepsilon\eta}{\varepsilon+\rho}}]\bar{k}dt.$$

It means that

$$\varphi=A-\delta-\frac{(1-\alpha)A}{\rho b+\frac{\varepsilon\eta}{\varepsilon+\rho}}. \tag{28}$$

Solving the equations system consisting of (25), (26), and (28), we can obtain the values of b, h, and φ as follows.

$$b=1/\rho,$$
$$h=\frac{\eta}{\varepsilon+\rho},$$
$$\varphi=A-\delta-\frac{(1-\alpha)A(\varepsilon+\rho)}{\varepsilon+\rho+\varepsilon\eta}.$$

Substituing the results into equation "17", we obtain that

$$\begin{aligned}0 \quad = \quad & \max_{c,\iota}\{\log\frac{(1-\alpha)A(\varepsilon+\rho)}{(\varepsilon+\rho)+\varepsilon\eta}+\log k-\log k \\ & +\eta\log[\frac{\eta(1-\alpha)A}{(\varepsilon+\rho)+\varepsilon\eta}k]-\eta\log[\frac{\eta(1-\alpha)A}{(\varepsilon+\rho)+\varepsilon\eta}k] \\ & +(b+\psi+h)\varphi-\rho a+\varepsilon\psi\log k+\varepsilon h\log k\}.\end{aligned}$$

It is easy to check that the sum of coefficients of $\log k$ term is zero. When χ takes the following value

$$\frac{1}{\rho}\log\frac{(1-\alpha)A(\varepsilon+\rho)}{\varepsilon+\rho+\varepsilon\eta}+\frac{1}{\rho^2}[A-\delta-\frac{(1-\alpha)A(\varepsilon+\rho)}{\varepsilon+\rho+\varepsilon\eta}],$$

the sum of constant terms is also zero.

From above analysis, we can find that the stochastic bubble exists only if the value of ε, which measures the probability that the bubble bursts, is less than the upper limit,

$$\bar{\varepsilon}\equiv\frac{\eta(1-\alpha)A-\rho}{\eta+1}. \tag{29}$$

In order to guarantee the value of $\bar{\varepsilon}$ is positive, the parameters restriction (16) should be satisfied. Since the value of $\bar{\varepsilon}$ is just equal to the difference between the growth rate of bubbleless economy and the real interest rate, a positive $\bar{\varepsilon}$ implies that the bubbleless economy is dynamically inefficient.

If the bubble lasts, the stochastic bubbly economy can be described by following three equations.

$$c=\frac{(1-\alpha)A(\rho+\varepsilon)}{\varepsilon\eta+\rho+\varepsilon}k, \tag{30}$$

$$q=[\frac{\eta(1-\alpha)A}{\varepsilon\eta+\rho+\varepsilon}-1]k, \tag{31}$$

$$\begin{aligned}\frac{\dot{c}}{c} &= \frac{\dot{k}}{k}=\frac{\dot{q}}{q} \\ &= A-\delta-\frac{(1-\alpha)A(\rho+\varepsilon)}{\varepsilon\eta+\rho+\varepsilon}\end{aligned} \tag{32}$$

Once the bubble bursts, the economy jumps to the bubbleless balanced growth path described by part (a) of proposition 3.

From equation (31), we can find that the relationship between the size of bubble and the capital stock in the stochastic economy is similar to what suggested by Weil (1987), which is based on Tirole's OLG framework. However, the explicit solution permits us to analyze other economic issues more intuitively.

By comparing the explicit solutions of the deterministic bubbly economy, the deterministic bubbleless economy, and the stochastic bubbly economy, we can easily find the impact of the uncertainty resulting from the stochastic bubble. Given the fact that

$$\frac{(1-\alpha)A+\rho}{\eta+1} < \frac{(1-\alpha)A(\rho+\varepsilon)}{\varepsilon\eta+\rho+\varepsilon} < (1-\alpha)A,$$

it is obvious that consumption in the stochastic economy is larger than that in bubbleless economy but less than that in the deterministic bubbly economy. Since the aggregate investment is equal to $Ak - c$, the investment in the stochastic economy is less than that in the deterministic bubbleless economy but larger than that in the deterministic bubbly economy. Thus, stochastic bubbles stimulate consumption and crowd out investment. The impact of the stochastic bubble is similar to what we obtain in the deterministic case, {but to a weaker extent}. The comparison also suggests that the growth rate in the stochastic economy is higher than that in the bubbly economy without uncertainty and lower than that in the deterministic bubbleless case.

From the explicit example, we can also see directly how the probability that the bubble bursts, measured by ε, affects the real economy. The fact that

$$\frac{\partial(c/k)}{\partial\varepsilon} < 0$$

illustrates a negative relationship between consumption and {the probability of bursting}. Since the growth rate of the real economy is equal to $A-\delta-\frac{c}{k}$, there is a positive relationship between the probability measured by ε and economic growth rate. A smaller probability of bursting leads to more consumption and a lower growth rate; while it is just the opposite with a higher probability.

The size of the stochastic bubble is also affected by the probability that the bubble collapses. A higher probability would reduce the size of bubble; and, a smaller possibility to burst allows for a larger size of the bubble. As the burst probability of bubbles, measured by ε, converges to zero, the stochastic bubbly economy approaches the deterministic bubbly economy; while, as the value of ε converges to its upper limit, $\bar{\varepsilon}$, the stochastic bubbly economy converges to the deterministic bubbleless economy.

The relationships obtained above are consistent with our intuitions. When the bubble has a higher possibility to burst, the expected wealth decreases. By the wealth effect, consumption will also decrease. Investors would adjust their portfolios and put more weight on physical capital. Thus, the value of financial assets would be even lower. Since the fundmental value of the financial asset always be zero, the size of bubble would be reduced. At the same time, higher investment stimulates economic growth.

Furthermore, how the economic environment affects the upper limit of the probability that the bubble bursts, measured by $\bar{\varepsilon}$, gives us a hint on what situation this type of stochastic bubbles more possibly arise in. From equation (29), we can find that higher values of η and A, and lower value of ρ, will raise the upper limit, $\bar{\varepsilon}$. The higher upper bar of the probability of bursting means a higher probability for the existence of stochastic bubbles. Thus, this type of stochastic bubbles more possibly appear in an economy where people more care about their status, or, the technology level is higher, or, the society is more patient. It is consistent with our finding from the deterministic case.

The value of $\bar{\varepsilon}$ is equal to the difference between the growth rate of bubbleless economy and the real interest rate. The difference usually measures the size of the dynamic inefficiency of the bubbleless economy. Therefore, the stochastic bubbles more likely emerge in an economy with the heavier extent of dynamic inefficiency. The result is consistent with the finding by Weil (1987) that bases on Tirole's OLG framework.

In addition, the explicit example of stochastic bubble also verifies the existence of a series of stochastic bubbly equilibria. From equation(31), we can obtain that

$$\varepsilon = \frac{\frac{\eta(1-\alpha)A}{\frac{q}{k}+1} - \rho}{\eta + 1}.$$

To satisfy the following restriction

$$0 \leq \varepsilon < \bar{\varepsilon},$$

together with equation (29), we need that

$$0 < q \leq [\frac{\eta(1-\alpha)A}{\rho} - 1]k.$$

Since the growth rate of the bubble is equal to the growth rate of the real economy at this stochastic bubbly equilibrium, the inequality above will hold at any time. Thus, the following proposition about stochastic bubbly equilibrium can be naturally obtained.

Proposition 5 *In an economy with "the spirit of capitalism", the preference function and production function are given by (9) and (14), respectively. Given the parameters restriction (16), for any positive innitial value of bubble, q_0, which is not larger than the threshold value of $[\frac{\eta(1-\alpha)A}{\rho} - 1]k_0$, there exists a stochastic bubbly equilibrium, where the process of the bubble is given by*

$$dq = \langle \begin{array}{c} \varphi q dt, \text{ with probability } 1 - \varepsilon dt \\ -q, \text{ with probability } \varepsilon dt \end{array},$$

and the real economy grows at the rate of φ, which is equal to

$$A - \delta - \frac{(1-\alpha)A(\rho+\varepsilon)}{\varepsilon\eta + \rho + \varepsilon}.$$

The value of ε, is given by

$$\frac{\frac{\eta(1-\alpha)A}{q_0/k_0+1} - \rho}{\eta + 1}.$$

5 Discussions on Optimal Policy

In this section, we first explore the social optimal solution for an economy with "the spirit of capitalism". Then, we discuss two policies which can help the competitive economy approach to the social optimal.

5.1 Social Optimal

Given the form of utility function as (9), the optimal question of the central planner is given by

$$\max \int_0^\infty e^{-\rho t} \log c dt$$

subjective to the real resource constraint

$$\dot{k} = f(k) - \delta k - c. \tag{33}$$

Here, I consider a general form of the production function.

The social optimal of the economy can be described by above real resource constraint (33), the Euler equation

$$-\frac{\dot{c}}{c} = \rho - (f'(k) - \delta), \tag{34}$$

and the transversality condition

$$\lim_{t \to \infty} e^{-\rho t} \frac{k}{c} = 0.$$

It is obvious that no bubble exists at the social optimal. Since the marginal utility of wealth is always zero, the central planer has no incentive to chase wealth for any other purpuses except for consumption flows in the furture. Therefore, there is also no overaccumulation of physical capital at the social optimal.

By comparing the social optimal with the competitive economy, we can easily have following findings. It is the status seeking in the preference of individuals that distorts the competive economy. In the bubbleless economy, physical capital is accumulated too much. The emergence of bubble relieves the situation but cannot yet help the economy approach to the social optimal. Because bubbles only can substitute but cannot totally take the place of the physical capital in the process of status seeking.

5.2 Discussions on Policies

Given above anaysis, it is natural to ask whether a policy to limit capital accumulation can push the competitive economy to the social optimal. To answer the question, we consider the following case about the tax of investment return.

5.2.1 Tax of Investment Return

Suppose the tax rate on investment return is denoted by τ. The total income of the government from the tax is paid back to the households by the lump-sum transfer, T. Thus, the budget constraint of the representive individual can be written as

$$\dot{a} = (r - \tau)k - c + \dot{q}s + \Pi + T.$$

From the first order conditions of the representive individual's optimal question, we obtain that the non-arbitrage condition

$$\frac{\dot{q}}{q} = r - \tau, \tag{35}$$

and the Euler equation

$$\frac{\dot{\mu}}{\mu} = \rho - \eta\frac{c}{a} - (r - \tau). \tag{36}$$

To let the economy approach to the social optimal, the government should set the tax rate of investment return to be equal to the ratio of marginal utility of wealth over the marginal utility of consumption, i.e.,

$$\tau^E = \eta\frac{c}{a}.$$

However, under above optimal tax rate, bubble might still exist. Because

$$\frac{\dot{\mu}}{\mu} + \frac{\dot{q}}{q} = \rho - \tau^E < \rho,$$

which implies that the transversality condition (7) is statisfied. To illustrate the point, we consider the following example. Assume the production function is given by Ak^α. At the steady state, we obtain that

$$\alpha A k^{\alpha-1} - \delta = \eta\frac{c}{a} = \rho.$$

Here, I use an variable with an upper bar to denote its value at the bubbly steady state. It is easy to check that the capital, k, and, the consumption, c, are both on the economic efficient level. However, with the parameters restriction (13), we can obtain that

$$\begin{aligned}
\frac{a}{k} &= \frac{\eta}{\rho}\frac{c}{k} = \frac{\eta}{\rho}(Ak^{\alpha-1} - \delta) \\
&= \frac{\eta}{\rho}\frac{\rho + (1-\alpha)\delta}{\alpha} \\
&> \frac{\eta}{\rho}\frac{(1-\alpha)\delta}{\alpha} > 1.
\end{aligned}$$

It implies that

$$q > 0.$$

Therefore, a bubbly economy with the optimal tax rate, τ^*, can approach to the social optimal.

5.2.2 Tax of Wealth

However, bubbles might burst at any time. The burst of bubbles increases the economic inefficiency and distorts the economy despite an optimal tax rate on investment return. Because the government need spend time in detecting the new ratio of marginal utility of wealth over the marginal utility of consumption. Thus, the policy that we desire is the one that pushes the economy to the social optimal and also expels bubbles. Here, I demonstrate that the tax of wealth is what we desired.

Suppose the tax rate of wealth is denoted by θ. The total income of the government from the tax is paid back to the households by the lump-sum transfer, T. The budget constraint of households is given by

$$\dot{a} = rk - c + \dot{q}s + \Pi - \theta a + T.$$

Solveing the optimal question of households, we obtain that the non-arbitrage condition

$$\frac{\dot{q}}{q} = r,$$

and the Euler equation

$$\frac{\dot{\mu}}{\mu} = \rho - \eta\frac{c}{a} - r + \theta.$$

Obviously, the optimal tax rate of wealth, θ^E, should also be the ratio of marginal utility of wealth over the marginal utility of consumption, $\eta\frac{c}{a}$. With the optimal tax rate, θ^E, we can obtain that

$$\frac{\dot{\mu}}{\mu} + \frac{\dot{q}}{q} = \rho.$$

It means that the transversality condition (7) will rule out bubbles.

6 The Review of Methodology

This section provides a brief survey of the methodologies to introduce rational bubbles into a general equilibrium framwork. We can broadly divide the literature on the methodologies into three types. Each type is briefly reviewed below.

Largely influenced by the non-existence result presented in Tirole (1982) in the infinite horizon framework, the early research turned to finite-horizon models, particularly the overlapping generations models. This branch starts from Tirole (1985). He introduces intrinsically useless paper into the Diamond model and argues that the existence of bubbles is dependent on the inefficiency of the bubbleless equilibrium, i.e, the real interest rate is less than the growth rate of output. The inefficiency condition can also be interpreted to imply that bubbles exist only if the present value of aggregate income (or aggregate consumption) is infinite. Weil (1987) complements Tirole's (1985) analysis by studying a stochastic bubble, which is believed to collapse with a constant probability. When the probability of the persistence of a bubble is larger than a threshold level, the so called "minimum rate of confidence", this type of stochastic bubbles will exist. It is also proved that this minimum rate of confidence depends on the degree of inefficiency of the bubbleless economy: the more inefficient the bubbleless economy, the lower the value of the minimum rate of confidence.

The intuition behind the above findings is simple. The existence of either deterministic bubbles or stochastic bubbles crowds out productive investment which in turn decreases the capital level, and raises the real interest rate. Given the fact that bubbles grow at the same rate as the real interest rate, if the bubbleless economy is already efficient, then bubbles should grow at a higher speed, which cannot be supported by economic growth. Thus, bubbles will be ruled out by real resource constraint.

Within the setting of endogenous growth, this framework also can be used to explore the relationship between bubbles and economic growth. Saint-Paul (1992), Yanagawa and Grossman (1992), and King and Ferguson (1993) reported that bubbles would retard growth when endogenous growth was introduced by externality in capital accumulations. On the other hand, Olivier (2000) argued that bubbles on equity would encourage the creation of firms and promote economic growth if endogenous growth was due to research and development (R&D).

Recently, under the assumption of imperfect financial markets, Tirole's framework has been used to explain a number of issues. For example, Caballero and Krishnamurthy (2006) explores emerging market crisis resulting from the bursting of bubbles; Caballero, Farhi, and Hammour (2006) provides a framework for understanding the "speculative growth" episodes in the U.S.; Farhi and Tirole (2010) analyzes the relationship between bubbles and liquidity; and Martin and Ventura (2010) revisits economic growth with bubbles.

Another branch of this literature emphasizes the importance of no-Ponzi-game conditions (constraints on debt accumulation) for the existence of bubbles in infinite-horizon models. Kocherlakota (1992) first pointed this out by showing that an individual cannnot reduce his asset position permanently when facing constraints on debt accumulation. Technically, these constraints help to guarantee transversality conditions not to be violated when asset price has a bubble term. If this constraint is a wealth constraint, the sufficient and necessary conditions for the existence of a bubble is zero net supply of the asset. On the other hand, if this constraint is an exogenous short sales constraint, bubbles can arise if and only if the growth rate of individual's income is not less than the real interest rate. As Kocherlakota (2008) stressed, with short sales constraints, bubbles can arise even if the present value of aggregate consumption is finite.

Based on this finding, Kocherlakota (2009) modeled a stochastic bubble in the price of collateral, which is intrinsically worthless, by introducing borrowing constraints faced by infinitely-lived entrepreneurs. The effects of bursting bubbles and the discussions of policies after the collapse of bubbles are provided. Wang and Wen (2009) took the analysis a step further by studying bubbles that may arise on assets with positive intrinsic values. Miao and Wang (2012) provides a theory on credit-driven bubble in the pricing of equity.

The third method of modeling rational bubbles is by assuming that wealth has a direct effect on the preference function. This is modeled in the same way as "the spirit of capitalism" models.

Kamihigashi (2008) first introduced rational bubbles on assets by this method. In this paper, he argued that bubbles may exist if "the marginal utility of wealth does not decline to zero as wealth goes to infinity". The relationship between bubbles and output, or, capital stock, depends on the property of the production function. For a production function with decreasing returns to scale, this relationship is negative. On the other hand, it might be positive for a production function with increasing returns to scale. However, all of these analyses are under the restrictive assumption of linear utility in consumption.

Kamihigashi (2009) discussed the existence of asset price bubbles in an exchange economy with status seeking. When status is modeled by the ratio of individual wealth to aggregate wealth, bubbles are ruled out by the transversality condition. This is because the marginal utility of individual wealth converges to zero along with a growing price path. This means that the effect of status seeking disappears. However, if the status is formulated by the difference of individual wealth and aggregate wealth, then bubbles might exist since the marginal utility of wealth remains as a positive constant.

7 Conclusion

This paper focuses on rational bubbles driven by the pursuit of status in an infinite-horizon model. In an economy with "the spirit of capitalism", as long as, eventually, the marginal utility of holding wealth is not trivial relative to the marginal utility of consumption, rational bubbles can emerge. The analysis of dynamics suggests similar results as what given by Tirole (1985). However, my infinite-horizon model eliminates the concern of incomplete market generated by the structure of overlapping generations. Since infinite-horizon framework is the common base for a vast literature on asset pricing and macroeconomics, many economic issues about bubbles can be discussed on the basis of my framework.

In addition, this paper also discusses an economy where a bubble might burst with an exogenous probability. It gives a simple theoretical foundation to discuss economic implications of the collapse of a bubble. As an interesting further direction, issues about financial crisis can be explored by introducing a banking sector into this framework.

References

[1] Abel, Andrew B., "Asset Prices under Habit Formation and Catching up with the Joneses", The American Economic Review Papers and Proceedings, Vol. 80, No. 2, pp.38-42, May, 1990

[2] Allen, F., and D. Gale, "Bubbles and Crises", The Economic Journal, Vol. 110, No. 460, pp. 236-255, Jan., 2000

[3] Allen, F., and G. Gorton, "Churning Bubbles", The Review of Economic Studies, Vol. 60, No. 4., pp. 813-836, Oct., 1993

[4] Bakshi Gurdip S. and Zhiwu Chen, "The Spirit of Capitalism and Stock-Market Prices", The American Economic Review, Vol. 86, No. 1, pp. 133-157, Mar., 1996

[5] Blanchard, O. J., and M. W. Watson, "Bubbles, Rational Expectations and Financial Markets", NBER Working Paper Series, July 1982

[6] Brunnermeier, "Bubbles", unpublished manuscript, 2009

[7] Caballero, R. J., E. Farhi, and M. L. Hammour, "Speculative Growth: Hints from the U.S. Economy", The American Economic Review, Vol. 96, No. 4 (Sep., 2006), pp. 1159-1192

[8] Caballero, R. J. and A. Krishnamurthy, "Bubbles and capital flow volatility: Causes and risk management", Journal of Monetary Economics 53 (2006) 35–53

[9] Conlon, J. R., "Simple Finite Horizon Bubbles Robust to Higher Order Knowledge", Econometrica, Vol. 72, No. 3 (May, 2004), pp. 927-936

[10] Cozzi, G., "Culture as a Bubble", The Journal of Political Economy, Vol. 106, No. 2 (Apr., 1998), pp. 376-394

[11] Diamond, P. A., "National Debt in a Neoclassical Growth Model", The American Economic Review, Vol. 55, No. 5, Part 1 (Dec., 1965), pp. 1126-1150

[12] Diba, B. T., and H. I. Grossman, “On the Inception of Rational Bubbles”, The Quarterly Journal of Economics, Vol. 102, No. 3 (Aug., 1987), pp. 697-700

[13] Diba, B. T., and H. I. Grossman, “The Theory of Rational Bubbles in Stock Prices”, The Economic Journal, Vol. 98, No. 392 (Sep., 1988), pp. 746-754

[14] Farhi, E., and J. Tirole, “Bubbly Liquidity”, The Review of Economic Studies, vol. 79, n. 2, 2012, p. 678-706.

[15] Kamihigashi, T., “The spirit of capitalism, stock market bubbles and output fluctuations”, International Journal of Economic Theory 4, 3-28, 2008

[16] Kamihigashi, T., “Status Seeking and Bubbles”, International Trade and Economic Dynamics – Essays in Memory of Koji Shimomura, 383-392, 2009

[17] Kiyotaki, Nobuhiro and John Moore, “Credit Cycles”, Journal of Political Economy 105, 211-248, 1997

[18] Kocherlakota, N. R., “Bubbles and Constraints on Debt Accumulation”, Journal of Economic Theory 57, 245-256, 1992

[19] Kocherlakota, N. R., “Bursting Bubbles: Consquences and Cures”, unpublished manuscript, January 2009

[20] Kurz, M., “Optimal Economic Growth and Wealth Effects”, International Economic Review, Vol. 9, No. 3 (Oct., 1968), pp. 348-357

[21] LeRoy, S. F., “Rational Exuberance”, Journal of Economic Literature, Vol. 42, No. 3 (Sep., 2004), pp. 783-804

[22] Martin, A., and J. Ventura, “Economic Growth with Bubbles”, American Economic Review, 2012, 102(6): 3033-3058

[23] Miao Jianjun and Pengfei Wang, “Bubbles and Credit Constraints”, unpublished manuscript, September 2012

[24] O’Hara, M., “Bubbles: Some Perspectives (and Loose Talk) from History”, The Review of Financial Studies v21 n1 2008

[25] Olivier, J., “Growth-Enhancing Bubbles”, International Economic Review, Vol. 41, No. 1 (Feb., 2000), pp. 133-151

[26] Rebelo, S., and D. Xie, “On the optimality of interest rate smoothing”, Journal of Monetary Economics 43, 263-282, 1999

[27] Romer, Paul M., “Increasing Returns and Long-Run Growth”, The Journal of Political Economy, Vol.94, No. 5, 1002-1037, Oct., 1986

[28] Santos, M. S., and M. Woodford, “Rational Asset Pricing Bubbles”, Econometrica, Vol. 65, No. 1, 19-57, January 1997

[29] Sidrauski, M., “Rational Choice and Patterns of Growth in A Monetary Economy”, The American Economic Review, Volume 57, Issue 2, Papers and Proceedings, 534-544, May 1967

[30] Stiglitz, J. E., “Symposium on Bubbles”, The Journal of Economic Perspectives, Vol. 4, No. 2. (Spring, 1990), pp. 13-18.

[31] Tirole, J., “On the Possibility of Speculation under Rational Expectations”, Econometrica, Vol. 50, No. 5, Sepetember 1982

[32] Tirole, J., “Asset Bubbles and Overlapping Generations”, Econometrica, Vol. 53, No. 6, November 1985

[33] Web, Max, “The Protestant Ethic and The Spirit of Capitalism”, New York: Charles Scribner’s Sons, 1958

[34] Weil, P., “Confidence and the Real Value of Money in an Overlapping Generations Economy”, Quarterly Journal of Economics, Vol. CII, Issue 1, February 1987

[35] Weil, P., “On the Possibility of Price Decreasing Bubbles”, Econometrica, Vol. 58, No. 6 (Nov., 1990), pp. 1467-1474

[36] Yanagawa, N., and G. M. Grossman, “Asset Bubbles and Endogenous Growth”, NBER Working Paper, February 1992

[37] Zhou, G., “Rational Bubbles and The Spirit of Capitalism”, MPRA Paper 33988, 2011

[38] Zhou, G., “Rational Equity Bubbles”, Forthcoming in Annals of Economics and Finance, Nov, 2013

[39] Zou, H., “‘The Spirit of Capitalism’ and Long-Run Growth”, European Journal of Political Economy Vol. 10 (1994) 279-293

Appendix

A Sufficiency of FOCs and TVCs for Dynamic Optimization

In this appendix, I prove that the first order conditions and transversality conditons are sufficient to solve the representive individual's optimal question. It also demonstrates that the forms of transversality conditions given by (3) and (4) are correct.

Individual's optimality question can be given by

$$\max \int_0^\infty e^{-\rho t} U(c,a) dt,$$

$$\begin{aligned} s.t. \quad &: \quad a = qs + k, \\ \dot{a} &= f(k) - \delta k - c + \dot{q}s, \\ a &\geq 0, \\ & \quad a_0 \text{ is given.} \end{aligned}$$

Here, $U(c,a)$ is concave, and $U''_{ca} = U''_{ac} = 0$, $f''(k) \leq 0$. The process of dq is exogenous for any individual.

Suppose that $\{c^*, k^*, s^*, a^*\}$ is the solution which satisfies the FOCs and TVCs, and $\{c, k, s, a\}$ is another possible choice. The difference of utilities evaluated at $\{c^*, k^*, s^*, a^*\}$ and at $\{c, k, s, a\}$ is given below.

$$\begin{aligned} D \equiv & \int_0^\infty e^{-\rho t}\{U(c^*, a^*) + \lambda^*(a^* - qs^* - k^*) + \mu^*[f(k^*) - \delta k^* - c^* + \dot{q}s^* - \dot{a}^*] \\ & - U(c,a) - \lambda^*(a - qs - k) - \mu^*[f(k) - \delta k - c + \dot{q}s - \dot{a}]\} dt, \end{aligned}$$

where λ^* and μ^* are the multipliers that satisfy the FOCs and TVCs.

By the fact that

$$\mu^* \dot{q} = \lambda^* q,$$

we obtain that

$$
\begin{aligned}
D &= \int_0^\infty e^{-\rho t}\{U(c^*,a^*)-U(c,a)+\mu^*[f(k^*)-f(k)+\delta(k^*-k)] \\
&\quad +\lambda^*(a^*-a)-\mu^*(c^*-c)-\lambda^*(k^*-k)-\mu^*(\dot{a}^*-\dot{a})\}dt.
\end{aligned}
$$

From the concavity of $U(c,a)$ and $f^{''}(k)\leq 0$, we can find that

$$
\begin{aligned}
D &\geq \int_0^\infty e^{-\rho t}\{U_c^{'}(c^*,a^*)(c^*-c)-\mu^*(c^*-c) \\
&\quad +\mu^*[f'(k^*)-\delta](k^*-k)-\lambda^*(k^*-k) \\
&\quad +U_a^{'}(c^*,a^*)(a^*-a)+\lambda^*(a^*-a)-\mu^*(\dot{a}^*-\dot{a})\}dt.
\end{aligned}
$$

By

$$U_c^{'}(c^*,a^*)=\mu^*,$$

and

$$\mu^*[f'(k^*)-\delta]=\lambda^*,$$

we obtain that

$$
\begin{aligned}
D &\geq \int_0^\infty e^{-\rho t}[U_a^{'}(c^*,a^*)+\lambda^*](a^*-a)dt-\int_0^\infty e^{-\rho t}\mu^*(\dot{a}^*-\dot{a})dt, \\
&= \int_0^\infty e^{-\rho t}[U_a^{'}(c^*,a^*)+\lambda^*+\dot{\mu}^*-\rho\mu^*](a^*-a)dt-e^{-\rho t}\mu^*(a^*-a)|_0^\infty.
\end{aligned}
$$

Since

$$U_a^{'}(c^*,a^*)+\lambda^*+\dot{\mu}^*=\rho\mu^*,$$

we obtain that

$$D\geq -\lim_{t\to\infty}e^{-\rho t}\mu^*a^*+\lim_{t\to\infty}e^{-\rho t}\mu^*a.$$

By the fact of

$$a\geq 0,$$

and the tranversality condition

$$\lim_{t\to\infty}e^{-\rho t}\mu^*a^*=0,$$

it is easy to find that

$$D \geq 0.$$

This means that the solution that satisfies with the FOCs and TVCs is optimal.

B Rational Bubbles Introduced by A Binding Credit Constraint

In this appendix, I use a simple infinite-horizon model to illustrate that a binding credit constraint works similarly as "the spirit of capitalism" from the point of view of introducing bubbles. Both methods prevent the transverslity condition ruling out bubbles.

The economy is composed of two sectors, the savers and the investors. Since the sector of savers, who only provide funds to investors, is trivial, I focus on the sector of investors. The representive investor wishes to maximize the sum of time discounted utility flows

$$\int_0^{\infty} e^{-\rho t} U(c) dt, \ \rho > 0$$

subjective to the budget constraint

$$\dot{a} = f(k) - \delta k - c + \dot{q}s - rL,$$

and also facing the credit constraint

$$\xi(qs + k) \geq L,$$

which is similar as the type of Kiyotaki and Moore (1997). Here, $a \equiv qs + k - L$, L denotes the loan borrowed from savers, r is the interest rate for the loans, $0 < \xi < 1$ is the pledge ratio of the investor's wealth.

The Hamiltonian of the optimal problem can be written as

$$\begin{aligned}\mathcal{H} &= U(c) + \lambda(a - qs - k + L) \\ &\quad + \zeta[\xi(qs + k) - L] \\ &\quad + \mu(f(k) - \delta k - c + \dot{q}s - rL).\end{aligned}$$

From the first order conditions, we obtain that the non-arbitrage condition

$$\frac{\dot{q}}{q} = \frac{\lambda - \zeta\xi}{\mu},$$

and the intertemporal substitiute condition

$$\frac{\dot{\mu}}{\mu} = \rho - \frac{\lambda}{\mu}.$$

At equilibrium, the transversality condition is given by

$$\lim_{t\to\infty} e^{-\rho t}\mu q = 0.$$

It is easy to find that

$$\frac{\dot{\mu}}{\mu} + \frac{\dot{q}}{q} = \rho - \frac{\zeta\xi}{\mu}.$$

The condition of

$$\lim_{t\to\infty} \frac{\zeta\xi}{\mu} > 0 \tag{37}$$

prevents the transversality condition ruleing out bubbles. When credit is scarce so that the credit constraint is binding, i.e.,

$$\zeta > 0,$$

the condition (37) is natually satisfied. Therefore, bubbles might emerge in an economy with a binding credit constraint.

The result is inconsistent with the argument by Miao and Wang (2012). In their paper, Kiyotaki-Moore-type collateral constraint will rule out bubbles. The reason is that the collateral in Kiyotaki-Moore-type constraint is just the physical capital while it is actually the bubble on equity of firm that is discussed in the baseline model of Miao and Wang (2012). When they adoptes the Kiyotaki-Moore-type collateral constraint, the equity of firm is not a collateral. The bubble on equity provides no "dividend yields" and grows at the rate of real interest. Thus, the transverality condition will be violated when bubbles arise. Intuitively, with Kiyotaki-Moore-type credit constraint, no one has any incentive to hold an equity with bubble forever since the equity cannot be the collateral. Sooner or later, the equity with bubble will be sold out for the purpose of material rewards. It is the behavior that expels the equity bubbles.

C Proof of Propositon 1

The neoclassical growth economy can be described by following equations system.

$$\dot{k} = Ak^{\alpha} - \delta k - c, \tag{38}$$

$$\dot{q} = (\alpha A k^{\alpha-1} - \delta)q, \tag{39}$$

$$\dot{c} = c[\eta \frac{c}{k+q} + \alpha A k^{\alpha-1} - \delta - \rho]. \tag{40}$$

At the bubbleless steady state, the value of bubble is equal to zero, i.e.,

$$q^* = 0.$$

From equation (38) and equation (40), we can obtain below two equations at the steady state.

$$Ak^{\alpha} - \delta k = c,$$

$$\delta + \rho = \eta \frac{c}{k} + \alpha A k^{\alpha-1}.$$

It is easy to slove above equations system. The the unique solution is given by

$$k^* = [\frac{\rho + (1+\eta)\delta}{(\eta+\alpha)A}]^{\frac{1}{\alpha-1}},$$

$$c^* = [\frac{\rho + (1+\eta)\delta}{(\eta+\alpha)A}]^{\frac{1}{\alpha-1}}[\frac{\rho + (1+\eta)\delta}{\eta+\alpha} - \delta].$$

In addition,

$$0 < \alpha < 1$$

makes sure that $c^* > 0$.

At the bubbly steady state, the value of bubble is some positive constant. From equation (39), we can solve that

$$k^{**} = (\frac{\delta}{\alpha A})^{\frac{1}{\alpha-1}}.$$

With equation (38), we obtain that

$$c^{**} = (1-\alpha)(\frac{\delta}{\alpha})^{\frac{\alpha}{\alpha-1}} A^{\frac{1}{1-\alpha}}.$$

By equation (40), we get that

$$q^{**} = [\frac{\eta\delta(1-\alpha)}{\rho\alpha} - 1](\frac{\delta}{\alpha A})^{\frac{1}{\alpha-1}}.$$

In order to gurantee $q^{**} > 0$, we need a parameter restriction given by

$$\eta\delta(1-\alpha) > \rho\alpha.$$

The restriction implies that $aAk^{*\alpha-1} - \delta < 0$. While the real interest rate at the bubbly steady state, $aAk^{**\alpha-1} - \delta$, is equal to zero. Given the decreasing return of production function Ak^α, we know that

$$k^{**} < k^*.$$

At any steady state,

$$c = Ak^\alpha - \delta k.$$

Since $Ak^\alpha - \delta k$ is a strictly concave function, it is easy to find that

$$c^{**} > c^*.$$

D Proof of Proposition 2

The proof mainly follows the method of Miao and Wang (2012).

D.1 Around the Bubbly Steady State

In order to check the stability of the bubbly steady state, firstly, we need to log-linearize the dynamic system of (38), (39), and (40) around the steady state $\{k^{**}, q^{**}, c^{**}\}$.

Here, we define

$$\hat{x} \equiv \log \frac{x}{x^{**}}.$$

It implies that

$$\frac{\dot{x}}{x} = \dot{\hat{x}}.$$

The log-linearized system is given by

$$\begin{pmatrix} \dot{\hat{k}} \\ \dot{\hat{q}} \\ \dot{\hat{c}} \end{pmatrix} = \Psi \begin{pmatrix} \hat{k} \\ \hat{q} \\ \hat{c} \end{pmatrix},$$

where

$$\Psi \equiv \begin{pmatrix} (\alpha-1)A(k^{**})^{\alpha-1} + \frac{c^{**}}{k^{**}} & 0 & -\frac{c^{**}}{k^{**}} \\ \alpha(\alpha-1)A(k^{**})^{\alpha-1} & 0 & 0 \\ \alpha(\alpha-1)A(k^{**})^{\alpha-1} - \frac{\eta k^{**}}{k^{**}+q^{**}} & -\frac{\eta q^{**}}{k^{**}+q^{**}} & \eta \end{pmatrix}.$$

Here,

$$\begin{aligned} \Psi_{1,1} &= (\alpha-1)A(k^{**})^{\alpha-1} + \frac{c^{**}}{k^{**}} = 0, \\ \Psi_{1,3} &< 0, \Psi_{2,1} < 0, \Psi_{3,1} < 0, \Psi_{3,2} < 0. \end{aligned}$$

The characteristic equation of matrix Ψ is given by

$$F(\lambda) = |\Psi - \lambda I| = -\lambda^3 + \eta\lambda^2 + \Psi_{1,3}\Psi_{3,1}\lambda + \Psi_{1,3}\Psi_{2,1}\Psi_{3,2}.$$

It can be rewritten as

$$
\begin{aligned}
F(\lambda) &= -(\lambda-\psi_1)(\lambda-\psi_2)(\lambda-\psi_3) \\
&= -\lambda^3+(\psi_1+\psi_2+\psi_3)\lambda^2 \\
&\quad -(\psi_1\psi_2+\psi_1\psi_3+\psi_2\psi_3)\lambda+\psi_1\psi_2\psi_3.
\end{aligned}
$$

Thus, we can know that characteristic roots ψ_1, ψ_2, ψ_3, have to satisfy below conditions.

$$\psi_1+\psi_2+\psi_3=\eta>0, \tag{41}$$

$$\psi_1\psi_2+\psi_1\psi_3+\psi_2\psi_3=-\Psi_{1,3}\Psi_{3,1}<0, \tag{42}$$

$$\psi_1\psi_2\psi_3=\Psi_{1,3}\Psi_{2,1}\Psi_{3,2}<0. \tag{43}$$

Since $F(0)<0, F(-\infty)=+\infty$, matrix Ψ has at least one negative characteristic root, which I denote by ψ_1. We discuss the other two characteristic roots, ψ_2 and ψ_3 by two cases.

Case 6 *ψ_2 and ψ_3 are both real numbers. From equation (43), we can obtain that either $\psi_2>0, \psi_3>0$, or $\psi_2<0, \psi_3<0$. By equation (41) and $\psi_1<0$, it is impossible that $\psi_2<0, \psi_3<0$. Therefore, $\psi_2>0, \psi_3>0$.*

Case 7 *ψ_2 and ψ_3 are one pair of complex numbers. I denote them by $\psi_2 \equiv a+bi, \psi_3 \equiv a-bi$. It is easy to find that $\psi_1+\psi_2+\psi_3=2a+\psi_1$. Given equation (41) and $\psi_1<0$, the value of a must be positive.*

Based on above discussions, we are sure that matrix Ψ has only one nagetive eigenvalue. It means that the bubbly steady state is a local saddle point since the dynamic system has only one state varable, k.

D.2 Around the Bubbleless Steady State

At first, we linearize q and loglinearize k and c around the bubbleless steady state $\{k^*, 0, c^*\}$. The dynamic system of (38), (39), and (40) can be rewritten as

$$\begin{pmatrix} \dot{\hat{k}} \\ \dot{q} \\ \dot{\hat{c}} \end{pmatrix} = \Phi \begin{pmatrix} \hat{k} \\ q \\ \hat{c} \end{pmatrix},$$

where

$$\Phi \equiv \begin{pmatrix} (\alpha-1)A(k^*)^{\alpha-1}+\frac{c^*}{k^*} & 0 & -\frac{c^*}{k^*} \\ 0 & \alpha A(k^*)^{\alpha-1}-\delta & 0 \\ \alpha(\alpha-1)A(k^*)^{\alpha-1}-\eta\frac{c^*}{k^*} & -\eta\frac{c^*}{(k^*)^2} & \eta\frac{c^*}{k^*} \end{pmatrix}.$$

Given the parameters restriction (13), we can find that

$$\begin{aligned} \Phi_{1,1} &= (\alpha-1)A(k^*)^{\alpha-1}+\frac{c^*}{k^*} = \alpha A(k^*)^{\alpha-1}-\delta < 0, \\ \Phi_{1,3} &< 0, \Phi_{2,2} < 0, \Phi_{3,1} < 0, \Phi_{3,2} < 0, \Phi_{3,3} > 0. \end{aligned}$$

The characteristic equation of matrix Φ is given by

$$\begin{aligned} F(\lambda) &= |\Phi-\lambda I| \\ &= -\lambda^3+(\Phi_{1,1}+\Phi_{2,2}+\Phi_{3,3})\lambda^2 \\ &\quad -(\Phi_{2,2}\Phi_{3,3}+\Phi_{1,1}\Phi_{3,3}+\Phi_{1,1}\Phi_{2,2})\lambda \\ &\quad +\Phi_{1,1}\Phi_{2,2}\Phi_{3,3}-\Phi_{1,3}\Phi_{2,2}\Phi_{3,1}, \end{aligned}$$

where

$$\Phi_{1,1}\Phi_{2,2}\Phi_{3,3}-\Phi_{1,3}\Phi_{2,2}\Phi_{3,1} > 0.$$

It can be rewritten as

$$\begin{aligned} F(\lambda) &= -(\lambda-\phi_1)(\lambda-\phi_2)(\lambda-\phi_3) \\ &= -\lambda^3+(\phi_1+\phi_2+\phi_3)\lambda^2 \\ &\quad -(\phi_1\phi_2+\phi_1\phi_3+\phi_2\phi_3)\lambda+\phi_1\phi_2\phi_3, \end{aligned}$$

where ϕ_1, ϕ_2, and ϕ_3, are characteristic roots. We can find that

$$\phi_1\phi_2\phi_3 > 0.$$

Since $F(0) > 0, F(-\infty) = -\infty$, matrix Φ has at least one negative real characteristic root, which denoted by ϕ_1. Given $\phi_1\phi_2\phi_3 > 0$, we can get that $\phi_2\phi_3 < 0$. It means that the other two eigenvalues must be two real number with opposite signs. Therefore, the bubbleless steady state is local saddle point with two dimensional stable manifold.

E Proof of Proposition 3

The endogenous growth economy can be described by the following equations system.

$$\dot{q} = (\alpha A - \delta)q, \tag{44}$$

$$-\frac{\dot{c}}{c} = \rho - \frac{\eta c}{q+k} - (\alpha A - \delta), \tag{45}$$

$$\dot{k} = (A - \delta)k - c, \tag{46}$$

together with transversality conditions (3) and (7).

E.1 Bubbleless Balanced Growth Path

At the bubbleless balanced growth path, the value of bubble is zero, i.e.,

$$q = 0.$$

By the real resource constraint, we obtain that

$$\frac{\dot{c}}{c} \leq \frac{\dot{k}}{k}.$$

If the growth rate of consumption is less than the growth rate of capital, i.e.,

$$\frac{\dot{c}}{c} < \frac{\dot{k}}{k},$$

then

$$\frac{c}{k} \to 0.$$

From equation (45) and equation (46), we can obtain that

$$\frac{\dot{c}}{c} \to \alpha A - \delta - \rho,$$

and

$$\frac{\dot{k}}{k} \to A - \delta.$$

Thus,

$$\frac{\dot{k}}{k} - \frac{\dot{c}}{c} \to (1-\alpha)A + \rho > \rho.$$

It means the transversality condition (3) is violated. Therefore, it must be that

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k}$$

at the bubbleless balanced growth path.

From equation (45) and equation (46), it is easy to obtain that

$$\frac{c}{k} = \frac{(1-\alpha)A + \rho}{\eta + 1},$$

and

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = A - \delta - \frac{(1-\alpha)A + \rho}{\eta + 1}.$$

E.2 Quasi-Bubbleless Balanced Growth Path

At the quasi-bubbleless balanced growth path, the value of the bubble is not zero at all. But, the growth rate of the bubble is less than the growth rate of the real economy. Finally, the value of the bubble will be trivial.

By the real resource constraint, we obtain that

$$\frac{\dot{c}}{c} \leq \frac{\dot{k}}{k}.$$

This implies that at least the growth rate of capital is larger than the growth rate of the bubble, i.e.,

$$\frac{\dot{k}}{k} > \frac{\dot{q}}{q} = \alpha A - \delta.$$

If the growth rate of consumption is less than the growth rate of capital, i.e.,

$$\frac{\dot{c}}{c} < \frac{\dot{k}}{k},$$

then the term $\frac{\eta c}{q+k}$ will converge to zero. Thus, the transversality condition (7) is violated. Therefore, at the quasi-bubbleless balanced growth path,

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} > \frac{\dot{q}}{q} = \alpha A - \delta.$$

Suppose the term $\frac{\eta c}{q+k}$ eventually converges to a positive constant, θ. We can obtain that

$$\frac{c}{k} \to \frac{\theta}{\eta}.$$

Combining equation (45) with equation (46), we obtain that

$$\theta = \frac{\eta[(1-\alpha)A + \rho]}{\eta + 1},$$

and

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} \to A - \delta - \frac{(1-\alpha)A + \rho}{\eta + 1}.$$

Given the restriction (16), it is easy to find that

$$A - \delta - \frac{(1-\alpha)A + \rho}{\eta + 1} > \alpha A - \delta.$$

Thus, there exists the quasi-bubbleless balanced growth path.

E.3 Bubbly Balanced Growth Path

At the bubbly balanced growth path, the growth rate of the bubble should not be less than the growth rate of the real economy. Otherwise, the value of the bubble would be trivial relative to the real economy. It implies that

$$\frac{\dot{q}}{q} \geq \frac{\dot{k}}{k}.$$

By the real resource constraint, we can obtain that

$$\frac{\dot{c}}{c} \leq \frac{\dot{k}}{k}.$$

However, the growth rate of consumption should not be less than the growth rate of capital. Otherwise, by equation (46), the growth rate of capital eventually converges to $A-\delta$, which is larger than the growth rate of bubble, $\alpha A-\delta$. Thus, at the bubbly balanced growth path, consumption and physical capital have the same growth rate, which is not larger than the growth rate of the bubble, i.e.,

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} \leq \frac{\dot{q}}{q}.$$

In order to make sure that the condition (10) is not violated, the term of $\frac{\eta c}{q+k}$ should eventually convergs to some positive constant. It means that eventually

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = \frac{\dot{q}}{q} = \alpha A - \delta.$$

From equation (45) and (46), respectively, we obtain that

$$\frac{\eta c}{q+k} = \rho,$$

and

$$\frac{c}{k} = (1-\alpha)A.$$

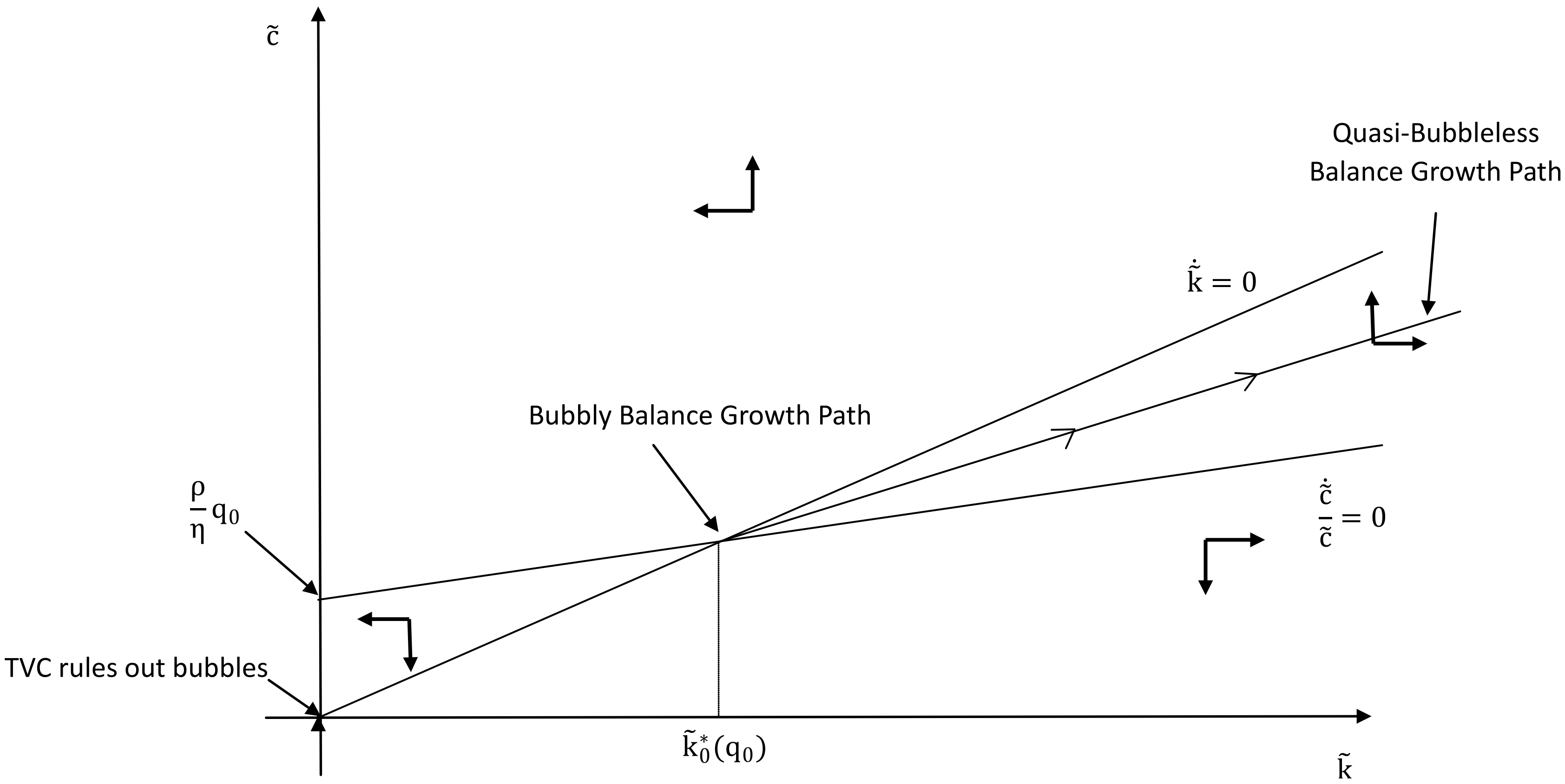

Figure: Endogenous Growth Model