Forgive, or Award, Your Debtor? - A Barrier Option Approach

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Abstract

We introduced in this study a model of sovereign debt with an embedded Down-and-In Put (DIP) to capture the discontinuity in sovereign debt pricing. This study suggests that debt forgiveness is a more effective solution in debt crisis, as repayment bonus or award or capital market exclusion penalty invites moral hazard and push up yields. Although a debtor's current repayment capability reflects its current levels of repayment award, debt forgiveness or default threshold. While forgiveness works unconditionally, a debtor can only receive repayment award conditional on full debt repayment, which could result in unfavored consequences due to moral hazard. A creditor should therefore avoid offering repayment award to, or attempting to lower default threshold on, a debtor. Granting more forgiveness is, however, beneficial always. Overall, a strong GDP growth is still the most effective solution to lower long-run sovereign yields. Reexamining the argument of Krugman (1988) verifies that extra financing is indeed inferior to forgiveness. The forecasting errors in our model are only at fractions of those produced by other related studies, as our unscented Kalman filter procedure is free of potential econometric problems. Our model of default threshold for sovereign debt is more tractable than existing works in literature, especially in the calibration of sovereign yields under various debt load levels, economic cycles, time to maturity and forecasting capability. Our model can be applied in the operation of risk management, as well as portfolio investments, for investors of sovereign debt instruments.

Keywords: Forgiveness, default threshold, default risks, sovereign debt, government bonds, fixed income, barrier option, Kalman filter.
JEL codes: C14, D81, D87, F34, G01, G11, G12

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I. Introduction

Sovereign default has been widely studied to explain what causes it, what costs it brings, how it can be avoided and what can be done to mitigate its consequences. Earlier works focus more on defaults by less developed countries (LDC) or smaller economies, and many argue that countries choose to default in their interests. Default probabilities are forecasted according to regressive findings between sovereign yields and observed default risks attributable usually unilaterally to sovereign debtors. However, events or factors arising in the debt negotiation process, which involve also choices of the creditors, have not been considered as often when forecasting future sovereign yields.

Recent sovereign debt crisis in the European Union indicates that the expectation of such factor as debt forgiveness affects significantly sovereign yields not only of the subject country, but also of other countries with similar risk profiles. To this extent, such factors are no less important than sovereign credit risks as forgiveness according Bulow and Rogoff (1989b) and repayment 'bonus' argued by Fernandez and Rosenthal (1990) both have potentially wide-spread contagious influences over the international capital market. Prior to the case of Greece, previous sovereign defaults never involved a debtor with its home currency being a major international reserve currency. With this becoming more probable, the understanding of the potentially contagious factors mentioned above would be more important than ever for both international capital market and countries with outstanding, or plan to continue issuing, sovereign debt. In the past, empirical values of forgiveness and repayment 'bonus' were documented[1] after default events. However, estimating and forecasting these factors before actual default happens becomes crucial now, as the impact of sovereign default may no longer be isolated if it happens in, say, the Euro zone.

Schwartz and Zurita (1992) show in a rational expectations model how forgiveness, as well as potential default penalty, is taken into account in the initial sovereign loan contract. As they argue that the penalty can be considered in the context of international regulation, it is similar in essence to the repayment bonus in Fernandez and Rosenthal (1990). Debt forgiveness and repayment bonus are in this context forward-looking and unobservable, or unknown at the time of loan pricing. They also need to be learned over time, from the perspective of sub-game equilibrium in a multi-stage renegotiation process (see Francois, Hubner, Sibille, 2011; Hayri, 2000; Fernandez and

[1] Hayri (2000) compares estimated and actual debt reductions under the Brady Plan, where the average reduction percentage is around 30% for seven countries opting for reductions. Neumayer (2002) discussed debt forgiveness, compiled from the Global Development Finance database of the World Bank, and its influencing variables. Borensztein and Panizza (2009), among others, report the imputed rise of borrowing costs.

Rosenthal, 1990; Bulow and Rogoff, 1989a; Eaton and Gersovitz, 1981). The time-varying nature of these unobservable variables prompts us to adopt a state space model, such as Kalman filter, for estimation and updating. Revenue flow (e.g. Hayri, 2000), foreign exchange reserves (e.g. Karmann and Maltritz, 2009) or GDP (e.g. Francois, Hubner and Sibille , 2011) have been used as the state variable in a model like this. We also choose GDP as the state variable in our Kalman filter model.

The structural approach of Merton (1974) has long been applied on the valuation of sovereign debt. Karmann and Maltriz (2003,2009) employ directly the Black-Scholes put option formula. The model of Gray, Merton and Bodie (2007) apply a reduced-form version of contingent claims model in the context of sovereign balance sheet with a distress barrier modeled as certain triggering levels of liabilities. Francois, et al. (2011) extend it to a structural approach framework to show how sovereign credit spread react to a nation's GDP and default threshold. Jeanneret (2013) builds his model also on a structural contingent claims approach, where a default boundary depends on tax revenue. In order to obtain reliable estimates of unobservable debt forgiveness and repayment bonus, we will also adopt a structural approach with an explicit barrier option formula. This model includes a third unobservable variable, default threshold, to reflect the strategic nature embedded in sovereign debt pricing.

Our barrier option model with an embedded Down-and-In Put (DIP) written by the creditor to the debtor, based on the original specification of Merton (1973), employs the explicit formula of Reiner and Rubinstein (1991) and Rich (1994). The effective debt-load is the amount of borrowing, net of forgiveness expected from the creditor, which serves as the *de facto* exercise value[2]. A rebate for the debtor, if the put is not knocked in, can be considered as a default penalty on the debtor at default or a repayment bonus for not defaulting. This model is set up to utilize information commonly available but without measurement imperfections or ambiguity in terms of interpretation, such as observed sovereign yields, GDP and debt to GDP ratio, to extract implied expectations on debt forgiveness, repayment bonus and default threshold. The path-dependent nature of its valuation is compatible with game-theoretic equilibrium of sovereign debt negotiations, and the results are easily comparable across sovereign debtors. It is also straightforward to project sovereign yields with a closed-form valuation formula.

Analysis of this study suggests that granting debt forgiveness to a debtor is more in the

[2] Considering the amount after forgiveness as the exercise value of the put has in essence all the features of using the borrowing amount directly, but offers more implications in the next section.

interest of the creditor than offering repayment award. While forgiveness works unconditionally, a debtor can only receive repayment award conditional on full debt repayment, which could result in unfavored consequences due to moral hazard. We show in our sensitivity analysis that forgiveness makes sovereign debt more valuable, and the effect is more prominent as time to maturity shortens or repayment ability weakens. However, contrary to the prior belief of many, we find that debt value goes down with the magnitude of repayment bonus, as it actually raises the value of the DIP held by the sovereign debtor and lowers the value of debt perceived by the creditor. This seeming punishment to the creditor is especially stronger when the debtor performs better in the economy. The default threshold, which activates the DIP, does not seem to affect debt value monotonously. When the threshold is relatively high, a drop of it could lower the debt value due to more valuable DIP, which is more significant for longer maturities than shorter ones. Only when the threshold is relatively low, then its decrease helps to increase the value of the debt.

Our Monte Carlo simulations results suggest that sustained growth in national income or asset values is the most effective way to lower financing costs for sovereign debt. At given levels of forgiveness, repayment bonus and default threshold are expected to fall with time to maturity, a sovereign debtor with only moderate economic growth may not be enough to escape from rising debt yields over time. On a trajectory with relatively low income growth, increasing debt forgiveness or repayment bonus is the only way to possibly lower future debt yields. However, on a path with high income growth, the effect of slightly falling forgiveness and bonus would be overwhelmed by income and debt yields could potentially fall. Even with high income growth, a default threshold falling moderately would still lower the debt value in the early part of such trajectory, and start raising the debt value later on. We also reexamine the argument of Krugman (1988), which suggests that extra financing to a heavily indebted country is distorting as compared to forgiveness. Simulations suggest that sovereign yields rise immediately following extra financing and would stay significantly higher than in the case of debt forgiveness. Extra financing literally pushes up the default threshold and hence the resulting yields. The knock-in feature of our model contributes much to the advantage of forgiveness over financing as debt value is highly sensitive to the 'distance to default'.

We adopt a Kalman filter algorithm to estimate the unobservable variables in our model, whose path-dependent nature also requires Bayesian updating. As our pricing model is highly nonlinear, we choose to apply an augmented unscented Kalman filter version according to Julier and Uhlmann (2004). Estimation results suggest that higher sovereign yields imply higher expected repayment bonus, forgiveness and default threshold. All three unobservable variables do fall with time to maturity. As a robustness check, we apply the same algorithm on yields simulated from our barrier option pricing model for various time to maturity and find results compatible with those from real market data. Additionally, the forecasting errors in our model are only at fractions of those produced by other related studies, as our unscented Kalman filter procedure is free of potential econometric problems with other traditional methods.

Our model of default threshold for sovereign debt is more tractable than existing works in literature, especially in the calibration of sovereign yields under various debt load levels, economic cycles, time to maturity and forecasting capability. Empirical analysis can be carried out easily based on our model as bond price is present in a closed forms. The model can be applied in the operation of risk management, as well as portfolio investments, for investors of sovereign debt instruments. Sovereigns can also benefit from this model in managing national debt load and in optimizing maturity, size and floating time of new issues. A brief literature review and the model of risk threshold is given in Section II. Data and empirical results are laid out in Section III. Conclusion is given in Section IV.

II. A Default Threshold Model for Sovereign Debt

Related Literature

Literature on sovereign debt focuses in general on global risks, credit risks and liquidity risks. The main purpose of this study is on the credit risk component as the other two are either exogenous or determined by long-term factors. Recent evidence shows that the sharp increases of government bond yield spreads during the financial crisis were not caused by changes in macroeconomic factors, but more possibly by rising default risks. One reason for the extremely volatile sovereign spreads is that holders of sovereign debt often do not have recourse to a bankruptcy code at sovereign defaults, which in many cases are not determined by economic factors. The costs of sovereign defaults include reputation costs as argued in Eaton and Gersovitz (1981). François, Hübner, and Sibille (2011) discuss how sovereigns can negotiate with creditors exogenously while maintaining endogenous optimizing default decisions. Many sovereign defaults are prevented through negotiations or restructuring, therefore pricing issue is sensitive especially to risks induced by the extended long period of time.

Forgiveness, according to the seminal study of Bulow and Rogoff (1989b), is one remedy, on the basis of another key element, an enforceable default penalty. Fernandez and Rosenthal (1990) take an approach stressing only possible awards to a sovereign debtor with game-theoretic models, where default penalty is not considered as a credible choice. The debtor receives, on full repayment, a 'bonus', in the form of improved access to subsequent financing, from international capital market rather than from the creditor. In their model, a creditor would pledge first on certain level of debt forgiveness, in place of default penalty, to encourage the debtor not choosing to default. In an equilibrium, the debtor chooses repayment willingly taking into account the magnitude of forgiveness as well as expected repayment bonus.

Following the early literature on debt reduction, many works of pricing formula are built on continuous time model with strategic default and debt reduction (see Cohen, 1993; Classens and van Wijnbergen, 1993; Hayri, 2000; Andrade, 2009). Further on, many works employ a reduced-form model with exogenously specified process of default intensity process, such as Merrick (2001), Duffie, Pederson and Singleton (2003) and Pan and Singleton (2008). They all assume that holders of sovereign debt face a single credit event of default, in the sense of Fischer, Heinkel, and Zechner (1989) and Leland (1994), and liquidation follows it.

Among those adopting a structural model, some employ a 'balance sheet' contingent claims approach (e.g., Gapen, Gray, Lim and Xiao, 2005; Gray, Merton and Bodie, 2007; Francois, Hubner and Sibille , 2011), which consider public, private and banking sectors in analyzing time and distance to a sovereign default. Gibson and Sundaresan (2001), Westphalen (2002), and Andrade (2009) also apply a contingent claims framework with strategic defaults. While default is modeled as a choice of the debtor, depending on domestic and international macroeconomic environment, it is not practical to assume the debtor's optimization scheme without allowing the investors to update their expectation as in models of Yue (2010) and Jeanneret (2013).

Sovereign debt limit is first formally introduced by Bulow and Rogoff (1989a), where a country's debt capacity is limited by the extent to which creditor can impose default penalty. Bulow and Rogoff (1989a) argue that issuing sovereigns possess legal or political advantages in deterring creditors' reclaiming moves, including payment rescheduling and renegotiation of payments. Hayri (2000) indicate with a continuous time framework that, whenever possible, creditors tend to choose larger and later debt reductions while debtors would choose smaller and earlier reductions. Guillard and Kempf (2012) discuss further about both default and no-default thresholds corresponding to the lower and upper critical values of debt levels. Beyond the default threshold or the upper critical debt level, ‘effective’ debt load appears to be explosive. These models of debt threshold or limit, however, have few implications about how creditors can alter the effect of the threshold to improve their welfare.

Model

The economy we consider is open one with a representative sovereign borrower and investors with certain endowment and facing consumption-saving choices. Investors could either lend domestic or foreign currency to the sovereign borrower. The technology of currency exchange and the exchange rate are assumed to be given and do not change with amount or currency of borrowing. Both sovereigns and creditors can refer to an observable signal to achieve each party’s decision, rather than relying on sets of expectations formed before the issuance of debt. They can expect to modify each other’s decision over the life of the debt depending on changes in environment and that observable signal. It is natural to use debt price as the signal, and a satisfactory valuation scheme should be easy to understand and conveniently available. We would like to focus on factors influencing the expectations on the time or distance to a likely default. The process has to be, conforming to practice in the financial markets, irreversible such that given the information of this process debt price would start assuming an entirely different path. This is one feature different from many existing models, but it is much needed by issuing sovereigns and investors.

In our model, a *Down-and-In Put* (*DIP*) is considered, according to Merton (1973) Reiner and Rubinstein (1991) and Rich (1994), in a framework where the issuing sovereign or the debtor purchases, conceptually, a *DIP* with exercise value at *X* from the investor or the creditor. In the literature of applying barrier option on corporate securities, a Down-and-In Call is often employed (e.g., Brockman and Turtle, 2003; Giesecke, 2004) as creditors of corporate debt are usually the ones initiating the related preservation process. However, in considering a sovereign debt Bulow and Rogoff (1989b) stress the active role played by the sovereign debtor. Other literature (e.g., Fernandez and Rosenthal, 1990; Hayri, 2000; Francois, et al., 2011) suggests that a sovereign debtor often makes decisions as to how much to borrow and when to or not to repay its creditors. In order to conduct analysis in a setting where a debt issuer takes initiative, we chose a model with *DIP* where creditors as the writer of the put are just passive in terms of altering the value of a sovereign debt.

We assume *D* to be the amount the sovereign agrees to repay the investor *T* years from now while X is the actual value that sovereign ends up repaying after an expected forgiveness is taken into account as

$$X = g \cdot D \tag{1}$$

Also, there exists a default threshold H which 'knocks in' the *DIP*. The asset value of underlying the *DIP*, V, is assumed to be the GDP value of the issuing country. Similar to the argument of Francois, et al. (2011), V can be considered as the only relevant part of the country's assets which can possibly be used in debt repayment. So the actual value of national wealth should be replaced, in the spirit of the model[3], by the nation's GDP. As a relevant value should be compared against the amount of repayment, we argue that V should be the relevant measure to monitor, and if it falls below H, then the *DIP* starts taking the value of a standard put as formulated in Black and Sholes (1973). The value of the debt would be the sum of discounted present value of X plus the value of a short put with parameters specified above. If V stays above H before maturity, then *DIP* maintain its value without changes, and the debt value would include in it a short *DIP* rather than a standard put. The value of the *DIP* follows, if $H<X$[4],

$$\begin{aligned} DIP &= -Ve^{(\delta-r)T}N(-x_1) + Xe^{-rT}N(-x_1+\sigma\sqrt{T}) + Ve^{-\delta T}(H/V)^{2\lambda}[N(y_1) - N(y_2)] \\ &- Xe^{-rT}(H/V)^{2\lambda-2}[N(y_1-\sigma\sqrt{T}) - N(y_2-\sigma\sqrt{T})] + Ke^{-rT}[N(x_1-\sigma\sqrt{T}) - (H/V)^{2\lambda-2}N(y_2-\sigma\sqrt{T})], \end{aligned} \tag{2}$$

where δ is current discount rate, r is the continuously compounded interest rate. K is, in the practice of option trading, the *rebate* offered by the writer if *DIP* is not knocked in before maturity, or in our context the expected repayment bonus from the creditor. Additionally,

$$\lambda = \frac{r-\delta}{\sigma^2} + \frac{1}{2},$$

$$x_1 = \frac{\ln(V/X)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T},$$

3 Among others, Claessens and Wijnbergen (1993) use expected non-oil current account, and Karmann and Maltriz (2009) consider foreign exchange reserves to be the 'ability-to-pay' measure. It is also apparent that only a small percentage of the measure is relevant to actual repayment.

4 In the literature of corporate securities, H is usually assumed to be above X for simplicity, as in Brockman and Turtle (2003), Giesecke (2004) and Lin and Sun (2009), and it follows that

$$DIP = -Ve^{(\delta-r)T}N(-x_1) + Xe^{-rT}N(-x_1+\sigma\sqrt{T}) + Ke^{-rT}[N(x_2-\sigma\sqrt{T}) - (H/V)^{2\mu}N(y_2-\sigma\sqrt{T})], \tag{3}$$

where

$$x_2 = \frac{\ln(V/H)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}.$$

$$y_1 = \frac{\ln(H^2/VX)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}$$ and

$$y_2 = \frac{\ln(H/V)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}$$.

The first two terms in (2) denote the value of a long position on a standard put. For a sovereign debt this value amounts to, in the event of a sovereign default, how the borrowing country evaluates its option of surrender its national income or fiscal autonomy to the creditors' scrutiny in place of subsequent repayment. The following two terms are the value of the DIP before being knocked in, and the last term is the value of the repayment bonus promised by the creditor, which could also considered as a default penalty on the debtor. Although in (2) the values of *K*, *g*, and *H* have been assumed for simplicity as constant parameters regardless of values of the national income or debt load, the probability distribution associated with them can still affect the values of the DIP.

We employ this model primarily for its explicit valuation formula and straightforward implications. There are two advantages of adopting this model. First, the nature of *DIP* brings along a threshold, in the spirit of Bulow and Rogoff (1989b), Hayri (2000), Bi (2010) and Guillard and Kempf (2012), to explain the explosively rising sovereign yields of countries with imminent default concerns. Another advantage for incorporating *DIP* debt pricing is that observed market yields can be fitted in this model to facilitate risk management for issuers or portfolio investors in sovereign debt. As creditors of the sovereign debt write to the issuer a *DIP* whose value is expressed in (2), so the creditor's claim *B* would be valued as

$$B(D,X,H,V,K;r,\sigma,T,\delta) = De^{-rT} - DIP(X,H,V,K;r,\sigma,T,\delta) \quad (4)$$

Although in our model the debtor receives an award for not defaulting prior to maturity, in the form of either a straight cash payment, lower borrowing rate in the future or a better credit rating, it could still be perceived relatively as a default penalty imposed on the defaulting sovereign debt issuers. The larger *K* is, the more the defaulting debtor would be penalized on the yield implied from (4), which would in turn raise the yield on its debt to be auctioned next time.

According to the binomial approximations of Baldi, Caramellino and Iovino (1999) and Barone-Adesi, Fusari and Theal (2008), the approximated price of a barrier option should be positively related to an exit probability. The value of *X* in (2) affects the debtor's exit probability, which according to (4) would have a different implication for the creditor. If the borrowing amount

D is the same as *X* and the debtor defaults, then higher *X* gives the debtor a more favorable exit and hence a higher *DIP* value. The creditor's claim by (4) is apparently lower given this defaulting situation. However if, before the debtor's income fall below the default threshold *H*, the creditor grants the debtor more forgiveness by lowering X, then it may be beneficial for both parties involved in this contract. The value of *DIP* could be higher due to a higher possibility of getting a repayment award, or lower due to the prospect of a less favorable exit at default[5]. In the latter case, the creditor's claim in (2) is higher, while the lower yield implied by that would benefit the debtor on subsequent borrowing.

Sensitive Analysis

【Figure 1】

Figure 1 plots, for the maturities of 1 year, 5, 10 and 20 years, sovereign yields according to (4) against values of expected repayment bonus at three given levels of GDP. Each of them can be considered respectively as representing a state with low risk (GDP=100), a state with medium risk (GDP=80) and a state with high risk (GDP=70). The variable *k*, expressed as a fraction of *D*, is the independent variable in this comparative statics analysis. Across the four sub-plots with different maturities, levels of expected forgiveness *g*, also expressed as a fraction of *D*, is parameterized such that it falls with time to maturity to fit reality. Similar treatment is done to default threshold *h*, which is expressed as a fraction of *X*. It is shown in Figure 1, contrary to the expectations of many, that sovereign yield gets higher if a larger repayment award is offered to the debtor. An explicit or implicit commitment to a larger repayment award the debtor increases the debtor's incentive to default, as the expected vale of exercising the DIP becomes higher due to an increase of this award. So the inclusion of the DIP in our model for sovereign debt leads us to question the effectiveness of a repayment award to the debtor in the sense of Fernandez and Rosenthal (1990).

Given different GDP levels, yields in the low risk state climb up faster than yields in the states with higher risks, and that gap widens as time to maturity shortens. For a maturity of 20 years in the low risk state, an increase of repayment award from an equivalent of 0.1% to 28.9% of *D* would raise the sovereign yields by 49 basis points, from 3.35% to 3.84%. In the high risk state, that difference is 14 basis points. If the maturity shortens to 10 years, the gap between two states

[5] Hayri (2000) show that a single-round strategic debt reduction game is capable of predicting non-forgiving exits always cause the creditor 150% more losses than debt forgiveness.

goes up to 111 basis points, and further up to 287 basis points for an issue with 5-year maturity. The difference reaches two thousand basis points for the maturity of 1 year. An imminent 28.9% repayment award could push the yield, in a low risk state, up to a level of 39.2%, 3,760 basis points higher than at the repayment award of 0.1%. While in a high risk state, that lift is only 1,755 basis points.

【Figure 2】

Figure 2 plots, for various maturities, sovereign yields according against values of expected forgiveness in three risk states as specified above. The variable *g* is the independent variable in this comparative statics analysis. Across the sub-plots with different maturities, levels of expected forgiveness *g* and *h* are also parameterized to fall with time to maturity. Here sovereign yields drops with increases in expected debt forgiveness. As in the comparative statics analysis of *k*, yields at a certain level of *g* are higher for the high risk state, with an exception in the 20-year maturity. In a high risk state, sovereign yields are actually lower than the other two states with lower risks, for low values of *g*. This is due to an assumed higher default threshold making the GDP level stay below *h*. DIP is exercised and takes on a higher value of standard put.

Here yields in the more risky states are also more sensitive to *g*, dropping faster as *g* increases, and more so as time to maturity shortens. As more forgiveness is expected by the market, a debt issue in the more risky state would benefit more. The exception appears when the time to maturity is only one year, where yields in the safest state increase, rather than decrease, in *g*. It happens because DIP is literally worthless and most of the value comes from the expected repayment award. When the forgiveness is expected to be over 20%, even the yields in the more risky states start increasing in *g*. For the maturities of 20, 10 and 5 years, yields in the most risky state drops respectively 40%, 75% and 137% faster than in the safest state. The differences between yields in the most and the least risky state are, respectively, 59, 95 and 171 basis points.

【Figure 3】

Yields against expected default thresholds are plotted in Figure 3, also separately in each of the three risk states. Levels of expected repayment award *k* and forgiveness *g* are still parameterized to fall with time to maturity. Sovereign yields decrease in expected default threshold at lower *h* values, and then start increasing in *h*. At relatively lower values of *h*, a higher threshold value makes DIP more valuable and the debt less valuable. As h gets beyond certain levels, a higher threshold value makes DIP less valuable because the debtor gets more reluctant to surrender its national income. However, for shorter maturities such as 5-year and 1-year, yields are insensitive at very low values of default threshold. As the DIP is highly unlikely to be knocked in, its value is simply the sum of a standard put and the value of the expected repayment award. Over all, yields are lower for a default threshold at 100% of current GDP than at, say, 70% or 50%. This pattern is less pronounced as time to maturity shortens.

【Figure 4】

In Figure 4, yields at given values of k are plotted against levels of GDP separately for each of the four maturities. In order to see more clearly how repayment award affects the responsiveness, both g and h are assumed to be fixed rather than falling with time to maturity. Although yields do fall as GDP increases, larger k makes yields less sensitive to GDP and also brings about higher yields at reasonable GDP levels, uniformly across all maturities. As GDP falls, the DIP becomes more likely to be knocked in, so a larger k committed to the debtor causes the value of the DIP to a bigger drop on the benefits expected from k, resulting in a higher debt value and lower implied yield.

In the case of 1-year maturity, yields become insensitive to increases in GDP, at higher GDP levels. This reflects the fact that exercising the DIP is practically unlikely and the only value left in the DIP is the expected value of the pre-committed repayment award, which is related only to the face value of debt rather than GDP values. For all other maturities, the likelihood of exercising the DIP remains, a higher GDP value makes the DIP less valuable. But the decrease in DIP is weaker because a larger repayment award adds more value to the DIP.

【Figure 5】

Figure 5, plots yields at given values of g against GDP for various maturities. Both k and h are also assumed to be the same for all maturities. Higher g results in lower yields at given income levels and makes yields less sensitive to GDP for shorter maturities. As GDP rises, the difference in yields caused by g decreases, as the debtor's benefits on DIP falls due to a lower likelihood of default in the case of shorter maturities. Given indirect effect of g on yields, the direct effect is much stronger for shorter maturities because the same amount of forgiveness is worth much more. When the maturity is at 20 years, raising forgiveness from 2.24% f original debt amount to 33.54% lowers yields by 82 basis points, at 80% of current GDP. The benefits for the 10-year, 5-year and

1-year debt are 145, 247 and 345 respectively. At 50% of current GDP, the differences would rise sharply to 108, 223, 479 and 3,455 basis points.

【Figure 6】

Figure 6 gives plots of yields against GDP at given values of *h*. Both *g* and *k* are assumed to be the same for all maturities. In general, higher default threshold does not benefit the debt in its value, a phenomenon contrary to the belief of many. The difference is caused by the existence of DIP. As a debtor's GDP stay further above the level of expected threshold, DIP becomes less valuable. For all maturities, if the threshold is 89% then the effective debt amount would be 52% of current GDP, where the embedded DIP gets exercised and yields rise sharply below that level of GDP. If *h* equals 0.67, then DIP knocks in only below 39% of current GDP. As the value of a standard put decreases with time to maturity, so does the knock-in value of DIP. The differences in yields caused by *h* thus goes down with time to maturity as well.

For the maturity of 20 years, if the default threshold goes down from 89.44% of effective debt amount to 53.67%, yields would be raised by 35 basis points, at 80% of current GDP. The consequence on the 10-year and 5-year maturities would be, respectively, 58, and 66. It drops to 9 basis points only for the 1-year maturity due to the extreme loss of time value. At 54% of current GDP, the differences would go up to 43, 93, 169 and 231 basis points for the four maturities.

Simulations

The sensitive analysis above gives static relations among sovereign yields and repayment award, debt forgiveness and default threshold. In order to conduct a dynamic exploration, preliminary to our subsequent empirical analysis, on the variables of interest, we simulate projected paths for yields of a sovereign debt with 20 years to maturity over the next 60 months. Raw yields are calculated using (4) and randomized to obtain simulated yields, which are also generated with a lognormal random generator with standard deviations falling with time to maturity. For the first month on the path, the standard deviation of the yield is set at 0.54%, and it is set at 0.51% for the 60th month. Simulation is carried out one thousand times and the average is reported. For the three variables of interest, we conduct simulations twice, with one assuming a 6% growth of GDP and 1.5% growth for the other.

【Figure 7】

Trajectories of simulated yields with 6% GDP growth are plotted in Figure 7 for each of the three scenarios with different paths of *k*. The first is one where increases rapidly at an annual pace of 33% from its current level of 0.074, while in the second scenario *k* rises steadily at roughly 17% a year. The last one is when *k* goes up slowly at a pace of 5.6% annually. Each of the three paths is simulated with a lognormal distribution with the current level of *k* as mean and standard deviation being 25% of it. It is shown in Figure 7(a) that the effect of rising k, which is supposed to drive up yields, is offset by income growth. When *k* is barely rising, the average simulated yields actually drop throughout the first 45 months before starting to go up slowly. It is the strong growth of GDP that overwhelms *k* over two-thirds of projected paths. Yields are able to maintain unchanged if *k* increases at the moderate pace, which is just enough to offset income growth. Even if *k* is expected to rise, for the purpose of exploration, in the first scenario at an incredible pace of 33% annually, yields could not start rising before the 25^{th} month due to the strong income growth.

The bottom panel of Figure 7 plots projected yields given a mere GDP growth of 1.5%. As weaker income is not enough anymore to overwhelm the effect of rising *k*, all three scenarios exhibit rising yields. Yield in the first scenario ends up with a gain of 19%, about 10% short of its counterpart in 7(a) due to a 30% lower ending income level. But it goes all the way up there without going down first. The same results hold for the other two scenarios, where one gains 7.7% and the other is up by 2.8%, with neither going through apparent periods of falling yields.

【Figure 8】

Figure 8 gives the plots of three simulated scenarios, two with falling *g*, and one with increased debt amount to verify the conclusion of Krugman (1988). The first is one where *g* rises slowly at 8.6% annually from its current level of 0.071, while in the second scenario *g* increases rapidly at an annual pace of 42%. In the last one, in addition to *g* falling at 42%, we let the debt amount go up by 5% whenever the default threshold happens to increase 20% over its value in the previous month. This scenario is set up to explore the extra financing alternative against simple forgiveness, as part of our verification of Krugman (1988). Each of the three paths is simulated with a lognormal distribution with the current level of *k* as mean and standard deviation being 15% of it.

The top panel Figure 7(a) contains plots on a 6% GDP growth. It is shown in that the effect of falling *g*, which is supposed to drive up yields, is offset by the strong income growth. When the forgiveness is expected to be go down over time in the first scenario, instead of rising, the average simulated yield drops by 4% in the end. Even when the forgiveness expectation falls rapidly in the second scenario, average yields manage to increase just slightly by a mere 4%. In the last scenario, we simulate under a scheme where 5% extra financing is granted whenever default threshold is 20% higher the in the previous month. Without any change on forgiveness, rising debt amount would drive yields up by 5.6%, which is even worse for the debtor than forgiveness being withdrawn 42% annually. This is another support for the Krugman (1988) argument that forgiveness works better than extra financing, for the creditor's consideration.

The bottom panel of Figure 8 plots projected yields given a mere GDP growth of 1.5%. As weaker income is not enough anymore to overwhelm the effect of falling *g*, an 8.75% annual drop of *g* is able to drive up the yield by 6.6%, compared with a drop of 4% in 8(a) under strong income growth. If *g* goes down much faster at 42% annually then the yield would go up by 18%. When we apply the same analysis as in 8(a) on the Krugman argument, we find the ending yield to be just about the same as in the second scenario. The effect of extra financing is equivalent to withdrawing forgiveness significantly on the debtor, from 26% to 3% of debt amount, within a five-year period.

【Figure 9】

To see how simulated yields are driven by changes of default threshold, we plot the results from three scenarios about *h* in Figure 9. In the first one, *h* is assumed to rise slowly at 1.7% annually from its current level of 0.71. In the second scenario *h* goes down moderately at 8.8% per year, and more rapidly at 33% in the third one. The top panel is based on a 6% GDP growth, where rising *h* in the first scenario causes the yields to drop by 10%, verifying the predictions given by comparative statics in Figure 3 and 6. Even when *h* falls moderately, due to the strong growth of GDP, yield would still drop and end up almost 5% lower than where it starts. Only when *h* goes down rapidly as in the third scenario could the yield increase by about 6% before declining after the thirty-sixth month and drop to 1% below the starting yield level. So the results in this panel indicates that the effect of an embedded DIP makes it extremely hard for the default threshold to push yields up, even when the threshold falls extremely fast over time.

The bottom panel of Figure 9 plots projected yields given a mere GDP growth of 1.5%. As weaker income growth is not enough anymore to overwhelm the effect of falling *h*, yields for all three scenarios increase over time. Along with the weak income growth, a slowly rising *h* at 1.2% annually would result in a 2% higher ending yield. But if *h* goes down at an annual rate of 8.6%, the yield would rise to 6.9% higher than at the start, a difference of 37 basis points. Furthermore, if *h* should fall rapidly in the third scenario, then Figure 9(a) shows that yields could go up by around 15% up to the 40th month before dropping a little afterwards, when higher GDP makes DIP less valuable and pushes down yields.

The simulation results suggest that GDP growth plays a crucial role in the paths of future sovereign yields. A strong growth often overwhelms potential influences especially from either repayment award or default threshold. In the absence of potential positive contribution of income growth, sovereign yields tend to go up with *k* and *h*, but go down with *g*. Forgiveness seems to be the most effective approach to prevent deteriorating sovereign yields, while the effect of the other two depends on their own dynamics and also income growth.

III. Empirical Analysis

The Data

Our sample comprises monthly bond yield data obtained from *Datastream* from 2003 to 2013, covering 7 countries and 698 yield series. The shortest series covers around four years, while the longest spans across 11 years. All the sovereign yields are calculated from month-end prices of zero-coupon government bonds traded in the data collection period. The average time to maturity is between 9 to 15 years. Some of them are OECD countries while others are not. All but Mexico and Israel are from the European Union area. Annual GDP growth data is from World Bank. Summary statistics are reported in Table I.

【Table I】

Table I shows that, for the average yields of Mexico are the highest among all. On observed standard deviations, yields from Spain and Italy appear to be the most volatile during this period. Apparently, compared with Italy, Spain has a problem of sharply declining GDP, which could be reflected by our simulation carried out earlier. While for Italy, short term volatility on its debt-to-GDP ratio rapidly pushes default threshold toward current income level and causes panicking pricing responses in 2011.

Unscented Kalman Filter estimation

The theoretical model introduced above in (2) and (3) in non-linear in a complex way.

Although k, g, and h are the main driving variables in (4), they are unobservable. So regression models are not feasible choice. To cope with corporate bond yields modeled as nonlinear functions of underlying state variables, Duffee (1999) adopt Extended Kalman Filter (EKF) recursions for estimation. Julier and Uhlmann (2004) argue, however, that EKF may suffer serious problems while executing algorithms, which include error propagation (Lerro and Bar-Shalom, 1993; Costa, 1994), the non-existence and the calculation of Jacobian matrix (Kastella, Kouritzin and Zatezalo, 1996). We therefore adopt the Unscented Kalman Filter (UKF) introduced by Julier and Uhlmann (2004) to overcome the potential technical difficulties in applying EKF. Information on the model is in the Appendix.

Table II reports the results of an augmented UKF algorithm on the barrier-option model in (2). The scheme of creating sigma points according to Julier and Uhlmann (2004) has been modified to accommodate the non-negativity requirement of yields. The state dynamic function is a simple mean-reverting process, while the measurement process applies directly from (2). The discount rate δ in (2) is assumed to be at 2%. The continuous interest rate is approximated by the short rate of each respective country. The volatility parameter is calculated with quarterly GDP of each country. GDP values are deflated first by figures for the year of 2002 and then scaled to 100. The face value of debt is the government debt to GDP ratio times 100. The theoretical values of the three unknown state variables are generated randomly, with priors filtered on simulated yields through similar UKF procedures whose results are presented in Table V, and are also assumed to be mean-reverting.

【Table II】

The augmented UKF procedure is executed for each country all the issues available in the data set for the years between 2003 and 2012. For instance, there are 208 issues from France, 90 from Italy and 47 from Spain, among others. Estimates for k, g and h are averaged over the 120 months in the estimation period and also over all the issues of a given country, and standard deviations and other statistics are calculated accordingly. The starting values of k, g, and h are therefore set at 0.05, 0.1 and 0.5 respectively.

It is shown that in general the yield data as we have collected appear to suggest that safer sovereign debt would produce lower estimates on the unobservable state variables of repayment award, forgiveness as well as default threshold. Germany appears to have the lowest average estimates, with k, g, and h at 0.0221, 0.0374 and 0.5641 respectively. Spain and Mexico appear to have the highest estimates, where on the implied repayment award of the former is about three and half times that of Germany and around seven times for the latter. While the estimates for Mexico on the implied forgiveness is close to eight times that of Germany, the implied default threshold is, however, only around two and half times. Comparison of information in Table I and II indicates that the debt-to-GDP ratio is not an ideal indicator of sovereign risks. Italy has a ratio much higher than Spain, but their sovereign yields are quite close to each other. Our UKF estimates suggest that Spain have higher risks in all three aspects. Another similar comparison is between Netherlands and Israel. Although the latter has a higher debt ratio, its average sovereign yield is lower, in part due to shorter average maturity. The UKF estimates in Table II indicates that Netherlands appear to be less risky between the two.

Also reported in Table II are the population standard deviations of the three state variables, from **P** in (A.6). The averages of estimated state variable are in general five to 6 times larger for all countries, indicating their statistical significances. It is also shown in Table II that the standard errors for individual estimates are only five to ten percent of the averages. As the parameters for these sovereign debt are not stable by nature, resorting to methods such as regressions would produce biased observations on how to perceive the pricing and risks of sovereign debt. With the kind of efficiency obtained in the UKF algorithms, we would be able to generate further simulated forecasts and provide meaningful inferences using this new model of sovereign debt pricing.

【Table III】

In the forgoing analysis, UKF estimation is carried out separately for every single unmatured issue of each country in the data set. As these concurrent issues of a given country share certain systematic characteristics, it would be more desirable to conduct an estimation on the states of all issues simultaneously. The Interactive Multiple Model (IMM) version of Kalman filter, proposed by Bar-Shalom, Li and Kirubarajan (2001), is one model available to us. Although similar to a panel regression model, it deals with unobservable independent variables model, and is rarely seen in literature[6] of fixed income. This method is especially ideal for analyzing bond market, where overlapping issues coexist. The IMM algorithm, along with the UKF modification for a standard Kalman filter, is specified in the Appendix. For each country, we apply the IMM-UKF method, using the same parameters for Table II, on all the issues simultaneously.

The IMM-UKF filter produces, in each month for a given country, combined or probability-weighted estimates for state mean and covariance matrix. Table III reports, also for the

[6] Duffee (1999) estimates 169extended Kalman filter problems for default-free and defaultable term structure. The estimations are carried separately although some independent variables are common to all.

years between 2003 and 2012, the combined estimates for *k*, *g* and *h*, averaged over the 120 months in the estimation period. The estimated standard deviation of *k* and the scatteredness of estimates across time are both smaller under IMM-UKF, confirming its efficiency in information utilization. But the average estimated mean of *k* becomes larger under the IMM-UKF filter, indicating that the repayment award appears to be more influential when all outstanding issues are jointly considered. In the cases of *g* and *h*, estimated means are both smaller under IMM-UKF, suggesting the individual UKF procedure could have overestimated them in Table II. The variations of the estimates are, however, still uniformly lower under IMM-UKF than individual UKF.

IV. Robustness analysis and related discussions

Performance gains with UKF and IMM-UKF

To justify the use of UKF in this study, instead of a more commonly adopted Extended Kalman Filter (EKF) estimation method, we apply both on the same set of data and compare the differences in forecasting performances. One-period forecasting performances of UKF and IMM-UKF are compared against those form an Extended Kalman Filter (EKF) algorithm. Estimations are produced, for each month in 2013, using the previous 60 months, and one-month ahead forecasted yields for each month in 2013 are compared against observed yields. 22 yield series with maturity less than a year are excluded. The remaining 676 series are divided into 4 maturity groups, with 96 having the maturity between 1 and 5, 188 between 6 and 10, 274 between 11 and 15 and 218 between 16 and above. For each maturity group, Root Mean Squared Errors (RMSE) and Mean Absolute Errors (MAE) are calculated monthly for all issues over 2013, and expressed in terms of basis points in Table IV.

It can be seen from Table IV that UKF and IMM-UKF are both superior to EKF in forecasting performance, either based on RMSE or MAE. Particularly, IMM-UKF has an edge of 9% over UKF, and 21% over EKF for the longest maturity, while for the shortest maturity the edges are 3.5% and 12.7% respectively. Both forecasting performance measures also indicate all three filtering algorithms produce smaller forecasting errors compared with Jeanneret (2013), Yue (2010) and Francois, et al. (2011). Examining a period similar to ours, Jeanneret (2013) report an average RMSE between observed and model sovereign spreads at 170 basis points for emerging markets, and 72 for 5 European countries. While the differences in credit spreads between observed and modeled Argentina yields in Yue (2010) are between 200 to 400 basis points, the model-implied Brazilian yields in Francois, et al. (2011) are about 150 basis points away from the observed spreads. The average RMSE for maturities between 6 and 15 years is only 55 bps under the IMM-UKF algorithm, and 59 bps under UKF. This indicates that, with the help of efficient estimation procedures, the three state variables help capturing well the dynamics of sovereign yields through a realistic valuation model.

Reliability of estimates using market data

The UKF procedures introduced in the previous section are carried out are primarily on debt issues all with different time to maturity, which creates a high level of noise in results. As the values of all three state variables should decrease in time to maturity, the averages reported in Table II and III are affected essentially by the average time to maturity of the country of interest. Comparisons across countries may not reflect differences in sovereign risks. As a robustness check on the results reported in Table II and III, we have created separately simulated yield data using (4) for given time to maturity and carry out the UKF procedure for alternative estimates. In this stage of simulation, the discount rate is also assumed to be at 2%, and short rate is set at 1.5%, GDP growth volatility at 25% and debt to GDP ratio at 75%. For the simulation, 0.1, 0.2 and 0.7 are assumed as priors for *k*, *g* and *h* to start the filtering process with. They are further randomized and substituted into (4) to obtain raw yields. For each maturity, twenty thousand simulated yields are generated with a lognormal distribution having the raw yield of respective maturity as its mean. For the maturity of 20 years, simulated yields average at 5.48% with a standard error at 0.74%, compared with lower average yields from Italy and Spain and their much higher standard errors. The average simulated yields are 3.71%, 2.97% and 1.34% for the 10-year, 5-year and 1-year debt. The standard errors are 0.62% , 0.38% and 0.19% respectively.

【Table V】

In Table V, it is shown that the estimated repayment award for the maturity of 20 years is 0.08 and 0.085 under UKF and IMM-UKF respectively, which are higher than the estimates for Italy and Spain, but lower than those for Mexico, in Table II and III. Considering the average time to maturity of around 14 years for issues from the two countries, the estimates in Table V is comparably reasonable. Similar situations apply for the estimates of debt forgiveness and default threshold. With the overwhelmingly large data size, this exploration suggests that the actual values of the three unobservable variables should be within certain range of the levels given by Table II and III. The average maturity, the number of issues outstanding and the actual market conditions all contribute to the variability of them.

【Table VI】

To verify the relevancy of the *k*, *g* and *h* in our analysis, we further regress the changes of 5-year sovereign CDS spreads on the changes in averaged estimates for *k*, *g* and *h* under UKF, and the combined estimates under IMM-UKF. The results are reported in Table VI, which suggest that, under both UKF and IMM-UKF, higher expected debt forgiveness for a given county pushes up CDS spreads. In the case of France, each additional percentage of expected debt forgiveness could make its 5-year CDS go up by 8 bps. While for Mexico, the effect goes down to only 4 bps, probably related to the fact that the expected debt forgiveness is already as high as 30%. Under IMM-UKF, both the expected repayment award and default threshold also influence CDS spreads in a significant way, only with smaller magnitudes.

Alternative Default Threshold

Our current model assumes the threshold to be below the exercise value of the debt. We have conducted related analysis on the case where the threshold is above the exercise value. The resulting knock-in effect becomes less apparent compared with the current model. The benefits of repayment award, forgiveness and default threshold are also less significant and become less conclusive in certain situations. The general outcome still holds in that forgiveness is the most effective in relieving the debtor's burden especially when default is more imminent. Also, forgiveness would still work as better alternative than financing.

V. Conclusion

We introduced in this study a model of sovereign debt with an embedded DIP to capture the discontinuity of pricing response along observed debt-to-GDP ratios, and also to explain the behavior of yield volatility with the incorporation of repayment award, debt forgiveness and default threshold. We show that, in the context of debt negotiation, debt forgiveness is the most effective alternative in preventing sovereign yields from exploding over the course of debt crisis. Repayment bonus or award in the sense of Fernandez and Rosenthal (1990) or the capital market exclusion penalty as in Bulow and Rogoff (1989a) would not be preferred. The existence of DIP explains how that would invite moral hazard and push up yields, contrary to the creditor's initial intention.

Our comparative statics and simulated projections indicate that while forgiveness helps lowering sovereign yields, neither raising repayment nor lowering default threshold is constructive in keeping yields down. Our estimates obtained from an Unscented Kalman Filter suggest, however, that smaller repayment award, debt forgiveness or default threshold coexist with lower sovereign risks[7]. The repayment capability of a less risky country or debtor is reflected with the fact that market as a whole would assign low values of repayment award, debt forgiveness or default threshold. While forgiveness works unconditionally, a debtor can only receive repayment award conditional on full debt repayment, which could result in unfavored consequences due to moral hazard. As a forward-looking perspective, offering repayment award to, or attempting to lower default threshold on, a debtor would not be in the interest of a creditor. But granting more forgiveness is beneficial always.

Overall, our analysis shows that a strong growth in GDP is, beyond any negotiation strategies, the most effective solution to lower long-run sovereign yields. In the absence of that, we reexamine the argument of Krugman (1988), which suggests that extra financing to a heavily indebted country is distorting as compared to forgiveness. We show in our analysis that extra financing literally pushes the default threshold toward the country's current income level and results in yields always higher than those under forgiveness. The knock-in feature of our model contributes much to the advantage of forgiveness over financing as debt value is highly sensitive to any income drop when default is imminent. To the debtor, forgiveness is better than extra financing granted from the creditors, regardless of the distance between current income and default threshold.

The forecasting errors in our model are only at fractions of those produced by other related studies, as our unscented Kalman filter procedure is free of potential econometric problems with other traditional methods. Our model of default threshold for sovereign debt is more tractable than existing works in literature, especially in the calibration of sovereign yields under various debt load levels, economic cycles, time to maturity and forecasting capability. Empirical analysis can be carried out easily based on our model as bond price is presented in a closed form. The model can be applied in the operation of risk management, as well as portfolio investments, for investors of sovereign debt instruments. Sovereigns can also benefit from this model in managing national debt load and in optimizing maturity, size and floating time of new issues.

[7] This is consistent with results of Yue (2010), whose model predicts that increasing bargaining power on the debtor would raise credit spreads on its sovereign debt.

References

1. Andrade, S. C., (2009), “A Model of Asset Pricing under Country Risk,” *Journal of InternationalMoney and Finance* 28(4), 671–695.
2. Baldi, P., Caramellino, L., & Iovino, M.G. (1999). Pricing general barrier options: a numerical approach using sharp large deviations, *Mathematical Finance*, 9, 293-322.
3. Barone-Adesi, G., Fusari, N. & Theal, J. (2008). Barrier option pricing using adjusted transitionprobabilities. *Journal of Derivatives*, Winter, 2008.
4. Bi, H., (2011). “Sovereign default risk premia, .scal limits, and .scal policy,” Working Paper 2011-10, Bank of Canada.
5. Black, F., and Scholes, M. (1973), "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy* 81,637-654.
6. Brockman P., and Turtle, H. J. (2003). “A barrier option framework for corporate security valuation,” *Journal of Financial Economics* 67, 511-529.
7. Bulow, J., and Rogoff, K.. (1989a), “A Constant Recontracting Model of Sovereign Debt” *Journal of Political Economy* 97, 155-178.
8. Bulow, J., and Rogoff, K.. (1989b), “LDC Debt: Is to Forgive to Forget?” *American Economic Review* 79, 43-50.
9. Claessens, S., and van Wijnbergen, S. (1993), “Secondary market prices and Mexico’s Brady deal,” *Quarterly Journal of Economics* 108, 965–982.
10. Cohen, D. (1993), “A valuation formula for LDC debt,” *Journal of International Economics* 34, 167–180.
11. Costa, P.J., (1994). “Adaptive Model Architecture and Extended Kalman-Bucy Filters,’ *IEEE Trans. Aersp. Electron. Syst.*, 30, 525-533.
12. Duffee, G.D., (1999), “Estimating the Price of Default Risk,” *Review of Financial Studies*, 12(1), 197-226.
13. Duffie, D., L. Pedersen, and K. Singleton (2003). “Modeling Sovereign Yield Spreads: A Case Study of Russian Debt,” *Journal of Finance* 58(1), 119–160.
14. Eaton, J., Gersovitz, M., (1981). “Debt with potential repudiation: theoretical and empirical analysis,“ *Review of Economic Studies* 48(2), 289 – 309.
15. Fischer, E., Heinkel, R., and Zechner, J. (1989) “Dynamic Capital Structure Choice: Theory and Tests,” *Journal of Finance 44*, 19–40.
16. François, P., G. Hübner, and J. Sibille (2011). “A Structural Balance Sheet Model of Sovereign Credit Risk,” Finance, 32(2), 137-165.
17. Gapen M., Gray, D., Lim, C. H., and Xiao, Y. (2005) “Measuring and Analyzing Sovereign Risk with Contingent Claims,” IMF Working Paper #05/155.
18. Gibson, R. and S. Sundaresan, (2001), “A Model of Sovereign Borrowing and Sovereign Yield Spreads,” Working Paper, Columbia University.
19. Giesecke, K. (2004). “Credit risk modeling and valuation: An Introduction,” *Journal of Financial Risk Management* 2(1), 1-40.

20. Gray, D.F., Merton, R.C., and Bodie, Z., (2007). "Contingent Claims Approach to Measuring and Managing Sovereign Credit Risk," *Journal of Investment Management* 5(4), 1-24.

21. Guillard, M., and Kempf, H., (2012). "Defaulting on Sovereign Debt: A Macroeconomic Analysis," Working Paper, University d'Evry val d'Essonne.

22. Hayri, A. (2000). "Debt Relief," *Journal of International Economics* 52, 137–152.

23. Jeanneret, A.. (2013). "The Dynamics of Sovereign Credit Risk," *Journal of Financial and Quantitative Analysis*, forthcoming.

24. Julier, S.J., and Uhlmann, J.K., (2004). "Unscented Filtering and Nonlinear Estimation," *Proceedings of IEEE*, 92(3), 401-422.

25. Karmann, A., and Maltritz, D., (2003). "Sovereign Risk in a Structural Approach: evaluating sovereign ability-to-pay and probability of default," Dresden Discussion Paper in Economics No. 7/03, Dresden University of Technology.

26. Karmann, A., and Maltritz, D., (2009). "Estimating Sovereign Risk by a Structural Approach: the role of Forex reserves for emerging market countries," *Investment Management and Financial Innovations* 6(3), 194-200.

27. Kastella, K., Kouritzin, M.A., and Zatezalo, A., (1996). "A Nonlinear Filter for Altitude Tracking," *Proceedings of Air Traffic Control Association*, 1-5.

28. Krugman, P., (1988). "Financing vs. Forgiving a Debt Overhang," *Journal of Development Economics* 29, 253-268.

29. Leland, H. E. (1994) "Corporate Debt Value, Bond Covenants, and Optimal Capital Structure," *Journal of Finance* 49, 1213–1252.

30. Lerro, D. and Bar-Shalom, Y.K., (1993). "Tracking with Debiased Consistent Converted Measurements vs. EKF," *IEEE Trans. Aersp. Electron. Syst.*, AES-29, 1015-1022.

31. Lin, W. T., and Sun, D. S.. (2009), "Are Credit Spreads Too Low or Too High? A Hybrid Barrier Option Approach for Financial Distress," *Journal of Futures Markets* 29(12), 1161−1189.

32. Merrick, J., (2001). "Crisis dynamics of implied default recovery ratios: evidence from Russia and Argentina," *Journal of Banking and Finance* 25(10), 1921-1939.

33. Merton, R. C. (1973). Theory of rational option pricing, *Bell Journal of Economics and Management Science*, 4, 141-183.

34. Pan, J. and K. Singleton, (2008), "Default and Recovery Implicit in the Term Structure of Sovereign CDS Spreads," *Journal of Finance* 63(5), 2345–2384.

35. Reiner, E., and Rubinstein, M. (1991), "Breaking Down the Barriers," *Risk Magazine,* 4(8).

36. Rich, D., (1994) "The Mathematical Foundation of Barrier Option-Pricing Theory," *Advances in Futures and Options Research,* 7, 267-311.

37. Schwartz, E.S., and Zurita, S., (1992) "Sovereign Debt: Optimal Contract, Underinvestment and Forgiveness," *Journal of Finance*, 47(3), 981-1004.

38. Wan, E.A., and van der Merwe, R. (2001). "The Unscented Kalman Filter," in Hayking S., editor, Kalman Filtering and Neural Networks, Wiley.

39. Westphalen, M., (2002), "Valuation of Sovereign Debt with Strategic Defaulting and

Rescheduling," Working Paper, University of Lausanne.

40. Yue, V. Z., (2010), "Sovereign Default and Debt Renegotiation," *Journal of International Economics* 80(2), 176–187.

Appendix

The Unscented Kalman filter (UKF)

The unscented transformation (UT) (Julier and Uhlmann, 2004;Wan and van der Merwe,2001) can be used for forming a Gaussian approximation to the joint distribution of random variables x and y, which are defined with equations (A.1),

$$\mathbf{x} \sim \mathrm{N}(\mathbf{m}, \mathbf{P}) \text{ and}$$
$$\mathbf{y} = \mathbf{g}(\mathbf{x}), \tag{A.1}$$

where $\mathbf{x} \in \mathfrak{R}^n$, $\mathbf{y} \in \mathfrak{R}^m$, and $\mathbf{g}$: $\mathfrak{R}^n, \rightarrow \mathfrak{R}^m$ is a nonlinear function. In UT, a fixed number of sigma points are deterministically choose to capture the desired moments (at least mean and covariance) of the original distribution of x exactly. The sigma points are propagated subsequently through the non-linear function g and moments of the transformed variable are estimated accordingly. The advantage of UT over the Taylor series based approximation is that the former is better at capturing the higher order moments caused by the non-linear transform. Under UT, the Jacobian and Hessian matrices are not needed, and therefore the estimation is easier and more error-free.

The unscented Kalman filter (UKF) (Julier et al., 1995; Julier and Uhlmann, 2004b; Wan and van der Merwe, 2001) makes use of UT described above to give a Gaussian approximation to the filtering solutions of non-linear optimal filtering problems of form

$$\mathbf{x}_{\boldsymbol{k}} = \mathbf{f}(\mathbf{x}_{\boldsymbol{k-1}}, \mathrm{k}-1) + \mathbf{q}_{\boldsymbol{k-1}}$$
$$\mathbf{y}_{\boldsymbol{k}} = \mathbf{h}(\mathbf{x}_{\boldsymbol{k}}, \mathrm{k}) + \mathbf{r}_{\boldsymbol{k}}, \tag{A.2}$$

where $\mathbf{x}_{\boldsymbol{k}} \in \mathfrak{R}^n$ is the vector of unobservable state variables. In the context of our estimation, $\mathbf{x}$ = (k, g, h). $\mathbf{y}_{\boldsymbol{k}} \in \mathfrak{R}^m$ is the measurement vector, denoting yields implied by (1) through (3) combined. $\mathbf{q}_{\boldsymbol{k}\text{-}\mathbf{1}}$ ~N($\mathbf{0}$,$\mathbf{Q}_{k\text{-}1}$) is the Gaussian process noise and $\mathbf{r}_{\boldsymbol{k}}$ ~N($\mathbf{0}$,$\mathbf{R}_k$) is the Gaussian measurement noise.

According to UT, the UKF algorithm consists of the *prediction* and *update* phases. In the *prediction* phase, predicted state mean $\mathbf{m}_k^-$ and predicted covariance matrix of state variables $\mathbf{P}_k^-$ are computed as

$$\mathbf{X}_{\boldsymbol{k-1}} = [\mathbf{m}_{k-1} \ \dots \ \mathbf{m}_{k-1}] + \sqrt{\mathrm{c}}[\sqrt{\mathbf{P}_{k-1}} \ \dots \ \sqrt{\mathbf{P}_{k-1}}]$$

$$\widehat{\mathbf{X}}_{\boldsymbol{k}} = \mathrm{f}(\mathbf{X}_{k-1}, k-1)$$

$$\mathbf{m}_k^- = \widehat{\mathbf{X}}_k \mathbf{w}_{\mathrm{m}}$$

$$\mathbf{P}_k^- = \hat{\mathbf{X}}_k \mathbf{W} [\hat{\mathbf{X}}_k]^T + \mathbf{Q}_{k-1}. \tag{A.3}$$

In the *update* phase, predicted state mean μ_k, measurement covariance matrix $\mathbf{S}_k$ and cross-covariance of the state and measurement $\mathbf{C}_k$ are computed as

$$\mathbf{X}_k^- = [\mathbf{m}_k^- \ \dots \ \mathbf{m}_k^-] + \sqrt{\mathrm{c}}[\sqrt{\mathbf{P}_k^-} \ \dots \ \sqrt{\mathbf{P}_k^-}]$$

$$\mathbf{Y}_k^- = \mathbf{h}(\mathbf{X}_k^-, k)$$

$$\mu_k = \mathbf{Y}_k^- \mathbf{w}_{\mathrm{m}}$$

$$\mathbf{S}_k = \mathbf{Y}_k^- \mathbf{W} [\mathbf{Y}_k^-]^T + \mathbf{R}_k$$

$$\mathbf{C}_k = \mathbf{X}_k^- \mathbf{W} [\mathbf{Y}_k^-]^T + \mathbf{R}_k. \tag{A.4}$$

To finish, the algorithm would compute the filter gain $\mathbf{K}_k$ and updated state mean $\mathbf{m}_k$, as well as the state covariance matrix $\mathbf{P}_k$ as

$$\mathbf{K}_k = \mathbf{C}_k \mathbf{S}_k^{-1}$$

$$\mathbf{m}_k = \mathbf{m}_k^- + \mathbf{K}_k [\boldsymbol{y}_k - \boldsymbol{\mu}_k]$$

$$\mathbf{P}_k = \mathbf{P}_k^- - \mathbf{K}_k \mathbf{S}_k \mathbf{K}_k^T. \tag{A.5}$$

The model above can also be modified to form an augmented state variable, which concatenates the state and noise components together, so that the effect of process and measurement noises can be used to better capture the odd-order moment information. If new sigma points are generated in the update step the augmented approach would give the same results as the nonaugmented, if we had assumed that the noises were additive. If the noises are not additive the augmented version should produce more accurate estimates than the nonaugmented version, even if new sigma points are created during the update step. The differences between the augmented algorithm and the original model is as follows,

$$\tilde{\mathbf{x}}_{k-1} = [\mathbf{x}_{k-1}^T \ \mathbf{q}_{k-1}^T \ \mathbf{r}_{k-1}^T] \ \text{(augmented state variable)}$$

$$\tilde{\mathbf{X}}_{k-1} = [\tilde{\mathbf{m}}_{k-1} \dots \tilde{\mathbf{m}}_{k-1}] + \sqrt{\mathrm{c}} \left[\sqrt{\tilde{\mathbf{P}}_{k-1}} \ \dots \ \sqrt{\tilde{\mathbf{P}}_{k-1}}\right] \text{(matrix of sigma points)}, \tag{A.6}$$

where

$$\tilde{\mathbf{m}}_{k-1} = [\mathbf{m}_{k-1}^T \ \mathbf{0} \ \mathbf{0}] \ \text{ and } \ \tilde{\mathbf{P}}_{k-1} = \begin{bmatrix} \mathbf{P}_{k-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_{k-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{R}_{k-1} \end{bmatrix}. \tag{A.7}$$

The predicted state mean $\mathbf{m}_k^-$ and covariance matrix $\mathbf{P}_k^-$ would then be computed as

$$\hat{\mathbf{X}}_k = \mathrm{f}(\mathbf{X}_{k-1}^x, \mathbf{X}_{k-1}^q, k-1)$$

$$\mathbf{m}_k^- = \widehat{\mathbf{X}}_k \mathbf{w}_{\mathrm{m}}$$

$$\mathbf{P}_k^- = \widehat{\mathbf{X}}_k \mathbf{W} [\widehat{\mathbf{X}}_k]^T, \tag{A.8}$$

where the sigma points of actual state variables and process noise are $\mathbf{X}_{k-1}^{\boldsymbol{x}}$ and $\mathbf{X}_{k-1}^{\boldsymbol{q}}$ respectively. In the update stage, except that μ_k retains the same functional form,we have instead

$$\mathbf{Y}_k^- = \mathbf{h}(\widehat{\mathbf{X}}_k, \mathbf{X}_{k-1}^r, k)$$

$$\mathbf{S}_k = \mathbf{Y}_k^- \mathbf{W} [\mathbf{Y}_k^-]^T$$

$$\mathbf{C}_k = \mathbf{X}_k^- \mathbf{W} [\mathbf{Y}_k^-]^T + \mathbf{R}_k. \tag{A.9}$$

The filter gain, update state mean and covariance matrix would retain the same functional from as (A.6).

The Interactive Multiple Model (IMM) filter

If we are to estimate a system with multiple models, which takes the form

$$\mathbf{x}_{\boldsymbol{k}} = \mathbf{f}_{\boldsymbol{j}}(\mathbf{x}_{\boldsymbol{k-1}}, \mathrm{k}-1) + \mathbf{q}_{\boldsymbol{k-1}}^{\boldsymbol{j}}$$

$$\mathbf{y}_{\boldsymbol{k}} = \mathbf{h}_{\boldsymbol{j}}(\mathbf{x}_{\boldsymbol{k}}, \mathrm{k}) + \mathbf{r}_k^j, \tag{A.10}$$

where j denotes a specific model within the system. This system could be estimated with the Interactive Multiple Model (IMM) filter according to Bar-Shalom et al (2001). For a Markovian switching system like (A.10), IMM is considered an efficient algorithm for simultaneously estimating multiple dynamic processes. There are three phases in IMM for each time step, with the first one being the interaction or mixing phase. Here state estimates produced by all filters from the previous time step are mixed to yield the initial conditions of this step, assuming that this mixed model is the correct model at this step. In the second phase, regular Kalman filtering for each individual model is estimated, and followed by computing a weighted sum of updated individual state estimates by respective filters, with weights conforming to probabilities of individual models obtained in the filtering phase. The weighted computation, or the combination phase, then gives, for this particular time step, the final estimate for the mean and covariance of the Gaussian density.

In the *interaction* phase, the mixing probabilities $\mu_k^{i|j}$ for each model M^i and M^j ,at the current step, are calculated as

$$\overline{c_j} = \sum_{i=1}^n p_{ij} \mu_{k-1}^i, \tag{A.11}$$

$$\mu_k^{i|j} = \frac{1}{\bar{c}_j} p_{ij} \mu_{k-1}^i, \tag{A.12}$$

where μ_{k-1}^i is the probability of model M^i in the time step k-1 and $\bar{c}_j$ is a normalization factor. Mixing the means and covariances each filter j is done by

$$\mathbf{m}_{k-1}^{0j} = \sum_{i=1}^{n} \mu_k^{i|j} \mathbf{m}_{k-1}^i, \tag{A.13}$$

$$\mathbf{P}_{k-1}^{0j} = \sum_{i=1}^{n} \mu_k^{i|j} \times \left\{ \mathbf{P}_{k-1}^i + \left[\mathbf{m}_{k-1}^i - \mathbf{m}_{k-1}^{0j}\right]\left[\mathbf{m}_{k-1}^i - \mathbf{m}_{k-1}^{0j}\right]^T \right\}, \tag{A.14}$$

where m_{k-1}^i and $\mathbf{P}_{k-1}^i$ are the updated mean and covariance for model i at time step k-1.

The *filtering* phase for UKF requires, for each model M^i, filtering is carried out as

$$\left[\mathbf{m}_k^{-,i}, \mathbf{P}_k^{-,i}\right] = \mathrm{KF}_p\left(\mathbf{m}_{k-1}^{0j}, \mathbf{P}_{k-1}^{0j}, \mathbf{Q}_{k-1}^i\right) \tag{A.13}$$

$$\left[\mathbf{m}_k^i, \mathbf{P}_k^i\right] = \mathrm{KF}_u\left(\mathbf{m}_k^{-,i}, \mathbf{P}_k^{-,i}, y_k, \mathbf{R}_k^i\right), \tag{A.14}$$

where KF_p and KF_u are the prediction and update steps according to (A.3) and (A.4) respectively. The likelihood of the measurement for each filter is also computed as

$$\Lambda_k^i = N\left(\mathbf{v}_k^i; 0, \mathbf{S}_k^i\right), \tag{A.15}$$

where $\mathbf{v}_k^i$ is the residual of the measurement and $\mathbf{S}_k^i$ is the covariance matrix for model M^i in (A.14). At the current time step k, the probability μ_k^i for model M^i is calculated as

$$\mu_k^i = \frac{1}{c} \sum_{i=1}^{n} \Lambda_k^i \bar{c}_i, \tag{A.16}$$

where c is a normalizing factor and given by

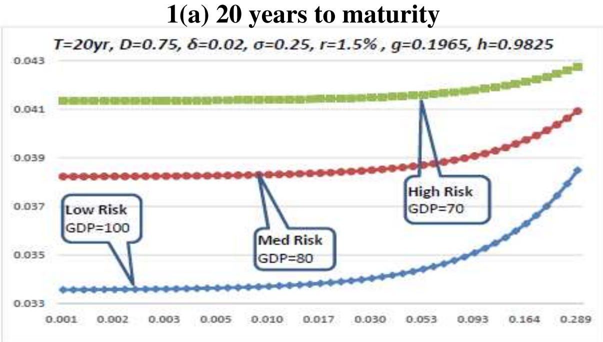
$$\mathrm{c} = \sum_{i=1}^{n} \Lambda_k^i \bar{c}_i, \tag{A.17}$$

In the *combination* phase, the probability-weighted estimate for the state mean and covariance are given by

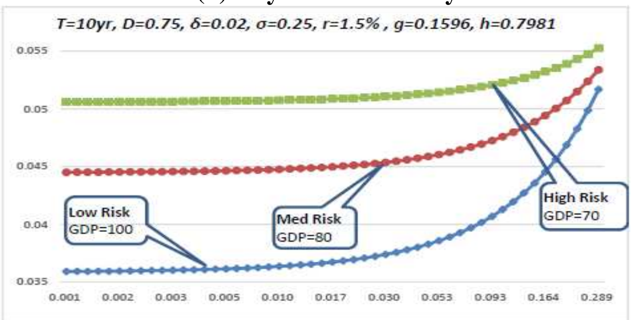
$$\mathbf{m}_k = \sum_{i=1}^{n} \mu_k^i \mathbf{m}_k^i, \tag{A.18}$$

$$\mathbf{P}_k = \sum_{i=1}^{n} \mu_k^i \times \left\{ \mathbf{P}_k^i \left[\mathbf{m}_k^i - \mathbf{m}_k\right]\left[\mathbf{m}_k^i - \mathbf{m}_k\right]^T \right\}. \tag{A.19}$$

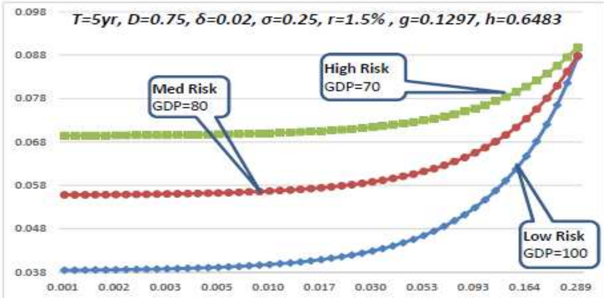
1(a) 20 years to maturity



1(b) 10 years to maturity



1(c) 5 years to maturity



1(d) 1year to maturity

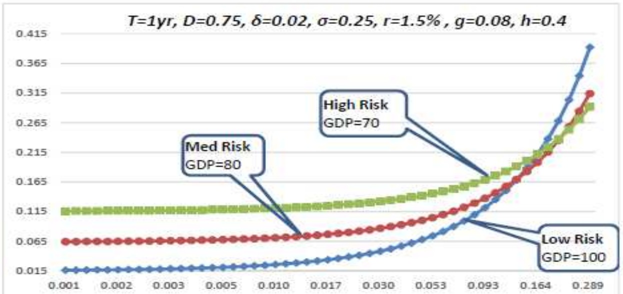
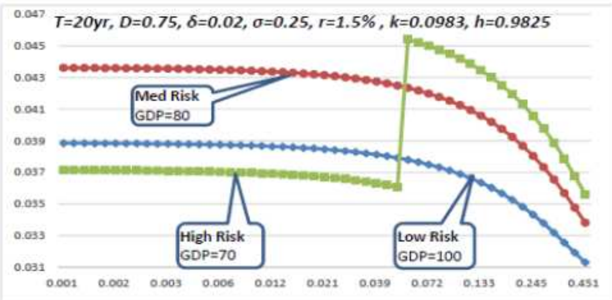


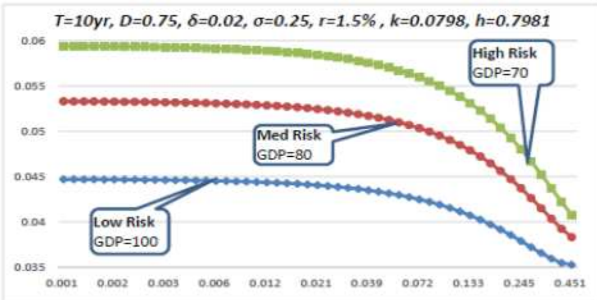
Figure 1: Sovereign debt yields against repayment bonus values at various levels of GDP

The level of repayment bonus (*k*) is expressed as proportions of original debt amount (*D*). The level of expected debt forgiveness (*g*) is expressed as proportions of *D*, while default threshold (*h*) is a proportion of *X*, the amount of debt after taking expected forgiveness into account. Levels of GDP are percentages of current value. Levels of *g* and *h* have been parameterized such that they decrease with time to maturity.

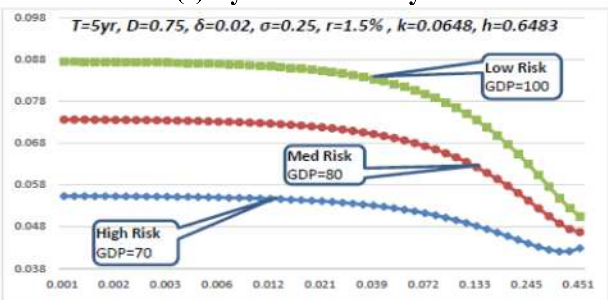
2(a) 20 years to maturity



2(b) 10 years to maturity



2(c) 5 years to maturity



2(d) 1year to maturity

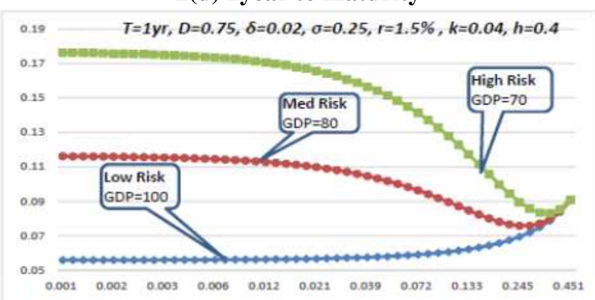
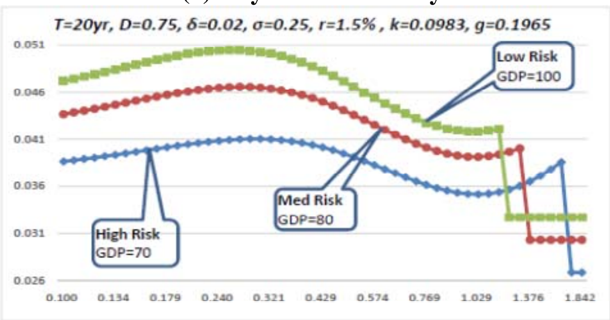


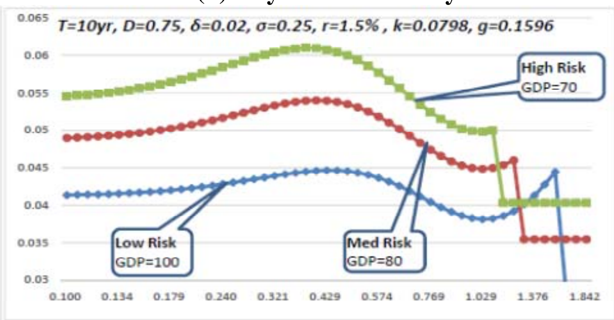
Figure 2: Sovereign debt yields against expected forgiveness values at various levels of GDP

The level of expected forgiveness (*g*) is expressed as proportions of original debt amount (*D*). Levels of *k* and *h* have been parameterized such that they decrease with time to maturity.

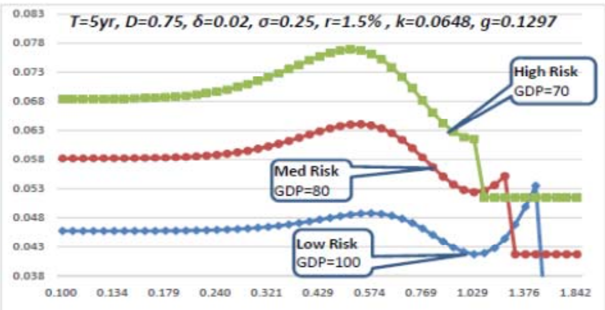
3(a) 20 years to maturity



3(b) 10 years to maturity



3(c) 5 years to maturity



3(d) 1year to maturity

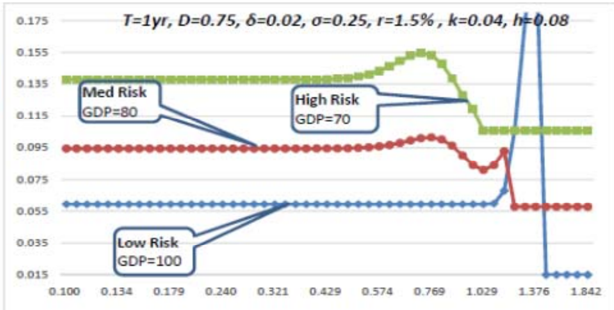
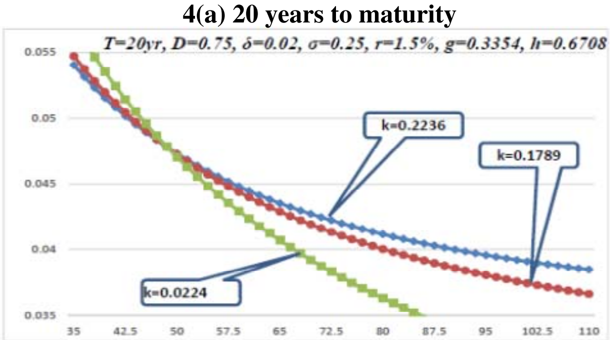


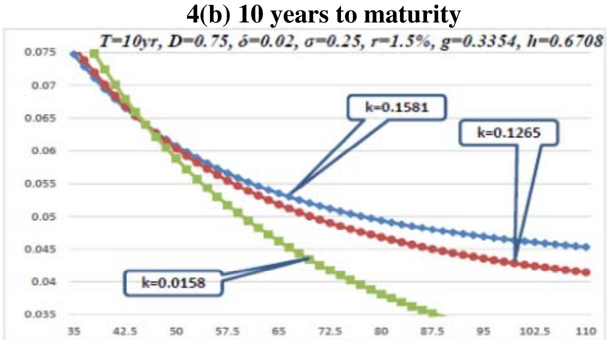
Figure 3: Sovereign debt yields against default threshold values at various levels of GDP

The level of default threshold (h) is expressed as proportions of original debt amount (D). Levels of k and g have been parameterized such that they decrease with time to maturity.

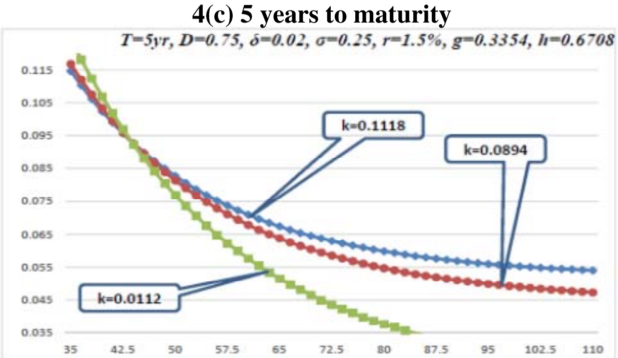
4(a) 20 years to maturity



4(b) 10 years to maturity



4(c) 5 years to maturity



4(d) 1year to maturity

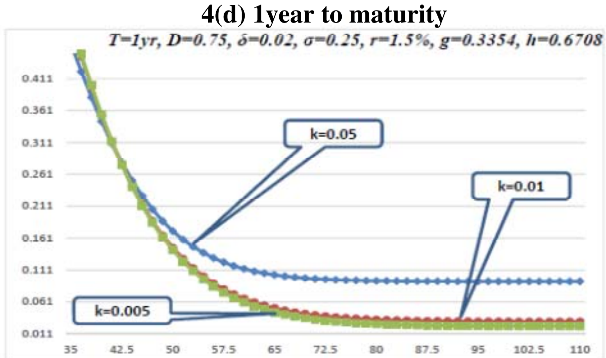


Figure 4: Sovereign debt yields against GDP values at various levels of repayment bonus

Levels of GDP on the horizontal axis are relative to a current level of 100. Levels of g and h have been parameterized such that they decrease with time to maturity.

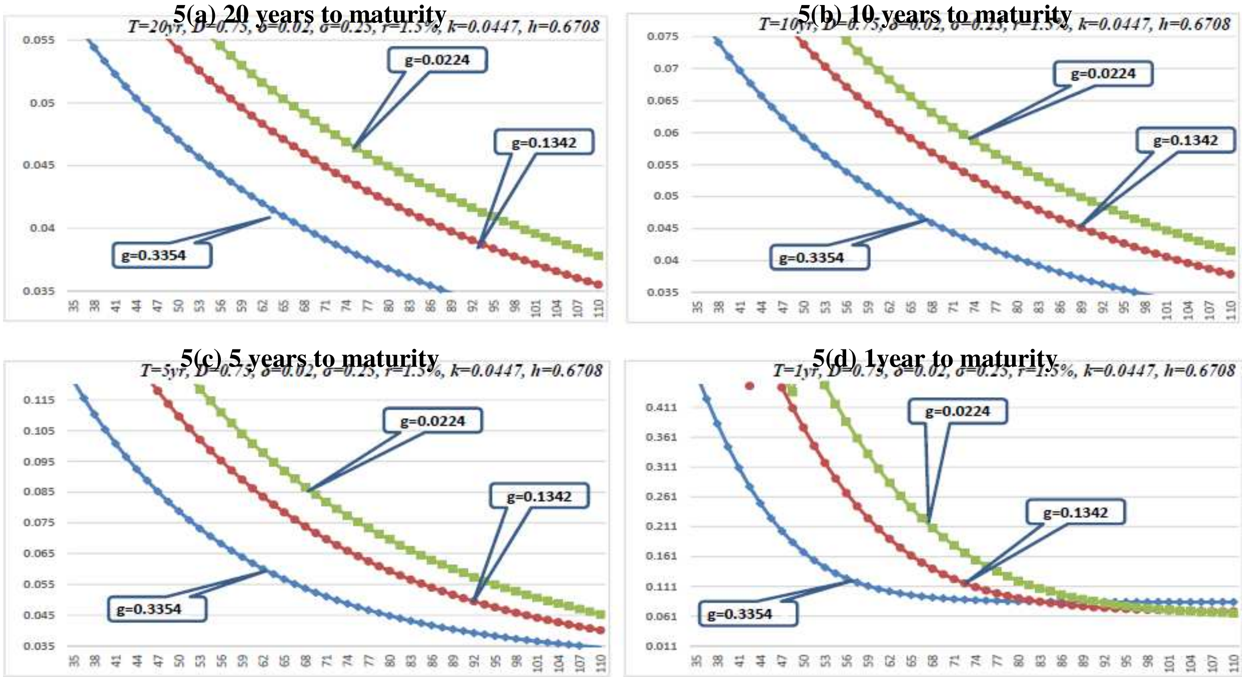


Figure 5: Sovereign debt yields against GDP values at various levels of expected forgiveness
Levels of GDP on the horizontal axis are relative to a current level of 100. Levels of *k* and *h* have been parameterized such that they decrease with time to maturity.

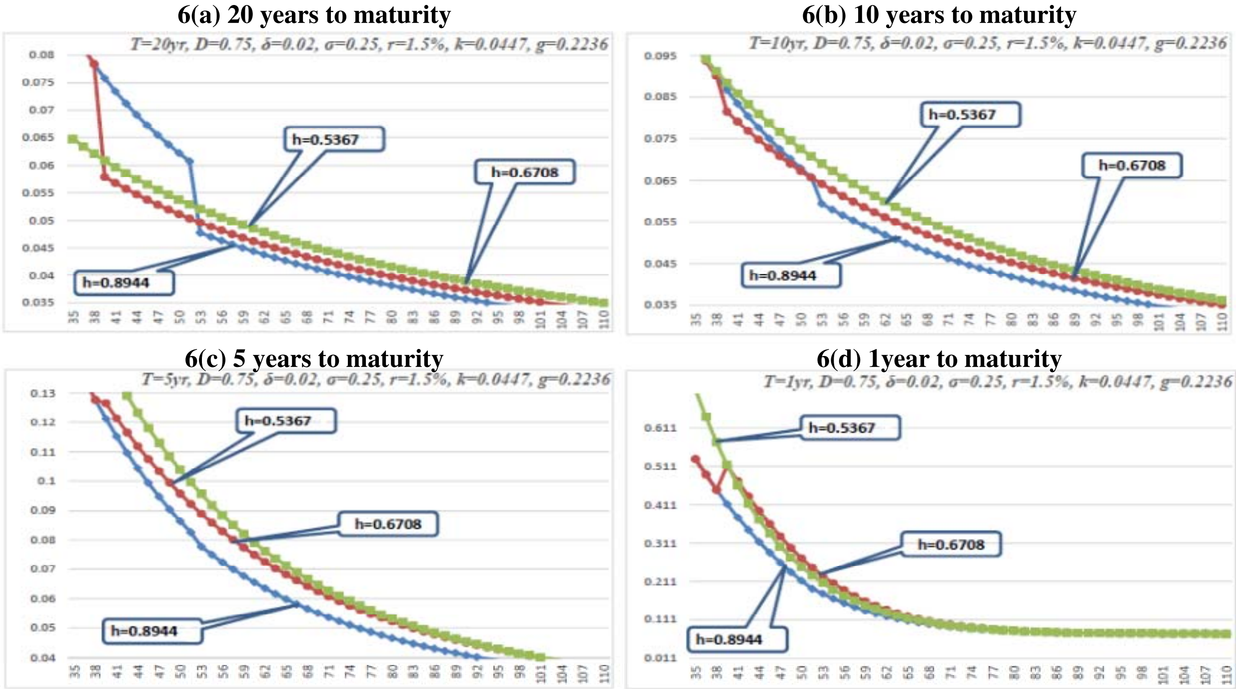
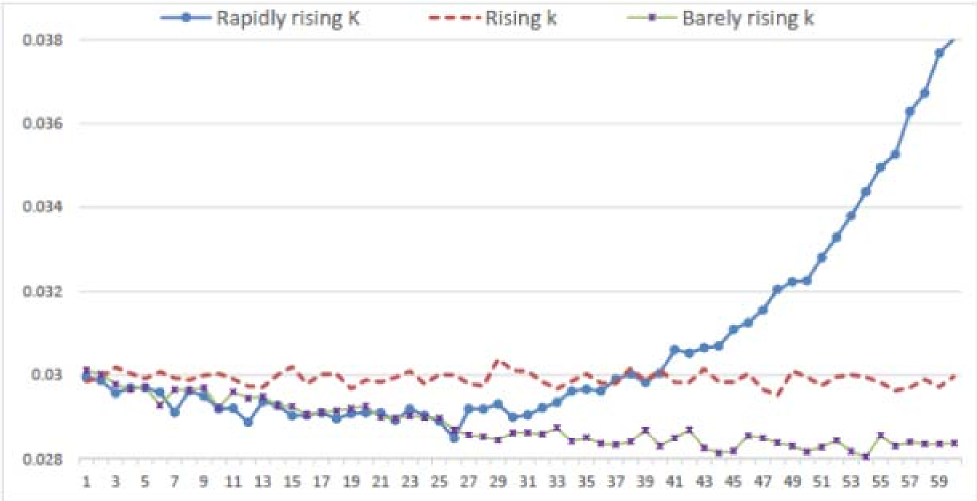


Figure 6: Sovereign debt yields against GDP values at various levels of default threshold
Levels of GDP on the horizontal axis are relative to a current level of 100. Levels of *k* and *g* have been parameterized such that they decrease with time to maturity.

20 Years to Maturity with GDP growing at 6%, g=0.27, h=0.71



20 Years to Maturity with GDP growing at 1.5%, g=0.27, h=0.71

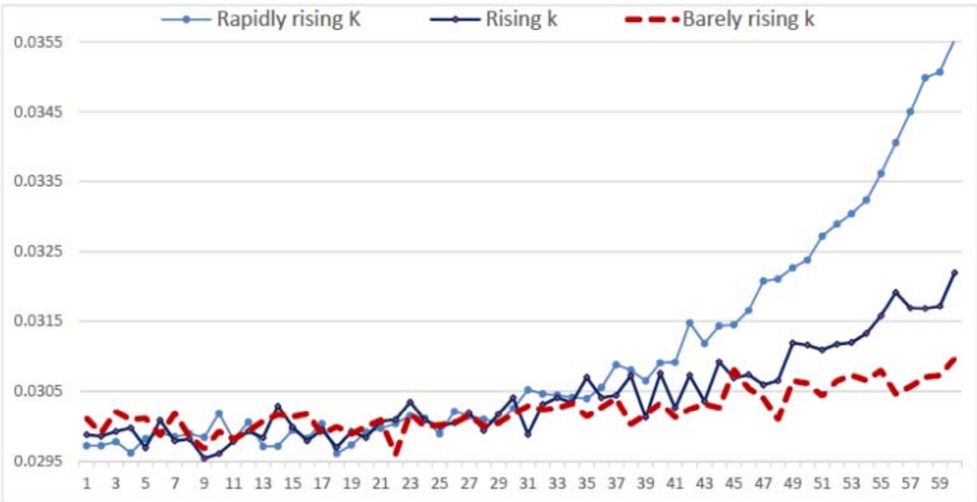
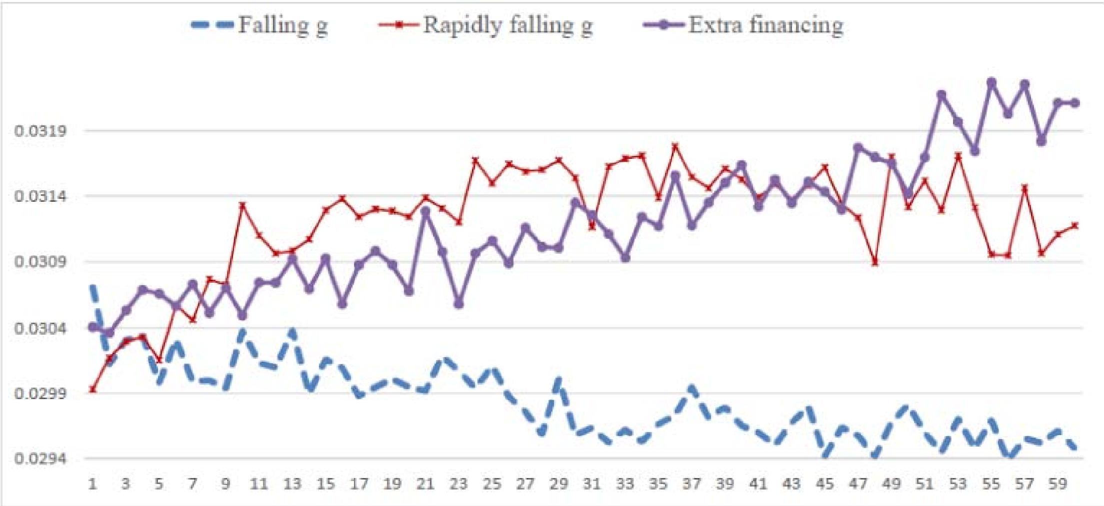


Figure 7: Sixty-month simulations of yields in response to changes of repayment award

Trajectories of simulated debt yields are plotted for the next 60 months for three scenarios with different paths of k. The first is one where increases rapidly at an annual pace of 33% from its current level of 0.074, while in the second scenario k rises steadily at roughly 17% a year. The last one is when k goes up slowly at a pace of 5.6% annually. Each of the three paths is simulated with a lognormal distribution with the current level of k as mean and standard deviation being 25% of it. Raw yields are calculated using (4) and randomized to obtain simulated yields, which are also generated with a lognormal random generator with standard deviations falling with time to maturity. For the first month on the path, the standard deviation of the yield is set at 0.54%, and it is set at 0.51% for the 60th month. Simulation is carried out one thousand times and the average is reported.

20 Years to Maturity, with GDP growing at 6%, *k*=0.071, *h*=0.71



20 Years to Maturity, with GDP growing at 1.5%, *k*=0.071, *h*=0.71

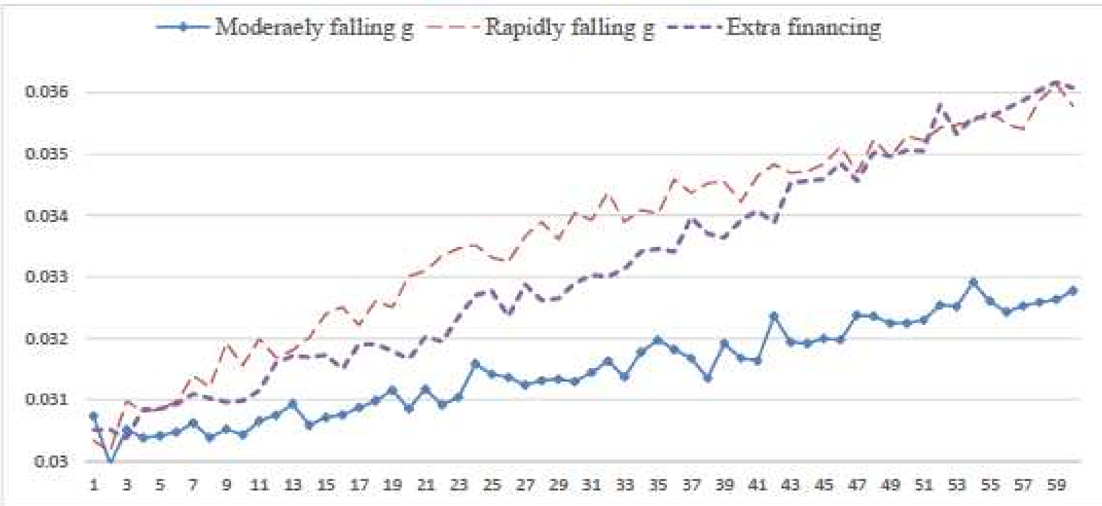
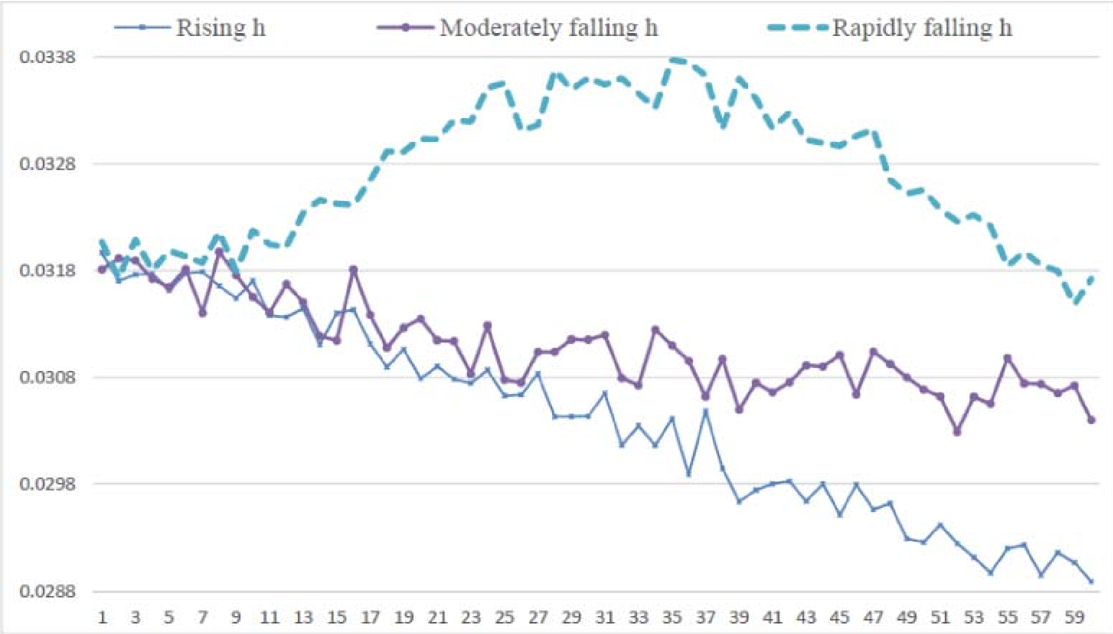


Figure 8: Sixty-month simulations of yields in response to changes of debt forgiveness

Trajectories of simulated debt yields are plotted for the next 60 months for three scenarios with different paths of *g*. The first is one where g rises slowly at 8.6% annually from its current level of 0.071, while in the second scenario *g* increases rapidly at an annual pace of 42%. In the last one, in addition to g falling at 42%, we let the debt amount go up by 5% whenever the default threshold happens to increase 20% over its value in the previous month. This scenario is set up to explore the extra financing alternative against simple forgiveness, as part of our verification of Krugman (1988). Each of the three paths is simulated with a lognormal distribution with the current level of *k* as mean and standard deviation being 15% of it. Raw yields are calculated using (4) and randomized further to obtain simulated yields with a lognormal random generator with standard deviations specified as for the simulations plotted in Figure 7.

20 Years to Maturity, with GDP growing at 6%, k=0.071, g=0.22



20 Years to Maturity, with GDP growing at 1.5%, k=0.071, g=0.22

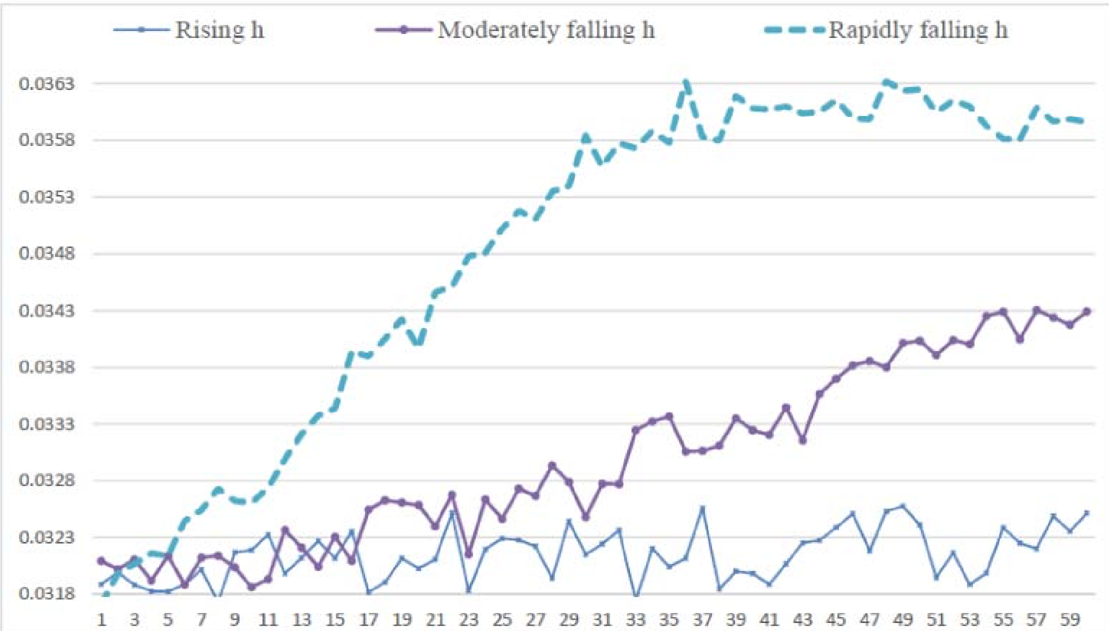


Figure 9: Sixty-month simulations of yields in response to changes of default threshold

Trajectories of simulated debt yields are plotted for the next 60 months for three scenarios with different paths of h. In the first one, h is assumed to rise slowly at 1.7% annually from its current level of 0.71. In the second scenario h goes down moderately at 8.8% per year, and more rapidly at 33% in the third one. Each of the three paths is simulated with a lognormal distribution with the current level of h as mean and standard deviation being 30% of it. Raw yields are calculated using (4) and randomized further to obtain simulated yields with a lognormal random generator with standard deviations specified as for the simulations plotted in Figure 7.

Table I Summary Statistics of Zero-coupon Government Bond Yields

Monthly implied zero-coupon bond yields are obtained from Datastream for the period between January 2003 and December 2012, covering 7 countries and 698 yield series.The shortest series covers around four years, while the longest spans across 11 years. Annual GDP growth data is from World Bank.

	France	Italy	Spain	Netherlands	Israel	Germany	Mexico
				Implied Yields (monthly closes)			
Average	2.45%	4.22%	4.34%	3.86%	3.14%	2.02%	6.54%
Maximum	5.17%	6.83%	7.21%	6.29%	6.30%	4.93%	10.67%
Minimum	0.09%	0.30%	0.36%	0.07%	0.02%	0.08%	0.95%
Std. Dev.	1.28%	2.26%	2.55%	1.17%	1.20%	1.03%	1.89%
				Time to Maturity			
Average	12.90	13.76	14.42	13.76	9.60	12.84	9.34
Maximum	25.87	28.58	27.66	25.75	18.82	26.29	15.12
Minimum	1.23	0.92	0.96	0.73	0.14	1.06	2.04
Std. Dev.	4.31	5.65	5.15	5.44	4.42	5.10	3.19
				Government Debt to GDP Ratios			
Average	66.70%	109.30%	50.45%	54.90%	77.51%	70.59%	41.62%
Maximum	85.80%	120.10%	68.50%	69.77%	79.41%	82.40%	44.60%
Minimum	56.90%	103.60%	36.10%	45.30%	76.05%	60.40%	37.80%
				GDP Growth Rates			
Average	1.69%	1.32%	2.11%	0.27%	0.86%	0.22%	2.03%
Maximum	2.92%	3.04%	4.04%	0.85%	1.05%	0.98%	5.08%
Minimum	-3.91%	-6.54%	-4.42%	-0.93%	0.05%	-0.70%	-4.68%

Table II Estimated Unknown State Variables of Penalty, Forgiveness and Threshold
under an Unscented Kalman Filter (UKF) model

This table reports the results for a UKF algorithm on the barrier-option model in (2) for the years between 2003 and 2012. The scheme of creating sigma points according to Julier and Uhlmann (2004) has been modified to accommodate the non-negativity requirement of yields. The state dynamic function is a simple mean-reverting process, while the measurement process applies directly from (2). The discount rate δ in (2) is assumed to be at 2%. The continuous interest rate is approximated by the short rate of each respective country. The volatility parameter is calculated with quarterly GDP of each country. GDP values are deflated first by the 2002 figures and then scaled to 100. The face value of debt is the government debt to GDP ratio times 100. The theoretical values of the three unknown state variables are generated randomly, and are also assumed to be mean-reverting. The starting values of k, g, and h are therefore set respectively at 0.05, 0.1 and 0.5.

	France	Italy	Spain	Netherlands	Israel	Germany	Mexico
				Repayment Award (k)			
Average	0.0318	0.0621	0.0762	0.0495	0.0587	0.0221	0.1446
Estimates for std. dev.	0.0046	0.0129	0.0147	0.0081	0.0089	0.0036	0.0227
Maximum	0.0444	0.1238	0.1379	0.0915	0.0862	0.0331	0.2021
Minimum	0.0190	0.0302	0.0294	0.0269	0.0304	0.0135	0.0940
Std. err. of estimates	0.0032	0.0038	0.0041	0.0035	0.0034	0.0025	0.0033
				Forgiveness (g)			
Average	0.0601	0.1390	0.1510	0.0893	0.1015	0.0374	0.2845
Estimates for std. dev.	0.0113	0.0211	0.0249	0.0176	0.0155	0.0054	0.0433
Maximum	0.0969	0.2583	0.3005	0.1839	0.1321	0.0502	0.5428
Minimum	0.0305	0.0359	0.0401	0.0343	0.0414	0.0226	0.1579
Std. err. of estimates	0.0031	0.0035	0.0035	0.0032	0.0032	0.0029	0.0034
				Default Threshold (h)			
Average	0.6534	0.8452	0.9130	0.7274	0.7565	0.5641	1.4044
Estimates for std. dev.	0.1088	0.1343	0.1577	0.1151	0.1209	0.0856	0.2161
Maximum	0.8168	1.2039	1.4445	0.8847	0.9129	0.7423	1.8539
Minimum	0.4318	0.6875	0.6641	0.5086	0.5669	0.3848	1.1035
Std. err. of estimates	0.0097	0.0108	0.0105	0.0101	0.0100	0.0096	0.0114

Table III Estimated Unknown State Variables of Penalty, Forgiveness and Threshold

under an Interactive Multiple Model with UKF (IMM-UKF) Algorithm

The IMM-UKF is applied on the same input data for Table II with the same parameters assumed. UKF is used in the filtering stage in every step instead of the standard Kalman filter. In each month, for a given country, combined or probability-weighted estimates for state mean and covariance matrix are calculated. Reported in this table are the averages of estimated standard deviations and means for the three state variables across the 120 months.

	France	Italy	Spain	Netherlands	Israel	Germany	Mexico
				Repayment Award (k)			
Average	0.0349	0.0657	0.0803	0.0536	0.0609	0.0255	0.1493
Estimates for std. dev.	0.0041	0.0115	0.0131	0.0076	0.0082	0.0032	0.0183
Maximum	0.0413	0.1039	0.1119	0.0861	0.0848	0.0302	0.1912
Minimum	0.0285	0.0375	0.0403	0.0396	0.0353	0.0192	0.1005
Std. err. of estimates	0.0030	0.0034	0.0036	0.0032	0.0032	0.0023	0.0032
				Forgiveness (g)			
Average	0.0535	0.1226	0.1318	0.0802	0.0885	0.0337	0.2451
Estimates for std. dev.	0.0097	0.0167	0.0189	0.0129	0.0113	0.0049	0.0349
Maximum	0.0874	0.1941	0.2216	0.1240	0.1185	0.0477	0.3949
Minimum	0.0492	0.0697	0.0754	0.0522	0.0538	0.0291	0.1764
Std. err. of estimates	0.0029	0.0032	0.0033	0.0030	0.0030	0.0027	0.0032
				Default Threshold (h)			
Average	0.5801	0.7339	0.7992	0.6536	0.6860	0.5113	1.1308
Estimates for std. dev.	0.0952	0.1175	0.1284	0.1044	0.1072	0.0768	0.1759
Maximum	0.7633	1.0424	1.1936	0.8134	0.8229	0.6809	1.5523
Minimum	0.4827	0.6811	0.6820	0.5601	0.5732	0.4266	1.0211
Std. err. of estimates	0.0092	0.0101	0.0102	0.0095	0.0095	0.0091	0.0107

Table IV Comparisons between Forecasting Performances against EKF

for given time to maturity

One-period forecasting performances of UKF and IMM-UKF are compared against those form an Extended Kalman Filter (EKF) algorithm. Estimations are produced, for each month in 2013, using the previous 60 months, and one-month ahead forecasted yields for each month in 2013 are compared against observed yields. 22 yield series with maturity less than a year are excluded. The remaining 676 series are divided into 4 maturity groups, with 96 having the maturity between 1 and 5, 188 between 6 and 10, 274 between 11 and 15 and 218 between 16 and above. For each maturity group, Root Mean Squared Errors (RMSE) and Mean Absolute Errors (MAE) are calculated monthly for all issues over 2013.

(*in basis points*)	16 and above	11-15	6-10	1-5
		UKF Estimations		
RMSE	115.91	75.82	42.08	13.42
MAE	93.63	61.25	34.24	11.79
		IMM-UKF Estimations		
RMSE	106.13	70.98	40.40	12.96
MAE	91.81	60.05	33.57	10.34
		EKF Estimations		
RMSE	134.88	86.23	47.11	14.83
MAE	103.10	67.44	37.70	13.18

Table V Estimations on Simulated Yields for Unobservable State Variables

under an Unscented Kalman Filter model

This table reports the results of Unscented Kalman Filter algorithm applied on simulated yields of various maturities based on (2). The discount rate is assumed to be at 2%, short rate at 1.5%, GDP growth volatility at 25% and debt to GDP ratio at 75%. For the simulation, 0.1, 0.2 and 0.7 are assumed for k, g and h. They are further randomized and substituted into (4) to obtain raw yields. For each maturity, 20,000 simulated yields are generated with a lognormal distribution having the raw yield of respective maturity as its mean. For the maturity of 20 years, simulated yields average at 5.48% with a standard error at 0.74%. The average simulated yields are 3.71%, 2.97% and 1.34% for the 10-year, 5-year and 1-year debt. The standard errors are 0.62% , 0.38% and 0.19% respectively.

	20-year		10-year		5-year		1-year	
	UKF	IMM-UKF	UKF	IMM-UKF	UKF	IMM-UKF	UKF	IMM-UKF
				Repayment Award (k)				
Average	0.0805	0.0849	0.0571	0.0602	0.0395	0.0411	0.0185	0.0195
Estimates for std. dev.	0.0104	0.0722	0.0080	0.0058	0.0062	0.0046	0.0031	0.0024
Maximum	0.0971	0.1033	0.0607	0.0749	0.0469	0.0571	0.0332	0.0420
Minimum	0.0353	0.0536	0.0182	0.0309	0.0144	0.0253	0.0108	0.0165
Std. err. of estimates	0.0009	0.0007	0.0012	0.0010	0.0016	0.0013	0.0021	0.0016
				Forgiveness (g)				
Average	0.1830	0.1598	0.1245	0.1073	0.0884	0.0767	0.0319	0.0285
Estimates for std. dev.	0.0321	0.0269	0.0226	0.0188	0.0155	0.0129	0.0064	0.0054
Maximum	0.2771	0.2136	0.1893	0.1477	0.1105	0.1020	0.0528	0.0429
Minimum	0.0648	0.0885	0.0482	0.0613	0.0344	0.0552	0.0127	0.0161
Std. err. of estimates	0.0007	0.0007	0.0008	0.0007	0.0010	0.0009	0.0013	0.0012
				Default Threshold (h)				
Average	0.7163	0.6249	0.5351	0.4641	0.3815	0.3301	0.2529	0.2213
Estimates for std. dev.	0.1045	0.0837	0.0784	0.0658	0.0569	0.0494	0.0388	0.0351
Maximum	1.1306	0.9323	0.8528	0.7012	0.6110	0.5130	0.4207	0.3439
Minimum	0.2725	0.3462	0.2261	0.2781	0.1852	0.2159	0.1613	0.1878
Std. err. of estimates	0.0093	0.0074	0.0077	0.0063	0.0054	0.0045	0.0036	0.0030

Table VI Regressions of Sovereign CDS on State Variables

Estimated from UKF and IMM-UKF

For each country, changes of monthly five-year sovereign CDS spreads observations taken from *Datastream* are regressed on changes of estimated monthly *k*, *g* and *h* values in Table II and III respectively.

	France	Italy	Spain	Netherlands	Israel	Germany	Mexico
				UKF			
k	0.0354	0.0412	0.0463*	0.0388	0.0376	0.0311	0.0449*
	(0.0287)	(0.0255)	(0.0263)	(0.0248)	(0.0251)	(0.0290)	(0.0246)
g	0.0807*	0.0735**	0.0727**	0.0784*	0.0792*	0.0889	0.0486**
	(0.0494)	(0.0323)	(0.0341)	(0.0423)	(0.0441)	(0.0515)	(0.0242)
h	0.0013	0.0017	0.0019	0.0015	0.0015	0.0011	0.0021*
	(0.0008)	(0.0010)	(0.0012)	(0.0009)	(0.0009)	(0.0006)	(0.0012)
R-*squared*	0.1521	0.1774	0.2035	0.1709	0.1813	0.1110	0.2332
				IMM-UKF			
k	0.0376	0.0455*	0.0486**	0.0438*	0.0429*	0.0393	0.0467**
	(0.0245)	(0.0216)	(0.0224)	(0.0252)	(0.0261)	(0.0272)	(0.0219)
g	0.0831**	0.0748**	0.0762**	0.0802**	0.0810**	0.0919**	0.0502**
	(0.0335)	(0.0301)	(0.0319)	(0.0293)	(0.0287)	(0.0366)	(0.0265)
h	0.0014**	0.0019**	0.0020**	0.0016**	0.0016**	0.0012**	0.0022**
	(0.0006)	(0.0007)	(0.0009)	(0.0007)	(0.0007)	(0.0006)	(0.0010)
R-*squared*	0.1733	0.2128	0.2467	0.1921	0.2033	0.1554	0.2617