Heterogeneous product and process innovations for a multi-product monopolist under finite life-cycles of production technologies.

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Abstract

Current paper analyses the influence of the length of production technologies life-cycles on the investments of a multi-product monopolist into different types of innovations. This monopolist is developing new versions of the basic product continuously and simultaneously invests into the production technologies of all these new products. In the paper the finite character of these products' life-cycles is assumed. With the assumption of heterogeneous products the influence of the length of life-cycles on generation of innovations is ambiguous. The level of innovations of both types is maximized for infinite life-cycles, but the range of existing products as well as the total mass of process innovations for these existing products may depend negatively on the length of the life-cycle for mature stage of industry development.
Keywords: Heterogeneous Innovations, Economic Dynamics, Multi-product Monopoly, Product Life-cycle, Distributed Control
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1 Introduction

The main focus of this paper is on the optimal behaviour of a multi-product monopolist firm in the field of its innovative policy in the presence of time constraints on its activities. Namely, what are the optimal relations between investments into the generation of innovations of different types if such an innovative activity may be carried out only during some limited time.

The multi-product monopolist is modelled as a single planning agent in the industry (market). He/she decides upon investments into the development of new products (variety expansion process in the sense of (Grossman and Helpman 1994)) and into the improvement of production technology or quality for every of already introduced products (quality improvement in the sense of (Aghion and Howitt 1992)). The process of variety expansion is described as continuous in time thus yielding infinitely many new potential products. The range of such products is a continuum. The special feature of the framework is that every such a product possesses its own quality characteristic and the quality-improving process is described on two levels: as an aggregate process, which depends on variety expansion (product innovations) and as a collection of separate quality processes for each of the products, which does not depend on variety expansion but in turn influences this last.

Such a description of multi-product innovations follows the ideas of (Lambertini and Orsini 2001), (Lin 2004) and (Belyakov, Tsachev, and Veliov 2011) where the single-agent dynamic optimization problem is treated.

In the current paper the infinite planning horizon is adopted. This allows for modelling long-run behaviour of the agent and for explicit solution for the dynamics of both types of innovations. Current paper differs from these by one other additional assumption. Namely, the improvement of quality for every new product is limited by a finite life-cycle of the product (length of which is identical across products). This finite life-cycle is considered as to describe the effect of creative destruction in the sense of (Schumpeter 1942): upon the invention of the product, the monopolist is granted the exclusive right to develop and sell this product during some fixed time, τ. After this time passes, the product development becomes common knowledge and no further economic profit may be derived from it. This is equivalent to the disappearance of the product from the optimization problem of the innovator. In the paper the question of how the investments of a monopolist would be distributed across development of qualities of different products and variety enhancing innovations is studied. It turns out, that the longer is the life-cycle (patent) of each of these new products, the more incentives an agent has to invest both into quality improvements for all the products and into the development and introduction of new products. Thus the infinite-time life-cycle is the optimal case for the innovator which is not very surprising.

When one considers the effective range of products in existence at each point in time as well as the total mass of ongoing quality innovations, it turns out, that this mass ambiguously depends on the length of the life-cycle. Namely, one observes two effects of opposite directions of the length of the life-cycle on innovations intensity, referred to as potential profit effect and compensation effect. Depending on the stage of development of the industry one or the other effect dominates. It is shown that at mature stages of industry development, when more than a half of all potential range of products is already covered by product innovations, the increase in the length of the life-cycle actually decreases the effective range of products and the total mass of quality innovations. Thus limited life-cycles stimulate the intensity of innovations of both types for mature industries.

To stimulate the introduction of new products at this later stage of development the life-cycle should be limited. Such an argument replicates the argument of Nordhaus in his seminal paper (**?**) but from completely different grounds: life-cycles should be limited because of the interplay between quality-improving and variety-enhancing innovations and not to stimulate innovations from other agents as in this classic paper.

Such a dynamics depends crucially on the heterogeneity of new products. In particular, it is necessary for investment efficiency into new products technologies to be decreasing in the index of the product for compensation effect (negative influence of the life-cycle length) to appear.

Main findings of the paper may be summarized as following:

- The level of product and process innovations is increasing in life-cycles length.
- For heterogeneous products with decreasing efficiency of investments across products two opposite effects of the life-cycles length onto the intensity of innovations of opposite direction are present which are labelled potential profit effect and compensation effect.
- The effective range of products in existence grows with life-cycle increase at initial stage of development but decreases at mature stages, thus decreasing intensity of introduction of new products.
- Process innovations for any given product depend positively on the products life-cycle length, but the aggregate process for the whole range of existing products follows the same pattern as product innovations. Namely, the density of process innovations increase with

life-cycle length at the initial stage, but decreases at mature stage of development of the industry.

- With infinite-time life-cycles the total mass of product and process innovations is maximized, but the intensity of innovations declines with longer life-cycles at mature stages of industry development.

The rest of this paper is organised as following: in the next section a brief review of the state of the arts is made. Then the basic framework of heterogeneous innovations without life-cycles is described. After that the model is extended to allow for limited life-cycles. The following section analyses the influence of the length of life-cycle on both types of innovations. Infinite and finite time life-cycles are also compared. It is demonstrated that the infinite-time model is a limiting case for finite life-cycles model. The paper concludes with the analysis of distribution of investments into innovations of different types, where the argument of limited patents is discussed.

2 Related Literature

The question of how patents and patenting policy influence the dynamics of innovations and more generally, rates of technological change has a long history in economics. This question has been considered already in (**?**). In this paper the first formal model of the optimal patent's length has been considered. It has been argued that the patent need not to be of infinite length to stimulate innovative activity. The basic idea behind this statement was that one needs patents to protect and stimulate innovators, but these patents need not to be very long to stimulate further innovative activity from other innovators. In the current paper some progress towards establishing the similar argument for heterogeneous innovations is made. Similar to (**?**) paper one has a stand alone model of innovator in this framework while there are no competitors or potential entrants into the industry. On the other hand, one has the stream of two types of innovations here not the single one and patent has to be granted to every single product.

This form of innovative activity has also been considered in the patent literature under the name of sequential or cumulative innovations, where every next innovation is built up on the results of the previous one, (Denicolo 2002). There the question of optimal patent's length becomes more complicated as the sequential character of innovations rises questions not only of the length but also of the breadth of the patent. These questions have been extensively studied in (Gilbert and Shapiro 1990; Farrell and Shapiro 2004; Scotchmer 1996; Menell and Scotchmer 2005), and others.

One of the other approaches to the patenting problem is known under the name of patent races. Under this approach two or more agents are competing to be the first one to invent some product in order to obtain a prize which is the patent on this product and associated stream of profits from it, (Denicolo 1996; de Laat 1996). The suggested framework assumes a stand alone innovator and the model does not have any notion of patent races in it. Rather it is concentrated on optimal patent length for cumulative streams of innovations. Concerning this last there is also literature on variable length of patents for different products, as (Cornelli and Schankerman 1999). The suggested framework although assuming identical length of patents for all products may be modified to consider this also.

The model has an uncountable number of such cumulative streams of innovations represented by every single product's quality growth process. More then this the underlying process of variety expansion is also modelled. So the question arises what is the level (scope) of patents one would like to consider in such a framework? One variant is patenting of every level of quality of every product. Such an approach would generate non-smooth quality dynamics since the agent would not have stimulus to increase quality of a given product until the patent on the preceding level of quality will not expire. Then one has to assume patenting on the level of variety expansion process. Such a modification to the basic model has a clear interpretation, since every level of $n(t)$ is associated with invention of the new product. Every such product is then granted a patent, or, alternatively has a limited life-cycle within each the agent is free to develop its quality without external pressure from the market. Such a formulation of patenting problem is in line with works on cumulative innovations with patents, as in (Chang 1995; Gilbert and Shapiro 1990), but has some differences from it. In the suggested model one has not a single stream of innovations each of which is then patented or not, as in cumulative innovations literature, but rather one has stream of patented innovations and additionally dynamics of quality growth for each of this patented newly invented products (for each level of variety expansion process). One also accounts for the role of heterogeneity of these products, as it is done in (Hopenhayn and Mitchell 2001), but in a given model it would influence not only the rate of innovations (which is the rate of variety expansion in suggested framework) but also the growth rate of qualities of all these products. This makes the suggested approach richer then the preceding ones.

3 Finite life-cycles model

In this section the departure from infinitely living products model of (Bondarev 2012) is made that allows for finite life-cycles, which may be treated as patents for policy analysis. The model of that paper is referred to as a benchmark model throughout the text. In the same way as in the benchmark model, assume there is one multiproduct firm, which maximizes its profits by maximizing the results of innovating activities. This innovating activities has two dimensions: improvements of existing products (I use interchangeably terms vertical innovations, process innovations, quality innovations) and creation of new products (referred to as variety expansion, horizontal innovations, product innovations). These new products are some newer versions of the basic product, which defines the industry and thus the potential space of new products is bounded. This distinguishes this space from the unbounded space of ideas. However, this bounded space is the real one and continuum of new products may be introduced into the market. Each such product has zero technology (quality) level upon its invention and is subject to further process innovations. More on the general setup of the framework may be found in the benchmark model.

The main question of interest in this model is: whether the introduction of limited time life-cycles would stimulate or depress innovative activity of both types. For that purpose one may assume that after the expiration of the life-cycle time the agent cannot use his/her achieved quality (technology) level of the given product for profitable activities (e.g. sell this product). This of course is not true in real economies, but one can imagine the high density of competition on the product market which approaches the perfect one. As soon as the patent expires, all quality development of this product becomes the common knowledge to all the competitors and hence the agent in the model is no longer able to derive non-zero economic profit from it and thus she/he is no longer interested in quality investments in this product. In terms of the model this means that every product from $n(t)$ range has a limited time life-cycle (determined by the patent's length) during which its quality is developed by the agent. After this time development stops. At the same time such a framework would make sense only if at any given time the agent may switch his/her activities from technology improvement to the variety expansion investments thus acquiring additional range of new products to be developed. Hence the product innovations process is assumed to be not time-limited. The agent has infinite time planning horizon with respect to the overall model and thus the dynamics of variety expansion should not be very different from the benchmark case.

I make the simplifying assumption concerning life-cycles lengths:

Assumption 1 *Length of life-cycles for all production technologies from the potential products range N is the same: $\tau(i) = \tau$.*

The benchmark model is obtained by letting $\tau = \infty$.

The objective functional of the agent includes the length of each product's technology life-cycle, τ as a parameter of process innovations:

$$J^{patent} \stackrel{\text{def}}{=}$$

$$\stackrel{\text{def}}{=} \max_{u(\bullet),g(\bullet)} \int_0^\infty \mathrm{e}^{-rt} \left(\int_{n(t-\tau)}^{n(t)} (q(i,t) - \frac{1}{2} g(i,t)^2) di - \frac{1}{2} u(t)^2 \right) dt \tag{1}$$

where:

- $n(t)$ is the total range of products being invented up to the time t;
- $q(i,t)$ is the quality level (productivity) being achieved up to time t for product i within this range of invented products;
- $u(t)$ are investments into variety expansion at time t;
- $g(i,t)$ are investments into the development of product i at time t;
- r is the discount factor;
- τ is the duration (length) of the patent (life-cycle) being equal for all the products.

Such an objective functional reduces to the one of the benchmark case with infinite length of the patent. The difference lies in the fact, that only those products, which patents are still effective, might be developed further and not all the mass of invented products. This mass of products, which are covered by the patent at time t, is measured by the inner integral. At each point in time, the agent tries to maximize the value of quality-improving innovations for these products by investing into their qualities and the value of such innovations consists of the total achieved qualities for all the products minus total investments into process innovations. Variety expansion process, in contrast, does not increase the value of the innovator directly. Rather it increases the range of available products and expands the opportunities for further innovations. This is reflected in the dynamic upper limit of the inner integral, $n(t)$. This variety expansion requires specific investments and the overall value is measured along the infinite time horizon.

The dynamics of both types of innovations are the same as in the benchmark model. Variety expansion is proportional to investments being made into this kind of innovations and the achieved level of variety cannot decrease in time as long as investments are non-negative:

$$\dot{n(t)} = \alpha u(t); \tag{2}$$

At the same time, each of the products within the range of $n(t)$ has the quality dimension, which increases through innovations, but may decrease in their absence. It is natural to require all yet uninvented products to have zero quality:

$$\begin{aligned} &\dot{q(i,t)} = \gamma(i)g(i,t) - \beta q(i,t); \\ &q(i,t)|_{i=n(t)} = 0. \end{aligned} \tag{3}$$

The dynamics of all the products qualities is different, since the efficiency of investments, $\gamma(i)$, is different. This is the foundation of heterogeneity of products in this model. For simplicity the particular form of this efficiency function is assumed. Namely, it is assumed, that every next product is more complicated than all the preceding ones and thus its development is more difficult. Thus efficiency of investments for each next product i is decreasing:

$$\gamma(i) = \gamma \cdot \sqrt{N - i} \tag{4}$$

where N is the maximal achievable variety of products for the given industry.

This model follows the same structure as the benchmark one. The explicit constraint for zero process innovations for older products (for which the patent is already expired) is not necessary and is automatically satisfied due to the form of objective functional, (1). I now describe the solution of the model through successive 2-step implementation of Hamilton-Jacobi-Bellman approach to dynamic problems.

4 Solution

For solution of this model it is useful to observe, that for each product the evolution of innovations is independent of all other products. It depends only on the variety expansion process, since no innovations into any product can take place before its invention. At the same time, variety expansion innovations has the only value in the expected value generated by quality-improving innovations for the next to be discovered product. Thus the objective functional (1) may be rewritten in the following way:

$J^{patent} \stackrel{\text{def}}{=}$

(5)

$$\stackrel{\text{def}}{=} \max_{u(\bullet),g(\bullet)} \int_0^\infty e^{-rt} \left(\alpha u(t) \int_{t_i(0)}^{t_i(0)+\tau} \{q(n, s+t) - \frac{1}{2} g(n, s+t)^2\} ds - \frac{1}{2} u(t)^2 \right) dt$$

where $t_i(0)$ is the time of emergence (invention) of the product i.

With this reformulation the problem may be decomposed into the quality growth part, where maximization of value of innovations for each product i is carried out and variety expansion part, which is governed by the expected value of development of next product to be invented.

4.1 Solution for quality growth

Start with the quality growth part. In the benchmark model every new product has infinite time life-cycle, $\tau = \infty$ and hence the value function for the quality growth of each product i is defined from 0 to infinity. Under finite life-cycles assumption the period of development of every new product i is limited by $\tau < \infty$. Hence the value function of quality growth is time-dependent:

$$V^{pat}(q_i, t) = \max_{g_i(\bullet)} \int_{t_i(0)}^{t_i(0)+\tau} e^{-rs} \{q_i(s) - \frac{1}{2} g_i(s)^2\} ds \tag{6}$$

Note, that value function now depends not only on the number of the product (which implicitly defines dependence on the time of emergence $t_i(0)$ also as the inverse function of i) but also on the length of the life-cycle, which is assumed to be the same for all products.

The solution of the problem of quality growth for the finite life-cycles case follows the same steps as for the benchmark model. First the Hamilton-Jacobi-Bellman equation for the development of every product i is derived, then assuming the polynomial form of the associated value function the optimal investments are calculated. These are then used to solve for the optimal dynamics of quality of product innovation i *within the duration of its life-cycle*, τ.

The Hamilton-Jacobi-Bellman equation for the development of every product i depends on t and not only on the quality level itself:

$$rV^{pat}(q_i, t) + \frac{\partial V^{pat}(q_i, t)}{\partial t} = \max_{g_i(\bullet)} \Big\{ q_i - \frac{1}{2} g_i^2 + \frac{\partial V^{pat}(q_i, t)}{\partial q_i} \times (\gamma \sqrt{(N-i)} g_i - \beta q_i) \Big\},$$

(7)

$t \in [t_i(0), .., t_i(0) + \tau]$

One may assume the same linear form of value function for this problem as in the basic model, but with time-varying coefficients:

$$V^{ass}(q_i, t) = A_i(t)q_i + B_i(t). \tag{8}$$

Then the first-order condition for every product's quality growth is:

$$-g_i + \frac{\partial V(q_i, t)}{\partial q_i} \times (\gamma \sqrt{(N-i)}) = 0;$$

$$g_i^{pat} = A_i(t) \times (\gamma \sqrt{(N-i)}). \tag{9}$$

However, due to the limited life-cycle, one have a system of 2 differential equations on value function coefficients rather then algebraic equations. This system and its solution may be found in the Appendix.

The resulting coefficients being inserted into the first order condition (9) yields optimal investments into quality growth which now do depend on time but only within the limits of the patent's length τ ($t \in [t_i(0); t_i(0) + \tau]$):

$$g_i^{pat}(t) = \gamma \sqrt{(N-i)} \Big(\frac{1 - e^{(r+\beta)(t - t_i(0) - \tau)}}{(r+\beta)} \Big) \tag{10}$$

Finally one obtains ODE for quality growth:

$$\begin{aligned} \dot{q_i}(t) &= \gamma^2 (N-i) \Big(\frac{1 - e^{(r+\beta)(t - t_i(0) - \tau)}}{(r+\beta)} \Big) - \beta q_i(t); \\ q_i(t_i(0)) &= 0. \end{aligned} \tag{11}$$

which is the first-order linear ODE with the unique solution:

$$\begin{aligned} &q_i^{pat}(t) = \\ &= \frac{\gamma^2 (N-i)}{(r+\beta)(r+2\beta)\beta} \times \\ &\times \Big(\beta (e^{-(r+\beta)(t_i(0) - \tau - t)} - e^{(r+\beta)(t - t_i(0) - \tau)}) - (r + 2\beta)(e^{-\beta(t - t_i(0))} - 1) \Big) \end{aligned} \tag{12}$$

As it can be seen from (12), quality growth for each product now depends on time, but only within the boundaries of the patent length and from the patent length itself, and from the time of emergence of the product, $t_i(0)$. One may treat this solution as suitable for *any* product i, including the boundary product $i = n(t)$. However, to define the location of the evolution path for the certain product in the product's space and in the overall quality improving process, one has to define the time of emergence from the variety expansion part of the problem. Observe also, that the solution for infinite-time case is the limiting case of this quality evolution path with $\tau \to \infty$. Indeed, this may be demonstrated by taking the respective limit of the (12).

As a result of limited life-cycle of the product development, each product's quality decays to zero after the time of expiration of the patent, as Figure 1 shows. To produce the quality evolution paths at this Figure the same set of parameters has been used as for infinite-time case above but with $\tau = 1$ to make the dynamics more clear. It has to be noted, that every next product

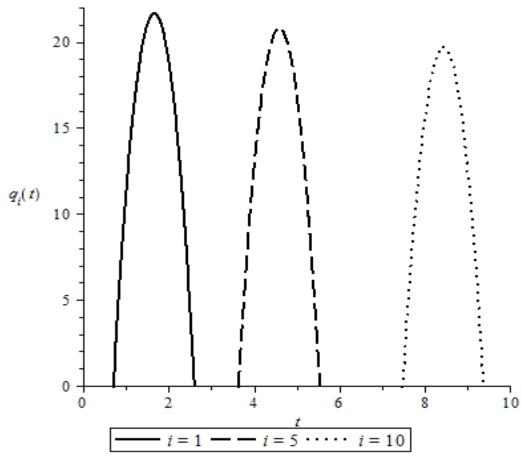

Figure 1: Quality innovations under limited life-cycles for different products i

has slightly lower maximal quality, then all the preceding products. This can be seen from the form of the solution (12): the greater is the index of the product i, the lower is its quality:

$$\frac{\partial q(i,t)}{\partial i} < 0. \tag{13}$$

For displayed on Figure 1 i values it also can be observed even with small changes of i.

At any given point in time the innovator has under his/her control the range (bounded continuum) of products to develop. This range, $n(t) - n(t - \tau)$, denotes those products which are already introduced to the market and their life-cycle (patent) have not yet expired. Further on this quantity is referred to as *effective range of products*. It is defined from the evolution of variety expansion path as well as emergence times of all the products development. This creates the link from variety expansion to the quality development, which is stronger then for the infinite-time case above. As it is discussed further in the paper, the process of out-dating of older products significantly changes innovator's behaviour. Main results on process innovations dynamics are summarized in the Proposition below.

Proposition 1 ***On process innovations under finite life-cycles.***
1. For every new product i at any time t process innovations under infinite life-cycles may be derived from the solution for finite life-cycles with $\tau \to \infty$. These infinite life-cycle innovations are maximal rates of process innovations possible.
2. Finitely living process innovations never reach maximal (steady-state) level of $\bar{q}_i$ and decreases to 0 after the life-cycle ends.
3. Other things equal, every next product i has lower process innovations level than all the preceding ones, $\frac{\partial q(i,t)}{\partial i} < 0$.

4.2 Solution for variety expansion

The final step of the solution of quality growth problem is the calculation of the value function. Then we take this value function with zero quality level as an input for variety expansion problem. Applying the same logic as for infinite-time horizon model, one may note that for variety expansion problem only the value of quality growth model at zero time is relevant because the agent estimates his potential profit from the expansion of the range of products available for him to develop only and this is done at the moment of the emergence of this good, $t_i(0)$. Hence one need to know only $V^{pat}(q_i, t)_{|q_i=0, t=t_i(0)} = V^{pat}(0, \tau)_{n(t)}$, which is:

$$V^{pat}(0, \tau)_{n(t)} = \frac{\gamma^2 (N - n(t))}{r\beta(r + 2\beta)(r + \beta)^2} \times$$

$$\times \Big(r(r + 2\beta)e^{-(r+\beta)\tau} - \frac{1}{2} r\beta e^{-2(r+\beta)\tau} - (r + \beta)^2 e^{-r\tau} + \frac{1}{2}\beta(r + 2\beta) \Big). \quad (14)$$

Also note that the time horizon for variety expansion model is infinite which gives time-autonomous HJB equation for this part of the problem. Denote

$$V(\tau) = \frac{V^{pat}(0,\tau)_{n(t)}}{(N - n(t))} = \frac{\gamma^2}{r\beta(r+2\beta)(r+\beta)^2}\times$$

$$\times\Big(r(r+2\beta)e^{-(r+\beta)\tau} - \frac{1}{2}r\beta e^{-2(r+\beta)\tau} - (r+\beta)^2 e^{-r\tau} + \frac{1}{2}\beta(r+2\beta)\Big). \tag{15}$$

This does not depend on $n(t)$.
Now the HJB equation for variety expansion problem takes the form:

$$rV^{pat}(n) = \max_{u(\bullet)}\Big\{\alpha u(t)\times V(\tau) - \frac{1}{2}u(t)^2 + \alpha u(t)\times\frac{\partial V_n}{\partial n}\Big\}. \tag{16}$$

Assuming quadratic form of the value function for this problem one have first order condition for the optimal control which depends on value function for quality problem:

$$u^{pat}(t) = \alpha\Big(V(\tau)(N - n(t)) + 2Cn(t) + F\Big) \tag{17}$$

with $V^{ass}(n) = Cn(t)^2 + Fn(t) + E$.

The system of algebraic equations for coefficients C, F, E is solved by inserting expression for $u^{opt}(t)$ into the (16) above and regrouping coefficients at equal powers of $n(t)$. Hence one arrives to the system of 3 equations with three unknown coefficients, which has a straightforward solution. Substitution for these coefficients into the first order condition (17) yields optimal investments into variety expansion process as a function of $V(\tau), n(t)$:

$$u^{pat}(t) = \frac{2\alpha r(N - n(t))V(\tau)}{r + \sqrt{4\alpha^2 rV(\tau) + r^2}} \tag{18}$$

Then dynamic constraint (2) yields the first-order ODE for $n(t)$:

$$\dot{n(t)} = \frac{2\alpha^2 rV(\tau)}{r + \sqrt{4\alpha^2 r\times V(\tau) + r^2}}(N - n(t)) \tag{19}$$

which has the solution

$$n^{pat}(t) = N + e^{-\frac{2\alpha^2 rtV(\tau)}{r+\sqrt{4\alpha^2 rV(\tau)+r^2}}}(n_0 - N) \tag{20}$$

The shape of dynamics of variety expansion demonstrates convergence of evolution paths with different initial ranges, as it is shown at the Figure 2.

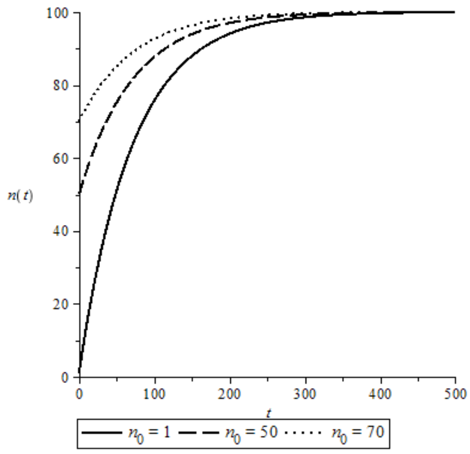

Figure 2: Products variety expansion for products with limited life-cycles

The last point which is necessary to obtain the full characterization of dynamics of the model is the emergence time, $t_i(0)$ for all products $i \in \mathbf{N}$. This is an inverse function of variety expansion process, since it is defined from the condition $i = n(t)$ in each case. It is calculated by substitution i for n and $t_i(0)$ for t into the variety expansion and then finding the inverse. Formally:

$$t_i(0) : i \to f(i);$$

$$f(i) = n(t)^{-1}|_{n=i};$$

(21) $$t_i(0) = -ln\Big(\frac{(N-i)}{(N-n_0)}\Big) \times \frac{r + \sqrt{4\alpha^2 rV(\tau) + r^2}}{2\alpha^2 rV(\tau)}$$

This function, demonstrated on Figure 3, shows, that the higher is the index of a product, i, the more time is needed for the introduction of the next product after this one. This is the direct consequence of the slowing rates of variety expansion, as Figure 2 shows. As a result, *density* of quality innovations, $q_i(t)$, is decreasing with time, as Figure 1 shows: the distance between evolution paths of technologies for products in the beginning of the products range is shorter, then in the end of it. Note, however, that this is not the effect of the finiteness of life-cycles of products but rather of their heterogeneity. Shorter life-cycles lead to slower emergence of products.

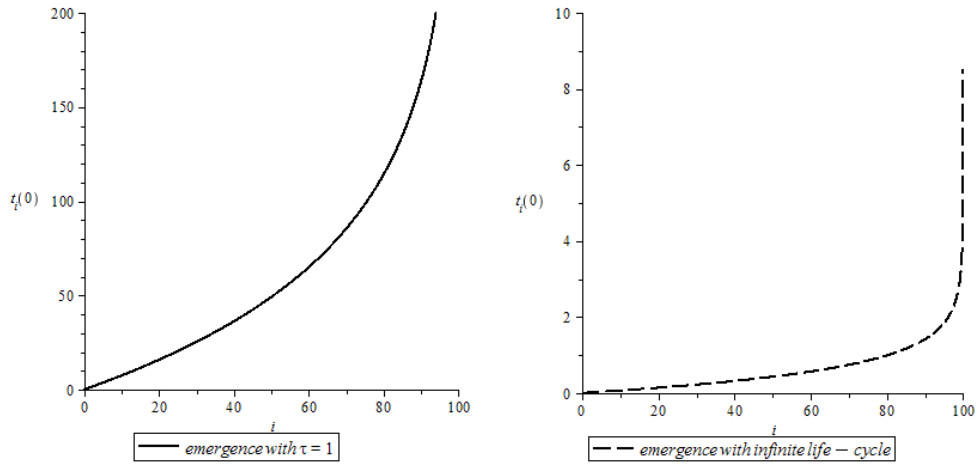

Figure 3: Time of emergence as function of product position in the product's space.

Compare two graphs of the Figure 3 to see that. Main points concerning the variety expansion are summarized in the Proposition below.

Proposition 2 ***On product innovations under finite life-cycles.***
1. Product innovations under infinite life-cycles may be derived from the solution for finite life-cycles with $\tau \to \infty$. They are maximal among product innovations for any τ.
2. Product innovations are convergent for different initial ranges of products available.
3. Product innovations speed is decreasing in time. This is the direct consequence of decreasing efficiency of investments into process innovations, $\gamma(i)$. As a result density of $q(i,t)$ function decreases in i.

5 Analysis of the model

5.1 Product innovations dynamics under changing life-cycles length

Now one may ask whether the introduction of patents stimulate quality growth and variety expansion processes or not relative to infinite-time version of the model.

First consider effects of the length of life-cycles on the variety expansion. A priori one may encounter two opposite effects in this part of the model.

The first one should be negative: the shorter is the length of patent (life-cycle), the lesser is the range of products effectively at the agent's disposal at each point in time. This range is given by $n^{pat}(t) - n^{pat}(t-\tau)$ since only products introduced during this time are covered by patents at the time t. Then to maximize the range of products under control at each point in time the agent should invest more in variety expansion with shortening patent length. This effect is referred to as *compensation effect*, because the agent has to compensate the decrease in the effective product's range with additional investments into variety expansion.

At the same time shorter length of the life-cycle limits agent's opportunities to develop products' qualities and thus, decreases incentives to develop new products. This effect is referred to as *potential profit effect*, as it is the changes in potential profit expected from new product, which creates it.

To formally define these two effects, consider first the derivative of the effective product's range with respect to the length of the life-cycle, which includes both effects:

$$\frac{\partial[n^{pat}(t) - n^{pat}(t-\tau)]}{\partial\tau} = -\frac{2\alpha r^2(r + \sqrt{4\alpha^2 rV(\tau) + r^2} + 2\alpha^2 V(\tau))}{\sqrt{4\alpha^2 rV(\tau) + r^2} \times (r + \sqrt{4\alpha^2 rV(\tau) + r^2})^2} \times$$

$$(N - n_0)\frac{dV(\tau)}{d\tau} e^{-\frac{2\alpha rV(\tau)}{r+\sqrt{4\alpha^2 rV(\tau)+r^2}}t}((t-1)e^{\frac{2\alpha rV(\tau)\tau}{r+\sqrt{4\alpha^2 rV(\tau)+r^2}}} - 1 - t). \tag{22}$$

The sign of this derivative depends on the sign of expression

$$\mathbf{A} = ((t-1)e^{\frac{2\alpha rV(\tau)\tau}{r+\sqrt{4\alpha^2 rV(\tau)+r^2}}} - 1 - t). \tag{23}$$

This last may be positive or negative depending on relative size of the value function $V(\tau)$. It depends on the length of the life-cycle, τ. For longer life-cycles it is greater then one and the subsequent expression (23) is positive for almost all t's, yielding negative derivative sign, for shorter life-cycles it is negative for most t's, yielding positive derivative sign. Observe also that since this expression depends on time there is always some initial period when it is negative for $t \to 0$ and always positive for $t \to \infty$. In effect this means that changes in patents' length may influence the effective range of products in different directions. This last phenomena is illustrated in Figure 4.

The effective product's range is first increasing with patent's length, but afterwards it decreases. This point to the fact that this effective range is subject to effects of the length of the life-cycle. To consider them, decompose the above derivative:

$$\frac{\partial[n^{pat}(t) - n^{pat}(t-\tau)]}{\partial\tau} = \frac{\partial n^{pat}(t)}{\partial\tau} - \frac{\partial n^{pat}(t-\tau)}{\partial\tau} = PPE + CE. \tag{24}$$

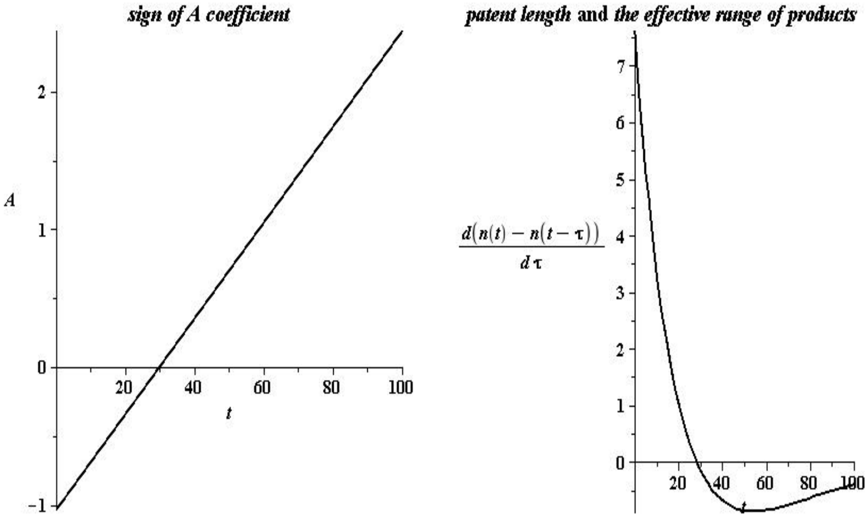

Figure 4: Varying sign of the effect of patents length

We identify the first component of (24) with the potential profit effect (PPE) and the second - with compensation effect (CE). Shorter patent length means that the agent is able to derive less profit from the usage of the given product which in turn lowers his incentives to invest into the variety expansion. This is the first effect, as this directly influences the total range $n(t)$ through rate of investments. To observe it, consider the derivative $\frac{\partial n^{pat}(t)}{\partial \tau}$ which amounts to:

$$PPE =$$
$$= \frac{\partial n^{pat}(t)}{\partial \tau} = (n_0 - N) \times \frac{\partial [e^{-X(\tau)t}]}{\partial \tau} = (N - n_0)e^{-X(\tau)t} \times \frac{\partial [X(\tau)]}{\partial \tau} \times t > 0;$$
(25)
$$X(\tau) > 0.$$

Here $X(\tau)$ denotes some expression of exogenous model's parameters and is function of τ only, not of time t.
This effect is quiet standard and describes how the evolution of products range changes with changes in the length of the life-cycle of every product in this range. The longer is the life-cycle, the more incentives the agent has to invest into the introduction of new products. It is the decreasing function of time (since the form of variety expansion path), but always positive.

The second effect is calculated in a similar fashion by substituting $t - \tau$

for t:

$$CE = -\frac{\partial n^{pat}(t-\tau)}{\partial \tau} =$$
(26)
$$= -(n_0 - N) \times \frac{\partial [e^{-X(\tau)(t-\tau)}]}{\partial \tau} = (N - n_0)e^{-X(\tau)(t-\tau)} \times (X(\tau) + (t-\tau)\frac{\partial [X(\tau)]}{\partial \tau}).$$

Unlike the first component this one is (almost always) negative. As time flows it becomes bigger in size and gradually offsets the influence of the first effect. This is illustrated by the Figure 5.

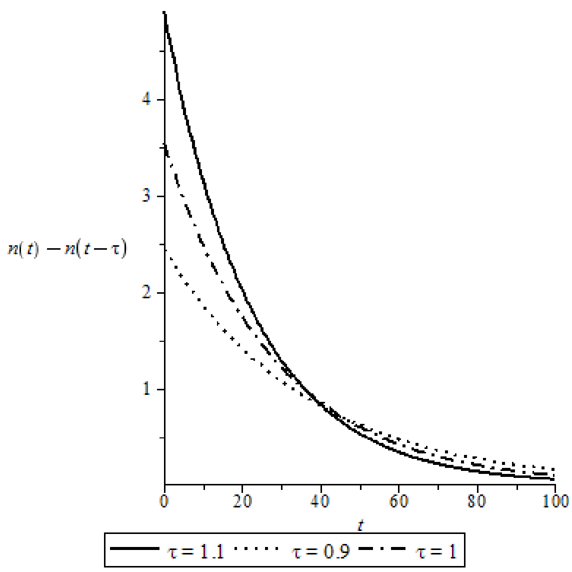

Figure 5: Changes in the effective products ranges in time

It can be seen from this figure that as length of the life-cycle increases, the effective range is increasing at initial stage and decreases afterwards. The time at which the compensation effect outperforms the potential profit effect does not depend on the parameter τ and is defined from the very form of the variety expansion process. The above observations may be summarized in the following Proposition:

Proposition 3 ***Life-cycles influence on variety expansion.***
Under finite life-cycles of technologies $q(i,t)$ the life-cycle length τ has two effects of different directions on the effective range of products $n^{pat}(t)$ –

$n^{pat}(t-\tau)$. *As a result the overall effect of life-cycle length on effective products range is non-monotonic. At the initial stage potential profit effect is greater in absolute value then the compensation effect and overall influence of life-cycle length is positive, while on mature stage the compensation effect dominates the potential profit effect and overall influence is negative. This result is valid for decreasing efficiency of process innovations across products,* $\gamma(i) : \frac{\partial \gamma(i)}{\partial i} < 0$.

Economic intuition behind this result is clear: at initial stage of development there are a lot of opportunities to develop new versions of the basic product for the monopolist. Hence, increase in the length of the life-cycle gives him/her more possibilities to develop all these new products and derive profit from them. Thus potential profit effect is high. As time flows, more products are introduced into the market, but effective range decreases, as the variety expansion process slows down. Then the effect of expected profit from all of the new products in the additional range from the increase in the length of the life-cycle wears down, as there is lesser mass of potential products, $N - n(t)$, in this range. Simultaneously it becomes less important to sustain the given effective range, as it increases from the length of the patent, hence compensation effect is larger.

It has to be noted, that the above discussion does not mean that the variety expansion process may negatively depend on the length of the life-cycle. As it has been seen, its derivative is always positive. Hence the process of variety expansion is always boosted by the increasing length of the life-cycle. This happens because the effective products range is a characteristic of a speed of variety expansion, not of its overall level. Thus with longer life-cycles the level (stock) of products variety is always increasing, while the rate of their introduction and as a result, the effective patented range, not always increases but only at the initial stage of development while decreasing afterwards. Infinite-time horizon model may be considered as a patent model with infinite patent length in this respect. It can be shown that the patent model is equivalent to the infinite-time horizon model with $\tau \to \infty$. The comparison of $n(t)$ dynamics with the same initial range for infinite-time horizon and patent model at Figure 6 illustrates the ideas of higher stock and lower dynamics.

In this figure the infinite-time variety has higher level then limited life-cycle variety at every point, but the intensity of addition of new products is higher for infinite time case at the beginning, while lower afterwards.

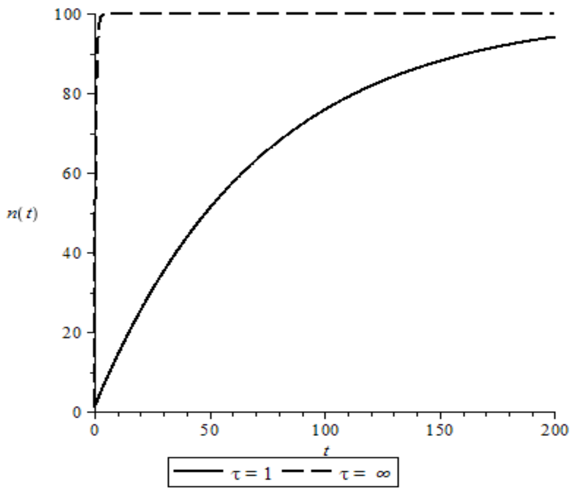

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Figure 6: Finite and infinite life-cycles and variety expansion

5.2 Process innovations dynamics under changes of the length of life-cycles of products.

Quality growth essentially depends on the length of the life-cycle of products also. The longer the life-cycle, the closer patent's model quality dynamics is to the infinite-time one. The quality growth displays only the potential profit effect as long as one consider single product quality investments: the longer the life-cycle of the product, the higher is the maximal attainable quality of this product and thus the higher is the expected stream of profits from the development of this product. Hence,

$$\frac{\partial q_i^{pat}(t)}{\partial \tau} > 0. \tag{27}$$

This can be checked by directly computing the derivative of (12) w.r.t. to τ. There is no ambiguity in the effect of the length of the life-cycle onto the development of every separate product i from the effective range of products.

However there are two different effects on the aggregate level of quality development. Observe that at any given time t there is a mass of products under the control of the innovator which qualities might be developed. This mass is given by the effective products range defined above. The level of aggregate (across products) quality development is then given by the quantity, denoted $\mathbf{Q}$:

(28) $$\mathbf{Q} = \int_{n(t-\tau)}^{n(t)} q_i(t)di.$$

We proceed in the same fashion as for the case of variety expansion: calculate the derivative and decompose it. For this we use usual rules of integration and derivative w.r.t. to the parameter. Expressions for this case are very cumbersome and not displayed. The quantity being computed is:

(29) $$\frac{\partial \mathbf{Q}}{\partial \tau} = \int\limits_{n(t-\tau)}^{n(t)} \frac{\partial q_i^{pat}(t)}{\partial \tau} di.$$

where we make use of the interchange of the order of differentiation and integration due to Fubini's theorem (and its extensions).

Surprisingly enough, the effect of the products life-cycle on this aggregate measure of quality growth follows the same pattern as for variety expansion, described in Proposition 3. The derivative sign is initially positive but changes to negative at the mature stage of products range development (for higher t). The compensation effect for qualities influences the range of the integral itself, that is, the total quantity of quality improving innovations.

Indeed the total effect of changes in patents length on the overall qualities development is defined from 2 sources: potential profit effect for quality of every single product within the effective range and by the scale of the effective range itself. It is known from above discussion, that this effective range tends to shrink along increase of the life-cycle's length for mature stages of development; thus the total range of quality investments shrinks also. The effect of the range's length outweighs the effect of potential profit for every single product and thus the overall behaviour of the $\mathbf{Q}$ is determined by the changes in effective range of products. This is illustrated at Figure 7.

Proposition 4 ***Life-cycles influence on quality innovations.*** *The total mass of process innovations,* $\boldsymbol{Q}$ *is subject to compensation effect and potential profit effect of opposite directions. The first affects the range of the mass through changes in effective products range, while the second influences the dynamics of process innovations for separate products* i *and is always positive.*

It should be noted, that for any $\tau < \infty$ quality growth for every separate product is less then the maximal level for infinite-time case, as Proposition 1 stresses. Figure 8 displays quality innovations for the finite ($\tau = 50$) an

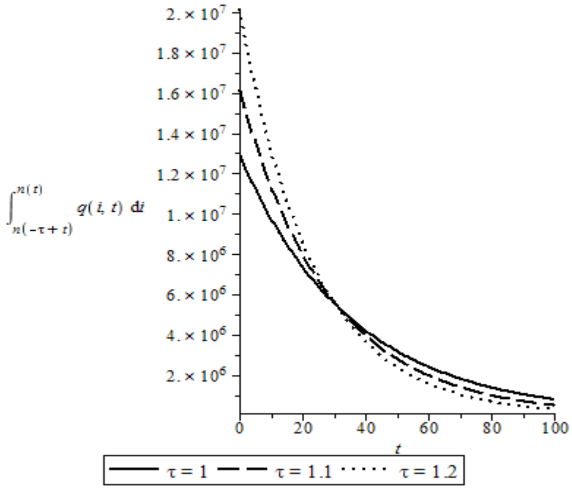

Figure 7: Length of life-cycles and overall quality innovations

infinite life-cycles for products $i = 1, i = 50$ with the same parameter values as before (left picture) and the influence of the changes in life-cycle length on the quality development of any single product (picture to the right, $i = 1$).

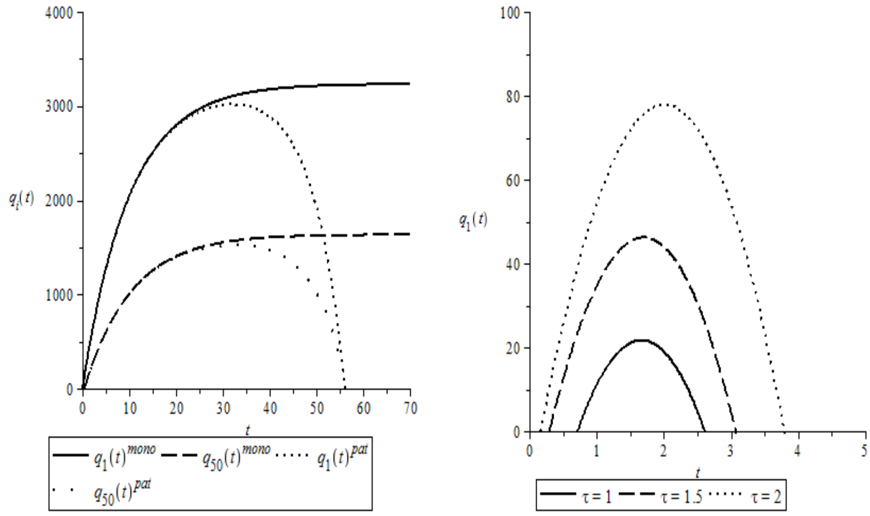

Figure 8: Effect of the length of life-cycles on the development of separate products

The above discussion demonstrates the important feature of the model: process innovations (quality) depend on product innovations (variety expansion) only as an aggregate $q(i,t)$ process, while for each separate product i process innovations are independent from the introduction of new products.

In conclusion of the analysis of the model one may observe the main difference between infinite-time and limited life-cycles dynamics. Under infinite life-cycles the level of both product and process innovations is maximized in the model, but the effective range of product and process innovations may decrease with increasing life-cycle length at mature stages of the industry development. This is described by Propositions 3 and 4. Hence one may establish the following final Proposition:

Proposition 5 ***Innovations dynamics under finite and infinite life-cycles of technologies.***
1. Under infinite life-cycles the output of both product and process innovations is maximized.
2. Under finite life-cycles the range of existing products and the total mass of process innovations into them decreases with life-cycle length increase at mature stage of industry development.

6 Discussion

In this paper the model which allows to consider finite-time life-cycles (patents) of products together with the infinite-time process of products' generation is developed. Every product has an effective life-time within which it has a substantial demand associated with it and hence is capable of generating profit for the innovator. Although not very much products literally disappear from the market if to take some reasonable scope of analysis, rather big portion of existing variety of products is renewed within some periods of time. This means one has to formulate a logic of behaviour of an innovating agent which would combine infinite planning horizon with finite life-time of products.

Limited life-cycles of products in the model of simultaneous product and process innovations attempts to unify the approaches from (Aghion and Howitt 1992) and (Grossman and Helpman 1994) and put the notion of creative destruction from (Schumpeter 1942) in them to a more formalized basis. If in (Aghion and Howitt 1992) creative destruction means actual replacement of the product by another, newer one, while the range of products is constant and in (Grossman and Helpman 1994) there is no replacement of products, in the current paper one has the destruction of products while new products are positioned higher in the products range than older ones and their refinement is more complicated due to assumed heterogeneity of investment characteristics of them.

Introduction of limited life-cycles into the model creates new compensation effect for both product and process innovations. This effect pushes the innovator to invest more into the development and introduction of new products with shorter life-cycle length. This effect becomes more important for mature industries, when more than a half of available products range is already used for creation of new versions of the basic product. For industries with wider range of available new innovations this effect is smaller and thus the usual positive effect of longer life-cycles is observed. Such a dynamics point to the fact, that process of out-dating of technologies, known as creative destruction may have positive as well as negative influence on the intensity of technological change in the industry depending on its stage of development.

The same compensation effect influences the process innovations as well, but only on the aggregate level. Each individual technology does not depend on products innovations process and as such responds positively on the increase in the life-cycle length. However, due to the possible decrease in effective range of technologies which might be developed at each point in time in mature stage, the overall mass of process innovations may decrease with increase in life-cycle length. This happens because less technologies are developed, while each of them is developed to a higher level with longer life-cycles. Thus, longer life-cycles weaken the creative destruction process, but improve the quality of existing products (innovations).

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Appendix

Solution for coefficients of value function for quality growth problem

The linear value function yields a system of two differential equations on coefficients:

$$\begin{aligned}
&\dot{A_i}(t) = (r+\beta)A_i(t) - 1;\\
&\dot{B_i}(t) = rB_i(t) - \frac{1}{2}\gamma^2(N-i)A_i(t)^2\\
&A_i(\tau + t_i(0)) = 0;\\
&B_i(\tau + t_i(0)) = 0.
\end{aligned} \tag{30}$$

Observe that for every product i the value function is different, as coefficients are different due to different boundary conditions and $N-i$ term in the second equation.

This is a system of first order equations which can be readily solved. First the solution for $A_i(t)$ coefficient as a function of emergence time $t_i(0)$ is obtained:

$$A_i(t) = \frac{1}{(r+\beta)}(1 - e^{(r+\beta)(t-t_i(0)-\tau)}) \tag{31}$$

Substitution of this into the equation for $B_i(t)$ term yields the second coefficient as function of emergence time and the position of the product within the products range, i:

$$\dot{B_i}(t) = rB_i(t) - \frac{1}{2}\gamma^2(N-i)\Big(\frac{1}{(r+\beta)}(1 - e^{(r+\beta)(t-t_i(0)-\tau)})\Big)^2$$

$$\begin{aligned}
&B_i(t) = \frac{\gamma^2(N-i)}{r(r+\beta)^2}\times\\
&\times\Big(\frac{r}{\beta}e^{(r+\beta)(t-(\tau+t_i(0)))} - \frac{1}{2(r+\beta)}e^{2(r+\beta)(t-(\tau+t_i(0)))} - \frac{(r+\beta)^2}{\beta(r+2\beta)}e^{r(t-(\tau+t_i(0)))} + \frac{1}{2}\Big).
\end{aligned} \tag{32}$$

These calculations provide the form of the value function for quality growth for every product i:

$$V^{pat}(q_i,t) = \frac{1}{(r+\beta)}(1-e^{(r+\beta)(t-t(0)_i-\tau)}) \times q_i(t) +$$

$$+\frac{\gamma^2(N-i)}{r(r+\beta)^2}\times$$

(33)

$$\times\Big(\frac{r}{\beta}e^{(r+\beta)(t-(\tau+t_i(0)))} - \frac{1}{2(r+\beta)}e^{2(r+\beta)(t-(\tau+t_i(0)))} - \frac{(r+\beta)^2}{\beta(r+2\beta)}e^{r(t-(\tau+t_i(0)))} + \frac{1}{2}\Big).$$