Banking, Inside Money and Outside Money

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Abstract

This paper presents an integrated theory of money and banking. I address the following question: when both individuals and banks have private information, what is the optimal way to settle debts? I develop a dynamic model with micro-founded roles for banks and a medium of exchange. I establish two main results: first, markets can improve upon the optimal dynamic contract at the presence of private information. Market prices fully reveal the aggregate states and help solve the incentive problem of the bank. Secondly, it is optimal for the bank to require loans be settled with short-term inside money, i.e. bank money that expires immediately after the settlement of debts. Short-term inside money dominates outside money because the former makes it less costly to induce truthful revelation and achieve more efficient risk sharing.

Key words: banking, inside money, outside money
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1 Introduction

The main goal of this paper is to integrate the banking theory with the monetary theory. I address the following question: given that both individuals and banks have private information, what is the optimal way to settle debts? This is a fundamental question concerning any modern economy, where both outside money (fiat money) and inside money (created by banks and payments systems) are used to facilitate trades. How to settle debts efficiently is critical for the performance of the banking system as a major source of lending. There are several aspects to this issue. For example, why should debts be settled with money? Which is a better instrument for settlements, inside money or outside money?

To answer these questions, I develop a dynamic model with micro-founded roles for banks and a medium of exchange. There are two types of frictions in the economy. The first one is lack of intertemporal double coincidence of wants. This, along with spatial separation and limited communication, gives rise to a role of money as the medium of exchange. The second friction is two-layered private information. On one hand, agents have private information about their random endowments. Hence banking has a role in providing risk-sharing. In particular, bankers can offer dynamic contracts to help agents smooth consumption over time. However, the contracts must be incentive compatible for individuals to truthfully make payments. On the other hand, bankers have private information about the uncertain aggregate endowments because they can filter out the idiosyncratic shocks by aggregating the reports of individual agents. This creates a role for markets to help solve the incentive problem on the bank's side. Indeed, markets at the settlement stages generate information-revealing prices such that bankers cannot lie about the aggregate states.

In the model, a banking sector arises endogenously at the beginning of time and provides dynamic contracts to agents. According to the contract, bankers lend money to agents at the beginning of a period and agents settle the current debt with bankers as they receive endowments at the end of the period. Each period, the amount of the loan entitlement of an agent depends on the individual's history of past settlements (i.e. his history of reported endowments) and the sequence of prices at settlement stages.

I establish two main results in this paper. First, markets can improve upon the optimal dynamic contract in the presence of private information on the bank's side. Markets of goods for money at the settlement stages generate prices that fully reveal the aggregate states. This costlessly solves the incentive problem of bankers. However, if debts are required to be settled with real goods, no market will arise at the settlement stages. Therefore, debt settlements must involve money in order to efficiently discipline bankers.

Second, the optimal instrument for settlements is the kind of inside money that expires immediately after each settlement. I call it one-period inside money. Induction of truthful revelation is less costly with one-period inside money than with outside money or inside money of any longer durations, which leaves agents better insured against idiosyncratic risks. Agents cannot benefit from holding one-period inside money across periods because it expires right after a settlement (which happens at the end of a period). In this case, the only profitable way for one to default is to save and consume one's own endowments, which is not very desirable. In contrast, when outside money is valued, an agent finds it more profitable to default by carrying outside money across periods than saving endowments. The reason is that the agent can use the hidden outside money to buy his preferred consumption goods. Thus the gain of default is higher with outside money than with one-period inside money. The same argument applies to inside money of longer durations. Longer-termed inside money functions similarly to outside money and involves higher incentives to misrepresent in periods when the current issue of money does not expire. Therefore, one-period inside money helps the optimal dynamic contracts implement better allocations. In equilibrium, more efficient risk-sharing is achieved and welfare is improved.

The key to the above result is the timing of the expiration of inside money, which is exactly when each settlement of debts is done. Once an agent obtains such inside money for the settlement, making the payments to the bank is nothing but giving up some worthless objects. However, this is not true if outside money is required for settlements. Outside money will still be valuable to the agent after the settlement stage. Hence the incentives to default are much stronger with outside money. Not surprisingly, inflation of outside money can be used to correct incentives. With outside money getting less valuable as time goes on, induction of truthful revelation tends to get less costly.

The model of this paper is built upon Andolfatto and Nosal (2003) and Sun (JME, forthcoming). Andolfatto and Nosal (2003) construct a model with spatial separation, limited communication friction and limited information friction. They explain why money creation is typically associated with banking. Sun (JME, forthcoming) addresses the problem of monitoring banks with undiversifiable risks and shows that there is no need to monitor a bank if it requires loans to be repaid partly with money. A market arises at the repayment stage and generates information-revealing prices that perfectly discipline the bank. This result is strengthened in the current paper of mine, which features an enduring relationship between bankers and the contracted agents. In contrast to the static contract studied in Sun (JME, forthcoming), here I show that even the more sophisticated contract form, dynamic contracts, can use the help of markets to deal with the incentive problem of bankers.

My work is complementary to the literature that examines the functioning of inside money and outside money, e.g. Cavalcanti and Wallace (1999), Williamson (2004), He, Huang and Wright (2005, 2006) and Sun (JME, forthcoming). Cavalcanti and Wallace (1999) study a random matching model of money and prove that inside money has the advantage of facilitating trades between bankers and non-bankers because with inside money bankers are not constrained by trading histories. One of the issues addressed by Williamson (2004) is the implication of private money issue for the role of outside money. Inside money has the advantage of being flexible and it responds to unanticipated shocks better than outside money. He, Huang and Wright (2005, 2006) study money and banking in a money search model. Bank liabilities are identified as a safer instrument than cash while cash is less expensive to hold. In equilibrium, agents may find it optimal to hold a mix of both. Sun (JME, forthcoming) establishes that with multiple banks, inside money helps achieve better outcomes than outside money does. The reason is that the competition of private monies drives up the equilibrium returns of money and improves welfare. A prohibition on inside money issue not only eliminates money competition but also triggers free-rider problems among bankers, which decreases welfare. All the above papers focus on the roles of inside money and outside money as alternative instruments to facilitate trades. In contrast, this paper of mine takes a new yet no less important perspective, which is the efficiency of alternative monetary instruments for settling debts.

This paper develops an integrated theory of money, banking and dynamic contracts, which is by far a rare effort in the literature. A related previous work is by Aiyagari and Williamson (2000). They study money, credit and dynamic contracts. In their model, financial intermediaries write long-term contracts with consumers. Money is essential because of limited participation in the financial market. There are incentive problems due to private information and limited commitment. With limited commitment, inflation has a large impact on the distribution of welfare and consumption. In contrast, here incentive problems are caused by private information and aggregate uncertainty. It is essential to have contracts that require settlements be made with money, in order to cope with the incentive problems of bankers. Both inside money and outside money are examined to derive the most efficient payment system for induction of truthful revelation.

The remainder of the paper is organized as follows. Section 2 describes the environment of the model. Section 3 studies banking with outside money. Section 4 examines banking with inside money. Section 5 explores banking with co-circulation of inside money and outside money. Section 6 studies the existence and uniqueness of the banking equilibrium. Section 7 concludes the paper.

2 The environment

Time is discrete and has infinite horizons, $t = 0, 1, ..., \infty$. Each period t consists of three sub-periods, indexed by $\tau = 1, 2, 3$. There are three islands indexed by $i = a, b, c$. Each island is populated by a continuum of agents who have unit mass, live forever and discount across time t with factor $\beta \in (0, 1)$. At any point in time, there are only two islands in communication, from which agents can freely visit each other. The sequence of communication at any date t is the following: islands a and b at $\tau = 1$, islands b and c at $\tau = 2$, and islands c and a at $\tau = 3$. Traveling agents return to their native islands at the end of the sub-period.

Agents on island i receive endowments of type i goods. Type b goods are endowed at $\tau = 1$ of all t, type c goods at $\tau = 2$ of all t and type a goods at $\tau = 3$ of all t. For individual type b and type c agents, the endowment is deterministic at $\overline{y}$ for all t, where $0 < \overline{y} < 1$. However, the endowment of a type a agent is stochastic: $y_t = s_t \theta_t$, where s_t and θ_t are both random variables and $E(y_t) = \overline{y}$. Here s_t is an aggregate shock, which is common to all type a agents. It is i.i.d. across time according to the probability density function $f(s)$ and the cumulative distribution function $F(s)$. The variable θ_t is an idiosyncratic shock. It is i.i.d. over time and drawn in such a way that the law of large numbers applies across type a agents, according to PDF $g(\theta)$ and CDF $G(\theta)$. Both $f(\cdot)$ and $g(\cdot)$ have support $[0, 1]$. Let $h(y)$ and $H(y)$ denote the PDF and CDF of y_t, respectively. By Rohatgi's well-known result,[1] $h(y) = \int_y^1 f(s)\, g\left(\frac{y}{s}\right) \frac{1}{s} ds$. The realization of y_t, not s_t or θ_t specifically, is private information of the agent. All agents know about $f(s)$ and $g(\theta)$. The aggregate endowment of type a goods is not publicly observable.

Endowments are received prior to the arrival of any traveling agent at the start of each τ. All goods are perishable. In particular, type b and type c goods can last for only one sub-period and cannot be stored across sub-periods. Type a goods, however, can last for two sub-periods. That is, the endowment of type a goods at $\tau = 3$ of t becomes inconsumable

[1] For the distribution of the product of two continuous random variables, see Rohatgi (1976).

starting $\tau = 2$ of $t+1$.

Agents' preferences are as follows:

$$\begin{aligned} U_a &= \underset{0}{E} \sum_{t=0}^{\infty} \beta^t u\left(C^a_{t,b} + \varepsilon C^a_{t-1,a}\right) \\ U_b &= \underset{0}{E} \sum_{t=0}^{\infty} \beta^t \left(C^b_{t,c} + C^b_{t,b}\right) \\ U_c &= \underset{0}{E} \sum_{t=0}^{\infty} \beta^t \left(C^c_{t,a} + C^c_{t,c}\right) \end{aligned}$$

where the function $u : \mathbb{R}_+ \to \mathbb{R}$ is twice continuously differentiable with $u' > 0$ and $u'' < 0$, and $C^i_{t,j}$ denotes a type $i = a, b, c$ agent's consumption of date-t type $j = a, b, c$ goods. That is, the superscript characterizes the agent and the subscripts describe the consumption goods. It is given that $C^a_{-1,a} = 0$. Note that agents can either consume their own endowments or another particular type of goods. In contrast to type $i = b, c$ agents, type a agents only consume their own endowments at one sub-period over.[2] The preference parameter ε is a very small positive number, i.e. $0 < \varepsilon \ll 1$. That is, type a agents strongly prefer type b goods to their own endowments.

There is lack of intertemporal double coincidence of wants among various types of agents. In particular, type a agents would like to trade endowments for type b goods. However, type b agents do not value type a goods. Type b agents can consume type c goods, but type c agents do not value type b goods. Similarly for type c and type a agents. This lack of double coincidence of wants, together with the limited communication friction, generates a role for money. At the beginning of time, each type a agent is endowed with M units of storable fiat objects called *outside money*. Agents can trade money for goods other than their own endowments (see Figure 1). With random endowments, type a agents' money incomes will also be random. Banking has a role in providing risk-sharing so as to

[2] This assumption, along with the assumption that type a goods can last for two sub-periods, is intended to simplify analysis but is not critical for the main results. As a result of these assumptions, a type a agent's current-period decision of truthfully settling debts is independent of his consumption of type b goods earlier this period.

efficiently insure type a agents against the idiosyncratic risks.

[Insert Figure 1]

3 The banking arrangement

A banking sector arises endogenously at the beginning of $t = 0$. Each type a agent chooses to be a banker or a non-banker. Bankers offer long-term contracts to non-bankers, to help them smooth consumption over time. Banking is competitive and the bankers end up offering the same equilibrium contract. Because of the free entry to banking, the equilibrium contract is such that individual bankers and non-bankers earn the same expected life-time utility. Without loss of generality, it is convenient to think of bankers work together as one intermediary, i.e. the *bank*. Both the bank and non-bankers commit to the contract. All terms of the contract are public information. Market trades are competitive.

The bank aims to insure type a agents against the idiosyncratic endowment shocks. Perhaps the most straightforward banking arrangement is as follows. At each $\tau = 1$, the bank offers money in exchange for the endowments of type b agents and then allocates type b goods efficiently among type a agents. Then at each $\tau = 3$, the bank collects type a endowments, gives the endowments to type c agents in exchange for money, and then allocates the rest of the type a goods (if any) efficiently among type a agents.

There are two-sided incentive problems associated with a banking arrangement as described above. On one hand, incentive problems arise due to private information at the individual level. For type a agents, none of the individual endowment, consumption and money holdings is observable. I focus on incentive compatible allocations. That is, any banking arrangement must be such that individual type a agents (both bankers and non-bankers) will truthfully reveal their endowments throughout time. On the other hand, the bank has the incentive to lie about the aggregate state. Note that the bank collects type

a endowments and hence gets to know exactly what the aggregate endowment is based on the reports of individual endowments. In other words, the aggregate endowment becomes private information of the bank. Therefore, the bank always has the incentive to misrepresent the aggregate information unless otherwise disciplined. For example, the bank can claim an adverse aggregate state and keep the hidden goods to benefit its bankers, instead of transferring the goods to type c and type a agents as it should. The incentive problem on the bank's side is known as the problem of monitoring the monitor.

Note that the bank cannot be actually monitored here because there is no state verification technology in this model. (Even if there was, state verification would be costly.) One way to induce truthful revelation of the bank is to design a contract that makes the banking profits depend on the aggregate state announced by the bank. That is, to reward the bank (with higher profits) as it announces a high aggregate state and to punish it (with lower profits) for claiming a low state. However, this mechanism will also be costly because it distorts the optimal allocations.

In a finite horizon model of banking, Sun (JME, forthcoming) shows that the bank is perfectly disciplined if loans are required to be repayed with money. This result can be readily applied here in the current model. Instead of the bank managing all the allocations of goods, the optimal contract requires that at least part of the allocations are done through monetary payments (from the bank to non-bankers and vice versa). As agents are obliged to make monetary payments, they must trade endowments for money first. I will show later that markets arise accordingly on island a and generate prices that fully reveal the aggregate states. As a result, the incentive problem of the bank is solved costlessly.

The bank can issue private money, which is also known as *inside money.* Between inside money and outside money, the bank chooses the optimal instrument for settling debts. In what follows, I study different banking arrangements which involve alternative kinds of money. Then I compare the results of the various arrangements and characterize the optimal banking contract.

4 Banking with outside money

For now, assume that private money issue is prohibited. The banking contract requires that monetary payments be made with only outside money. The contract specifies that (i) at the beginning of each $t \geq 1$ the bank pays the non-banker $m_t \in \mathbb{R}^+$ units of outside money to finance his date-t consumption of type b goods; (ii) at $\tau = 3$ of each $t \geq 0$, the non-banker must sell a fraction z of his endowments y_t for outside money and then contributes to the bank his money income $p_t^a z y_t$ and the rest of his endowments $(1-z)\, y_t$, where p_t^a is the market price of type a goods for outside money. Then the bank reallocates the collected type a goods among type a agents. Trivially, a non-banker's date-0 consumption of type b goods is financed by his endowment of M units of outside money.

After money payments to non-bankers, the bankers use the residual money balance to finance their own consumptions of type b goods. Each banker is allocated $m_t^B \in \mathbb{R}^+$ units of outside money at the onset of each period. At each $\tau = 3$, each banker must also sell zy_t units of endowments and contribute the income $p_t^a z y_t$ and the rest of his endowments $(1-z)\, y_t$. Then bankers divide the type a goods among themselves after the allocations to non-bankers.

4.1 Timing of events

Timing of events is illustrated by Figure 2. In any t, at the beginning of $\tau = 1$, the bank allocates money among non-bankers and its bankers. Then type a agents visit island b and trade money for type b goods. At $\tau = 2$, type b agents trade money for type c goods. At $\tau = 3$, first type c agents trade money for type a goods. Then type a agents make payments to the bank, which is called the *settlement*. The bank reallocates the collected type a goods (if any) among type a agents. The above procedure is repeated for all t.

[Insert Figure 2]

4.2 The banking equilibrium

Let v_0 be a non-banker's expected life-time utility prescribed by the contract. Correspondingly, W_0 is a banker's expected life-time utility. Let $\alpha \in [0,1]$ be the equilibrium measure of bankers (i.e. the size of the bank) and hence $1-\alpha$ the equilibrium measure of non-bankers.

Definition 1 *A banking equilibrium consists of a contract with the initial promised value v_0 to a representative non-banker and the associated initial value W_0 to a representative banker, an aggregate measure α, allocations $\left\{C_{t,c}^b, C_{t,b}^b, C_{t,a}^c, C_{t,c}^c\right\}_{t=0}^{\infty}$, market prices $\left\{p_t^a, p_t^b, p_t^c\right\}_{t=0}^{\infty}$ such that: (i) given v_0 and α, the contract maximizes W_0 while delivering the promised v_0; (ii) α clears the market of contracts, that is, $W_0 = v_0$; (iii) given prices and the contract, allocations $\left\{C_{t,c}^b, C_{t,b}^b, C_{t,a}^c, C_{t,c}^c\right\}_{t=0}^{\infty}$ maximize type b and type c agents' utilities; (iv) prices $\left\{p_t^a, p_t^b, p_t^c\right\}$ clear goods markets for all $t \geq 0$.*

Before examining the banking contract, it is helpful to first study the equilibrium decisions of type b and type c agents. Consider type c agents' best responses. Taking (p_t^c, p_t^a) as given, a representative type c agent maximizes his expected life-time utility:

$$\begin{aligned} &\max_{\left(C_{t,a}^c, C_{t,c}^c, d_{t+1}^c\right)} \underset{0}{E} \sum_{t=0}^{\infty} \beta^t \left(C_{t,a}^c + C_{t,c}^c\right) \\ &s.t. \quad p_t^a C_{t,a}^c + d_{t+1}^c = d_t^c + p_t^c \left(\overline{y} - C_{t,c}^c\right) \end{aligned}$$

where d_t^c is the type c agent's beginning-of-t money holdings. Let $\left(C_{t,a}^{c*}, C_{t,c}^{c*}, d_{t+1}^{c*}\right)$ denote the optimal choices. Similarly, taking $\left(p_t^b, p_t^c\right)$ as given, a representative type b agent maximizes his expected life-time utility:

$$\begin{aligned} &\max_{\left(C_{t,c}^b, C_{t,b}^b, d_{t+1}^b\right)} \underset{0}{E} \sum_{t=0}^{\infty} \beta^t \left(C_{t,c}^b + C_{t,b}^b\right) \\ &s.t. \quad p_t^c C_{t,c}^b + d_{t+1}^b = d_t^b + p_t^b \left(\overline{y} - C_{t,b}^b\right) \end{aligned}$$

where d_t^b is the type b agent's beginning-of-t money holdings. Let $\left(C_{t,c}^{b*}, C_{t,b}^{b*}, d_{t+1}^{b*}\right)$ denote the optimal choices.

The equilibrium prices are $p_t^b = D_t^a / \left(\overline{y} - c_{t,b}^{b*}\right)$, $p_t^c = D_t^b / \left(\overline{y} - c_{t,c}^{c*}\right)$ and $p_t^a = D_t^c / Z_t$ for all t, where D_t^i is the aggregate money supply to the market by type i agents and $Z_t = zY_t$ is the aggregate supply of type a goods to the market when the aggregate endowment is Y_t. It is straightforward to derive that $d_{t+1}^{b*} = d_{t+1}^{c*} = 0$ and $C_{t,b}^{b*} = C_{t,c}^{c*} = \overline{y} - E\left[Z_t\right] = (1-z)\,\overline{y}$ for all t. Neither type b nor type c agents hold money across periods because they receive a constant stream of endowments.

Now I proceed to study the optimal banking contract. First the bank must decide the optimal fraction of the aggregate type a endowments to be traded in the market, z. *Ex ante* the expected amount of type a goods to be saved and consumed by type a agents every period is $(1-z)\,\overline{y}$, which is equivalent to consuming $\varepsilon\,(1-z)\,\overline{y}$ units of type b goods. Suppose instead of saving it up, the bank also requires the fraction $1-z$ of the aggregate endowment to be sold to type c agents. According to $C_{t,b}^{b*}$, this will get type b agents to sell $(1-z)\,\overline{y}$ more units of goods to type a agents. Since $\varepsilon < 1$, it is efficient for the bank to require $z = 1$. As a result, type a agents must sell all their endowments to type c agents. In return, the aggregate consumption of type a agents is maximized and equal to $\overline{y}$ units of type b goods every period.

Now let c_t denote a non-banker's date-t consumption financed by the contract. Thus, $c_t = \frac{m_t}{p_t^b}$ where p_t^b is the date-t price of type b goods for outside money and $m_0 = M$. Without loss of generality, Normalize $M = 1$. The contract prescribes $v_0 = E\sum\limits_{t=0}^{\infty}\beta^t u\left(c_t\right)$. Correspondingly, $c_t^B = \frac{m_t^B}{p_t^b}$ denote a banker's date-t consumption of type b goods and hence $W_0 = E\sum\limits_{t=0}^{\infty}\beta^t u\left(c_t^B\right)$. Again $m_0^B = 1$.

Due to private information of individuals, payments from the bank to a non-banker must be based on the latter's reported history of endowments. Recall that I focus on incentive compatible contracts. Unless otherwise stated, reported values also represent true values. Denote a non-banker's history of reported endowments up to period t as

$h_t = (y_0, y_1, \cdots, y_t) \in [0,1]^{t+1}$. Since $z_t = z_t^B = y_t$ for all t, the equilibrium price is $p_t^a = \frac{1}{Y_t} = \frac{1}{s_t \int_0^1 \theta_t g(\theta_t) d\theta_t}$ for all t. Hence the market price at the settlement stage ($\tau = 3$) fully reveals the aggregate state, i.e. $s_t = \frac{1}{p_t^a E[\theta_t]}$. In other words, agents can infer the true aggregate state simply by observing the market price. As a result, the bank cannot misrepresent the aggregate information to benefit its bankers. Denote the price sequence of settlement stages up to period t as $P_t = (p_0^a, p_1^a, \cdots, p_t^a) \in \mathbb{R}_+^{t+1}$. The banking contract can be formally defined as follows.

Definition 2 *A contract σ is a constant σ_0 and a sequence of functions $\{\sigma_t\}_{t=1}^{\infty}$ where $\sigma_t : [0,1]^t \times \mathbb{R}_+^t \to \mathbb{R}_+$. The consumption stream to a non-banker depends on his reported history of endowments and the price sequence of settlement stages. That is, $c_0 = \sigma_0$ and $c_t = \sigma_t (h_{t-1}, P_{t-1})$ for all $t \geq 1$.*

4.3 The contract design problem

The contract design problem of the bank can be formulated recursively. At the end of $\tau = 3$ of any $t \geq 0$, non-bankers report current endowments and make the corresponding payments to the bank. Then the bank makes decisions on future payments and promised values according to what non-bankers have reported. For any $t \geq 1$, each non-banker is identified with a number v_{t+1}, which is his discounted future value starting $t+1$ and it was promised to him by the bank at $t-1$. The bank delivers v_{t+1} by financing a state-dependent next-period consumption c_{t+1} and a promised value v_{t+2} starting period $t+2$. Let the density function $\mu_{t+1}(v_{t+1})$ characterize the distribution of the promised values made by the bank to be delivered starting $t+1$. Then μ_{t+1} is the state variable for the bank's recursive problem at the end of each period t. Note that the $t = 0$ consumption of type b goods is financed by the agent's endowment of outside money. Thus $c_0 = \overline{y}$. Since v_0 is the lifetime expected value promised by the contract, it follows that $v_1 = v_0 - u(\overline{y})$

and

$$\mu_1(v_1) = \begin{cases} 1, & \text{if } v_1 = v_0 - u(\overline{y}) \\ 0, & \text{otherwise} \end{cases} . \tag{1}$$

The objective of the recursive contract design problem is to maximize a representative banker's expected discounted value W_{t+1} starting $t+1$, while delivering the distribution of promised values μ_{t+1}. Dropping time subscripts and letting $+1$ denote $t+1$ and $+2$ denote $t+2$, the bank's end-of-period-t objective can be formulated by the following functional equation:

$$(TW_{+1})\left(\mu_{+1}\right) = \max_{\left(c^B_{+1}, c_{+1}, v_{+2}\right)} \int_0^1 \int_0^1 \left\{u\left[c^B_{+1}(y,s)\right] + \beta W_{+2}\left(\mu_{+2}\right)\right\} g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right) f(s)\, ds \tag{2}$$

The maximization problem is subject to the following conditions:

$$\begin{aligned} & u\left[c_{+1}(y,s,v_{+1})\right] + \beta v_{+2}(y,s,v_{+1}) \\ \geq \;& u\left[c_{+1}(\widetilde{y},s,v_{+1}) + \max_{\gamma\in[0,1]}\left\{\varepsilon\gamma\left(y-\widetilde{y}\right) + (1-\gamma)\left(y-\widetilde{y}\right)\frac{p^a(s)}{p^b_{+1}}\right\}\right] + \beta v_{+2}(\widetilde{y},s,v_{+1}) \\ & \qquad\qquad\qquad\qquad \forall \;\; s, v_{+1}, \;\; \forall \;\; \widetilde{y} < y \end{aligned} \tag{3}$$

$$\begin{aligned} u\left[c^B_{+1}(y,s)\right] \;\; \geq \;\; & u\left[c^B_{+1}(\widetilde{y},s) + \max_{\gamma^B\in[0,1]}\left\{\varepsilon\gamma^B\left(y-\widetilde{y}\right) + \left(1-\gamma^B\right)\left(y-\widetilde{y}\right)\frac{p^a(s)}{p^b_{+1}}\right\}\right] \\ & \qquad\qquad\qquad \forall \;\; s, \;\; \forall \;\; \widetilde{y} < y \end{aligned} \tag{4}$$

$$\int_0^1 \int_0^1 \left\{u\left[c_{+1}(y,s,v_{+1})\right] + \beta v_{+2}(y,s,v_{+1})\right\} g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right) f(s)\, ds = v_{+1}, \quad \forall \; v_{+1} \tag{5}$$

$$\mu_{+2}(w_{+2};s) = \iint_{\{(y,v_{+1}):w_{+2}=v_{+2}(y,s,v_{+1})\}} g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right) \mu_{+1}(v_{+1})\, dv_{+1}, \quad \forall \; s \tag{6}$$

$$\alpha \int_0^1 c^B_{+1}(y,s)\, g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right) + (1-\alpha)\int_0^1 \int_{-\infty}^{\overline{V}} c_{+1}(y,s,v_{+1})\, g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right) \mu_{+1}(v_{+1})\, dv_{+1} = \overline{y}, \;\; \forall \; s \tag{7}$$

$$c_{+1}(y,s,v_{+1}) \geq 0, \qquad \forall\, y,s,v_{+1} \tag{8}$$

$$c^{B}_{+1}(y,s) \geq 0, \qquad \forall\, y,s \tag{9}$$

$$v_{+2}(y,s,v_{+1}) \in \left(-\infty,\overline{V}\right], \quad \forall\, y,s,v_{+1} \tag{10}$$

where $\overline{V} = u(\overline{y})/(1-\beta)$ is the value of the unconstrained first-best contract that finances consumption of $\overline{y}$ units of type b goods every period.

Constraints (3) and (4) are the incentive compatibility constraints for a non-banker and a banker respectively. Incentive compatibility requires that both bankers and non-bankers are induced to sell the entire endowments and turn over the entire incomes every period. Here $c_{+1}(y,s,v_{+1})$ and $v_{+2}(y,s,v_{+1})$ are a non-banker's next period consumption and promised value starting the period after the next, given that he is currently promised v_{+1}, his current endowment is y and the current aggregate state is s. For a banker, $c^{B}_{+1}(y,s)$ is his next-period consumption given his current endowment y and the current aggregate state s.

For both parties, the payoff of truthful revelation must be no lower than the payoff of any possible deviation. The right-hand side of (3) is the payoff if the non-banker reports $\widetilde{y} < y$ instead of the truth y. (Note that it is not feasible for an agent to claim $\widetilde{y} > y$ because he would not have $p^{a}\widetilde{y} > p^{a}y$ units of money to submit to the bank when his true endowment is y.) The misreported endowment can either be stored for next-period consumption or be traded for money to buy type b goods. The non-banker chooses γ, the fraction of endowment to be stored, to maximize his gain of default. The first term in the maximization problem (on the right-hand side of [3]) is the extra consumption of stored endowment $\varepsilon\gamma(y-\widetilde{y})$. The second term is the extra consumption of type b goods purchased with the misreported money, which is $(1-\gamma)(y-\widetilde{y})\frac{p^{a}(s)}{p^{b}_{+1}}$. Similar logic for the right-hand side of (4). Given prices, an agent optimally chooses $\gamma^{*}=0$ (or $\gamma^{B*}=0$) if $\frac{p^{a}}{p^{b}_{+1}} > \varepsilon$. In equilibrium, $p^{a} = \frac{1}{sE(\theta)}$ and $p^{b}_{+1} = \frac{1}{\overline{y}} = \frac{1}{E(s)E(\theta)}$. Therefore, we have $\gamma^{*} = \gamma^{B*} = 0$ provided that $\varepsilon < E(s)$. Now constraints (3) and (4) can both be simplified:

$$
\begin{aligned}
u\left[c_{+1}(y,s,v_{+1})\right] + \beta v_{+2}(y,s,v_{+1}) &\geq u\left[c_{+1}(\widetilde{y},s,v_{+1}) + (y-\widetilde{y})\frac{p^a(s)}{p^b_{+1}}\right] + \beta v_{+2}(\widetilde{y},s,v_{+1}) \qquad (11)\\
&\qquad \forall\ s, v_{+1},\ \forall\ \widetilde{y} < y \\
u\left[c^B_{+1}(y,s)\right] &\geq u\left[c^B_{+1}(\widetilde{y},s) + (y-\widetilde{y})\frac{p^a(s)}{p^b_{+1}}\right] \qquad (12)\\
&\qquad \forall\ s,\ \forall\ \widetilde{y} < y
\end{aligned}
$$

Constraint (5) is the promise-keeping constraint. All the values promised to non-bankers must be delivered. Constraint (6) characterizes the law of motion of the state variable μ, i.e. the distribution of the promised values. Constraint (7) is the resource constraint. Consumptions of bankers and non-bankers exhaust $\overline{y}$ units of type b goods every period. Constraints (8)-(10) define the choice sets for the choice variables.

Let $W^*(\mu)$ denote the fixed point of T in (2). One can show that $W^*(\mu)$ is a strictly increasing, concave function from the fact that T is a contraction mapping that maps the space of increasing, concave functions to itself. The policy functions $\{c_{+1}(y,s,v_{+1}), v_{+2}(y,s,v_{+1})\}$, together with the initial consumption c_0 and the associated initial promised value v_0, completely characterize the lifetime contract to a non-banker. Hence, $v_0 = u(\overline{y}) + E\sum_{t=1}^{\infty}\beta^t u(c_t)$. Similarly, the policy function $c^B_{+1}(y,s)$ pins down the initial value of a representative banker, $W_0 = u(\overline{y}) + E\sum_{t=1}^{\infty}\beta^t u\left(c^B_t\right)$. The equilibrium condition $v_0 = W_0$ implies that given α,

$$
v_0 = u(\overline{y}) + W_1(\mu_1;\alpha), \qquad (13)
$$

where μ_1 is given by (1). The above condition defines v_0 as a function of α, i.e. the relationship between the initial value and the aggregate measure that clears the market of contracts. The equilibrium contract must be the one that offers the highest achievable v_0^*. Therefore, $v_0^* = \max_{\alpha\in[0,1]} v_0(\alpha)$.

So far I have set up the contract design problem and describe the banking equilibrium for the banking contract that requires payments of outside money. The following section examines the contract that requires payments made exclusively of inside money. Then I compare the implications of the two contracts and show that it matters whether inside money or outside money is used as the settlement instrument.

5 Banking with inside money

5.1 One-period inside money

Now assume private issue of money is permitted. The bank can finance consumptions through allocations of private money. In this section, I study the banking arrangement where outside money is not valued and the bank issues a particular kind of inside money, one-period inside money (OPIM). Namely, it is issued at the beginning of each $t \geq 0$ and expires at the end of t after the current-period settlements are done.[3] As before, let α be the equilibrium measure of bankers.

The contract specifies that (i) at the beginning of all $t \geq 0$ the bank pays the non-banker $m_t \in \mathbb{R}^+$ units of inside money to finance his date-t consumption of type b goods; (ii) at $\tau = 3$ of all $t \geq 0$ the non-banker must sell the entire endowment y_t for inside money and then contributes to the bank the money income $p_t^a y_t$, where p_t^a is now the market price of type a goods for inside money.

The same notations are used as in the previous section. In particular, let c_t denote a non-banker's date-t consumption of type b goods financed by the contract. That is, $c_t = \frac{m_t}{p_t^b}$ where p_t^b is the price of type b goods for inside money. Denote h_t as a non-banker's history of reported endowments up to period t and P_t as the price sequence of settlement stages up to period t. The contract and the banking equilibrium are still defined by Definition 1 and Definition 2, respectively.

The objective of the recursive contract design problem by implementing OPIM is given

[3] The expiration of inside money can be thought of as the object deteriorates after a certain amount of time. Or we can interpret it as an electronic account whose balance automatically becomes zero at the prescribed point of time. Accordingly, a new issue of money is simply an amount newly transferred into the account by bankers.

by (2) subject to the same constraints as (5)-(10). However, the incentive compatibility constraints are now different from (3) and (4):

$$u\left[c_{+1}\left(y,s,v_{+1}\right)\right]+\beta v_{+2}\left(y,s,v_{+1}\right) \geq u\left[c_{+1}\left(\widetilde{y},s,v_{+1}\right)+\varepsilon\left(y-\widetilde{y}\right)\right]+\beta v_{+2}\left(\widetilde{y},s,v_{+1}\right) \quad (14)$$

$$\forall \;\; s,v_{+1}, \;\; \forall \;\; \widetilde{y}<y$$

$$u\left[c_{+1}^{B}\left(y,s\right)\right] \geq u\left[c_{+1}^{B}\left(\widetilde{y},s\right)+\varepsilon\left(y-\widetilde{y}\right)\right] \quad (15)$$

$$\forall \;\; s, \;\; \forall \;\; \widetilde{y}<y$$

Constraint (14) is the incentive compatibility constraint for a representative non-banker and constraint (15) for a representative banker. The right-hand sides of the constraints are the payoffs of default. As required by the contract, type a agents must sell the entire endowments for inside money. As a result, outside money is not valued by type b or c agents. Moreover, it is not beneficial for a non-banker or a banker to sell any misreported endowment for inside money because it will expire before period $t+1$ comes. Thus the only profitable way to default is to save the hidden endowments for next-period consumption.

Since the initial allocation of inside money does not depend on any report of endowments, naturally $m_0 = 1$ and $c_0 = \overline{y}$. Again, $v_0 = u\left(\overline{y}\right) + E\sum_{t=1}^{\infty}\beta^t u\left(c_t\right)$. The equilibrium contract must be the one that offers the highest achievable v_0^*. Index values of banking with outside money by superscript o and values of banking with one-period inside money by superscript I. Provided that $\varepsilon < E\left(s\right)$, we have the following propositions:

Proposition 1 $W^o\left(\mu\right) < W^I\left(\mu\right)$ *for any given* μ.

Proposition 2 $W_0^o\left(v_0;\alpha\right) < W_0^I\left(v_0;\alpha\right)$ *for any given* v_0 *and* α.

Proposition 3 $v_0^{o*} < v_0^{I*}$. *Moreover,* $v_0^{I*} \rightarrow \overline{V}$ *as* $\varepsilon \rightarrow 0$ *while* v_0^{o*} *is independent of* ε.

Proofs of Propositions 1-3 are provided in the Appendix.

Propositions 1-2 establish that all else equal bankers can always achieve a higher utility by offering contracts with one-period inside money than with outside money. Accordingly, the bank will choose to implement the former contract. This result is driven by the fact that the incentives to default are weaker with one-period inside money than with outside money. When outside money is valued, agents expect it to carry value into the future. On evaluating the options to default, agents find it more profitable to sell endowments for outside money than saving them for consumption in the following period (given that $\varepsilon < E[s]$). One-period inside money, however, expires right after settlements. Thus, type a agents cannot benefit from selling the hidden endowments for inside money. The only benefit from default now is to save the endowments for next-period consumption, which is associated with a much lower utility gain. Thus it is less costly to induce truthful revelation with OPIM. This allows the bank to achieve more efficient risk-sharing and offer higher equilibrium promised values, which is established by Proposition 3. As a result, welfare of type a agents is improved by the contract that requires settlement be made with one-period inside money. The overall welfare of the economy is also improved because the expected life-time utility of a type b or type c agent is $\frac{\overline{y}}{1-\beta}$ regardless of their optimal decisions to trade.

Furthermore, the advantage of the OPIM contract gets stronger as type a agents value less of their own endowments. As $\varepsilon \to 0$, the utility gain of consuming their own endowments becomes negligibly small. With one-period inside money, the incentives to default diminish because neither saving endowments nor trading endowments for money is profitable. The result approaches the allocations achieved by the unconstrained first-best contract. That is, $c_t(y_{t-1}, s_{t-1}, v_t) = \overline{y}$, $c_t^B(y_{t-1}, s_{t-1}) = \overline{y}$ and $v_{t+1}(y_{t-1}, s_{t-1}, v_t) = \frac{u(\overline{y})}{1-\beta}$ for all (y_{t-1}, s_{t-1}, v_t). However, these policy functions obviously do not satisfy constraints (11)-(12) of the contract with outside money. With outside money, the incentives to default are merely driven by the gain of holding outside money to the following period. These incentives do not go away even if one does not value one's own endowments. Therefore, there is no way the contract with outside money can implement perfect risk-sharing, not even when $\varepsilon = 0$.

5.2 Inside Money with Longer Durations

The previous section studies a special kind of inside money, one-period inside money. Welfare is improved with one-period inside money than with outside money. Now I turn to inside money of more generalized forms and investigate the associated welfare implications. The bank issues inside money that has a duration of κ periods, where κ is an integer and $2 \leq \kappa < \infty$. (Note that if $\kappa = \infty$, inside money never expires, which is equivalent to outside money in this environment.) That is, each issuance of inside money is made at the beginning of period $t = 0, \kappa, 2\kappa, \cdots$, and expires at the end of period $t = \kappa - 1, 2\kappa - 1, \cdots$. Other than that, the bank functions in the same way as in Section 4. Definition 1 and Definition 2 still apply.

The objective of the recursive contract design problem with κ-period inside money is given by

$$\left(TW^{\kappa}_{+1}\right)\left(\mu_{+1}, \iota\right) = \max_{\left(c^B_{+1}, c_{+1}, v_{+2}\right)} \int_0^1 \int_0^1 \left\{u\left[c^B_{+1}\left(y, s\right)\right] + \beta W^{\kappa}_{+2}\left(\mu_{+2}, \iota_{+1}\right)\right\} g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right) f\left(s\right) ds \tag{16}$$

subject to the same constraints as (5)-(10). However, instead of constraints (3) and (4), here the incentive compatibility constraints are formulated by the following:

$$u\left[c_{+1}\left(y, s, v_{+1}\right)\right] + \beta v_{+2}\left(y, s, v_{+1}\right) \geq u\left[c_{+1}\left(\widetilde{y}, s, v_{+1}\right) + \iota\Phi_1 + \left(1 - \iota\right)\Phi_2\right] + \beta v_{+2}\left(\widetilde{y}, s, v_{+1}\right)$$
$$\forall \;\; s, v_{+1}, \;\; \forall \;\; \widetilde{y} < y \tag{17}$$

$$u\left[c^B_{+1}\left(y, s\right)\right] \geq u\left[c^B_{+1}\left(\widetilde{y}, s\right) + \iota\Phi_1 + \left(1 - \iota\right)\Phi_2\right] \tag{18}$$
$$\forall \;\; s, \;\; \forall \;\; \widetilde{y} < y$$

where

$$
\begin{aligned}
\iota_t &= \begin{cases} 1, & \text{if } t = \kappa - 1, 2\kappa - 1, \cdots \\ 0, & \text{otherwise} \end{cases} \\
\Phi_1 &= \varepsilon\left(y - \widetilde{y}\right) \\
\Phi_2 &= \max_{\gamma \in [0,1]} \left\{ \varepsilon\gamma\left(y - \widetilde{y}\right) + \left(1 - \gamma\right)\left(y - \widetilde{y}\right)\frac{p^a}{p^b_{+1}} \right\}. \qquad (19)
\end{aligned}
$$

Let v_0^κ and W_0^κ be the initial values of a representative non-banker and a representative banker, respectively. The banker's recursive contract design problem now differs in periods with and without expiration of money. In periods with expiration of money, that is, $t = \kappa - 1, 2\kappa - 1, \cdots$, the banker's problem is similar to the case with one-period inside money. Since the current issue of money expires at the end of the period, the only profitable way for type a agents to default is to save the endowments for next-period consumption. The incentive compatibility constraints are equivalent to (14)-(15). In periods without expiration of money, the problem is similar to the case with outside money. Agents would prefer to default by holding money into the next period. Accordingly, the IC constraints are equivalent to (3)-(4). Let $v_0^{\kappa *}$ be the equilibrium initial promised value with κ-period inside money.

Proposition 4 $v_0^{o*} < v_0^{\kappa *} < v_0^{I*}$.

The proof of Proposition 4 is provided in the Appendix.

Proposition 4 establishes that welfare is the highest with one-period inside money. Banking with κ-period inside money takes the second place while the outside money arrangement ranks the last. With κ-period inside money, incentives to default in periods without expiration of money are as strong as with outside money. It does provide more stringent discipline when there is expiration of money at the end of a period. However, overall agents are not always as disciplined as with one-period inside money. Not surprisingly, incentive compatibility is still more costly with κ-period inside money than with one-period inside money. Hence Proposition 4.

6 Co-circulation of inside money and outside money

In this section I study co-circulation of inside money and outside money. Previously, it has been established that one-period inside money is the best of all kinds of inside money in that it helps the banking contract achieve the highest welfare level. Therefore, it makes sense here to focus on the co-circulation of outside money and one-period inside money.

The contract specifies that (i) at the beginning of all $t \geq 0$ the bank pays the non-banker a portfolio of $\{m_t^I, m_t^o\}$ to finance his date-t consumption of type b goods, where $m_t^I \in \mathbb{R}^+$ is the amount of current period inside money and $m_t^o \in \mathbb{R}^+$ is the amount of outside money; (ii) at $\tau = 3$ of all t the non-banker must sell $(1-\phi)\, y_t$ units of endowment for current-period inside money and ϕy_t units of endowment for outside money, where $\phi \in [0,1]$ is a constant. Then the portfolio of money incomes $\left[p_t^{a,I}(1-\phi)\, y_t, p_t^{a,o}\phi y_t\right]$ must be contributed to the bank, where $p_t^{a,I}$ is the market price of type a goods for date-t inside money and $p_t^{a,o}$ is the market price of type a goods for outside money. Trivially, $m_0^o = 1$.

Note that if $\phi = 0$, the contract reduces to one with only one-period inside money; if $\phi = 1$, the contract becomes one with only outside money. In this section I focus on $\phi \in (0,1)$. Now define $P_t = \left(\left(p_0^{a,I}, p_0^{a,o}\right), \left(p_1^{a,I}, p_1^{a,o}\right), \cdots, \left(p_t^{a,I}, p_t^{a,o}\right)\right) \in (\mathbb{R}_+ \times \mathbb{R}_+)^{t+1}$ as the price sequences of settlement stages up to period t. Definition 1 still applies. Let $p_t^{b,o}$ and $p_t^{b,I}$ be the market prices of type b goods for outside money and date-t inside money, respectively. Then $c_t = \frac{m_t^I}{p_t^{b,I}} + \frac{m_t^o}{p_t^{b,o}}$ for all t. Let $\left\{m_t^{B,I}, m_t^{B,o}\right\}$ denote a banker's beginning-of-date-t portfolio, where $m_t^{B,I}, m_t^{B,o} \in \mathbb{R}_+$ and $m_0^{B,o} = 1$. It follows that $c_t^B = \frac{m_t^{B,I}}{p_t^{b,I}} + \frac{m_t^{B,o}}{p_t^{b,o}}$ for all t.

Definition 3 *A banking equilibrium with co-circulation of inside money and outside money consists of a contract with the initial promised value v_0 to a representative non-banker and* *the associated initial value W_0 to a representative banker, an aggregate measure α, allocations $\{C^b_{t,c}, C^b_{t,b}, C^c_{t,a}, C^c_{t,c}\}_{t=0}^{\infty}$, market prices $\left\{p_t^{a,I}, p_t^{a,o}, p_t^{b,I}, p_t^{b,o}, p_t^{c,I}, p_t^{c,o}\right\}_{t=0}^{\infty}$ such that: (i) given v_0 and α, the contract maximizes W_0 while delivering the promised v_0; (ii) α clears the market of contracts, that is, $W_0 = v_0$; (iii) given prices and the contract, allocations $\{C^b_{t,c}, C^b_{t,b}, C^c_{t,a}, C^c_{t,c}\}_{t=0}^{\infty}$ maximize type b and type c agents' utilities; (iv) prices clear goods markets for all $t \geq 0$.*

In equilibrium, $p_t^{a,I} = 1/\left[(1-\phi)\, s_t \int_0^1 \theta_t g(\theta_t)\, d\theta_t\right]$ and $p_t^{a,o} = 1/\left[\phi s_t \int_0^1 \theta_t g(\theta_t)\, d\theta_t\right]$ for all t. Obviously in equilibrium,

$$\frac{p_t^{a,o}}{p_t^{a,I}} = \frac{p_t^{b,o}}{p_t^{b,I}} = \frac{p_t^{c,o}}{p_t^{c,I}} = \frac{1-\phi}{\phi}, \quad \forall t.$$

That is, the value of outside money relative to inside money on island a is given by the ratio of the amounts of goods required to sell in respective markets. Expecting this, type b and type c agents value inside and outside monies by the same ratio.

The objective of the recursive contract design problem now is given by (2) subject to the same constraints as (5)-(10). Instead of constraints (3)-(4), here the incentive compatibility constraints are formulated by

$$\begin{aligned} & u\left[c_{+1}(y,s,v_{+1})\right] + \beta v_{+2}(y,s,v_{+1}) \\ \geq\ & u\left[c_{+1}(\widetilde{y},s,v_{+1}) + \max_{\gamma\in[0,1]}\left\{\varepsilon\gamma(y-\widetilde{y}) + (1-\gamma)(y-\widetilde{y})\frac{p^{a,o}(s)}{p^{b,o}_{+1}}\right\}\right] + \beta v_{+2}(\widetilde{y},s,v_{+1}) \\ & \forall\ s, v_{+1},\quad \forall\ \widetilde{y} < y \end{aligned} \tag{20}$$

$$\begin{aligned} u\left[c^B_{+1}(y,s)\right] \ \geq\ & u\left[c^B_{+1}(\widetilde{y},s) + \max_{\gamma^B\in[0,1]}\left\{\varepsilon\gamma^B(y-\widetilde{y}) + \left(1-\gamma^B\right)(y-\widetilde{y})\frac{p^{a,o}(s)}{p^{b,o}_{+1}}\right\}\right] \\ & \forall\ s,\quad \forall\ \widetilde{y} < y \end{aligned} \tag{21}$$

In fact, the above constraints are equivalent to (3)-(4) because $\frac{p^{a,o}}{p^{b,o}_{+1}} = \frac{\phi\overline{y}}{\phi y} = \frac{E(s)}{s} = \frac{p^a}{p^b_{+1}}$.

Similar to the case with exclusive circulation of outside money, here agents can default by selling endowments for outside money. The extra outside money obtained is used to purchase more type b goods. Each unit of hidden endowment can be converted into $\frac{p^{a,o}}{p^{b,o}_{+1}}$ units of next-period type b goods. Given prices, an agent optimally chooses $\gamma^* = 0$ (or $\gamma^{B*} = 0$) if $\frac{p^{a,o}}{p^{b,o}_{+1}} > \varepsilon$. Therefore, provided that $\varepsilon < E(s)$, we have $\gamma^* = \gamma^{B*} = 0$ for all equilibrium prices. This is exactly the same result as in the case with outside money only. Let v_0^{co*} denote the equilibrium initial promised value associated with co-circulation of one-period inside money and outside money. i.e. $\phi \in (0, 1)$. Hence the following proposition:

Proposition 5 $v_0^{co*} = v_0^{o*}$.

Proof of Proposition 5 is provided in the Appendix.

As a result, co-circulation of one-period inside money and outside money generates the same outcome as the sole circulation of outside money. The incorporation of inside money into the outside money system, $\phi \in (0, 1)$, has no impact on welfare at all. As long as outside money is valued, agents' incentives to default are just as high with or without inside money. The reason is that the profitability of carrying the misreported outside money to the succeeding period depends on the ratio of the prices of goods for outside money, $p_t^{a,o}/p_{t+1}^{b,o}$. With a constant outside money supply, the price ratio $p_t^{a,o}/p_{t+1}^{b,o}$ only depends on the ratio of aggregate market supplies of goods, $\overline{y}/y_t$. The parameter ϕ, however, only affects the relative value of outside money to inside money. Therefore, the incentives are as strong as ever unless outside money is not valued, $\phi = 0$.

6.1 Inflation and incentives

Thus far a constant money supply has been assumed. Now I relax this assumption and explore the effect of changes in the money supply on incentive compatibility and welfare. According to the previous results, the incentives to default are high when outside money is valued. Moreover, incorporation of inside money into the outside money system does not help weaken the incentives in any way. The value of carrying misreported outside money crucially depends on the ratio of the price of date-t type a goods for outside money relative to the price of date-$t+1$ type b goods for outside money, i.e. $p_t^{a,o}/p_{t+1}^{b,o}$. A change in outside money supply can affect $p_t^{a,o}/p_{t+1}^{b,o}$ and hence the equilibrium outcomes. In contrast, any change of the stock of inside money does not have an impact on $p_t^{a,o}/p_{t+1}^{b,o}$. Without loss of generality, the supply of inside money is still assumed to be constant and normalized to one.

Let M_t be the outside money supply at date t. Assume $M_t = (1+\pi) M_{t-1}$, where π is a constant. New money is injected as lump-sum transfers to type a agents at the beginning of $t \geq 1$. Now $\phi \in (0,1]$. Analogously, $c_t = \frac{m_t^I}{p_t^{b,I}} + \frac{m_t^o + T_t}{p_t^{b,o}}$ and $c_t^B = \frac{m_t^{B,I}}{p_t^{b,I}} + \frac{m_t^{B,o}+T_t}{p_t^{b,o}}$ for all t, where $T_t \in \mathbb{R}$ are the money transfers and taken as given by agents. Moreover, $m_t^I, m_t^{B,I} \in \mathbb{R}^+$ and $m_t^o, m_t^{B,o} \geq -T_t$. Note that now m_t^o and $m_t^{B,o}$ can be negative, which is interpreted as payments from a non-banker to a banker (m_t^o) or reallocation of money among bankers ($m_t^{B,o}$), right after the money transfer is received.

The equilibrium prices are $p_t^{a,o} = \frac{M_t}{\phi y_t}$ and $p_{t+1}^{b,o} = \frac{M_{t+1}}{\phi \overline{y}}$. Revisiting constraints (20)-(21), recall that given prices, an agent optimally chooses

$$\gamma_t^*, \gamma_t^{B*} \begin{cases} = 0, & \text{if } \frac{p_t^{a,o}}{p_{+1}^{b,o}} > \varepsilon \\ \in [0,1], & \text{if } \frac{p_t^{a,o}}{p_{+1}^{b,o}} = \varepsilon \\ = 1, & \text{otherwise} \end{cases} . \tag{22}$$

Let $v_0^{\inf *}$ denote the equilibrium initial promised value with inflation of outside money. Hence follows proposition:

Proposition 6 *If $\pi \leq E(s)/\varepsilon - 1$, $v_0^{\inf *}$ is constant in π and $v_0^{\inf *} = v_0^{co*} = v_0^{o*}$; if $\pi > E(s)/\varepsilon - 1$, $v_0^{\inf *}$ is strictly increasing in π. Also, $v_0^{\inf *} \to v_0^{I*}$ as $\pi \to +\infty$.*

Provided that $\varepsilon < E(s)$, Proposition 6 implies that a high enough positive inflation rate can correct incentives to some extent. As a result, outside money is getting less valuable as time goes on. If the aggregate endowment on island a is high, outside money is more costly to obtain. It is even more so considering that it will not be as valuable tomorrow as it is today. Therefore, for aggregate states above a certain threshold, i.e. $s_t > \frac{E(s)}{\varepsilon(1+\pi)}$, type a agents would choose to save endowments for next-period consumption if they were to default. Otherwise, they prefer to default by holding outside money across periods.

To sum up, with a positive inflation rate, from time to time type a agents may find it more profitable to default by saving endowments than carrying outside money across periods. In this case, agents get better disciplined as inflation goes higher.

7 Existence and uniqueness of equilibrium

Now it has been established that it is optimal for the bank to implement the contract with one-period inside money. This section studies the existence and uniqueness of a banking equilibrium. In the banking equilibrium, the bank makes allocations of money to finance a type a agent's consumptions according to $c_0 = \overline{y}$ and the optimal policy functions $c_t^*(y_{t-1}, s_{t-1}, v_t; \mu_t)$, $v_t^*(y_{t-1}, s_{t-1}, v_t; \mu_t)$, $c_t^{B*}(y_{t-1}, s_{t-1}; \mu_t)$ for all $t \geq 1$ that solve (2) subject to constraints (5)-(10) and (14)-(15). The aggregate measure α clears the market of contracts such that no one can offer a contract that achieves a higher initial value $v_0 > v_0^*$ that satisfies $W_0(v_0) = v_0$.

Proposition 7 *There exists a unique equilibrium initial value v_0^*.*

Proposition 7 shows that the banking equilibrium exists and is unique. Note that the equilibrium outcome is not the constrained efficient (i.e. second-best) outcome unless $\alpha^* = 0$ in the equilibrium. When $\alpha^* = 0$, the size of the bank is negligibly small. In this case, the bank's contract design problem is analogous to the efficiency problem addressed by Atkeson and Lucas (1992) and others, in which a planner endeavors to minimize the expected value of the total resources he allocates. The reason why the constrained efficiency is not necessarily achieved here is because in general the minimum resources needed to attain a given distribution of promised values may not exhaust all the resources available. In this model, there is no planner as the residual claimant. A utility-maximizing private banker can profit by retaining any positive residual. The competition in banking reaches equilibrium until the expected value of being a banker equals the expected value of a non-banker. The equilibrium outcome is not the second-best if the equilibrium measure of bankers is not negligible relative to that of non-bankers ($\alpha^* > 0$).

However, as established by Propositions 1-3, the main result of this paper is robust to any banking contract: one-period inside money can help the banking contract achieve better allocations for any α. That is to say, if the second-best allocations can be achieved in the banking equilibrium, it can only be achieved if the contract requires payments be made with one-period inside money.

8 Conclusion

This paper has developed an integrated theory of money, banking and dynamic contracts. The theory is used to evaluate inside money and outside money as alternative instruments for settling debts. The model has micro-founded roles for both banks and a medium of exchange. A banking sector arises endogenously and offers dynamic contracts to help agents smooth consumption over time. According to the contract, bankers lend money to agents at the beginning of a period and agents settle the current debt with bankers as they receive endowments at the end of the period. Each period, the amount of the loan entitlement of an agent depends on the individual's history of past settlements (that is, his history of reported endowments) and the sequence of prices at settlement stages.. The environment is characterized by a two-sided incentive problem. At the individual level, agents have private information about their random endowments. Contracts must be incentive compatible for individuals to report the true endowments. On the aggregate level, bankers have private information about the uncertain aggregate endowments. This incentive problem on the bank's side gives rise to a role for the market to generate information-revealing prices so that the bank cannot lie about the aggregate states.

I have shown that the optimal instrument for settlements is the kind of inside money that expires immediately after each settlement. With such one-period inside money, fewer resources are needed to reward truthful revelation and agents are better insured against idiosyncratic risks. Agents cannot benefit from holding one-period inside money across periods because it expires right after a settlement (which happens at the end of a period). As a result, the only profitable way for one to default is to save endowments for one's own consumption. However, when outside money is valued, an agent finds it more profitable to default by carrying outside money across periods than saving endowments. That is, the gain of default is higher with outside money than with one-period inside money. The same argument applies to inside money of longer durations. Longer-termed inside money functions similarly to outside money and involves higher incentives to misrepresent in periods when the current issue of money does not expire. Therefore, induction of truthful revelation is the least costly with one-period inside money, which helps the optimal dynamic contract implement better allocations. In equilibrium, more efficient risk-sharing is achieved and welfare is improved.

The key to the above result is the timing of the expiration of inside money, which is exactly when each settlement of current debts is done. Once an agent obtains such inside money for the settlement, making the payments to the bank is nothing but giving up some worthless objects. However, this is not true if outside money is required for settlements. Outside money will still be valuable to the agent after the settlement stage. Hence the incentives to default are much stronger with outside money. Not surprisingly, inflation of outside money can be used to correct incentives. With outside money getting less valuable as time goes on, induction of truthful revelation tends to get less costly.

Appendix

Proof of Propositions 1-2. Consider the contract problem (2) subject to constraints (5)-(10) and (14)-(15). Since u is strictly increasing in consumption, constraint (15) implies that

$$c_{+1}^{B}(y,s) > c_{+1}^{B}(\widetilde{y},s), \qquad \forall \ s, \ \forall \ \widetilde{y} < y. \tag{23}$$

Given μ_{+1}, let $\left\{\widehat{c}_{+1}(y,s,v_{+1};\mu_{+1}), \widehat{v}_{+2}(y,s,v_{+1};\mu_{+1}), \widehat{c}_{+1}^{B}(y,s;\mu_{+1})\right\}$ be the optimal policy functions for the banking contract with outside money. That is, they maximize the objective of (2) subject to constraints (5)-(12). Note that $\left\{\widehat{c}_{+1}, \widehat{v}_{+2}, \widehat{c}_{+1}^{B}\right\}$ also satisfy constraints (14)-(15). That is,

$$\begin{aligned} & u\left[\widehat{c}_{+1}(y,s,v_{+1})\right] + \beta\widehat{v}_{+2}(y,s,v_{+1}) \\ \geq \ & u\left[\widehat{c}_{+1}(\widetilde{y},s,v_{+1}) + (y-\widetilde{y})\frac{p^{a}(s)}{p_{+1}^{b}}\right] + \beta\widehat{v}_{+2}(\widetilde{y},s,v_{+1}) \\ > \ & u\left[\widehat{c}_{+1}(\widetilde{y},s,v_{+1}) + s(y-\widetilde{y})\right] + \beta\widehat{v}_{+2}(\widetilde{y},s,v_{+1}), \qquad \forall \ s, v_{+1}, \ \forall \ \widetilde{y} < y \end{aligned} \tag{24}$$

$$\begin{aligned} u\left[\widehat{c}_{+1}^{B}(y,s)\right] \ \geq \ & u\left[\widehat{c}_{+1}^{B}(\widetilde{y},s) + (y-\widetilde{y})\frac{p^{a}(s)}{p_{+1}^{b}}\right] \\ > \ & u\left[\widehat{c}_{+1}^{B}(\widetilde{y},s) + s(y-\widetilde{y})\right], \qquad \forall \ s, \ \forall \ \widetilde{y} < y \end{aligned} \tag{25}$$

The above two strict inequalities hold because $\varepsilon < E(s)$ and $s \in [0,1]$.

Now construct the following policy function such that (15) holds:

$$\widetilde{c}_{+1}^{B}(y,s;\mu_{+1}) = \begin{cases} \widehat{c}_{+1}^{B}(y,s;\mu_{+1}) + \delta_{y}, & \text{if} \ \ y \leq \frac{1}{2}s \\ \widehat{c}_{+1}^{B}(y,s;\mu_{+1}) - \Delta_{y}, & \text{if} \ \ y > \frac{1}{2}s \end{cases}, \tag{26}$$

where δ_y and Δ_y are infinitely small positive numbers and satisfy $\delta_y g\left(\frac{y}{s}\right) = \Delta_{s-y} g\left(1-\frac{y}{s}\right)$ for $\frac{y}{s} \leq \frac{1}{2}$ and all s. Values of $\delta_{s\theta}$ and $\Delta_{s\theta}$ exist by the strict inequality of (25). In what follows, it will be proven that $\left\{\widehat{c}_{+1}(y,s,v_{+1};\mu_{+1}), \widehat{v}_{+2}(y,s,v_{+1};\mu_{+1}), \widetilde{c}_{+1}^{B}(y,s;\mu_{+1})\right\}$ achieve a higher value of $W_{+1}^{I}(\mu_{+1})$ than $\left\{\widehat{c}_{+1}(y,s,v_{+1};\mu_{+1}), \widehat{v}_{+2}(y,s,v_{+1};\mu_{+1}), \widehat{c}_{+1}^{B}(y,s;\mu_{+1})\right\}$

do. First note that given s

$$
\begin{aligned}
&\alpha \int_0^1 \widetilde{c}_{+1}^{B}\left(y,s;\mu_{+1}\right) g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right) \\
= \; &\alpha \int_0^{1/2} \left[\widehat{c}_{+1}^{B}\left(y,s;\mu_{+1}\right)+\delta_y\right] g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right) + \alpha \int_{1/2}^1 \left[\widehat{c}_{+1}^{B}\left(y,s;\mu_{+1}\right)-\Delta_y\right] g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right) \\
= \; &\alpha \int_0^1 \widehat{c}_{+1}^{B}\left(y,s;\mu_{+1}\right) g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right) + \alpha \int_0^{1/2} \delta_y g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right) - \alpha \int_0^{1/2} \Delta_{s-y} g\left(\frac{s-y}{s}\right) d\left(\frac{y}{s}\right) \\
= \; &\alpha \int_0^1 \widehat{c}_{+1}^{B}\left(y,s;\mu_{+1}\right) g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right) + \alpha \int_0^{1/2} \delta_y g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right) - \alpha \int_0^{1/2} \delta_y g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right) \\
= \; &\alpha \int_0^1 \widehat{c}_{+1}^{B}\left(y,s;\mu_{+1}\right) g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right)
\end{aligned}
$$

Therefore, $\widetilde{c}_{+1}^{B}\left(y,s;\mu_{+1}\right)$ and $\widehat{c}_{+1}\left(y,s,v_{+1};\mu_{+1}\right)$ satisfy constraint (7). Rewrite the objective of (2) as the following:

$$
W_{+1}\left(\mu_{+1}\right) = \max_{\left(c_{+1}^{B},c_{+1},v_{+2}\right)} \int_0^1 \int_0^1 u\left[c_{+1}^{B}\left(y,s\right)\right] g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right) f\left(s\right) ds + \beta \int_0^1 W_{+2}\left(\mu_{+2}\right) f\left(s\right) ds \tag{27}
$$

Apply the first-order Taylor expansion on the first term of the above with $\widetilde{c}_{+1}^{B}\left(y,s;\mu_{+1}\right)$:

$$
\begin{aligned}
&\int_0^1 \int_0^1 u\left[\widetilde{c}_{+1}^{B}\left(y,s;\mu_{+1}\right)\right] g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right) f\left(s\right) ds \\
= \; &\int_0^1 \left\{ \begin{array}{c} \int_0^{1/2} u\left[\widehat{c}_{+1}^{B}\left(y,s;\mu_{+1}\right)+\delta_y\right] g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right) \\ +\int_{1/2}^1 u\left[\widehat{c}_{+1}^{B}\left(y,s;\mu_{+1}\right)-\Delta_y\right] g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right) \end{array} \right\} f\left(s\right) ds \\
= \; &\int_0^1 \left\{ \begin{array}{c} \int_0^{1/2} \left\{u\left[\widehat{c}_{+1}^{B}\left(y,s;\mu_{+1}\right)\right] + u'\left[\widehat{c}_{+1}^{B}\left(y,s;\mu_{+1}\right)\right]\delta_y\right\} g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right) \\ +\int_{1/2}^1 \left\{u\left[\widehat{c}_{+1}^{B}\left(y,s;\mu_{+1}\right)\right] - u'\left[\widehat{c}_{+1}^{B}\left(y,s;\mu_{+1}\right)\right]\Delta_y\right\} g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right) \end{array} \right\} f\left(s\right) ds \\
= \; &\int_0^1 \left\{ \begin{array}{c} \int_0^1 u\left[\widehat{c}_{+1}^{B}\left(y,s;\mu_{+1}\right)\right] g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right) \\ +\int_0^{1/2} u'\left[\widehat{c}_{+1}^{B}\left(y,s;\mu_{+1}\right)\right] \delta_y g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right) \\ -\int_0^{1/2} u'\left[\widehat{c}_{+1}^{B}\left(s-y,s;\mu_{+1}\right)\right] \Delta_{s-y} g\left(\frac{s-y}{s}\right) d\left(\frac{y}{s}\right) \end{array} \right\} f\left(s\right) ds
\end{aligned}
$$

$$= \int_0^1 \left\{ \begin{array}{c} \int_0^1 u\left[\widehat{c}_{+1}^B\left(y,s;\mu_{+1}\right)\right] g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right) \\ +\int_0^{1/2} \left\{u'\left[\widehat{c}_{+1}^B\left(y,s;\mu_{+1}\right)\right] - u'\left[\widehat{c}_{+1}^B\left(s-y,s;\mu_{+1}\right)\right]\right\} \delta_y g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right) \end{array} \right\} f(s)\,ds$$

$$> \int_0^1 \int_0^1 u\left[\widehat{c}_{+1}^B\left(y,s;\mu_{+1}\right)\right] g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right) f(s)\,ds \tag{28}$$

The strict inequality holds because (23) implies that $u'\left[\widehat{c}_{+1}^B\left(y,s;\mu_{+1}\right)\right] - u'\left[\widehat{c}_{+1}^B\left(s-y,s;\mu_{+1}\right)\right] \geq 0$ for all $\frac{y}{s} \in \left[0,\frac{1}{2}\right]$, with an equality if and only if $\frac{y}{s} = \frac{1}{2}$. As has been established, $\left\{\widehat{c}_{+1},\widehat{v}_{+2},\widetilde{c}_{+1}^B\right\}$ satisfy constraints (5)-(10) and (14)-(15). Note that $W_{+2}\left(\mu_{+2}\right)$ takes the same value for $\left\{\widehat{c}_{+1},\widehat{v}_{+2},\widetilde{c}_{+1}^B\right\}$ and $\left\{\widehat{c}_{+1},\widehat{v}_{+2},\widehat{c}_{+1}^B\right\}$ because of the same policy function $\widehat{v}_{+2}$. Thus the second term in (27) also takes the same value for $\left\{\widehat{c}_{+1},\widehat{v}_{+2},\widetilde{c}_{+1}^B\right\}$ and $\left\{\widehat{c}_{+1},\widehat{v}_{+2},\widehat{c}_{+1}^B\right\}$. The strict inequality in (28) shows that $\left\{\widehat{c}_{+1},\widehat{v}_{+2},\widetilde{c}_{+1}^B\right\}$ achieve a higher value of $W_{+1}^I\left(\mu_{+1}\right)$ than $\left\{\widehat{c}_{+1},\widehat{v}_{+2},\widehat{c}_{+1}^B\right\}$ do. Therefore, $\left\{\widehat{c}_{+1},\widehat{v}_{+2},\widehat{c}_{+1}^B\right\}$ cannot be the optimal policy functions for the contract problem of banking with one-period inside money. It follows that $W^I(\mu) > W^o(\mu)$ for any given μ. By (13), $W_0^I > W_0^o$ for any given v_0 and α.
■

Proof of Proposition 3. Given α, consider two optimal contracts with associated initial values and consumption streams of $\left\{\widetilde{v}_0, \{\widetilde{c}_t\}_{t=0}^{\infty}, \widetilde{W}_0, \left\{\widetilde{c}_t^B\right\}_{t=0}^{\infty}\right\}$ and $\left\{\widehat{v}_0, \{\widehat{c}_t\}_{t=0}^{\infty}, \widehat{W}_0, \left\{\widehat{c}_t^B\right\}_{t=0}^{\infty}\right\}$, respectively. Suppose $\widetilde{W}_0 \geq \widehat{W}_0$ for any $\widetilde{v}_0 > \widehat{v}_0$. This means the optimal contract $\left\{\widetilde{v}_0, \{\widetilde{c}_t\}_{t=0}^{\infty}, \widetilde{W}_0, \left\{\widetilde{c}_t^B\right\}_{t=0}^{\infty}\right\}$ achieves higher life-time utilities for both bankers and non-bankers than $\left\{\widehat{v}_0, \{\widehat{c}_t\}_{t=0}^{\infty}, \widehat{W}_0, \left\{\widehat{c}_t^B\right\}_{t=0}^{\infty}\right\}$ does. Hence the latter cannot be an optimal contract given α, which is a contradiction. Therefore, it must be true that W_0 is strictly decreasing in v_0 given α.

Given α, let v_0^o and v_0^I be the solutions to (13), respectively for the outside money arrangement and the one-period inside money arrangement. By Proposition 2, we have $v_0^o = W_0^o(v_0^o;\alpha) < W_0^I(v_0^o;\alpha)$. Obviously, $v_0^o \neq v_0^I$. Suppose $v_0^o > v_0^I$, then it follows that $W_0^o(v_0^o;\alpha) = v_0^o > v_0^I = W_0^I\left(v_0^I;\alpha\right)$. This is a contradiction because W_0 is strictly decreasing in v_0. Thus, $v_0^o < v_0^I$ all given α. Then it must be that $v_0^{o*} < v_0^{I*}$ because

$v_0^* = \max_{\alpha \in [0,1]} v_0(\alpha)$.

With outside money, the contract design problem is given by (2) subject to constraints (5)-(12). It is obvious that ε does not enter into the problem at all. Therefore, v_0^{o*} is independent of ε. With OPIM, ε only enters into the incentive compatibility constraints. Consider any $\varepsilon_1 < \varepsilon_2$. The values of the right-hand sides of the IC constraints are smaller with ε_1 than with ε_2. Analogous to the proof of Proposition 1, one can construct alternative policy functions that achieve a higher value for the problem with ε_1 than the optimal policy functions for the problem with ε_2. (Details are omitted for brevity.) Then it follows that v_0^{I*} strictly increases as ε decreases. When $\varepsilon \to 0$, the incentives to default diminish and the optimal contract approaches the unconstrained first-best contract. That is, $c_t(y_{t-1}, s_{t-1}, v_t) \to \overline{y}$, $c_t^B(y_{t-1}, s_{t-1}) \to \overline{y}$ and $v_{t+1}(y_{t-1}, s_{t-1}, v_t) \to \frac{u(\overline{y})}{1-\beta}$ for all (y_{t-1}, s_{t-1}, v_t), which concludes the proof. ■

Proof of Proposition 4. It is straightforward that $W_{t+1}^I(\mu_{t+1})$ is equivalent to $W_{t+1}^\kappa(\mu_{t+1}, \iota_t)$ with $\iota_t = 1$ for all t. Given μ_{+1}, let $\left\{\overline{c}_{+1}(y, s, v_{+1}; \mu_{+1}, \iota), \overline{v}_{+2}(y, s, v_{+1}; \mu_{+1}, \iota), \overline{c}_{+1}^B(y, s; \mu_{+1}, \iota)\right\}$ be the optimal policy functions for the banking contract with κ-period inside money. As the case with outside money, we have $\gamma^* = 0$ for the problem given by (19) and hence $\Phi_2 > \Phi_1$. It follows that when $\iota = 0$,

$$\begin{aligned} u[\overline{c}_{+1}(y, s, v_{+1})] + \beta \overline{v}_{+2}(y, s, v_{+1}) &\geq u[\overline{c}_{+1}(\widetilde{y}, s, v_{+1}) + \Phi_2] + \beta \overline{v}_{+2}(\widetilde{y}, s, v_{+1}) \\ &> u[\overline{c}_{+1}(\widetilde{y}, s, v_{+1}) + \Phi_1] + \beta \overline{v}_{+2}(\widetilde{y}, s, v_{+1}) \\ &\qquad \forall \; s, v_{+1}, \;\; \forall \; \widetilde{y} < y \end{aligned}$$

$$\begin{aligned} u\left[\overline{c}_{+1}^B(y, s)\right] &\geq u\left[\overline{c}_{+1}^B(\widetilde{y}, s) + \Phi_2\right] \\ &> u\left[\overline{c}_{+1}^B(\widetilde{y}, s) + \Phi_1\right], \qquad \forall \; s, \;\; \forall \; \widetilde{y} < y \end{aligned}$$

Therefore, $\left\{\overline{c}_{+1}, \overline{v}_{+2}, \overline{c}_{+1}^B\right\}$ satisfy all constraints (5)-(10) and (14)-(15). It follows that

$W^I_{t+1}(\mu_{t+1}) \geq W^\kappa_{t+1}(\mu_{t+1}, \iota_t)$ for any given (μ_{t+1}, ι_t). Analogous to the construction in (26), one can find other policy functions that achieve a higher value for $W^I_{t+1}(\mu_{t+1})$ than $\{\bar{c}_{+1}, \bar{v}_{+2}, \bar{c}^B_{+1}\}$ do, which implies $W^I_{t+1}(\mu_{t+1}) > W^\kappa_{t+1}(\mu_{t+1}, \iota_t)$ for any given (μ_{t+1}, ι_t). (Details are omitted for brevity.) This in turn implies that $W^I_0 > W^\kappa_0$ for any given v_0 and α. Analogous to the proof of Proposition 3, one can show that $v^{\kappa *}_0 < v^{I*}_0$. By the same token, $W^o_{t+1}(\mu_{t+1})$ is equivalent to $W^\kappa_{t+1}(\mu_{t+1}, \iota_t)$ where $\iota_t = 0$ for all t. Similarly, one can prove that $W^o_{t+1}(\mu_{t+1}) < W^\kappa_{t+1}(\mu_{t+1}, \iota_t)$ for any given (μ_{t+1}, ι_t) and hence $W^o_0 \leq W^\kappa_0$ for any given v_0 and α. It follows that $v^{o*}_0 < v^{\kappa *}_0$. ■

Proof of Proposition 5. Since constraints (3)-(4) are equivalent to constraints (20)-(21), the contract problem under co-circulation of inside money and outside money is exactly the same as under exclusive circulation of outside money. Hence $v^{co*}_0 = v^{o*}_0$. ■

Proof of Proposition 6. Plugging in the equilibrium prices, (22) becomes

$$\gamma^*_t, \gamma^{B*}_t \begin{cases} = 0, & \text{if } \pi < \frac{E(s)}{\varepsilon s_t} - 1 \\ \in [0,1], & \text{if } \pi = \frac{E(s)}{\varepsilon s_t} - 1 \\ = 1, & \text{if } \pi > \frac{E(s)}{\varepsilon s_t} - 1 \end{cases}.$$

It is straightforward to show that $\gamma^*, \gamma^{B*} = 0$ for any $s_t \in [0,1]$ if $\pi \leq E(s)/\varepsilon - 1$. The contract design problem is the same as given by (2) subject to the same constraints as (5)-(10) and (20)-(21). Hence $v^{\inf *}_0$ is constant in π and $v^{\inf *}_0 = v^{co*}_0 = v^{o*}_0$ by Proposition 5.

Provided that $\pi > E(s)/\varepsilon - 1$, then $\gamma^*_t, \gamma^{B*}_t = 1$ if $s_t > E(s)/[\varepsilon(1+\pi)]$ and $\gamma^*_t, \gamma^{B*}_t = 0$ if $s_t \leq E(s)/[\varepsilon(1+\pi)]$ for given s_t. Thus for high enough aggregate state, it becomes less profitable to default and carry outside money into the future. Now the recursive contract design problem is given by (2) subject to the same constraints as (5)-(10). Nevertheless, instead of constraints (3) and (4), the IC constraints become the following: for

$s > E(s) / [\varepsilon(1+\pi)]$,

$$u[c_{+1}(y,s,v_{+1})] + \beta v_{+2}(y,s,v_{+1}) \geq u[c_{+1}(\widetilde{y},s,v_{+1}) + \Psi_1] + \beta v_{+2}(\widetilde{y},s,v_{+1}) \tag{29}$$
$$\forall \;\; s, v_{+1}, \;\; \forall \;\; \widetilde{y} < y$$
$$u\left[c^B_{+1}(y,s)\right] \geq u\left[c^B_{+1}(\widetilde{y},s) + \Psi_1\right], \;\; \forall \;\; s, \;\; \forall \;\; \widetilde{y} < y \tag{30}$$

and for $s \leq E(s) / [\varepsilon(1+\pi)]$,

$$u[c_{+1}(y,s,v_{+1})] + \beta v_{+2}(y,s,v_{+1}) \geq u[c_{+1}(\widetilde{y},s,v_{+1}) + \Psi_2] + \beta v_{+2}(\widetilde{y},s,v_{+1}) \tag{31}$$
$$\forall \;\; s, v_{+1}, \;\; \forall \;\; \widetilde{y} < y$$
$$u\left[c^B_{+1}(y,s)\right] \geq u\left[c^B_{+1}(\widetilde{y},s) + \Psi_2\right], \;\; \forall \;\; s, \;\; \forall \;\; \widetilde{y} < y \tag{32}$$

where $\Psi_1 = \varepsilon(y - \widetilde{y})$ and $\Psi_2 = (y - \widetilde{y}) \frac{p_t^{a,o}}{p_{+1}^{b,o}}$. Let $\overline{s} = E(s) / [\varepsilon(1+\pi)]$. Then the objective of (2) can be rewritten as:

$$W^{\inf}_{+1}(\mu_{+1}) = \max_{\left(c^B_{+1}, c_{+1}, v_{+2}\right)} \left\{ \begin{array}{c} \int_0^{\overline{s}} \int_0^1 \left\{u\left[c^B_{+1}(y,s)\right] + \beta W^{\inf}_{+2}(\mu_{+2})\right\} g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right) f(s)\, ds \\ + \int_{\overline{s}}^1 \int_0^1 \left\{u\left[c^B_{+1}(y,s)\right] + \beta W^{\inf}_{+2}(\mu_{+2})\right\} g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right) f(s)\, ds \end{array} \right\}.$$

The above problem can be further decomposed into:

$$W^{\inf}_{+1}(\mu_{+1}) = W^{\inf}_{+1}(\mu_{+1}; \Psi_1) + W^{\inf}_{+1}(\mu_{+1}; \Psi_2) \tag{33}$$

where

$$W^{\inf}_{+1}(\mu_{+1}; \Psi_1) = \max_{\left(c^B_{+1}, c_{+1}, v_{+2}\right)} \int_{\overline{s}}^1 \int_0^1 \left\{u\left[c^B_{+1}(y,s)\right] + \beta W^{\inf}_{+2}(\mu_{+2})\right\} g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right) f(s)\, ds \tag{34}$$
$$s.t. \;\; (5)\text{-}(10) \text{ and } (29)\text{-}(30)$$

and

$$W_{+1}^{\inf}\left(\mu_{+1};\Psi_2\right) = \max_{\left(c_{+1}^B,c_{+1},v_{+2}\right)} \int_0^{\overline{s}}\int_0^1 \left\{u\left[c_{+1}^B\left(y,s\right)\right]+\beta W_{+2}^{\inf}\left(\mu_{+2}\right)\right\} g\left(\frac{y}{s}\right) d\left(\frac{y}{s}\right) f\left(s\right) ds \quad (35)$$
$$s.t.\ \ (5)\text{-}(10) \text{ and } (31)\text{-}(32)$$

Note that the problem in (34) is equivalent to the problem for the exclusive circulation of one-period inside money, except that the lower bound for s is now $\overline{s}$ instead of zero. Similarly, the problem of (35) is equivalent to the problem for circulation of outside money with a constant money supply, with the upper bound of s being $\overline{s}$ instead of one. It is obvious that $W_{+1}^{\inf}\left(\mu_{+1}\right)\to W_{+1}^{I}\left(\mu_{+1}\right)$ as $\overline{s}\to 0$ and $W_{+1}^{\inf}\left(\mu_{+1}\right)\to W_{+1}^{o}\left(\mu_{+1}\right)=W_{+1}^{co}\left(\mu_{+1}\right)$ as $\overline{s}\to 1$ given any μ_{+1}.

Consider any $s_1<s_2$. Given μ_{+1}, let $\left\{\widehat{c}_{+1}\left(y,s,v_{+1};\mu_{+1}\right),\widehat{v}_{+2}\left(y,s,v_{+1};\mu_{+1}\right),\widehat{c}_{+1}^B\left(y,s;\mu_{+1}\right)\right\}$ be the optimal policy functions for (33) given $\overline{s}=s_2$. It is straightforward to see that $\left\{\widehat{c}_{+1},\widehat{v}_{+2},\widehat{c}_{+1}^B\right\}$ also satisfy all the constraints for (33) given $\overline{s}=s_1$. Now construct the following policy function $\widetilde{c}_{+1}^B\left(y,s;\mu_{+1}\right)$ such that (30) and (32) hold:

$$\widetilde{c}_{+1}^B\left(y,s;\mu_{+1}\right)=\begin{cases}\widehat{c}_{+1}^B\left(y,s;\mu_{+1}\right) \text{ for } y\in[0,s], & \text{if } s\in[0,s_1)\cup(s_2,1]\\ \begin{cases}\widehat{c}_{+1}^B\left(y,s;\mu_{+1}\right)+\delta_y \text{ for } y\in\left[0,\frac{1}{2}s\right]\\ \widehat{c}_{+1}^B\left(y,s;\mu_{+1}\right)-\Delta_y \text{ for } y\in\left(\frac{1}{2}s,s\right]\end{cases}, & \text{if } s\in[s_1,s_2]\end{cases}$$

where δ_y and Δ_y are infinitely small positive numbers and satisfy $\delta_y g\left(\frac{y}{s}\right)=\Delta_{s-y} g\left(1-\frac{y}{s}\right)$ for $y\in\left[0,\frac{1}{2}s\right]$ and $s\in[s_1,s_2]$. Values of δ_y and Δ_y exist by the strict inequality of (30) given $\widehat{c}_{+1}^B\left(y,s;\mu_{+1}\right)$ for $s\in[s_1,s_2]$. Analogous to the proof of Proposition 1, it can be shown that given $\overline{s}=s_1$, $\left\{\widehat{c}_{+1}\left(y,s,v_{+1};\mu_{+1}\right),\widehat{v}_{+2}\left(y,s,v_{+1};\mu_{+1}\right),\widetilde{c}_{+1}^B\left(y,s;\mu_{+1}\right)\right\}$ achieve a higher value of $W_{+1}^{\inf}\left(\mu_{+1}\right)$ than $\left\{\widehat{c}_{+1}\left(y,s,v_{+1};\mu_{+1}\right),\widehat{v}_{+2}\left(y,s,v_{+1};\mu_{+1}\right),\widehat{c}_{+1}^B\left(y,s;\mu_{+1}\right)\right\}$ do. This implies that $W_{+1}^{\inf}\left(\mu_{+1}\right)\left(\overline{s}\right)$ is strictly decreasing in $\overline{s}$ for any given μ_{+1}. Hence by the same argument of the proof of Proposition 4, $v_0^{\inf *}$ is strictly decreasing in $\overline{s}$. Note that $\overline{s}$ is strictly decreasing in π. Thus $v_0^{\inf *}$ is strictly increasing in π. ■

Proof of Proposition 7. Consider the following policy functions: $\underline{c}_{+1}\left(y,s,v_{+1};\mu_{+1}\right)=\underline{c}_{+1}^{B}\left(y,s;\mu_{+1}\right)$ for any given v_{+1} and μ_{+1}. Trivially, $\underline{v}_{+2}\left(y,s,v_{+1};\mu_{+1}\right)=\frac{1}{1-\beta}\underset{y,s}{E}\left\{u\left[\underline{c}_{+1}^{B}\left(y,s;\mu_{+1}\right)\right]\right\}$. Given $\underline{c}_{+1}$ and $\underline{v}_{+1}$, $\underline{c}_{+1}^{B}$ solves the following maximization problem:

$$W_{+1}=\max_{c_{+1}^{B}}\left\{\frac{1}{1-\beta}\int_{0}^{1}\int_{0}^{1}u\left[c_{+1}^{B}\left(y,s\right)\right]g\left(\frac{y}{s}\right)d\left(\frac{y}{s}\right)f\left(s\right)ds\right\} \tag{36}$$

subject to

$$u\left[c_{+1}^{B}\left(y,s\right)\right]\geq u\left[c_{+1}^{B}\left(\widetilde{y},s\right)+\varepsilon\left(y-\widetilde{y}\right)\right],\quad\forall\ s,\ \ \forall\ \widetilde{y}<y \tag{37}$$

$$\int_{0}^{1}c_{+1}^{B}\left(y,s\right)g\left(\frac{y}{s}\right)d\left(\frac{y}{s}\right)=\overline{y},\quad\forall\ s \tag{38}$$

Since $y=s\theta$, the above problem can be rewritten as

$$W_{+1}=\max_{c_{+1}^{B}}\left\{\frac{1}{1-\beta}\int_{0}^{1}\int_{0}^{1}u\left[c_{+1}^{B}\left(\theta,s\right)\right]g\left(\theta\right)d\left(\theta\right)f\left(s\right)ds\right\} \tag{39}$$

subject to

$$u\left[c_{+1}^{B}\left(\theta,s\right)\right]\geq u\left[c_{+1}^{B}\left(\widetilde{\theta},s\right)+\varepsilon s\left(\theta-\widetilde{\theta}\right)\right],\quad\forall\ s,\ \ \forall\ \widetilde{\theta}<\theta \tag{40}$$

$$\int_{0}^{1}c_{+1}^{B}\left(\theta,s\right)g\left(\theta\right)d\left(\theta\right)=\overline{y},\quad\forall\ s \tag{41}$$

It is straightforward to show that the solution to this problem $\underline{c}_{+1}^{B}$ exists and is unique. The policy functions $\left\{\underline{c}_{+1},\underline{v}_{+2},\underline{c}_{+1}^{B}\right\}$ imply that $\underline{v}_{0}=\underline{W}_{0}\left(\underline{v}_{0}\right)$. This is true for any $\alpha\in[0,1]$.

By definition, the policy function $\underline{c}_{+1}^{B}$ is optimal given $\left\{\underline{c}_{+1},\underline{v}_{+2}\right\}$ and hence $\underline{v}_{0}$. But $\left\{\underline{c}_{+1},\underline{v}_{+2}\right\}$ may not be the optimal policy functions to achieve $\underline{v}_{0}$. If they are optimal, then it is trivial that the banking equilibrium exists and is unique. Suppose they are not optimal. Let $\{\underline{c}_{t}\}_{t=0}^{\infty}$ be the sequence of consumptions achieved by $c_{0}=\overline{y}$ and policy functions $\left\{\underline{c}_{+1},\underline{v}_{+2}\right\}$ for all $t\geq 0$. Since the goal of the bank is to maximize the life-time expected utility of a banker, it chooses functions c_{+1} and v_{+2} to minimize the expected value of the total resources it allocates to the non-bankers for any promised value v_{0}. Since $\left\{\underline{c}_{+1},\underline{v}_{+2}\right\}$ are not optimal by assumption, there must be a less costly sequence of allocations other than $\{\underline{c}_t\}_{t=0}^{\infty}$ that achieves $\underline{v}_0$. Put it another way, there must be allocations that achieves a higher value than $\underline{W}_0$ for a representative banker while delivering the promised $\underline{v}_0$. Formally, it must be true that $W_0(\underline{v}_0) > \underline{W}_0$ and $W_0\left(v_0^{'}\right) = \underline{W}_0$ for some $v_0^{'} > \underline{v}_0$. This holds for any α. Let $\varphi(v_0) = W_0(v_0)$. The Theorem of the Maximum delivers φ as a continuous function on $\left[\underline{v}_0, \overline{V}\right]$, where $\overline{V} = \frac{u(\overline{y})}{1-\beta}$ is the value achieved by the first-best contract. Recall from the proof of Proposition 3 that given α, the function $W_0(v_0)$ is strictly decreasing in v_0. Therefore, it must be true that there exists a unique $v_0 \in \left[\underline{v}_0, \overline{V}\right]$ that satisfies $W_0(v_0) = v_0$ for any given α. The uniqueness of the equilibrium value v_0^* follows because $v_0^* = \max\limits_{\alpha\in[0,1]} v_0(\alpha)$. ■

Figure 1 Monetary Trades

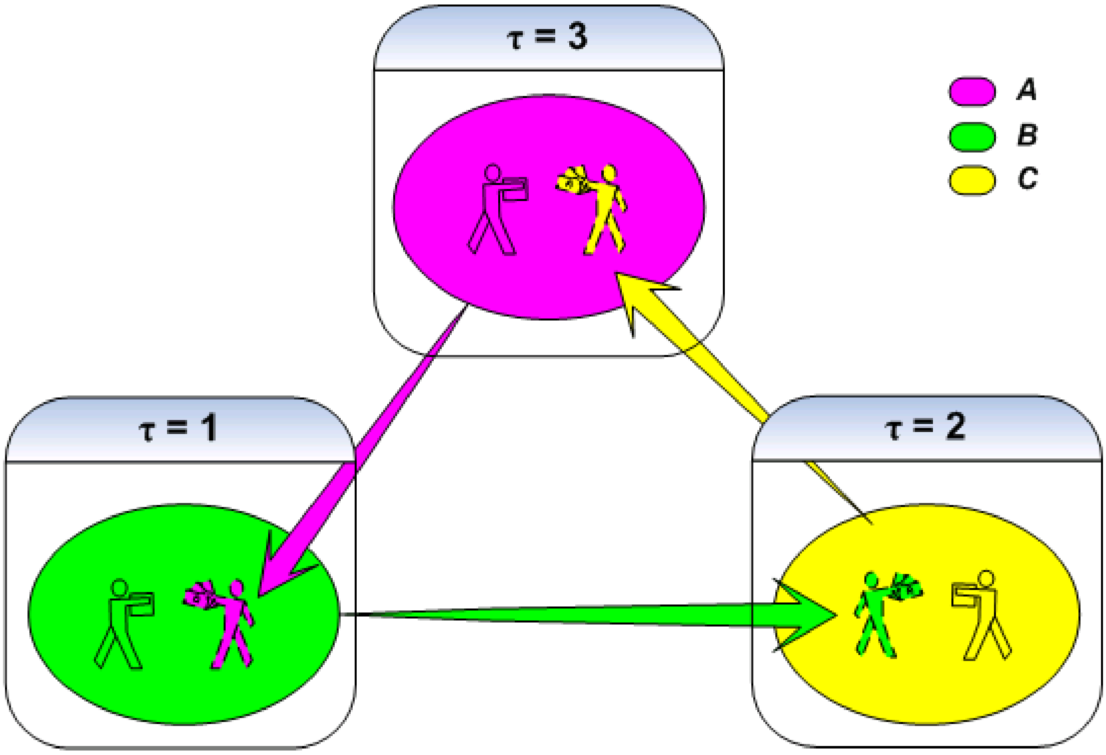
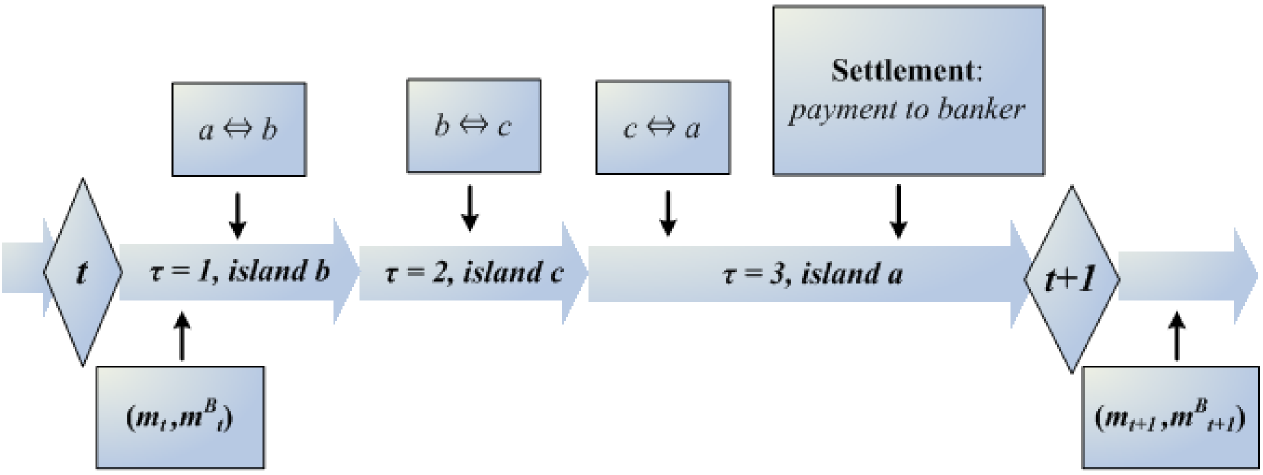


Figure 2 Timing of Events



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