On the theoretic and numeric problems of approximating the bond yield to maturity

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On the Theoretic and Numeric Problems of Approximating the Bond Yield to Maturity

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I. STATEMENT OF THE PROBLEM

The computation of the exact yield to maturity on a bond with a finite maturity of n periods requires solving a polynomial of degree n. Given the price of the bond, its face value and its coupon rate, the problem is to solve the equation:

$$P = \sum_{t=1}^{n} (1+r)^{-t} + F\,(1+r)^{-n} \tag{1}$$

where

- P = the price of the bond,
- F = the face value,
- C = the coupon value,
- n = the maturity,
- i = the coupon rate $(= C/F)$,
- r = the exact yield (defined in (1)),

and other symbols used are,

- a = the approximate yield (defined in (2)),
- k = the current yield (defined in (4)),
- e = the approximate yield differential (defined in (3)),
- e' = the current yield differential (defined in (5)).

There are no general algebraic solutions to equation (1) for n larger than four. To solve this equation, a trial-and-error method must be applied until the exact yield is found. Several finance text books suggest the use of an approximate yield

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to maturity commonly employed by financial analysts.[1] This approximate yield is computed according to:[2]

$$a = \frac{C + (F-P)/n}{(F+P)/2} \tag{2}$$

It is defined as the ratio of the average income per period to the average price of the bond. It is an approximation because it ignores the time-value of money. In the numerator the discount or the premium is not time-adjusted and is assumed to be equally and periodically distributed over the life of the bond. In the denominator the average price of the bond is assumed to be equal to the arithmetic mean of the bond's current price and its face value which is only correct if one ignores the time-value of money.

If the approximate yield must be employed, various questions are worth examining. Are there prices for which the approximate yield is equal to the exact yield? Over which price ranges will the approximate yield overstate or understate the exact yield? How large an error is committed when the approximate yield is used?[3] How does this error behave when either the term to maturity or the coupon rate changes? How does the bond's current yield, defined as the ratio of its coupon payment to its current price, compare to the approximate yield as an approximation of the exact yield? Are there cases for which the current yield provides a better approximation of the exact yield than does the approximate yield itself?

In general, an error is committed when the approximate yield is used instead of the exact yield and we can write:

$$e(P;\ n,i) = a(P;\ n,i) - r(P;\ n,i) \tag{3}$$

where the variables in parentheses indicate that yields are functions of the price of the bond for the given maturity and coupon rate. In this paper the error $e(P;\ n,i)$ will be referred to as the *approximate-yield differential*. The first concern of this article is to examine the sign and the magnitude of the approximate-yield differential as well as its response to changes in either the term to maturity or the coupon rate of the bond.

Alternatively, one can use the bond's current yield:

$$k = \frac{C}{P} = \frac{iF}{P} = k(P;\ i) \tag{4}$$

as an approximation of the exact yield to maturity. In doing so an error is committed such as

$$e'(P;\ n,i) = k(P;\ i) - r(P;\ n,i) \tag{5}$$

where $e'(P;\ n,i)$ is the *current-yield differential*. Its behavior is also examined in this paper and compared to that of the approximate-yield differential. We prove that under a simple condition the current yield provides a better approximation of the exact yield than does the approximate yield itself. It is worth noting that since the exact yield is a solution to an nth order polynomial and the approximate yield is a solution to a 2nd order polynomial, both yield differentials $e(P)$ and $e'(P)$ are solutions to an nth order polynomial and hence the solutions to the problems we wish to examine in this paper will involve nontrivial mathematical proofs.

Finally, it should be pointed out that the problems raised in this paper can be examined in a capital budgeting context. In this case the exact yield becomes the internal rate of return on the investment proposal, the approximate yield becomes the accounting rate of return on average investment and the current yield becomes the reciprocal of the payback period of the investment proposal.[4]

The remaining part of the paper is organized as follows. Section II investigates the behavior of the approximate-yield differential $e(P;\ n,i)$ in response to changing prices, holding the bond's coupon rate and its term to maturity fixed. Section III examines the price behavior of the current-yield differential $e'(P;\ n,i)$ and compares it to that of the approximate-yield differential. Section IV is devoted to the investigation of the behavior of both the approximate-yield differential and the current-yield differential in response to changes in the bond's term to maturity and its coupon rate, holding the bond's price fixed. Section V is a summary of the major results obtained in this paper.

II. THE PRICE BEHAVIOR OF THE APPROXIMATE-YIELD DIFFERENTIAL

In this section we examine the sign and the magnitude of the approximate-yield differential $e(P;\ n,i)$ when the price of the bond varies from zero to infinity, holding constant both the coupon rate and the term to maturity. The problem is illustrated in Figure 1. The horizontal axis indicates yields and the vertical axis prices. The solid curve is the approximate-yield curve drawn as a function of the bond's price. It cuts the yield axis at a point where the approximate yield is equal to $2(i+1/n)$, the value of the approximate yield for which the bond's price is zero and cuts the price axis at point B where the bond's price is equal to $(Cn+F)$, the price of the bond

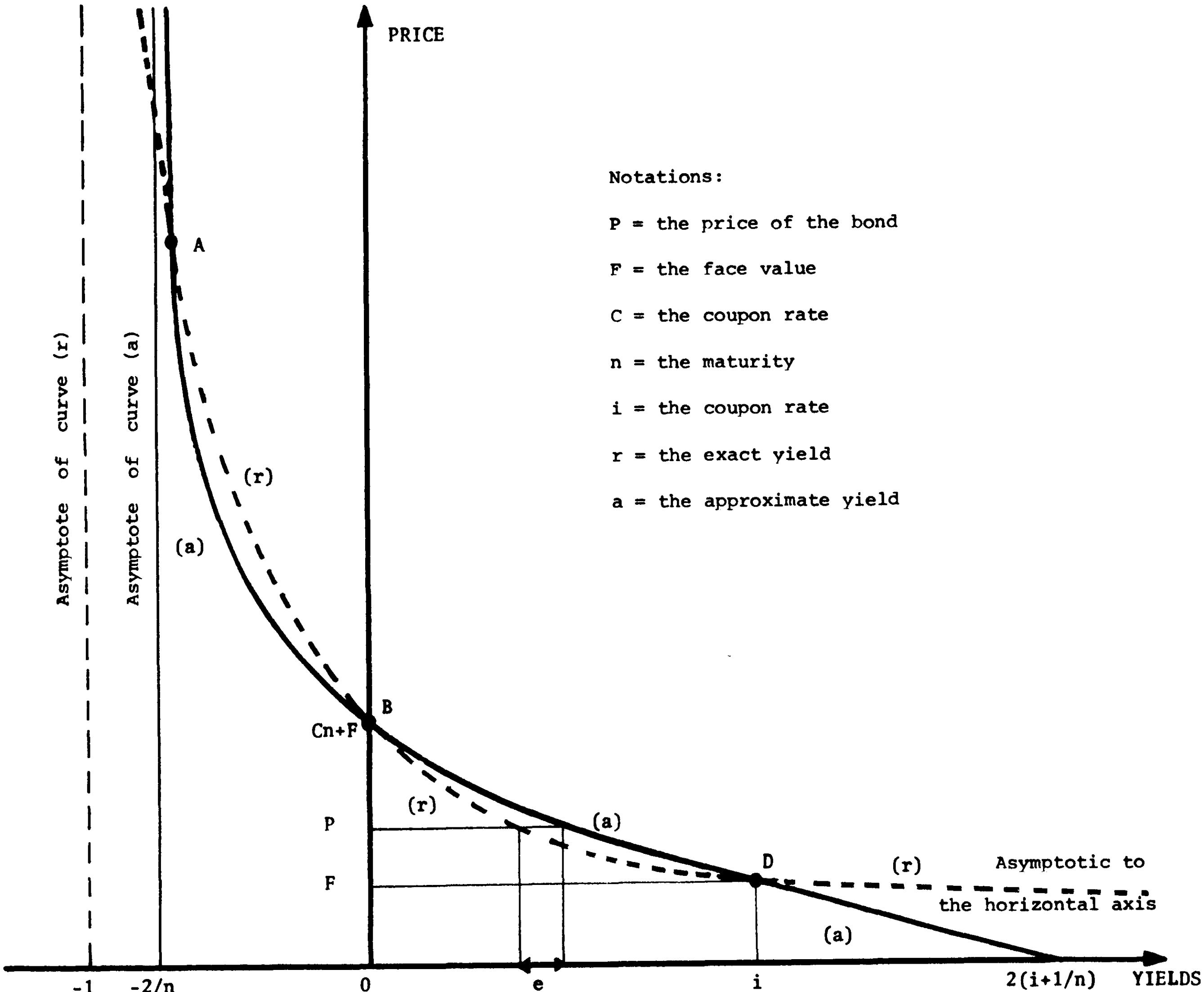

FIGURE 1
Behavior of the exact and approximate yield curves

for which the approximate yield is zero. For prices larger than $(Cn+F)$ the approximate yield is negative and has a vertical asymptote at $a = -2/n$ since $\lim_{P \to +\infty} a(P) = -2/n$.

The broken curve is the exact-yield curve drawn as a function of price. This curve has the yield axis as a horizontal asymptote since $\lim_{P \to 0} r(P) = +\infty$ and a vertical asymptote at $r = -1$ since $\lim_{P \to +\infty} r(P) = -1$. The exact-yield curve cuts the price axis at the same point B as the approximate-yield curve since $P = Cn+F$ for $r=0$ in equation (1). Finally, the two curves intersect at point D where both the approximate yield and the exact yield are equal to the coupon rate $(a=r=i)$.

We will prove that for non perpetual bonds $(n<\infty)$ the two yield curves have *only two* non-negative intersection points, point D for which $a=r=i$ and point B for which $a=r=0$, and at least one negative intersection such as point A, between $a = -2/n$ and zero, when $n>2$. Furthermore, we will demonstrate that the approximate-yield curve lies below the exact-yield curve when the bond is selling at a discount $(P<F)$, above the exact-yield curve when the bond is selling at a premium and yields are positive $(F<P<Cn+F)$, and again below the exact-yield curve when the bond is selling at $P>Cn+F$.[5]

We can see in Figure 1 that the horizontal distance separating the two yield curves is the approximate-yield differential $e(P;\, n,i)$. Alternatively, the approximate-yield differential can be examined with the help of Figure 2. The vertical axis indicates the approximate yield differential e and the horizontal axis the bond's price. The function $e(P)$ is drawn for non-negative yields only as a solid curve. Some interesting properties can be drawn from Figure 2. *When the bond sells at a discount $(P<F)$, the approximate yield understates the exact yield and the approximate-yield differential increases with the bond's price. When the bond sells at a premium $(P>F)$ and yields are positive, the approximate yield overstates the exact yield and the approximate-yield differential first increases with the bond's price, reaches a maximum and then decreases to become zero when yields are zero.* Thus, contrary to the case where the bond sells at a discount, there exists a maximum error when the bond sells at a premium. The result of equations (6) through (20) are illustrated numerically in the columns of Tables II and IV for various combinations of maturity (n) and the coupon rate (i). Tables I and III present the results related to the current yield. The price at which the maximum error is reached when the bond sells at a premium is given below the tables.[6] The error committed when using the approximate yield is smallest when the bond's price is closer to its face value or closer to its price at zero yields $(P=Cn+F)$.

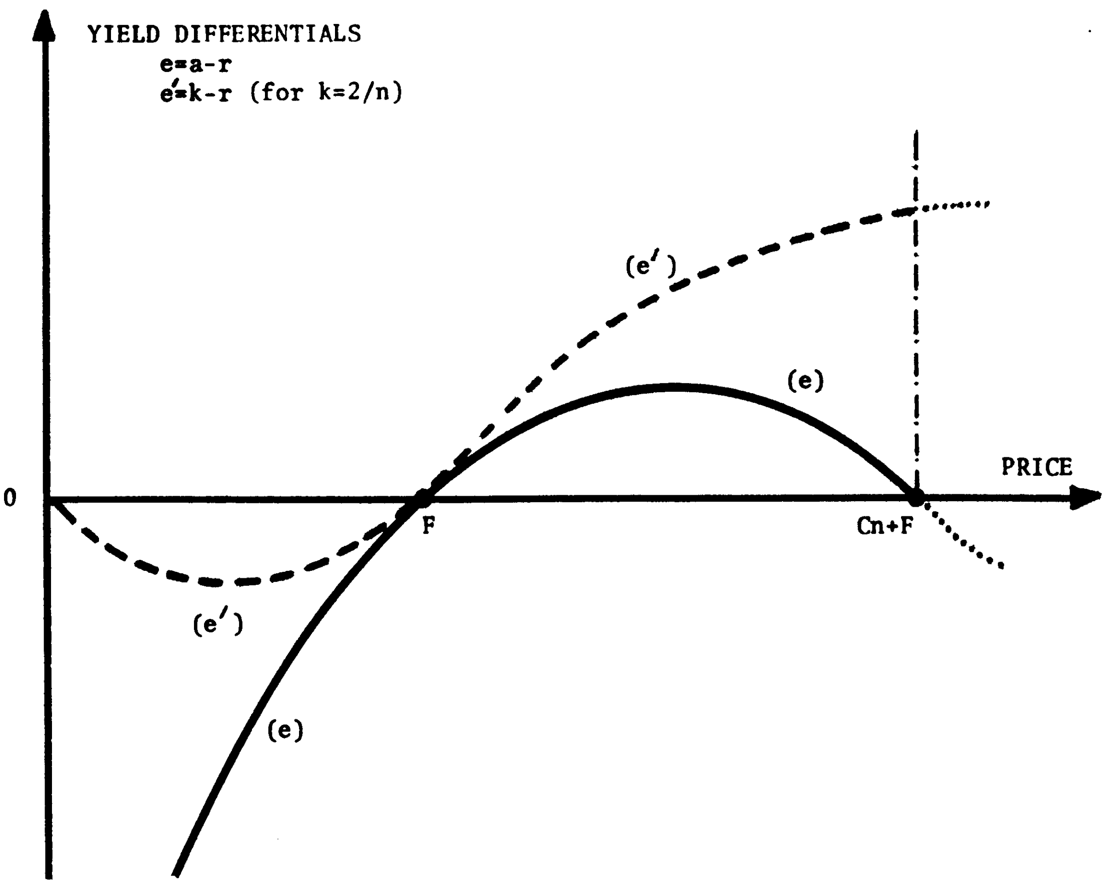

FIGURE 2
Behavior of the approximate and current yield differentials

We will now prove the conclusions discussed so far. We first demonstrate that for $2<n<\infty$ the two curves have only two non-negative intersections and at least one negative intersection. The price of the bond as a function of its exact yield, given by equation (1), can be expressed as:[7]

$$\frac{P}{F} = \frac{i}{r} + \left(1-\frac{i}{r}\right)\left(\frac{1}{1+r}\right)^{n} \qquad (6)$$

This price, expressed as a percentage of the bond's face value, can also be written as a function of the approximate yield using equation (2). We get:

$$\frac{P}{F} = \frac{2(ni+1) - na}{2+na} \tag{7}$$

Equating the price given by equation (6) to the price given by equation (7) and letting $r=a$ we obtain:

$$\frac{2 + 2ni - na}{2+na} - \frac{i}{a} - \frac{1}{(1+a)^n} + \frac{i}{a(1+a)^n} = 0 \tag{8}$$

The roots of equation (8) are the intersection points we wish to determine. Equation (8) can be rewritten as:

$$(i-a)\left[\frac{-2}{a(2+na)} + \frac{n}{(2+na)} + \frac{1}{a(1+a)^n}\right] = 0 \tag{9}$$

The first root of equation (8) is $a=i$ given that $a \neq -2/n$ and $a \neq -1$, the two vertical asymptotes discussed earlier. This is point D where both the exact yield and the approximate yield are equal to the coupon rate (i). The terms in brackets can be rearranged such as:

$$(i-a)\left[\frac{-2(1+a)^n + na(1+a)^n + (2+na)}{a(2+na)(1+a)^n}\right] = 0 \tag{10}$$

Using the binomial expansion we get:

$$\frac{(i-a)\left\{-2\left[\sum_{j=0}^{n}\binom{n}{j}a^j\right] + na\left[\sum_{j=0}^{n}\binom{n}{j}a^j\right] + 2+na\right\}}{a(2+na)(1+a)^n} = 0 \tag{11}$$

where

$$\binom{n}{j} = \frac{n!}{j!\,(n-j)!}$$

are the binomial coefficients. The terms in braces can be rewritten as:

$$\left\{-2 -2\left[\sum_{j=1}^{n}\binom{n}{j}a^j\right] + na\left[\sum_{j=0}^{n}\binom{n}{j}a^j\right] + 2 + na\right\}$$

$$= a\left\{-2\left[\sum_{j=1}^{n}\binom{n}{j}a^{j-1}\right] + n\left[\sum_{j=0}^{n}\binom{n}{j}a^j\right] + n\right.$$

TABLE I

THE MATURITY BEHAVIOR OF THE CURRENT-YIELD DIFFERENTIAL (e') FOR i=10%

(e' is in percentage points)

P/F	n=2	n=4	n=6	n=10	n=15	n=20	n=30	n=40
0.050	-279.583	-35.576	-4.855	-0.064	-0.0	-0.0	-0.0	-0.0
0.100	-185.410	-39.396	-11.368	-0.850	-0.027	-0.001	-0.0	-0.0
0.250	-90.713	-28.942	-13.336	-3.604	-0.731	-0.141	-0.005	-0.0
0.500	-38.661	-15.063	-8.199	-3.276	-1.250	-0.504	-0.083	-0.014
0.750	-14.623	-6.247	-3.624	-1.644	-0.755	-0.379	-0.104	-0.030
0.900	-5.138	-2.278	-1.353	-0.641	-0.312	-0.168	-0.054	-0.018
0.950	-2.471	-1.107	-0.662	-0.317	-0.157	-0.086	-0.029	-0.010
1.000	0	0	0	0	0	0	0	0
1.050	2.298	1.049	0.635	0.310	0.158	0.089	0.032	0.012
1.200	8.333	3.900	2.393	1.199	0.628	0.366	0.142	0.060
1.300	-	5.591	3.459	1.756	0.936	0.555	0.225	0.099
1.400	-	7.143	4.452	2.287	1.236	0.744	0.312	0.143
1.500	-	-	5.380	2.793	1.529	0.932	0.402	0.191
max e'	8.33333	7.14285	6.250	5.0	4.0	3.33333	2.50	2.0
P(max e')	1.200	1.400	1.600	2.000	2.500	3.000	4.000	5.000

TABLE II

THE MATURITY BEHAVIOR OF THE APPROXIMATE-YIELD DIFFERENTIAL (e) FOR i=10%

(e is in percentage points)

P/F	n=2	n=4	n=6	n=10	n=15	n=20	n=30	n=40
0.050	-370.059	-171.290	-155.648	-162.921	-168.889	-171.905	-174.920	-176.428
0.100	-185.410	-80.305	-65.913	-66.305	-70.936	-73.637	-76.364	-77.727
0.250	-54.713	-22.942	-17.336	-15.604	-16.731	-18.141	-20.005	-21.000
0.500	-11.994	-5.063	-3.755	-3.276	-3.472	-3.837	-4.528	-5.014
0.750	-2.242	-1.009	-0.767	-0.691	-0.755	-0.855	-1.057	-1.220
0.900	-0.460	-0.231	-0.184	-0.173	-0.195	-0.226	-0.288	-0.340
0.950	-0.177	-0.095	-0.077	-0.074	-0.085	-0.099	-0.128	-0.152
1.000	0	0	0	0	0	0	0	0
1.050	0.091	0.062	0.054	0.055	0.065	0.077	0.102	0.123
1.200	0	0.112	0.120	0.138	0.174	0.214	0.294	0.363
1.300	-	0.072	0.114	0.151	0.200	0.254	0.358	0.450
1.400	-	0	0.086	0.144	0.204	0.268	0.391	0.500
1.500	-	-	0.047	0.126	0.196	0.266	0.402	0.524
max e	0.11370	0.11445	0.12160	0.15080	0.20495	0.26861	0.40273	0.53031
P(max e)	1.090	1.170	1.230	1.310	1.380	1.430	1.520	1.590

TABLE III

THE COUPON BEHAVIOR OF THE CURRENT-YIELD DIFFERENTIAL (e') FOR n=10

(e' is in percentage points)

P/F	i=2%	i=4%	i=6%	i=8%	i=10%	i=15%	i=20%	i=25%
0.050	-13.916	-3.650	-0.833	-0.214	-0.064	-0.005	-0.001	-0.0
0.100	-15.827	-8.524	-4.102	-1.866	-0.850	-0.141	-0.030	-0.008
0.250	-11.719	-9.015	-6.774	-4.983	-3.604	-1.539	-0.655	-0.290
0.500	-6.211	-5.340	-4.563	-3.876	-3.276	-2.113	-1.341	-0.847
0.750	-2.616	-2.337	-2.083	-1.852	-1.644	-1.210	-0.883	-0.642
0.900	-0.961	-0.870	-0.787	-0.711	-0.641	-0.492	-0.376	-0.287
0.950	-0.468	-0.426	-0.386	-0.350	-0.317	-0.246	-0.191	-0.147
1.000	0	0	0	0	0	0	0	0
1.050	0.446	0.408	0.373	0.340	0.310	0.246	0.194	0.153
1.200	1.667	1.536	1.415	1.303	1.199	0.972	0.786	0.636
1.300	-	2.220	2.054	1.900	1.756	1.441	1.182	0.969
1.400	-	2.857	2.654	2.464	2.287	1.897	1.573	1.304
1.500	-	-	3.218	2.998	2.793	2.338	1.958	1.640

TABLE IV

THE COUPON BEHAVIOR OF THE APPROXIMATE-YIELD DIFFERENTIAL (e) FOR n=10

(e is in percentage points)

P/F	i=2%	i=4%	i=6%	i=8%	i=10%	i=15%	i=20%	i=25%
0.050	-32.01	-57.94	-91.31	-126.88	-162.92	-253.34	-343.81	-434.29
0.100	-15.83	-24.89	-36.83	-50.96	-66.30	-106.50	-147.30	-188.19
0.250	-4.52	-6.62	-9.17	-12.18	-15.60	-25.54	-36.66	-48.29
0.500	-0.88	-1.34	-1.90	-2.54	-3.28	-5.45	-8.01	-10.85
0.750	-0.14	-0.24	-0.37	-0.52	-0.69	-1.21	-1.84	-2.55
0.900	-0.03	-0.05	-0.09	-0.13	-0.17	-0.32	-0.49	-0.70
0.950	-0.01	-0.02	-0.04	-0.05	-0.07	-0.14	-0.22	-0.31
1.000	0	0	0	0	0	0	0	0
1.050	0.004	0.013	0.024	0.038	0.055	0.106	0.171	0.246
1.200	0	0.021	0.052	0.091	0.138	0.290	0.483	0.712
1.300	-	0.013	0.047	0.094	0.151	0.338	0.621	0.947
1.400	-	0	0.035	0.083	0.144	0.349	0.621	0.947
1.500	-	-	0.018	0.065	0.126	0.338	0.624	0.973
max e	0.00541	0.02222	0.05172	0.09451	0.15080	0.34947	0.62633	0.97331
P (max e)	1.090	1.160	1.220	1.260	1.310	1.390	1.460	1.520

$$= a\left\{n + \left[\sum_{j=0}^{n-1}\left\{n\binom{n}{j} - 2\binom{n}{j+1}\right\}a^j\right] + n\binom{n}{n}a^n\right\}$$

$$= a\left\{n + \left(n\binom{n}{0} - 2\binom{n}{1}\right)a^0 + \left[\sum_{j=1}^{n-1}\left(n\binom{n}{j} - 2\binom{n}{j+1}\right)a^j\right] + na^n\right\}$$

$$= a\left\{\left[\sum_{j=1}^{n-1}\left(n\binom{n}{j} - 2\binom{n}{j+1}\right)a^j\right] + na^n\right\}$$

$$= a^2\left\{\left[\sum_{j=1}^{n-1}\left(n\binom{n}{j} - 2\binom{n}{j+1}\right)a^{j-1}\right] + na^{n-1}\right\}$$

and equation (11) becomes

$$\frac{a(i-a)\left\{\sum_{j=1}^{n-1}\left[n\binom{n}{j} - 2\binom{n}{j+1}\right]a^{j-1} + na^{n-1}\right\}}{(2+na)(1+a)^n} = 0 \tag{12}$$

It follows from equation (12) that $a=0$ is the second root of equation (8). This is point B where both the exact and the approximate yield are equal to zero. We will show that the polynomial of degree $(n-1)$ in braces in equation (12) has at least one negative root for $2<n<\infty$ and no non-negative roots. The coefficients of this polynomial, excluding n for the last term, are equal to:

$$a_j = n\binom{n}{j} - 2\binom{n}{j+1} = \frac{n(j+1)}{n-j}\binom{n}{j+1} - 2\binom{n}{j+1}$$

$$a_j = \binom{n}{j+1}\left[\frac{n(j+1)}{(n-j)} - 2\right],\ 1 \leqq j \leqq (n-1),\ n>2. \tag{13}$$

For $j\geqq 1$, the coefficients a_j are positive and since n, the coefficient of the last term a^{n-1}, is also positive it follows that the polynomial in braces has only positive coefficients and therefore cannot have any real positive roots.[8] We will prove that the approximate-yield curve lies below the exact-yield curve between points B and A in figure 1, and since for $2<n<\infty$ the vertical asymptote of the approximate-yield curve is to the right of that of the exact-yield curve it follows that the curves must cut at least once between $(-2/n)$ and zero. When n equals either two or one, there is no negative intersection for the yield curves. This can be easily shown by noting that equation (12) for $n=2$ and $n=1$ becomes, respectively.

$$\frac{a(i-a)}{2(1+a)} = 0 \qquad \text{and} \qquad \frac{a(i-a)}{(2+a)} = 0 \tag{14}$$

Having established the position of the intersection points we must now demonstrate that the approximate-yield curve is below the exact-yield curve between points *A* and *B* and to the right of point *D* and above the exact-yield curve between points *B* and *D*. First note that both curves are concave to the origin and monotonically decreasing since:

$$\frac{dP}{da} = -\frac{2n(nC+2F)}{(2+na)^2} < 0 \qquad \frac{d^2P}{da^2} = \frac{4n^2(nC+2F)}{(2+na)^3} > 0 \tag{15}$$

$$\left.\begin{aligned} \frac{dP}{dr} &= -C\left[\sum_{j=1}^{n} \frac{j}{(1+r)^{j+1}}\right] - \frac{nF}{(1+r)^{n+1}} < 0 \\ \frac{d^2P}{dr^2} &= C\left[\sum_{j=1}^{n} \frac{j(j+1)}{(1+r)^{j+2}}\right] + \frac{n(n+1)F}{(1+r)^{n+2}} > 0 \end{aligned}\right\} \tag{16}$$

In order to establish the relative position of the two curves, we evaluate their respective derivative at point *B* where yields are zero. We have:

$$\left.\frac{dP}{da}\right|_{a=0} = -\frac{n}{2}(nC+2F) < 0 \tag{17}$$

$$\left.\frac{dP}{dr}\right|_{r=0} = -C\left[\sum_{j=1}^{n} j\right] - nF = -\frac{n}{2}\left[(n+1)C + 2F\right] < 0 \tag{18}$$

and

$$\left.\frac{dP}{da}\right|_{a=0} - \left.\frac{dP}{dr}\right|_{r=0} = \frac{nC}{2} > 0 \tag{19}$$

From equation (19) it follows that

$$\left.\frac{dP}{da}\right|_{a=0} > \left.\frac{dP}{dr}\right|_{r=0} \tag{20}$$

Since the slope of the approximate-yield curve is larger than the slope of the exact-yield curve at point *B*, and since both curve are monotonically decreasing and intersect again only at point *D* and point *A* it follows that the approximate-yield curve is above the exact-yield curve between points *B* and *D* and below between points *A* and *B* and to the right of point *D*. This establishes unambiguously the relative positions of the two curves and justifies the conclusions stated earlier. It should be pointed out that the proof we used takes a round-about way to establish the relative position of the two curves because the exact yield is an nth degree polynomial and even though the roots of the exact yield are parametric in the price of the bond, it is not possible to express the exact yield as an explicit function of the bond's price. Hence the approximate-yield differential cannot be expressed as an explicit function of price.

III. THE CURRENT YIELD AS AN APPROXIMATION OF THE EXACT YIELD

The current yield $k=C/P$ can also be employed as an approximation of the exact yield. In doing so an error $e'=k-r$ is committed. In this section we examine the price behavior of the current-yield differential e' and compare it to that of the approximate yield differential $e=a-r$. The current yield k is equal to iF/P and thus:

$$\frac{P}{F}=\frac{i}{k} \tag{21}$$

First, let us determine the intersection points of the current-yield curve with the exact-yield curve. At the intersections points the prices (P/F) are equal and $r=k$. From equations (6) and (21) we have:

$$\frac{i}{k}=\frac{i}{k}+\frac{1}{(1+k)^n}-\frac{1}{k(1+k)^n}$$

or

$$\frac{(i-k)}{k(1+k)^n}=0 \tag{22}$$

and it follows that the current-yield curve and the exact-yield curve have only one intersection point at $k=r=i$, which is point D as shown in figures 3a, 3b, and 3c. We now demonstrate that the current-yield curve lies above the exact-yield curve when the bond is selling at a premium and below the exact-yield curve when the bond is selling at a discount. Similarly to the exact-yield curve, the current-yield curve is monotonically decreasing since

$$\frac{dP}{dk}=-\frac{C}{k^2}<0 \quad \text{and} \quad \frac{d^2p}{dk^2}=\frac{2C}{k^3}>0 \tag{23}$$

Evaluating the slope of both curves at their unique intersection point D, where $k=r=i$, we get:

$$\left.\frac{dP}{dk}\right|_{k=i}=-\frac{F}{i} \quad \text{and} \quad \left.\frac{dP}{dr}\right|_{r=i}=-\frac{F}{i}\left[1-\frac{1}{(1+i)^n}\right] \tag{24}$$

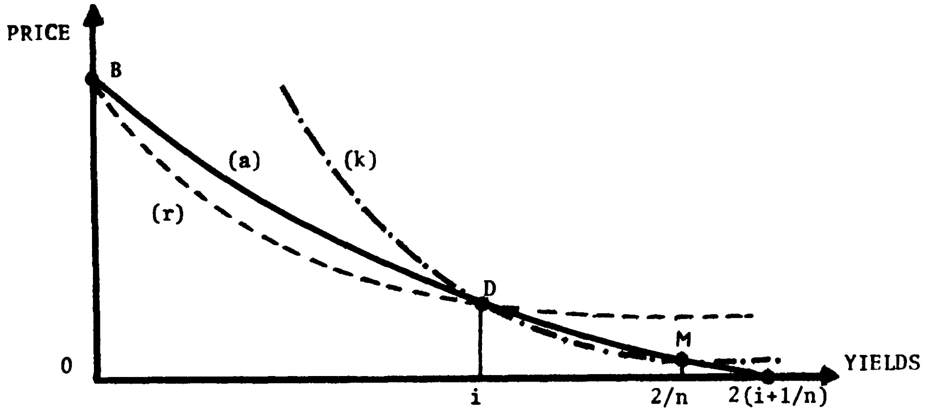

FIGURE 3-a
i<2/n

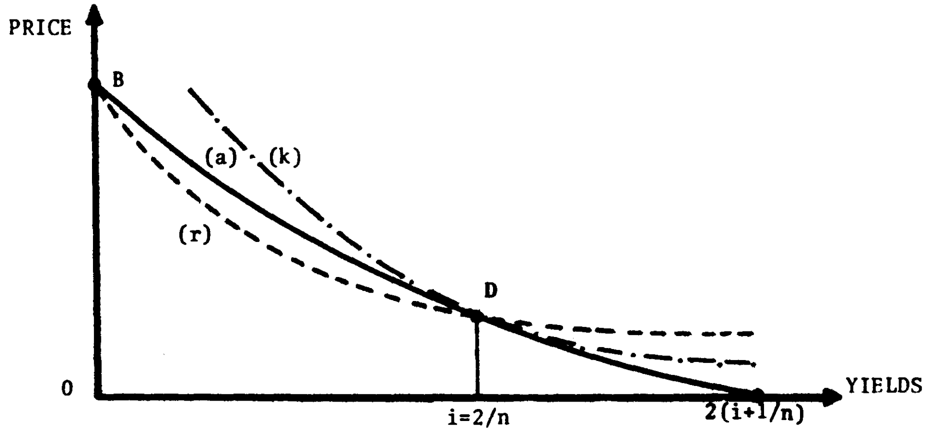

FIGURE 3-b
i=2/n

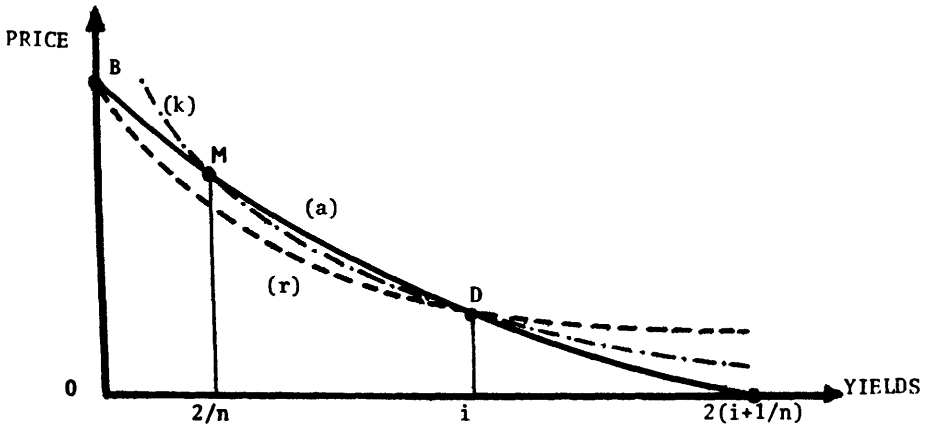

FIGURE 3-c
i>2/n

from equations (24) it is clear that

$$\left.\frac{dP}{dk}\right|_{k=i} < \left.\frac{dP}{dr}\right|_{r=i} \tag{25}$$

which proves our statement. Hence the *sign* of the current-yield differential $e'(P)$ is the same as the sign of the approximate-yield differential $e(P)$. Thus *both the current yield and the approximate yield overstate the exact yield when the bond sells at a premium and understates it when the bond sells at a discount.* The magnitude of the error, however, is different. To compare the magnitude of $e'(P)$ to that of $e(P)$ we can determine the intersection points between the current-yield curve and the approximate-yield curve. Again, at the intersection points, the prices (P/F) are equal and $k=a$. From equation (7) and (21) we get:

$$\frac{i}{k} = \frac{2 + 2ni - nk}{2 + nk}$$

or

$$\frac{(i-k)(2-nk)}{k(2 + nk)} = 0 \tag{26}$$

Equation (26) indicates the existence of two positive intersection points, one at $k=a=i$, which is point D, and the second at $k=a=2/n$. Three cases must be considered according as $i \gtreqless 2/n$. They are illustrated in Figure 3. To interpret Figure 3 we should first demonstrate that the intersection point M is to the right of point D when $i<2/n$ (Figure 3a), coincides with point D when $i=2/n$ (Figure 3b), and is to the left of point D where $i>2/n$ (Figure 3c). To do so, we evaluate the slope of the approximate-yield curve at point D where $a=i$. We have:

$$\left.\frac{dP}{da}\right|_{a=i} = -\left(\frac{F}{i}\right)\left(\frac{2ni}{ni+2}\right) \tag{27}$$

Comparing the slope of the approximate-yield curve in (27) with that of the current-yield curve in (24), it is clear that:

$$\left.\frac{dP}{da}\right|_{a=i} \gtreqless \left.\frac{dP}{dk}\right|_{k=i} \qquad \text{according as } i \lesseqgtr \frac{2}{n} \tag{28}$$

The above results determine the relative magnitude of the two slopes. Combined with the fact that the two curves intersect only at $k=a=i$ and at $k=a=2/n$ it follows that the approximate-yield curve and the current-yield curve can take either one of the three positions shown in Figure 3. An examination of the figure indicates that *when the current yield is larger than twice the reciprocal of the bond's maturity $(k>2/n)$, it provides a better approximation of the exact yield than does the approximate yield* *itself*. For example, consider a bond selling at $P/F = .6819$ with a mathrity $n=40$ years and a coupon rate $i = .08$. Its exact yield is $.1200$. The bond satisfies the condition $k>2/n$ since $(k = .1173)$ is larger than $(2/n = .0500)$. The approximate yield is equal to $.1046$ and, hence, the current yield provides a better approximation than the approximate yield itself. Consequently, the current yield (k) when larger than $(2/n)$, and the approximate yield (a) when k is smaller than $(2/n)$ can act as "greedy" algorithm (see Miller and Thatcher [5]) and may be used as a starting point for refined algorithms of Fisher [1] and Kaplan [3] for finding the exact yield to maturity.

Finally, the behavior of the current-yield differential $e'(P) = k-r$ as a function of price is illustrated in Figure 2. It is drawn for the case where $i=2/n$ and shown as a broken curve. When the bond's price is zero the error $e'(P)$ is zero since both the current yield (k) and the exact yield (r) are infinitely large. The error e' is zero again when the bond sells at face value. *Between a price of zero and face value the current-yield differential reaches a minimum for which the absolute value of the error term is maximum.* When the bond sells at a premium the error increases with price over the relevant price range. It is worth noting that the current-yield differential has an optimum when the bond sells at a discount whereas the approximate-yield differential has an optimum when the bond sells at a premium.[9] These results are further illustrated in the columns of Table I and Table III for various combinations of maturity (n) and coupon rate (i).

IV. THE MATURITY AND COUPON BEHAVIOR OF YIELD DIFFERENTIALS

This section is devoted to the examination of the behavior of the yield differentials $e(P;\, n,i)$ and $e'(P;\, n,i)$ in response to changes in either the bond's term to maturity (n) or its coupon rate (i), holding its price fixed.

A. The Maturity Behavior of Yield Differentials. As the bond's term to maturity changes, the response of the current-yield differential is given by:

$$\frac{\partial e'}{\partial n} = \frac{\partial k}{\partial n} - \frac{\partial r}{\partial n} = -\frac{\partial r}{\partial n} \lesseqgtr 0 \qquad \text{according as } P \gtreqless F, \tag{29}$$

since the current yield (k) is not a function of maturity and the partial derivative of the exact yield with respect to maturity is negative, zero, or positive according as the bond sells at a premium, face-value, or a discount.[10] Hence, at all prices *the absolute value of the error committed when using the current yield instead of the exact yield varies inversely with the bond's term to maturity.* This result is illustrated numerically in Table I. The negative error increases and its absolute value decreases when maturity rises for bonds selling at a discount and the positive error decreases when the bond is selling at a premium.

The response of the approximate-yield differential to changes in maturity is given by:

$$\frac{\partial e}{\partial n} = \frac{\partial a}{\partial n} - \frac{\partial r}{\partial n} \tag{30}$$

Since the derivatives $(\partial a/\partial n)$ and $(\partial r/\partial n)$ have the same sign for $P \gtreqless F$, the sign of the partial derivative of the approximate-yield differential with respect to maturity will depend on the magnitude of $(\partial a/\partial n)$ relative to that of $(\partial r/\partial n)$. Expressing $(1+r)^n$ as:

$$(1+r)^n = 1+rn+r^2 n\theta \tag{31}$$

where $$\theta = \left[\frac{(n-1)}{2!} + \frac{(n-1)(n-1)}{3!} r + \frac{(n-1)(n-2)(n-3)}{4!} r^2 + \cdots \right]$$

We show in the mathematical appendix that the ratio of the derivatives is equal to:

$$\frac{\partial a/\partial n}{\partial r/\partial n} = \frac{\left[\left(\frac{1+i}{1+r}\right) + i\theta\right] i}{\left[\left(\frac{1}{1+r\theta}\right) + \left(\frac{r+i}{2}\right) n\right] \ln(1+r)} \tag{32}$$

Suppose that $i>r$ and $\theta>n$, then $\frac{\partial a/\partial n}{\partial r/\partial n} > 1$ and $\frac{\partial e}{\partial n} > 0$. Alternatively suppose that $i>r$ and $\theta<n$ then $\frac{\partial e}{\partial n} \gtreqless 0$. Finally, for $i<r$ and $\theta \gtreqless n$ the sign of $\frac{\partial e}{\partial n}$ is again indeterminate. The conditions under which $\theta \gtreqless n$ are discussed in the mathematical appendix. It is clear that the sign of the derivative $\frac{\partial e}{\partial n}$ cannot be determined in general unless the bond sells at a premium $(i>r)$ *and* the condition $\theta>n$ is satisfied. Thus, contrary to the case of the current yield, *the absolute value of the error committed when using the approximate yield to calculate the exact yield is not unambiguously related to changes in the bond's maturity.*[11] As the maturity increases, the error may either increase, remain the same, or decrease. This result is illustrated numerically in Table II. As the bond's term to maturity increases for a given price, note that the absolute value of the error first drops, reaches a minimum and then rises.[12]

B. <u>The Coupon Behavior of Yield Differentials</u>. As the bond's coupon rate changes, the response of the yield differentials is given by:

$$\frac{\partial e'}{\partial i} = \frac{\partial k}{\partial i} - \frac{\partial r}{\partial i} = \frac{F}{P} - \frac{\partial r}{\partial i} \tag{33}$$

and

$$\frac{\partial e}{\partial i} = \frac{\partial a}{\partial i} - \frac{\partial r}{\partial i} = \frac{2F}{P+F} - \frac{\partial r}{\partial i} \tag{34}$$

The partial derivative $(\partial r/\partial i)$ is positive[13] and the sign of the partial derivatives of both yield differentials with respect to the coupon rate (i) depends on the magnitude of $(\partial r/\partial i)$ relative to $(\partial k/\partial i)$ and $(\partial a/\partial i)$. Using the same procedure as in the case of the maturity behavior of the approximate-yield differential, it is shown in the mathematical appendix that:

$$Sign \left\{\frac{\partial e'}{\partial i}\right\} = Sign \left\{(i-r)\left(1-n+\frac{1}{\theta^{-1}+r}\right)\right\} \tag{35}$$

and

$$Sign \left\{\frac{\partial e}{\partial i}\right\} = Sign \left\{(i-r)(2+n(1+r\theta))\left(r-\frac{1}{1+(r+1)\theta}\right)\right\} \tag{36}$$

Using a second order approximation $(\theta = (n-1)/2)$, a third order approximation $(\theta = (n-1)/2 + (n-1)(n-2)r/6)$, and so on, we show in the mathematical appendix that:

$$\frac{\partial e'}{\partial i} \lesseqgtr 0 \quad \text{according as} \quad P \gtreqless F \tag{37}$$

and

$$\frac{\partial e}{\partial i} \lesseqgtr 0 \quad \text{according as} \quad P \gtreqless F \tag{38}$$

(37) holds for all values of n and r and (38) holds for all values of $n>0$ and $r \leq 1$ and for most values of $n>0$ and $r<1$. Hence, at all prices, *the absolute value of the error committed when using the current yield instead of the exact yield varies inversely with the bond's coupon rate.* This result is illustrated numerically in the rows of Table III. Over the relevant range of yields $(r \leq 1)$, *the absolute value of the error committed when using the approximate yield instead of the exact yield varies directly with the bond's coupon rate.* This result is illustrated numerically in the rows of Table IV. Higher coupon rates produce a larger error when the approximate yield is employed but a smaller error when the current yield is used.

V. SUMMARY OF MAJOR RESULTS

In this paper we examined the behavior of the error committed when using either the approximate yield or the current yield instead of the bond's exact yield to maturity. We proved rigorously that both the approximate and the current yield overstate (understate) the exact yield when the bond is selling at a premium (discount).

However, when the current yield is larger than twice the reciprocal of the bond's term to maturity, the current yield is closer to the exact yield than is the approximate yield.

Turning to the behavior of the error we have shown that when the current yield is used, the error varies directly with the bond's maturity and inversely with the bond's coupon rate. When the approximate yield is used, the maturity behavior of the error is not unambiguously defined. It may either increase, remain constant or decrease. As the coupon rate changes, the error will, however, vary in the same direction.

MATHEMATICAL APPENDIX

1. The Coupon Behavior of Yield Differentials

We have from equation (2), equation (21), and equation (6), respectively:

$$\frac{\partial k}{\partial i} = \frac{F}{P} \tag{A.1}$$

$$\frac{\partial a}{\partial i} = \frac{2F}{F+P} \tag{A.2}$$

$$\frac{\partial r}{\partial i} = \frac{1}{\frac{i}{r} + \frac{n(r-i)}{1+r} \cdot \frac{1}{((1+r)^{n}-1)}} \tag{A.3}$$

$$(1+r)^{n} = 1 + rn + r^{2}n\theta$$

where $$\theta = \frac{n-1}{2!} + \frac{(n-1)(n-2)}{3!}r + \frac{(n-1)(n-2)(n-3)}{4!}r^{2} + \cdots\cdots \tag{A.4}$$

First substituting equation (6) in (A.1) and (A.2); and then equation (A.4) in (A.3) we get:

$$\frac{\partial k}{\partial i} = \left\{ \frac{1 + nr(1+r\theta)}{1 + ni(1+r\theta)} \right\}$$

$$\frac{\partial a}{\partial i} = \left\{ \frac{1 + nr(1+r\theta)}{1 + n((r+i)/2)(1+r\theta)} \right\}$$

$$\frac{\partial r}{\partial i} = \left\{ \frac{(1+r)(1+r\theta)}{(1+i) + i\theta(1+r)} \right\}$$

Note that $\theta>0$ requires that $n\geqq 2$. The above set of equations is not an approximation. It is just a substitution. Depending on how we truncate θ will determine the approximate functional value.

1.1 The Current-Yield Differential

$$\frac{\partial e'}{\partial i} = \frac{\partial k}{\partial i} - \frac{\partial r}{\partial i} = \left[\frac{1}{\{(1+i) + i\theta(1+r)\}\{1 + ni(1+r\theta)\}}\right] \cdot$$

$$\left[- \{(1+r)(1+r\theta)\}\{1 + ni(1+r\theta)\} + \{(1+i) + i\theta(1+r)\}\{1 + nr(1+r\theta)\}\right]$$

The sign of $\frac{\partial e'}{\partial i}$ will depend only on the second bracket, denoted X'. We have:

$$X' = (i-r)(1 + r\theta + \theta - n(1+r\theta)) \qquad \text{or}$$

$$Y' = \frac{X'}{1+r\theta} = (i-r)(1 - n + \frac{1}{\theta^{-1}+r}) \qquad \text{and}$$

$$Sign\ \{\frac{\partial e'}{\partial i}\} = Sign\left\{(i-r)(1 - n + \frac{1}{\theta^{-1}+r})\right\}$$

(i) Suppose $\theta = (n-1)/2,\ n>1$. Then

$$Sign\ \{\frac{\partial e'}{\partial i}\} = Sign\left\{(r-i)(n-1)(\frac{1+nr-r}{2+nr-r})\right\}$$

and thus $\frac{\partial e'}{\partial i} \lesseqqgtr 0$ according as $P \gtreqqless F$.

(ii) Suppose $\theta = \frac{3(n-1) + (n-1)(n-2)r}{6}$, $n>2$. Then

$$Sign\ \{\frac{\partial e'}{\partial i}\} = Sign\left\{(i-r)(n-1)(\frac{3 + (2n-1)r + (n-1)(n-2)r^2}{6 + 3(n-1)r + (n-1)(n-2)r^2}\right\}$$

and again $\frac{\partial e'}{\partial i} \lesseqqgtr 0$ according as $P \gtreqqless F$.

Using higher functional approximations we can demonstrate the generality of the above result.

1.2 The Approximate-Yield Differential

$$\frac{\partial e}{\partial i} = \frac{\partial a}{\partial i} - \frac{\partial r}{\partial i} = \left[\frac{1}{\{(1+i) + i\theta(1+r)\}\{2 + n(r+i)(1+r\theta)}\right] \cdot$$

$$\left[\{(1+i) + i\theta(1+r)\}\{2 + 2nr(1+r\theta)\} - \{(1+r)(1+r\theta)\}\{2 + n(r+i)(1+r\theta)\}\right]$$

The sign of $\frac{\partial e}{\partial i}$ will depend only on the second bracket, denoted X. We have:

$$X = (i-r)(1+r\theta+\theta)\left[2 + n(1+r\theta)(r - \frac{1}{1+r\theta+\theta})\right] \quad \text{or}$$

$$Y = \frac{X}{1+r\theta+\theta} = (i-r)\left[2 + n(1+r\theta)(r - \frac{1}{1+r\theta+\theta})\right] \quad \text{and}$$

$$Sign\ \{\frac{\partial e}{\partial i}\} = Sign\left\{(i-r)\left[2 + n(1+r\theta)(r - \frac{1}{1+r\theta+\theta})\right]\right\}$$

Suppose that $\theta = (n-1)/2$, $n>1$. Then

$$Sign\ \{\frac{\partial e}{\partial i}\} = Sign\left\{(i-r)\left[\frac{4(1-r) + 5nr + 3nr(1-r) + nr^3 + 2n^2r^2(1-r) + n^3r^2 + n^3r^3}{2(1 + nr - r + n)}\right]\right\}$$

Note that for all values of $n>0$ and $r\leqq 1$ the bracket is positive. For most values of $n>0$ and $r>1$ the bracket will be positive. Thus

$$\frac{\partial e}{\partial i} \gtreqqless 0 \qquad \text{according as} \qquad P \gtreqqless F.$$

The expression in brackets can be shown to be positive for higher approximation and the above result holds in general.

2. <u>The Maturity Behavior of the Approximate-Yield Differential</u>

From equation (2) and equation (6) we get, respectively:

$$\frac{\partial a}{\partial n} = \frac{(i-r)}{n\left[(1+r\ )^{-1} + \left(\frac{r+i}{2}\right)n\right]}$$

$$\frac{\partial r}{\partial n} = \frac{1}{n}\left[\frac{\ell n(1+r)}{r}\right]\left[\frac{(i-r)}{\left(\frac{1+i}{1+r}\right) + i\theta}\right]$$

and thus:

$$\frac{\partial a/\partial n}{\partial r/\partial n} = \frac{r\left[\left(\frac{1+i}{1+r}\right) + i\theta\right]}{\left[(1+r\ )^{-1} + \left(\frac{1+i}{2}\right)n\right]\ell n(1-r)}$$

<u>Case 1</u>: Let $i>r$ and $\theta>n$, then

$$\left(\frac{1+i}{1+r}\right) > 1, \qquad \frac{1}{1+r\theta} < 1, \qquad i\theta > \left(\frac{r+i}{2}\right)n$$

since $r>\ell n(1+r)$ by definition it follows that the ratio is larger than one and $\frac{\partial e}{\partial n} > 0$.

Case 2: Let $i>r$ and $\theta<n$, then

$$\left(\frac{1+i}{1+r}\right) > 1, \qquad \frac{1}{1+r\theta} < 1, \qquad \text{but} \qquad i\theta \gtreqless \left(\frac{r+i}{2}\right)$$

and consequently $\frac{\partial e}{\partial n} \gtreqless 0$.

Case 3 and Case 4: Let $i<r$ and $\theta>n$ or $\theta<n$, then $\frac{\partial e}{\partial n}$ is again indeterminate.

The meaning of the condition $\theta<n$:

$\theta<n$ implies that $(1+r)^n < 1 + rn + r^2n^2$. This condition is satisfied for $n\leqq 2$. It is satisfied for $n=3$, if $r<6$; for $n=4$, if $r<1.75$; for $n=5$, if $r<.95$, etc.

FOOTNOTES

[1]The formulation of the approximate yield in (2) is acceptable for the R.R. (registered representative) examination.

[2]See for example Francis [2], page 191.

[3]Note that even if the error (e) is very small and can be neglected without any consequence, we still have to prove that this is the case.

[4]Some of these questions have been examined by Sarnat and Levy [6]. However, their analysis is restrictive and assume that projects have zero terminal value. The analysis developed in this paper can be applied to a capital budgeting problem when the cash inflows are constant through time and there is a positive salvage value. If we denote P as the net initial cost of investment and F as the net salvage value, under most circumstances P/F will be much greater than unity. In addition, $i=C/F$ is a number of high magnitude, without strong economic meaning in the present context, and can be used in our analysis without any problem.

In the analysis that follows we will see that in this case the approximate return will overestimate the true internal rate of return. From Table IV we see that for high values of P/F and i the overestimation is by a wide margin. The current return (the reciprocal of the payback period), also, overestimates the true i.r.r. The magnitude of this overestimation, as compared to the former, is a function of the life of project as seen in equation (28) and Table III.

If the cash inflows varied through time, it will not be possible to use the mathematical analysis of this paper but the general conclusions will be similar.

[5]Note that the cases of an infinite maturity $(n=\infty)$ and a one year maturity $(n=1)$ are trivial. For example, for a perpetual bond we have $a_p = 2C/P+F$ and $r_p = C/P$. In this case the yield curves have *only one* intersection point at D where $a_p = r_p = i$. Both curves have the two axes as asymptotes and from the expressions for a_p and r_p it is clear that $a_p \gtreqless r_p$ and $P \gtreqless F$.

[6]Note that for premia and discounts of equal size, the absolute value of the yield differential is the largest for discount bonds.

[7]This standard transformation of equation (1) into equation (4) can be found in Malkiel [4].

[8]This is usually referred to as the Descartes Rule of Signs.

[9]We can define a new function to approximate the exact yield that would consist of two parts. For $0<P\leqq F$ it will be the current yield and for $P>F$ it will be the approximate yield.

[10]Holding the bond's price $P=P(i,n,r)$ constant, we get:

$$\frac{\partial r}{\partial n} = -\left[\frac{\partial P}{\partial n} \Big/ \frac{\partial P}{\partial r}\right]$$

$$\text{with } \frac{\partial P}{\partial r} < 0 \text{ and } \frac{\partial P}{\partial n} \gtreqless 0 \qquad \text{according as } P \gtreqless F$$

[11]Note, however, that we show in the mathematical appendix that the condition $\theta>n$ is usually satisfied. In that case the ambiguity disappears if the bond sells at a premium, and the error increases with an increase in maturity.

[12]It should not be concluded from Table II that the minimum absolute error is reached at higher maturities as the price of the bond increases toward its face value. A table with prices rising by incremental values of .005 will show that the minimum absolute error is reached for a maximum maturity of 10 years.

[13]Holding the bond's price $P = P(i,n,r)$ fixed we get:

$$\frac{\partial r}{\partial i} = -\left[\frac{\partial P}{\partial i} \Big/ \frac{\partial P}{\partial i}\right] > 0$$

$$\text{since } \frac{\partial P}{\partial r} < 0 \text{ and } \frac{\partial P}{\partial i} > 0$$

REFERENCES

1. Fisher, Lawrence, "An Algorithm for Finding Exact Rates of Return," *The Journal of Business*, 39, January, 1966, pp. 111-118.

2. Francis, Jack C., *Investments*, Second Edition, New York: McGraw-Hill Book Company, 1976.

3. Kaplan, Seymour, "Computer Algorithms for Finding Exact Rates of Return," *The Journal of Business*, 40, October, 1967, pp. 389-392.

4. Malkiel, Burton G., "Expectations, Bond Prices, and the Term Structure of Interest Rates," *Quarterly Journal of Economics*, 76, May, 1962, pp. 197-218.

5. Miller, Raymond E. and James W. Thatcher (Eds.), "Complexity of Computer Computations," Plenum Press, 1972.

6. Sarnat, Marshall and Haim Levy, "The Relationship of Rules of Thumb to the Internal Rate of Return: A Restatement and Generalization," *Journal of Finance*, 24, June, 1969, pp. 479-490.