Ordinal equivalence of values and Pigou-Dalton transfers in TU-games

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Abstract : The paper studies the ordinal equivalence of Linear, Efficient and Symmetry (LES) values in TU-games. It demonstrates that most of the results obtained by Carreras F, Freixas J (2008) in the case of semivalues and simple games are transposable on LES values and the whole TU-games set. In particular, linear and weakly linear games are analyzed. We characterize both values which are ordinal equivalent in all TU-games. Pigou-Dalton transfers are introduced for social comparison of values and to shed light on the way payoffs are redistributed from a value to another.

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1. Introduction and notations

A transferable utility game (or cooperative game or coalitional game with side pay-ments or simply a TU-game) is a pair (N, v), where N is a finite set of at least two players and $v : 2^N \to \mathbb{R}$ is a characteristic function satisfying $v(\emptyset) = 0$. An element of N and a nonempty subset S of N is called a player and coalition respectively, and the real number $v(S)$ is called the worth of coalition S. For a fixed N of cardinality n, we denote $\Gamma(N)$ the set of all transferable utility games v on N.

For each game in $\Gamma(N)$, we consider the complete preordering induced in the set of players by the application of values. A value on $\Gamma(N)$ is a function ψ that assigns a single payoff vector $\left(\psi_i(N, v)\right)_{i \in N} \in \mathbb{R}^n$ to every game (N, v). From the multiplicity of value in cooperative games it follows that comparisons between these values should be made. In some context the amount of gain given to a player in a game by a value may be less important than the rank assigned to the player, so that it is sufficient to base the comparison on the corresponding preordering instead of the numerical amount provided by each value. Both values are said to be ordinal equivalent in a game if their corresponding preordering coincide. The interest in ordinal equivalence of indices in cooperative games starts with the work of Tomiyama Y. (1987) who proved that, for every weighted majority game, the Shapley–Shubik (1954) and Penrose–Banzhaf–Coleman preorderings coincide. Next, Diffo Lambo and Moulen (2002) extended Tomiyama's result to all linear simple games. More recently, Carreras F, Freixas J (2008) extended Diffo Lambo and Moulen's results, in two senses: to all regular semivalues and to a large class of weakly linear simple games.

Throughout this paper we focus on the class, denoted LES, of values that satisfy linearity, symmetry and efficiency. A value ψ on $\Gamma(N)$ is said to be linear if $\psi_i(N, \alpha v + \beta w) = \alpha\psi_i(N, v) + \beta\psi_i(N, w)$ for all games (N, v), (N, w), for all player $i \in N$ and for all $\alpha, \beta \in \mathbb{R}$. ψ on Γ^n is symmetric if for all games (N, v) and for any automorphism π of v, $\psi_i(N, v) = \psi_{\pi(i)}(N, v)$. Finally a value ψ on Γ^n posses the efficiency property if $\sum_{i \in N} \psi_i(N, v) = v(N)$. Obviously, the class of LES values is very large and includes, among others, the Shapley value (1953) and the solidarity value (Nowak and Radzik, 1994) which differ of their payoffs from social considerations. However, it seems important to characterize and evaluate how large is the class of games in which both values preserve the same rank among the players.

The parametrization of LES values is the core topic in papers by Ruiz et al. (1998), Hernandez-Lamoneda et al. (2008), Chameni-Nembua and Andjiga (2008), more recently Chameni-Nembua (2012) and Malawski (2012) put forward a more economic interpretational one.

The aim of this paper is to study the ordinal equivalence of LES values not only in the case of simple games but when all TU-games are considered. It is shown that, most of the results obtained by Carreras F, Freixas J (2008) in the case of semi-values and simple games are transposable in the whole TU-games set and LES values. It is also shown that, two LES values are ordinal equivalent in all TU-game if and only if one is a positive convex combination of the second and the Egalitarian value. Next, for social comparison of LES values, we introduce Pigou-Dalton's progressive transfers (1912-1920). A sufficient condition is established to obtain, in weakly linear games, the payoffs of a LES value as Pigou-Dalton progressive transfer from another. Due to Symmetry and Linearity property of values, the sequences of progressive transfers obtained are selective and closer to the concept of concentration, which is a particular type of redistribution studied by kolm (1999) and Udo Ebert (2009, 2010).

The paper is organized as follows. The present section introduces the paper and reminds some basic definitions and results on LES values. Section 2 is concerned with ordinal equivalence of two values in the whole set of TU-games. While section 3 is devoted to the study of linear games, section 4 deals with the ordinal equivalence of LES values in the class of weakly linear games. Section 5 addresses the issue of obtaining the payoffs of a value from Pigou-Dalton progressive transfer of another value. Finally, the paper is concluded in section 6.

Theorem 1.1. Consider a set of player N of cardinality n and $\Gamma(N)$ the set of all transferable utility games v on N. Then the following statement, for a value ψ on $\Gamma(N)$, are equivalent:

(i) ψ is LES

(ii) There exists a unique sequence of $n-1$ real number $a(s)_{s=1}^{n-1}$ such that, for any $i \in N$,

$$\psi_i(N,v) = \frac{v(N)}{n} + \sum_{s=1}^{n-1} a(s) \left[\frac{(n-s)!(s-1)!}{n!} \sum_{S \ni i} v(S) - \frac{(n-s-1)!s!}{n!} \sum_{S \not\ni i} v(S)\right] \quad (1.1)$$

(iii) There exists a unique sequence of $n-1$ real number $b(s)_{s=2}^{n}$ such that, for any $i \in N$ and for any $S \subseteq N \backslash i$, if $A_i(\mathrm{S}) = b(s+1)[v(S+i) - v(S)] + \frac{1-b(s+1)}{s} \sum_{j \in S} [v(S) - v(S-j)]$,

$$\psi_i(N,v) = \frac{v(i)}{n} + \sum_{S \subseteq N \backslash i} \frac{(n-s-1)!s!}{n!} A_i(\mathrm{S}) \quad (1.2)$$

We refer the reader to [Hernandez and ali. 2008 ; Chameni 2012] for the proof of the theorem. While the parameterization given by (1.1) is hardly interpretable, Chameni (2012) proposes to interpret $A_i(\mathrm{S})$ defined in (1.2) as follow.When player i joins S to form $S+i$, he/she receives $b(s+1)[v(S+i) - v(S)]$ which, when $b(s+1) \in [0, 1]$, may be seen as a fraction of his/her marginal contribution while the incumbents in S receive the rest in equal shares. When player j joins $S+i \backslash j$ to form $S+i$, incumbent i receives additional individual share $\frac{1-b(s+1)}{s}[v(S) - v(S-j)]$. Thus $A_i(\mathrm{S})$ is i's conditional expected payoff given formation of S by the addition of any one player.

An additional remark here is the link between the two sequences. It is easy to check that,

$$a(s) = b(s+1), s = 1,2,\dots,n-1 \quad (1.3)$$

Another classical property yields the relationship about the difference between the value payoffs of two players.

Lemme 1.2 Consider two distinct players i and j in N. If ψ is any LES value on $\Gamma(N)$, then for all TU-game (N,v),

$$\psi_i(N,v) - \psi_j(N,v) = \sum_{S \subseteq N \backslash \{i,j\}} \frac{s!(n-2-s)!}{n!} a(s+1)[v(S+i) - v(S+j)] \quad (1.4)$$

Definition 1.3 A LES value ψ on $\Gamma(N)$ is considered to be:

- *Regular* whenever the associated sequence $a(s)_{s=1}^{n-1}$ or $b(s)_{s=2}^{n}$ are all strictly positive real.
- *Semi-regular* whenever the associated sequence $a(s)_{s=1}^{n-1}$ or $b(s)_{s=2}^{n}$ are all positive or nil real.

- *Strict semi-regular* whenever the associated sequence $a(s)_{s=1}^{n-1}$ or $b(s)_{s=2}^{n}$ are all positive or nil real with some of them strictly positive.

Note that, the regularity of a value implies its strict semi-regularity which implies its semi-regularity. Most of the classical values such as Shapley value, Solidarity value and Consensus (Yuan Ju, and P. Ruys 2007) value are regular. But the Egalitarian value E that allocates the same payoff to every player in all game (N, v) in that, $E_i(N, v) = \frac{v(N)}{n}$, is neither regular nor strict semi-regular but semi-regular as its associated sequence is $a_E(s) = b_E(s+1) = 0, s = 1,2, \dots, n-1$. Also, the so called Equal Surplus (or CIS) value (Driessen and Funaki, 1991) is not regular but strict semi-regular as $a_{CIS}(1) = n-1$ and $a_{CIS}(s) = 0$ for $s = 2, \dots, n-1$.

2. Ordinal equivalence of LES Values

In this section, we first provide the requirement condition for both LES values to be ordinal equivalent.

Definition 2.1 Let (N, v) be a TU-game, consider two values ψ *and* φ on $\Gamma(N)$. ψ *and* φ are said to be ordinal equivalent in the game (N, v) if for all $i, j \in N$,
$\psi_i(N, v) > \psi_j(N, v)$ iff $\varphi_i(N, v) > \varphi_j(N, v)$ and $\psi_i(N, v) = \psi_j(N, v)$ iff $\varphi_i(N, v) = \varphi_j(N, v)$.
ψ *and* φ are ordinal equivalent iff they are ordinal equivalent in all TU-game (N, v).

We remark that egalitarian value E is a singular case as it admits no ordinal equivalent value but itself. When a value ψ is different from E, it seems important to characterize the class of all values φ such that ψ *and* φ are ordinal equivalent; the next theorem deals with this issue.

Theorem 2.1 . Consider a set of players N of cardinality n and $\Gamma(N)$ the set of all transferable utility games v on N. Then the following statements for two LES value ψ *and* φ on $\Gamma(N)$ are equivalent:

1) ψ and φ (different from E) are ordinal equivalent.

2) Their associated sequence $a_\psi(s)_{s=1}^{n-1}$ and $a_\varphi(s)_{s=1}^{n-1}$ $\left(\text{or } b_\psi(s)_{s=2}^{n} \text{ and } b_\varphi(s)_{s=2}^{n}\right)$ defined by (1.1) or (1.2) are positively proportional i.e. there exists a strict positive real constant c such that $a_\psi(s) = c a_\varphi(s)$ for all $s = 1,2, \dots, n-1$.

3) There exists a strict positive constant c such that $\varphi_i(N, v) = c\psi_i(N, v) + (1-c)\frac{v(N)}{n}$ for all TU-game(N, v) and for all $i \in N$.

Proof. 1)⟹ 2)
Consider two distinct players i, j in N, for coalitions S and T such that S , $T \subseteq N\backslash\{i, j\}$, we define the games (N, v) and (N, w) as:

$$v(C)=\begin{cases}\frac{n!a_\psi(t+1)}{s!(n-2-s)!} & if\ C=S+i\\ \frac{n!a_\psi(s+1)}{t!(n-2-t)!} & if\ C=T+j\\ 0 & otherwise\end{cases}\qquad w(C)=\begin{cases}\frac{n!}{s!(n-2-s)!} & if\ C=S+i\\ 0 & otherwise\end{cases}$$

According to (1.4), $\varphi_i(N,v)-\varphi_j(N,v)=\ a_\psi(t+1)a_\varphi(s+1)-\ a_\psi(s+1)a_\varphi(t+1)$; thus for the value ψ ,

$$\psi_i(N,v)-\psi_j(N,v)=0$$

φ and ψ ordinal equivalent implies $\varphi_i(N,v)-\varphi_j(N,v)=0$,
therefore, $a_\psi(t+1)a_\varphi(s+1)-\ a_\psi(s+1)a_\varphi(t+1)=0$.
Since the relation is valid for all s and for all t from 0 $to\ n-2$, we obtain:

$$a_\psi(t)a_\varphi(s)-\ a_\psi(s)a_\varphi(t)=0 \text{ for all } s=1,2,\dots,n-1 \text{ and for all } t=1,2,\dots,n-1. \quad (1.5)$$

Suppose $a_\psi(t)=0$, then from (1.5), $a_\psi(s)a_\varphi(t)=0$ for all s=1,2,…,n-1.
if $a_\psi(s)=0$ for all s=1,2,…,n-1, then $\psi=E$; else $a_\varphi(t)$= 0;
Suppose $a_\psi(t)\neq 0$, then from (1.5), $a_\varphi(s)=\frac{a_\varphi(t)}{a_\psi(t)}a_\psi(s)$ for all s=1,2,…,n-1.
Hence, the both sequences a_ψ and a_φ are proportional i.e. there exist a constant $c\epsilon\mathbb{R}$ such that $a_\varphi(s)=ca_\psi(s)$, s=1,2,…,n-1.
To see that c is a strict positive constant, we use the game (N,w).
$\varphi_i(N,w)-\varphi_j(N,w)=\ a_\varphi(s+1)= ca_\psi(s+1)=c\left[\psi_i(N,w)-\psi_j(N,w)\right]$
Since φ and ψ are ordinal equivalent, $\varphi_i(N,w)-\varphi_j(N,w)$ and $\psi_i(N,w)-\psi_j(N,w)$have the same sign and consequently c is positive constant.

2)⟹ 3)
Suppose that $a_\varphi(s)=ca_\psi(s)$ for all s=1,2,…,n-1, then from (1.1)

$$\varphi_i(N,v)=\frac{v(N)}{n}+\sum_{s=1}^{n-1}a_\varphi(s)\left[\frac{(n-s)!(s-1)!}{n!}\sum_{S\ni i}v(S)-\frac{(n-s-1)!s!}{n!}\sum_{S\not\ni i}v(S)\right]$$

$$=\frac{v(N)}{n}+c\sum_{s=1}^{n-1}a_\psi(s)\left[\frac{(n-s)!(s-1)!}{n!}\sum_{S\ni i}v(S)-\frac{(n-s-1)!s!}{n!}\sum_{S\not\ni i}v(S)\right]+c\frac{v(N)}{n}-c\frac{v(N)}{n}$$

$$=(1-c)\frac{v(N)}{n}+c\left[\frac{v(N)}{n}+\sum_{s=1}^{n-1}a_\psi(s)\left[\frac{(n-s)!(s-1)!}{n!}\sum_{S\ni i}v(S)-\frac{(n-s-1)!s!}{n!}\sum_{S\not\ni i}v(S)\right]\right]$$

$$=\ (1-c)\frac{v(N)}{n}+c\psi_i(N,v)$$

3)⟹ 1) Obvious because φ is a strict positive affine transformation of ψ. ■

Applying the theorem in the particular case of the Shapley value yields the following result.

Corollary 2.2 : Consider a set of players N of cardinality n and $\Gamma(N)$ the set of all transferable utility games v on N. Then the following statement for a LES value ψ on $\Gamma(N)$ are equivalent :

1) ψ and the Shapley value ψ^{shap} are ordinal equivalent.

2) The ψ associated sequence $a(s)_{s=1}^{n-1}$ or $b(s)_{s=2}^{n-1}$ defined by (1.1) or (1.2) is a strict positive constant.

3) There exists a strict positive constant c such that,
$\psi_i(N,v) = c\psi_i^{shap}(N,v) + (1-c)\frac{v(N)}{n}$ for all TU-game(N,v) and for all $i \in N$.

When the constant c lies between 0 and 1, $\psi = c\psi^{shap} + (1-c)E$ is called the egalitarian Shapley value (Joosten 1996). This case has been recurrently studies in the literature and different authors (Nowak, A.S., and T. Radzik, (1996); Dragan, I., Driessen, T.S.H., and Y. Funaki, (1996)) propose different characterizations of the egalitarian Shapley value. Here we consider the redistribution of gain and introduce the concept of Pigou-Dalton transfer that is a well-known criterion in Economic inequality theory.

Definition 2.3 Let (N,v) be a TU-game, consider two values ψ *and* φ on $\Gamma(N)$. ψ is said to be obtained from φ, in the game (N,v), by a progressive Pigou-Dalton transfer if

1- ψ and φ are ordinal equivalent in the game (N,v);
2- there are $i,j \in N$, $\delta > 0$ such that $\varphi_k(N,v) = \psi_k(N,v)$ for $k \neq i$, $k \neq j$ and
$\varphi_i(N,v) < \psi_i(N,v) = \varphi_i(N,v) + \delta < \psi_j(N,v) = \varphi_j(N,v) - \delta < \varphi_j(N,v)$.

It will be noted $\psi(N,v) = PD[\varphi(N,v)]$ to say that ψ is obtained from φ, in the game (N,v), by a combination of progressive Pigou-Dalton transfers, while $\psi = PD[\varphi]$ means that the property holds in all game (N,v).

Clearly, a progressive Pigou-Dalton transfer from a value φ corresponds to a rank-preserving transfer of gain from a richer player to a poorer one. In particular, it requires the same treatment for two players having the same amount of gain relatively to the value φ.

Theorem 2.4: Consider a set of players N of cardinality n and $\Gamma(N)$ the set of all transferable utility games v on N. Then the following statements for two LES values ψ *and* φ on $\Gamma(N)$ are equivalent:

1) $\psi = PD[\varphi]$
2) There exists a strict positive constant $c \in]0\,;1]$ such that
$\psi_i(N,v) = c\varphi_i(N,v) + (1-c)\frac{v(N)}{n}$ for all TU-game(N,v) and for all $i \in N$.

Proof.

$\psi = PD[\varphi]$ implies in particular that ψ and φ are ordinal equivalent. In the view of theorem 2.1, there exist a constant $c > 0$ such that $\psi_i(N,v) = c\varphi_i(N,v) + (1-c)\frac{v(N)}{n}$ for all game (N,v) and for all player $i \in N$.

Suppose that $c > 1$, setting $c = 1 + \varepsilon$ implies $\psi_i(N,v) = \varphi_i(N,v) + \varepsilon\left(\varphi_i(N,v) - \frac{v(N)}{n}\right)$ therefore, $\psi_i(N,v)$ is obtained from $\varphi_i(N,v)$ as follow:

- If $\varphi_i(N,v) \geq \frac{v(N)}{n}$ then $\varphi_i(N,v)$ is increased and the amount increased is proportional to the distance between $\varphi_i(N,v)$ and the average $\frac{v(N)}{n}$.

- If $\varphi_i(N,v) < \frac{v(N)}{n}$ then $\varphi_i(N,v)$ is decreased and the diminished amount is proportional to the distance between $\varphi_i(N,v)$ and the average $\frac{v(N)}{n}$.

Of course, this sort of allocation is in contradiction with the Pigou-Dalton transfer. Hence $c \in]0\,;1]$.

Conversely, if $c \in]0\,;1]$, setting $c = 1 - \varepsilon,\ \ 0 \leq \varepsilon < 1$ leads to

$$\psi_i(N,v) = \varphi_i(N,v) + \varepsilon\left(\frac{v(N)}{n} - \varphi_i(N,v)\right)$$

Thus $\psi_i(N,v)$ is obtained from $\varphi_i(N,v)$ as follow:

- If $\varphi_i(N,v) \geq \frac{v(N)}{n}$ then $\varphi_i(N,v)$ is decreased and the diminished amount is proportional to the distance between $\varphi_i(N,v)$ and the average $\frac{v(N)}{n}$.
- If $\varphi_i(N,v) < \frac{v(N)}{n}$ then $\varphi_i(N,v)$ is increased and the amount increased is proportional to the distance between $\varphi_i(N,v)$ and the average $\frac{v(N)}{n}$.

  This kind of allocation is a particular case of a series of Pigou-Dalton transfer. ■

Clearly, the only ordinal equivalent values to a LES value ψ are those obtained as positive convex combinations of ψ with the egalitarian value E. Economic interpretation of such a value needs to shed more light on the convex combination terms.

When $\varphi_i(N,v) = c\psi_i(N,v) + (1-c)\frac{v(N)}{n}$ with $c > 0$, and for all $i \in N$, two cases may appear.

If, $0 < c \leq 1$, setting $c = 1 - \varepsilon,\ \ 0 \leq \varepsilon < 1$ and re-arrange the terms leads to

$$\varphi_i(N,v) = \psi_i(N,v) + \varepsilon\left(\frac{v(N)}{n} - \psi_i(N,v)\right) \qquad (1.6)$$

In this case, from theorem 2.4 it follows that φ is obtained from ψ, by a series of progressive transfers. In the light of (1.6), as consequences of linearity and symmetry, the concern transfers are in such a way that all payoff move toward the average payoff $\frac{v(N)}{n}$. The poor (here, players with payoff less than $\frac{v(N)}{n}$) receive an equiproportional increasing of the distance between their payoff and the average payoff $\frac{v(N)}{n}$, while the payoff of the rich (players with payoff greater than $\frac{v(N)}{n}$) decrease and move in the opposite way. In the literature, such a situation is qualified, when analyzing economic inequality, as concentration. A concentration[1] is a kind of redistribution of payoff which reduces the gap between each payoff and the average payoff in the same proportion. Of course, a concentration is a particular combination of progressive Pigou–Dalton transfers. Thus, for any good inequality measure I satisfying Pigou–Dalton transfer principle[2] we have $I\big(\varphi_i(N,v)_{i=1,,,n}\big) < I\big(\psi_i(N,v)_{i=1,,,n}\big)$.

If $c > 1$, a similar manipulation leads to

[1] We refer the reader to Udo Ebert (2010, 2009) for more details on the topic.

[2] Most of classical inequality indices such as Entropy family of indices (Cowell F.A. 1980), the standard Gini (1916) index and the α −Gini family (Chameni Nembua 2006; Elbert U. 2010) of indices satisfy the Pigou-Dalton transfer principle which requires that a rank-preserving transfer of income from a richer individual to a poorer one decreases inequality.

$$\varphi_i(N,v) = \psi_i(N,v) + \varepsilon\left(\psi_i(N,v) - \frac{v(N)}{n}\right).$$

Henceforth, φ is obtained from ψ by equiproportional increasing of the distance between each gains and the average gain $\frac{v(N)}{n}$ of rich players. Socially considerations might accuse φ for being anti redistributive i.e. benefiting the rich at the expense of the poor.

Unfortunately, for a given value ψ, the class of ordinal equivalence value is very thin as it is restrict to positive convex combinations of ψ with the egalitarian value E. To remedy this and enlarge the class, in the following sections, the condition of "all game" is relaxed and two particular categories of games are investigated.

3. The desirability relation and linear games

To start with, for any LES value ψ on $\Gamma(N)$, we denote $\succcurlyeq_\psi$ the complete preordering induces by ψ in N in that, $i \succ_\psi j$ iff $\psi_i(N,v) > \psi_j(N,v)$ and $i \backsim_\psi j$ iff $\psi_i(N,v) = \psi_j(N,v)$. Note that, the ordinal equivalence of two values corresponds to the coincidence of their preordering.

Définition 3.1: let (N,v) be a TU-game and consider in N the so called desirability binary relation defined as:

$$i \succcurlyeq_D j \text{ iff } v(S+i) \geq v(S+j) \text{ for all } S \subseteq N \setminus \{i,j\}.$$

Isbell JR (1958) proved that the desirability relation $\succcurlyeq_D$ is a preordering in N. There are many games of interest in which $\succcurlyeq_D$ turn out to be complete, but this is not generally the rule.

Definition 3.2: A TU-game (N,v) is said to be linear whenever the desirability relation $\succcurlyeq_D$ is a complete preordering.

For example any unanimity game is linear, and generally any weighted simple game is linear.

The following result holds.

Proposition 3.3: Let (N,v) be any linear game, then for any regular value ψ on $\Gamma(N)$, the preordering $\succcurlyeq_\psi$ and the desirability preordering coincide.
In other words, any two regular values $\psi \text{ and } \varphi$ on $\Gamma(N)$ are ordinal equivalent in all linear game (N,v).

Proof.
(N,v) linear $\Leftrightarrow$ $\succcurlyeq_D$ is a complete preordering in (N,v)

- $i \succcurlyeq_D j \Rightarrow v(S+i) \geq v(S+j)$ for all $S \subseteq N \setminus \{i,j\} \Rightarrow \frac{(n-s-2)!s!}{n} v(S+i) \geq \frac{(n-s-2)!s!}{n} v(S+j)$ for all $S \subseteq N \setminus \{i,j\} \Rightarrow$ $\frac{(n-s-2)!s!}{n} a(s+1) v(S+i) \geq \frac{(n-s-2)!s!}{n} a(s+1) v(S+j)$ for all $S \subseteq N \setminus \{i,j\}$ and for all $a(s+1) > 0$ $\Rightarrow \sum_{S\subseteq N\setminus\{i,j\}} \frac{s!(n-2-s)!}{n!} a(s+1)[v(S+i) - v(S+j)] \geq 0$ $\Rightarrow$ $i \succcurlyeq_\psi j$ for all regular value ψ.
- $i \succ_D j \Rightarrow v(S+i) \geq v(S+j)$ for all $S \subseteq N \setminus \{i,j\}$ with $v(S+i) > v(S+j)$ for some S, then applying the same process straightforward leads to $i \succ_\psi j$ for all regular value. ■

Notice that, from the first part of the proof of the proposition 3.3, it is clear that, in a linear game(N, v), $i \succcurlyeq_D j$ implies $i \succcurlyeq_\psi j$ for all semi regular value ψ. But the coincidence of $\succcurlyeq_D$ and $\succcurlyeq_\psi$ becomes awkward.

The converse of the assertion in the proposition 3.3 is not true in the sense that, a game in which all regular value are ordinal equivalent is not necessary a linear game. However, as we will see, the property is true if $n \leq 3$.

To remedy this, we propose to enlarge the class of linear games by replacing the linearity property of game with a weaker one. But before tackle the issue, let us introduce the class of strongly linear games in order to take into account the ordinal equivalence of strict semi-regular values.

Definition3.4: let (N, v) be a TU-game and consider in N the strong desirability binary relation $\succcurlyeq_{SD}$defined in N as:

First, the strict preordering: $i \succ_{SD} j$

iff

$v(S+i) \geq v(S+j)$ for all $S \subseteq N \setminus \{i, j\}$ and

for all $0 \leq k \leq$n-2 there exists $|S_k| = k$ such that $v(S_k + i) > v(S_k + j)$

Second, the equivalence: $i \backsim_{SD} j$

iff

$v(S+i) = v(S+j)$ for all $S \subseteq N \setminus \{i, j\}$.

Then, $i \succcurlyeq_{SD} j$ iff $i \succ_{SD} j$ or $i \backsim_{SD} j$

It is straightforward to check that $\succcurlyeq_{SD}$ is a preordering in N so that one may follow the linear game definition approach to obtain the so called strongly linear game. Furthermore, $\succcurlyeq_{SD}$ is a sub-preordering of $\succcurlyeq_D$ in the sense that $i \succcurlyeq_{SD} j \Longrightarrow i \succcurlyeq_D j$ and $i \backsim_{SD} j \Longrightarrow i \backsim_D j$.

Definition 3.5: A TU-game (N, v) is said to be strongly linear whenever the strong desirability relation $\succcurlyeq_{SD}$ is a complete preordering.

It is clear that the strong linearity of a game implies it linearity but the converse is not true, so that the class of strongly linear games is narrow compare to the class of linear games. This justifies the reason that the term "strong" is used.

For example the game (N, v) with $n = 3$:

$v(1) = v(2) = v(3) = 6$; $v(1,2) = 8$; $v(1,3) = 7$; $v(2,3) = 6$; $v(1,2,3) = 5$

(N, v) is linear :1 $\succ_D$ 2 $\succ_D$ 3 but not strongly linear 2 $\not\succcurlyeq_{SD}$ 3 and 3 $\not\succcurlyeq_{SD}$ 2.

Proposition 3.6: Let (N, v) be any strongly linear game, then for any strict semi-regular value ψ on $\Gamma(N)$, the preordering $\succcurlyeq_\psi$ and the strong desirability preordering coincide.

In other words, any both strict semi-regular values ψ and φ on $\Gamma(N)$ are ordinal equivalent in all strongly linear game (N, v).

Proof

(N, v) strongly linear $\Leftrightarrow$ $\succcurlyeq_{SD}$ is a complete preordering in (N, v).

- $i \succ_{SD} j \Rightarrow\ v(S+i) \geq v(S+j)$ for all $S \subseteq N \setminus \{i,j\}$ and for all s there exist at least one S with $|S| = s$ such that $v(S+i) > v(S+j)$.

  $\Rightarrow\ \frac{(n-s-2)!s!}{n} v(S+i) \geq \frac{(n-s-2)!s!}{n} v(S+j)$ for all $S \subseteq N \setminus \{i,j\}$ and for all s with the strict inequality for at least one S.
  $\Rightarrow \frac{(n-s-2)!s!}{n} a(s+1)v(S+i) \geq \frac{(n-s-2)!s!}{n} a(s+1)v(S+j)$ for all $S \subseteq N \setminus \{i,j\}$ and for all $a(s+1) \geq 0$ with the strict inequality for at least one s.

  $\Rightarrow \sum_{S\subseteq N\setminus\{i,j\}} \frac{s!(n-2-s)!}{n!} a(s+1)[v(S+i) - v(S+j)] > 0 \ \Rightarrow\ i \succ_{\psi} j$ for all strict semi- regular value ψ.
- $i \sim_{SD} j \Rightarrow\ v(S+i) = v(S+j)$ for all $S \subseteq N \setminus \{i,j\}$, then applying the same process straightforward leads to $i \sim_{\psi} j$ for all strict semi-regular value. ■

4. The weak desirability relation and the weakly linear games

We introduce an entity which reflects the importance level of a player in the game; the entity may also be interpreted as the player productivity in the game. To this purpose, we define, for a given TU-game (N, v), for each player $i \in N$ and for all $1 \leq k \leq n-1$, the quantity,

$$u_k(i) = \sum_{S\ni i;|S|=k} v(S) \quad \text{and} \quad u(i) = \left(u_k(i)_{k=1,2,\ldots,n-1}\right)$$

In words, $u_k(i)$ is the sum of worth of all coalition size k containing the player i, while $u(i)$ is the vector of $\mathbb{R}^{n-1}$ whose components are respectively $u_k(i)$. Clearly, the process defines a function $u: N \longrightarrow \mathbb{R}^{n-1}$ therefore the binary relation $i \succcurlyeq_u j$ iff $u_k(i) \geq u_k(j)$ is a preordering in N that is not always complete. The equivalence and the strict preordering induced by $\succcurlyeq_u$ are: $i \backsim_u j$ iff $u_k(i) = u_k(j)$ for all $k = 1,2,\ldots,n-1$ and $i \succ_u j$ iff $u_k(i) \geq u_k(j)$ for all $k = 1,2,\ldots,n-1$ but $u_k(i) > u_k(j)$ for some k.

Definition 4.1: let (N, v) be a TU-game and consider in N the so called weak desirability preordering relation defined as:

$$i \succcurlyeq_{WD} \text{j iff } i \succcurlyeq_u j$$

Following the definition process of the linear games, we use the weak desirability relation to define a new class of TU-games.

Definition 4.2: A TU-game (N, v) is said to be weakly linear whenever the weak desirability relation $\succcurlyeq_{WD}$ is a complete preordering.

Note that definition 4.2 is an extension to all TU-games of the weakly linearity defined by Carreras F, Freixas J (2008) in the case of simple games.

The following lemma elucidates the reason that the term “weakly” is used as it asserts that, the completeness of the desirability relation (linearity of the game) implies the completeness of the weak desirability relation, so that the second condition is weaker than the former one.

Lemma 4.3: let (N, v) be a TU-game and ψ be any regular value on $\Gamma(N)$ then:

1) The desirability relation $\succcurlyeq_D$ is a sub-preordering of the weak desirability preordering $\succcurlyeq_{WD}$ in that: $i \succcurlyeq_D$ j $\Longrightarrow$ $i \succcurlyeq_{WD}$ j and $i \succ_D j \Longrightarrow i \succ_{WD} j$.
In particular, if $\succcurlyeq_D$ is complete, so is $\succcurlyeq_{WD}$ and the both preordering coincide.

2) (N, v) linear implies that (N, v) is weakly linear but the converse is true only when $n \leq 3$.

3) The weak desirability relation is a sub-preordering of the complete preordering $\succcurlyeq_\psi$ induced by ψ in N.

Proof.

1) Consider a TU-game (N, v) and $i, j \in N$,

- $i \succcurlyeq_D j \Longrightarrow$ for all $S \subseteq N - \{i,j\}$, $v(S+i) \geq v(S+j) \Longrightarrow$ for all k, for all $S \subseteq N - \{i,j\}$ with $|S| = k-1$, $v(S+i) \geq v(S+j)$ therefore, for a fixed k

$\sum_{S \ni i; |S|=k} v(S) = \sum_{\substack{S \ni i; |S|=k \\ S \not\ni j}} v(S) + \sum_{\substack{S \ni i; |S|=k \\ S \ni j}} v(S) = \sum_{\substack{S \not\ni i; |S|=k-1 \\ S \not\ni j}} v(S+i) + \sum_{\substack{S \ni i; |S|=k \\ S \ni j}} v(S) \geq$
$\sum_{\substack{S \not\ni i; |S|=k-1 \\ S \not\ni j}} v(S+j) + \sum_{\substack{S \ni i; |S|=k \\ S \ni j}} v(S) \geq \sum_{\substack{S \not\ni i; |S|=k \\ S \ni j}} v(S) + \sum_{\substack{S \ni i; |S|=k \\ S \ni j}} v(S) \geq \sum_{S \ni j; |S|=k} v(S)$.
Hence $i \succcurlyeq_{WD} j$.

- $i \succ_D j \Longrightarrow$ for all $S \subseteq N - \{i,j\}$, $v(S+i) \geq v(S+j)$ with $v(S+i) > v(S+j)$ for some S.
$\Longrightarrow$ for all k, for all $S \subseteq N - \{i,j\}$ with $|S| = k-1$, $v(S+i) \geq v(S+j)$ with $v(S+i) > v(S+j)$ for some k.
Thus, The same operations clearly lead to $i \succ_{WD} j$.
- $\succcurlyeq_D$ complete $\Longrightarrow$ $\succcurlyeq_{WD}$ and the both preordering coincide is straightforward as $\succcurlyeq_D$ is a sub-preordering of $\succcurlyeq_{WD}$.

2) (N, v) linear $\Longrightarrow \succcurlyeq_D$ is complete in (N, v), according to 1) this implies $\succcurlyeq_{WD}$ is complete in $(N, v) \Longrightarrow (N, v)$ is weakly linear.

The converse is true only when $n \leq 3$:

- If $n = 3$, suppose that $i \succcurlyeq_D j \succcurlyeq_D h$, then $v\{i\} \geq v\{j\} \geq v\{h\}$, $v\{i,j\} \geq v\{i,h\} \geq v\{j,h\}$ thus, $\begin{pmatrix} v\{i\} \\ v\{i,j\} + v\{i,h\} \end{pmatrix} \geq \begin{pmatrix} v\{j\} \\ v\{i,j\} + v\{j,h\} \end{pmatrix} \geq \begin{pmatrix} v\{h\} \\ v\{j,h\} + v\{i,h\} \end{pmatrix} \Longrightarrow$
$$u(i) \geq u(j) \geq u(h) \Longrightarrow i \succcurlyeq_{WD} j \succcurlyeq_{WD} h$$

- If $n \geq 4$, let, for instance, (N, v) be the game defined as:
$$v\{1\} = 2;\ v\{2\} = 5;\ v\{3\} = 6; v\{4\} = 10$$
$$v\{1,2\} = 6; v\{1,3\} = 10;\ v\{1,4\} = 8\ ;\ v\{2,3\} = 8;\ v\{2,4\} = 10;\ v\{3,4\} = 15;$$
$$v\{1,2,3\} = 8;\ v\{1,2,4\} = 10;\ v\{1,3,4\} = 11;\ v\{2,3,4\} = 12$$
$$v\{1,2,3,4\} = a = v\{N\} \text{ and}$$
$$v\{S\} = 0 \text{ otherwise.}$$
It is clear that (N, v) is not linear since $v\{2\} > v\{1\}$ and $v\{2,3\} < v\{1,3\}$, thus player 2 and player 1 are not comparable and therefore $\succcurlyeq_D$ is not complete.

$$u(4)=\begin{pmatrix}10\\33\\33\\a\\0\\.\\.\\.\\0\\0\end{pmatrix}\quad u(3)=\begin{pmatrix}6\\33\\31\\a\\0\\.\\.\\.\\0\\0\end{pmatrix}\quad u(2)=\begin{pmatrix}5\\24\\30\\a\\0\\.\\.\\.\\0\\0\end{pmatrix}\quad u(1)=\begin{pmatrix}5\\24\\30\\a\\0\\.\\.\\.\\0\\0\end{pmatrix} u(5)=\begin{pmatrix}0\\0\\0\\0\\.\\.\\.\\0\\0\end{pmatrix}\ldots u(n)=\begin{pmatrix}0\\0\\0\\0\\.\\.\\.\\0\\0\end{pmatrix}$$

Thus, $4 \succ_{WD} 3 \succ_{WD} 2 \succ_{WD} 1 \succ_{WD} 5 \sim_{WD} 6 \sim_{WD} \ldots \sim_{WD} n$. Hence, (N, v) is weakly linear but not linear.

3) As $\sum_{S\ni i;|S|=k} v(S) + \sum_{S\not\ni i;|S|=k} v(S) = \sum_{S\ni j;|S|=k} v(S) + \sum_{S\not\ni j;|S|=k} v(S)$ for all k and for all $i, j \in N$,

$$i \succcurlyeq_{WD} j \implies \text{for all } k, \begin{cases} \sum_{S\ni i;|S|=k} v(S) \geq \sum_{S\ni j;|S|=k} v(S) \\ and \\ \sum_{S\not\ni i;|S|=k} v(S) \leq \sum_{S\not\ni j;|S|=k} v(S) \end{cases}$$

As ψ is a regular value, its associated sequence $a(k)_{k=1}^{n-1}$ satisfies $a(k) > 0$ for all k, hence

$$\text{for all } k, \begin{cases} \frac{(n-k)!(k-1)!}{n!} a(k) \sum_{S\ni i;|S|=k} v(S) \geq \frac{(n-k)!(k-1)!}{n!} a(k) \sum_{S\ni j;|S|=k} v(S) \\ and \\ -\frac{(n-k-1)!k!}{n!} a(k) \sum_{S\not\ni i;|S|=k} v(S) \geq -\frac{(n-k-1)!k!}{n!} a(k) \sum_{S\not\ni j;|S|=k} v(S) \end{cases} \implies$$

$$\frac{(n-k)!(k-1)!}{n!} a(k) \sum_{S\ni i;|S|=k} v(S) - \frac{(n-k-1)!k!}{n!} a(k) \sum_{S\not\ni i;|S|=k} v(S)$$

$$\geq \frac{(n-k)!(k-1)!}{n!} a(k) \sum_{S\ni j;|S|=k} v(S) - \frac{(n-k-1)!k!}{n!} a(k) \sum_{S\not\ni j;|S|=k} v(S) \text{ for all } k, \implies$$

$$\sum_{k=1}^{n-1}\left[\frac{(n-k)!\,(k-1)!}{n!} a(k) \sum_{S\ni i;|S|=k} v(S) - \frac{(n-k-1)!\,k!}{n!} a(k) \sum_{S\not\ni i;|S|=k} v(S)\right]$$

$$\geq \sum_{k=1}^{n-1}\left[\frac{(n-k)!\,(k-1)!}{n!} a(k) \sum_{S\ni j;|S|=k} v(S) - \frac{(n-k-1)!\,k!}{n!} a(k) \sum_{S\not\ni j;|S|=k} v(S)\right]$$

$\implies$ From (1.1), $\psi_i(N, v) \geq \psi_j(N, v)$

The same process clearly proves that $i \succ_{WD} j \implies i \succ_{\psi} j$ ■

Remark 4.4: Notice that, from the third part of the proof of the Lemma 4.3, it follows that, if ψ is a semi-regular value then $i \succcurlyeq_{WD} j \implies i \succcurlyeq_{\psi} j$. But $\succcurlyeq_{WD}$ is not necessary a sub-preordering of $\succcurlyeq_{\psi}$ as it is not true that $i \succ_{WD} j \implies i \succ_{\psi} j$.

Proposition 4.5: let (N,v) be a TU-game of $\Gamma(N)$ that is linear then:
(N,v) is weakly linear and the desirability relation $\succcurlyeq_D$, the weak desirability relation $\succcurlyeq_{WD}$ and the complete preordering $\succcurlyeq_\psi$induce by any regular value ψ on $\Gamma(N)$ coincide.

Proof. As (N,v) is linear, $\succcurlyeq_D$ is complete, according to lemma4.3 it coincides with $\succcurlyeq_{WD}$ which is then complete. From proposition3.3, if ψ is any regular value, $\succcurlyeq_\psi$ and $\succcurlyeq_D$ coincide, thus $\succcurlyeq_\psi$ and $\succcurlyeq_{WD}$ coincide. ■

We can now state the first characterization of the weakly linear game.

Theorem 4.6: let (N,v) be a TU-game of $\Gamma(N)$, if we set for any player $i \in N$ and for any $1 \leq k \leq n-1$,

$$t_k(i) = \frac{(n-k)!(k-1)!}{n!}\sum_{S\ni i;|S|=k} v(S) - \frac{(n-k-1)!k!}{n!}\sum_{S\not\ni i;|S|=k} v(S) \qquad (1.7)$$

And $$t(i) = \left(t_k(i)_{k=1,2,\dots,n-1}\right) \in \mathbb{R}^{n-1}$$

Then,

1) $i \succcurlyeq_{WD} j$ iff $i \succcurlyeq_t j$. In word, $\succcurlyeq_{WD}$ and $\succcurlyeq_t$ coincide.

2) (N,v) is weakly linear iff $\succcurlyeq_{WD}$, $\succcurlyeq_t$ and $\succcurlyeq_\psi$ the complete preordering induced by each regular value ψ, coincide.

3) If for all k, $\succcurlyeq_{t_k}$ coincide then (N,v) is weakly linear but the converse is not true.

Proof.

1) $$\begin{aligned} t_k(i) &= \frac{(n-k)!(k-1)!}{n!}\sum_{S\ni i;|S|=k} v(S) - \frac{(n-k-1)!k!}{n!}\sum_{S\not\ni i;|S|=k} v(S) \\ &= \frac{(n-k)!(k-1)!}{n!}\sum_{S\ni i;|S|=k} v(S) - \frac{(n-k-1)!k!}{n!}\left[\sum_{|S|=k} v(S) - \sum_{S\ni i;|S|=k} v(S)\right] \\ &= \frac{(n-k-1)!(k-1)!}{(n-1)!}\sum_{S\ni i;|S|=k} v(S) - \frac{(n-k-1)!k!}{n!}\sum_{|S|=k} v(S) \end{aligned}$$

   Hence, for any $i,j \in N$, $t_k(i) - t_k(j) = \frac{(n-k-1)!(k-1)!}{(n-1)!}\left(u_k(i) - u_k(j)\right)$
   Which proves the property.

2) Suppose that (N,v) is weakly linear, then from the lemma 4.3, $\succcurlyeq_{\mathrm{WD}}$ coincides with the complete preordering $\succcurlyeq_\psi$.
   Conversely, if the weak desirability relation coincides with the complete preordering $\succcurlyeq_\psi$ induced by a regular value ψ, then $\succcurlyeq_{\mathrm{WD}}$ is complete and hence (N,v) is weakly linear.

3) Obvious from 1) ■

Remark 4.7: From the first part of theorem 4.6, it follows that, for any TU-game (N,v) , for any players $i,j \in N$, and for any LES value ψ,

- $\psi_i(N,v) = \frac{v(N)}{n} + \sum_{k=1}^{n-1} a_\psi(k)t_k(i)$

- $\psi_i(N,v) - \psi_j(N,v) =$
$$\sum_{k=1}^{n-1} a_\psi(k)\big(t_k(i) - t_k(j)\big) \quad = \quad \sum_{k=1}^{n-1} \frac{(n-k-1)!(k-1)!}{(n-1)!} a_\psi(k)\big(u_k(i) - u_k(j)\big)$$

From the theorem 4.6, we can straightforwardly withdraw the following results.

Corollary 4.8: let (N, v) be a TU-game of $\Gamma(N)$

1) If the weak desirability relation coincide with the preordering $\succcurlyeq_\psi$ induced by a given LES value ψ (regular or not), then (N, v) is weakly linear.

2) The weak linearity of (N, v) does not imply the coincidence of weak desirability relation $\succcurlyeq_{WD}$ with $\succcurlyeq_\psi$ for a particular LES value.

Note that the second assertion of the corollary is particularly true for a semi-regular but non regular value.

The following theorem deals with the second characterization of the weakly linear games.

Theorem 4.9: let (N, v) be a TU-game of $\Gamma(N)$ then,
(N, v) is weakly linear iff all regular values are ordinal equivalent in (N, v).

Proof.

- (N, v) weakly linear $\Rightarrow \succcurlyeq_{WD}$ coincides with $\succcurlyeq_\psi$ for each regular value ψ (theorem 4.5).
- Conversely, suppose that $\succcurlyeq_{WD}$ coincides with $\succcurlyeq_\psi$ for all regular value ψ and let us assume that (N, v) is not weakly linear. Then there exist two distinct players i and j such that $i \not\succcurlyeq_{WD} j$ and $j \not\succcurlyeq_{WD} i$. Hence there exist k and l such that $u_k(i) > u_k(j)$ and $u_l(i) < u_l(j)$. For any $0 < \varepsilon < 1$, we consider two LES values ψ_ε and $\varphi_\varepsilon$with corresponding sequence $a_{\psi_\varepsilon}(s)_{s=1}^{n-1}$ and $a_{\varphi_\varepsilon}(s)_{s=1}^{n-1}$s such that:

$$a_{\psi_\varepsilon}(s) = \begin{cases} \frac{n!}{(n-s-1)!(s-1)!}\varepsilon & if\ s \neq k \\ \frac{n!}{(n-s-1)!(s-1)!}(1-\varepsilon) & if\ s = k \end{cases}$$

$$a_{\varphi_\varepsilon}(s+1) = \begin{cases} \frac{n!}{(n-s-1)!(s-1)!}\varepsilon & if\ s \neq l \\ \frac{n!}{(n-s-1)!(s-1)!}(1-\varepsilon) & if\ s = l \end{cases}$$

Then in the view of remark 4.7,

$$\psi_{\varepsilon i}(N,v) - \psi_{\varepsilon j}(N,v) = \sum_{p=1}^{n-1} \frac{(n-p-1)!(p-1)!}{(n-1)!} a_{\psi_\varepsilon}(p)\left(u_p(i) - u_p(j)\right)$$

$$= \sum_{p\neq k}^{n-1} \frac{(n-p-1)!(p-1)!}{(n-1)!} a_{\psi_\varepsilon}(p)\left(u_p(i) - u_p(j)\right) + \frac{(n-k-1)!(k-1)!}{(n-1)!} a_{\psi_\varepsilon}(k)\big(u_k(i) - u_k(j)\big)$$

$$= \varepsilon \sum_{p\neq k}^{n-1} \left(u_p(i) - u_p(j)\right) + (1-\varepsilon)\big(u_k(i) - u_k(j)\big)$$

Thus, when ε tends toward 0, the sign of $\psi_{\varepsilon i}(N,v) - \psi_{\varepsilon j}(N,v)$ is the same as the sign of $u_k(i) - u_k(j)$, therefore, $\psi_{\varepsilon i}(N,v) - \psi_{\varepsilon j}(N,v) > 0 \Leftrightarrow i \succ_{\psi_\varepsilon} j$.
As $\succcurlyeq_{WD}$ coincides with $\succcurlyeq_{\psi}$ for all LES value ψ, we obtain $i \succ_{WD} j$.
Using the same process and compute $\varphi_{\varepsilon i}(N,v) - \varphi_{\varepsilon j}(N,v)$ leads to
$\varphi_{\varepsilon i}(N,v) - \varphi_{\varepsilon j}(N,v) < 0 \Leftrightarrow j \succ_{\varphi_\varepsilon} i$, thus we obtain $i \succ_{WD} j$ and then directly fall into a contradiction. ■

Theorem4.9 is silent about the semi-regular and strict semi-regular values, to remedy this, we now introduce the strict weakly linear games. We need firstly to define the strict weak desirability preordering.

Definition 4.10: let (N,v) be a TU-game and consider in N the strict weak desirability relation $\succcurlyeq_{SWD}$ defined in N as:

First, the strict preordering: $i \succ_{SWD} j$
iff $u_k(i) = \sum_{S\ni i;|S|=k} v(S) > u_k(j) = \sum_{S\ni j;|S|=k} v(S)$ for all $1 \leq k \leq$n-1

Second, the equivalence: $i \backsim_{SWD} j$ iff $u_k(i) = u_k(j)$ for all $1 \leq k \leq$n-1

Then, $i \succcurlyeq_{SWD} j$ iff $i \succ_{SWD} j$ or $i \backsim_{SWD} j$

It is immediate that $\succcurlyeq_{SWD}$ is a preordering in N which is a sub-preordering of $\succcurlyeq_{WD}$ in the sense that $i \succcurlyeq_{SWD} j \Longrightarrow i \succcurlyeq_{WD} j$ and $i \succ_{SWD} j \Longrightarrow i \succ_{WD} j$.
Secondly, we define a strict weakly linear game as the one where the strict weak desirability relation is a complete preordering. Then the following results hold.

Theorem 4.11: let (N,v) be a TU-game of $\Gamma(N)$ then,
(N,v) is strict weakly linear iff all strict semi-regular values are ordinal equivalent in (N,v).

Proof. Suppose that (N,v) is strict weakly linear and consider a strict semi-regular value ψ.

- $i \succ_{SWD} j \Longrightarrow t_k(i) > t_k(j)$ for all $1 \leq k \leq$n-1
  $\Longrightarrow a_\psi(k)t_k(i) \geq a_\psi(k)t_k(j)$ for all $1 \leq k \leq$n-1 with the strict inequality for some k.
  $\Longrightarrow \sum_{k=1}^{n-1} a_\psi(k)t_k(i) > \sum_{k=1}^{n-1} a_\psi(k)t_k(j)$
  $\Longrightarrow \frac{v(N)}{n} + \sum_{k=1}^{n-1} a_\psi(k)t_k(i) > \frac{v(N)}{n} + \sum_{k=1}^{n-1} a_\psi(k)t_k(j)$
  $\Longrightarrow \psi_i(N,v) > \psi_j(N,v)$

- It is easy to check that $i \backsim_{SWD} j \Longrightarrow \psi_i(N,v) = \psi_j(N,v)$

Henceforth, $\succcurlyeq_{SWD}$ and $\succcurlyeq_{\psi}$ coincide for all strict semi-regular value ψ.

Conversely, consider $n-1$ strict semi-regular value $\psi_k$such as

$$a_{\psi_k}(s) = \begin{cases} \frac{n!}{(n-s-1)!(s-1)!} & if\ s = k \\ 0 & if\ s \neq k \end{cases}$$

for all $1 \leq k \leq$n-1 ψ_k are all ordinal equivalent in (N, v). Thus for any $i, j \in N$, we have $\psi_{ki}(N, v) - \psi_{kj}(N, v) = a_{\psi_k}(k)\big(t_k(i) - t_k(j)\big) = \big(u_k(i) - u_k(j)\big)$

Since $\Big(\psi_{ki}(N, v) - \psi_{kj}(N, v)\Big)$ have a constant sign for all k, it implies that $u_k(i) - u_k(j)$ have a constant sign for all k and for all $i, j \in N$.

Thus (N, v) is strict weakly linear. ■

Corollary 4.12: let (N, v) be a TU-game of $\Gamma(N)$, then

(N, v) is strict weakly linear iff $\succcurlyeq_{t_k}$coincide for all k

5. Pigou-Dalton transfers in weakly linear games

This section shed light on the way payoffs are redistributed from a value to another ordinal equivalent value in weakly linear games.

Proposition 5.1: Let (N, v) be any weakly linear game and suppose that players are numbered such that, $1 \preccurlyeq_{WD} 2 \preccurlyeq_{WD} 3, \dots, \preccurlyeq_{WD} n$. Consider two strict semi-regular values $\psi\ and\ \varphi$ on $\Gamma(N)$ with associated sequences $a_\psi(s)_{s=1}^{n-1}$ and $a_\varphi(s)_{s=1}^{n-1}$.

If $a_\varphi(s) \leq a_\psi(s)$ for all $1 \leq s \leq n-1$ then

There exists a sequence $(\varepsilon_i(N, v))_{i=1}^{n}$ and i_0 such that:

$$\big((\varphi_i)_{i=1,\dots,n}\big) = \big(\psi_1 + \varepsilon_1, \psi_2 + \varepsilon_2, \dots, \psi_{i_0} + \varepsilon_{i_0}, \psi_{i_0+1} + \varepsilon_{i_0+1}, \dots, \psi_n + \varepsilon_n\big)$$

With, $\varepsilon_{i+1} \leq \varepsilon_i$ for i =1,…, $n-1$;

$\varepsilon_i \geq 0$; for i =1,…, i_0; $\varepsilon_i \leq 0$ for all $i > i_0$;

$\sum_{k=1}^{n} \varepsilon_i = 0$

Proof. Suppose that $a_\varphi(s) \leq a_\psi(s)$ for all $1 \leq s \leq n-1$ with $a_\varphi(s) < a_\psi(s)$ for some s, thus there exist a real positive sequence $(c(s))_{s=1}^{n-1}$ such that

$a_\varphi(s) = a_\psi(s) - c(s)$ for all $1 \leq s \leq n-1$ with $c(s) > 0$ for some s.

Denote ϕ the LES value whose (1.1) sequence is $(c(s))_{s=1}^{n-1}$, it is clear that ϕ is strict semi-regular value (different from the egalitarian value E) and hence, in the view of remark 4.4, for any two players $i, j \in N$, $i \succcurlyeq_{WD} j \Longrightarrow i \succcurlyeq_\phi j$.

Consider that,

$$\varphi_i(N, v) = \frac{v(N)}{n} + \sum_{k=1}^{n-1} a_\varphi(k) t_k(i)$$

Then,

$$\varphi_i(N, v) = \frac{v(N)}{n} + \sum_{k=1}^{n-1}\big(a_\psi(k) - c(k)\big) t_k(i)$$

$$= \frac{v(N)}{n} + \sum_{k=1}^{n-1}\Big(a_\psi(k)\Big) t_k(i) - \sum_{k=1}^{n-1}(c(k)) t_k(i)$$

$$= \psi_i(N, v) - \Big(\phi_i(N, v) - \frac{v(N)}{n}\Big)$$

Setting $\varepsilon_i = \left(\frac{v(N)}{n} - \phi_i(N,v)\right)$, it is obvious that:

- $\varepsilon_{i+1} \leq \varepsilon_i$ for all $i = 1,2,\ldots,n-1$ as ϕ is semi-regular.
- $\varepsilon_1 \geq 0$ and $\varepsilon_n \leq 0$.
- $\sum_{i=1}^{n} \varepsilon_i = 0$ as ϕ is efficient.
- Therefore there exist i_0 such that $\varepsilon_i \geq 0$ for $i \leq i_0$ and $\varepsilon_i \leq 0$ for $i > i_0$.
  $i_0 = max\{i/\varepsilon_i \geq 0\}$ ■

Corollary 5.2: Let (N,v) be any weakly linear game and suppose that players are numbered such that, $1 \preccurlyeq_{WD} 2 \preccurlyeq_{WD} 3, \ldots, \preccurlyeq_{WD} n$. Consider two strict semi-regular values ψ *and* φ on $\Gamma(N)$ satisfying $a_\varphi(s) \leq a_\psi(s)$ for all $1 \leq s \leq n-1$ then,

1) $\sum_{i=1}^{k} \varphi_i(N,v) \geq \sum_{i=1}^{k} \psi_i(N,v)$ for all $1 \leq k \leq n$.
   i.e. φ generalized Lorenz dominates ψ in(N,v).

2) $\sum_{i=1}^{k} \frac{\varphi_i(N,v)}{v(N)} \geq \sum_{i=1}^{k} \frac{\psi_i(N,v)}{v(N)}$ for all $1 \leq k \leq n$, in the case $v(N) > 0$
   i.e. φ relative Lorenz dominates ψ in(N,v)

3) $\varphi_{i+1}(N,v) - \varphi_i(N,v) \leq \psi_{i+1}(N,v) - \psi_i(N,v)$ for all $1 \leq i \leq n-1$.
   i.e φ dominates ψ in absolute differentials in(N,v)

4) If $\psi_1(N,v) > 0$, $\frac{\varphi_i(N,v)}{\varphi_{i+1}(N,v)} \geq \frac{\psi_i(N,v)}{\psi_{i+1}(N,v)}$ for all $1 \leq i \leq n-1$.
   i.e φ dominates ψ in relative differentials in(N,v)

Proof

1) $\sum_{i=1}^{k} \varphi_i(N,v) = \sum_{i=1}^{k} \psi_i(N,v) + \varepsilon_i = \sum_{i=1}^{k} \psi_i(N,v) + \sum_{i=1}^{k} \varepsilon_i$.

Thus, we only need to prove that $\sum_{i=1}^{k} \varepsilon_i \geq 0$ for all $1 \leq k \leq n$.

- If $i \leq k \leq i_0$, then $\varepsilon_i \geq 0$ hence, $\sum_{i=1}^{k} \varepsilon_i \geq 0$
- If $k > i_0$, then $\sum_{i=1}^{k} \varepsilon_i = \sum_{i=1}^{i_0} \varepsilon_i + \sum_{i=i_0+1}^{k} \varepsilon_i \geq \sum_{i=1}^{i_0} \varepsilon_i + \sum_{i=i_0+1}^{n} \varepsilon_i = 0$.

2) Obvious from 1)
3) $\varphi_{i+1}(N,v) - \varphi_i(N,v) = [\psi_{i+1}(N,v) - \psi_i(N,v)] - (\varepsilon_i - \varepsilon_{i+1})$. The results holds since $\varepsilon_i - \varepsilon_{i+1} \geq 0$.
4) $\frac{\varphi_i(N,v)}{\varphi_{i+1}(N,v)} = \frac{\psi_i(N,v)+\varepsilon_i}{\psi_{i+1}(N,v)+\varepsilon_{i+1}}$

These results clearly demonstrate the effect on inequality of a decrease in the coefficients $a(s)_{s=1}^{n-1}$. It is well-known that the Lorenz dominance implies there is a series of progressive Pigou-Dalton transfers. The next theorem deals with this repercussion.

Theorem 5.3: Let (N,v) be any weakly linear game, and consider two distinct strict semi-regular values ψ *and* φ on $\Gamma(N)$.
If their associated sequences $a_\psi(s)_{s=1}^{n-1}$ and $a_\varphi(s)_{s=1}^{n-1}$ are such that $a_\varphi(s) \leq a_\psi(s)$ for all $1 \leq s \leq n-1$, then

$$\varphi(N,v) = PD[\psi(N,v)].$$

The converse of the theorem 5.3 assertion is not true. It is easy to find a (N,v) weakly linear game and both regular values ψ *and* φ with $\varphi(N,v) = PD[\psi(N,v)]$ but the condition

$a_\varphi(s) \leq a_\psi(s)$ for all $1 \leq s \leq n-1$ does not hold. In particular, if (N, v) is such that for some s, $\sum_{S \ni i; |S|=s} v(S)$ is independent to player i, $a_\psi(s)$, for any LES value ψ, has no impact on the computation of $\psi_i(N, v)$, hence a change in the amount of $a_\psi(s)$ changes nothing on $\psi_i(N, v)$. However, in a weakly linear game, $\varphi(N, v) = PD[\psi(N, v)]$ arrives only if $a_\varphi(s) \leq a_\psi(s)$ for at least one s.
Another interesting comment here is about the comparison of Shapley value and Solidarity value. It is well-kwon that they are respectively characterized by the sequence $a(s) = 1$ and $a(s) = \frac{1}{s+1}$ for all $1 \leq s \leq n-1$. Therefore, in a weakly linear game, Solidarity value not only Lorenz dominates Shapley value but also, it is always obtained from the Shapley value by a combination of Pigou-Dalton progressive transfers. This gives other reasons to consider the Solidarity value as more social behavior than Shapley value.

6. Concluding Remark

The paper has studied the ordinal equivalence of Linear, Efficient and Symmetry (LES) values in TU-games. Most of the results established in the case of semi-values and simple games have been re-obtained, in particular when the linear and weakly linear games are considered. The paper has characterized both values that are ordinal equivalent in all TU-games as values such that one is a positive convex combination of the other and the Egalitarian value. This therefore completely solves the problem of the ordinal equivalence of LES values in TU-games. Applying this result, we obtain for example that, Shapley value and Solidarity value are not ordinal equivalent in all TU-games and thus open the problem of finding the largest class of games in which the ordinal equivalence of both values holds. Pigou-Dalton transfers have been introduced for social comparison of values and to shed light on the way payoffs are redistributed from a value to another.

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