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# Stochastic Optimal Hedge Ratio: Theory and Evidence

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## Abstract

The minimum variance hedge ratio is widely used by investors to immunize against the price risk. This hedge ratio is usually assumed to be constant across time by practitioners, which might be too restrictive assumption because the optimal hedge ratio might vary across time. In this paper we put forward a proposition that a stochastic hedge ratio performs differently than a hedge ratio with constant structure even in the situations in which the mean value of the stochastic hedge ratio is equal to the constant hedge ratio. A mathematical proof is provided for this proposition combined with some simulation results and an application to the US stock market during 1999-2009 using weekly data.

**Running title:** Stochastic Optimal Hedge Ratio

**JEL Classification:** G10, C32

**Keywords:** Optimal Hedge Ratio, Stochastic Hedge Ratio, the US

## 1. Introduction

Finding an optimal hedge ratio is one of the vital issues pertinent to the investors risk management analysis. Optimal hedge ratio is of fundamental importance in order to neutralize price risk. There are a number of published papers on the calculation of the optimal hedge ratio (see, for example, Cechetti et al., 1988; Myers and Thompson, 1989; Baillie and Myers, 1991; Kroner and Sultan, 1991; Lien and Luo, 1993; Park and Switzer, 1995; and Hatemi-J and Roca, 2006). In the existing literature it is common practice to assume that the estimated parameter of the underlying hedge ratio is constant during the period of study. This assumption is clearly too restrictive as there are many underlying causes for the hedge ratio to be time-varying. The potential reasons behind this phenomenon might be technological progress, economic or financial crises, variations in the people's preferences and their behavior, policy alteration, and organizational or institutional evolutions. If the parameters are time-varying but in the estimations they are treated as constant then the underlying inference might not be accurate. In this paper we put forward a proposition that the performance of a stochastic hedge ratio is different than the performance of a constant hedge ratio even in the situations in which the mean value of the stochastic hedge ratio is equal to the hedge ratio with a constant structure.<sup>1</sup> This proposition is mathematically proved and some simulation results are also generated. The stochastic hedge ratio is applied to the US equity market and it is compared to a constant hedge ratio. Empirical results support a stochastic structure for the optimal hedge ratio.

The rest of this paper is organised as follows. Section 2 derives the optimal hedge ratio with a constant structure. This section also defines the optimal hedge ratio with a stochastic structure. In addition, some mathematical and simulation results regarding the properties of the stochastic hedge ratio are presented Section 2. Section 3 provides an application to the equity market of the US and the last section concludes the paper. An appendix at the end of the paper presents proofs as well as the description of the estimation methodology.

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<sup>1</sup> This issue has financial implications for any portfolio with a hedging strategy.

## 2. Optimal Hedge Ratio

A hedge ratio that minimizes the variance of the hedger's position is the optimal hedge ratio according to the literature. Assume that we expect to sell  $N_A$  units of an asset at time  $t_2$  and decide to hedge at time  $t_1$  via shorting futures contracts on  $N_F$  units of a similar asset. The hedge ratio,  $h$ , is

$$h = \frac{N_F}{N_A}. \quad (1)$$

We denote the total amount realized for the asset when the profit or loss on the hedge is taken into account by the variable  $Y$ , so that:

$$Y = S_2 N_A - (F_2 - F_1) N_F$$

Which can be reformulated as

$$Y = S_1 N_A + (S_2 - S_1) N_A - (F_2 - F_1) N_F \quad (2)$$

Where

$S_1$  = the asset price at time  $t_1$ ,

$S_2$  = the asset price at time  $t_2$ ,

$F_1$  = the futures price at time  $t_1$ ,

$F_2$  = the futures price at time  $t_2$ .

By substituting equation (1) into equation (2) we can express  $Y$  as the following:

$$Y = S_1 N_A + (\Delta S - h \Delta F) N_A \quad (3)$$

Where  $\Delta S = S_2 - S_1$  and  $\Delta F = F_2 - F_1$ . Because  $S_1$  and  $N_A$  are known at time  $t_1$ , the variance of  $Y$  in equation (3) is minimized if the variance of  $\Delta S - h \Delta F$  is minimized. This variance is expressed as

$$V[\Delta S - h \Delta F] = \sigma_S^2 + h^2 \sigma_F^2 - 2h\rho\sigma_S\sigma_F \quad (4)$$

Where

$\sigma_S^2$  = the variance of  $\Delta S$ , ( $\sigma_S$  = the standard deviation of  $\Delta S$ ),

$\sigma_F^2$  = the variance of  $\Delta F$ , ( $\sigma_F$  = the standard deviation of  $\Delta F$ ), and

$\rho$  = the correlation coefficient between  $\Delta S$  and  $\Delta F$ .

Thus, the minimum variance hedge ratio can be obtained by minimizing equation (4) with respect to  $h$ . By using the first order condition for optimization we find:

$$\frac{\partial V}{\partial h} = 2h\sigma_F - 2\rho\sigma_S\sigma_F = 0 \Rightarrow \quad h = \rho \frac{\sigma_S}{\sigma_F} \quad (5)^2$$

Notice that the second derivative with respect to  $h$  is positive, which is the sufficient condition for minimization. This optimal hedge ratio can also be obtained through the following regression:

$$\Delta S_t = c + h\Delta F_t + \varepsilon_t. \quad (6)$$

It is common practice to use the variables in the level format in order to avoid losing any long run information in case each variable has a unit root but a linear combination between them is stationary. This is the standard regression approach that is usually used nowadays to calculate the optimal hedge ratio. However, this paper discusses the cases where the hedge ratio follows a stochastic dynamic. We argue that this hypothesis accords better with the reality than the supposition of the constant hedge ratio. To illustrate this point let  $(B_t)_{t \in [0, T]}$  be a Brownian motion living on the probability space  $(\Omega, F_T, P)$  where  $F = (F_t)_{t \in [0, T]}$  is the natural filtration generated by  $(B_t)_{t \in [0, T]}$ . Assume that the hedge ratio follows the subsequent stochastic dynamic  $h := (h_t)_{t \in [0, T]}$ , where  $h_t = \alpha + B_t$ , defined as the stochastic process describing the hedge ratio. The parameter  $\alpha$  is a constant that is assumed to be equal to the constant hedge ratio as defined by equation (5). We put forward the following propositions:

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<sup>2</sup> The derivation of equation (5) comes mainly from Hull (2002).

**Proposition 1:**  $E(h_t) = \alpha$  for any time  $t \in [0, T]$ .

**Proposition 2:**  $e := \int_0^T |h_t - \alpha| dt = \int_0^T |B_t| dt > 0$ . Thus, the integral that is representing the difference between a stochastic optimal hedge ratio and a hedge ratio with constant structure across time is higher than zero in absolute terms. This is even the case when the expected value of both hedge ratio measures is the same.

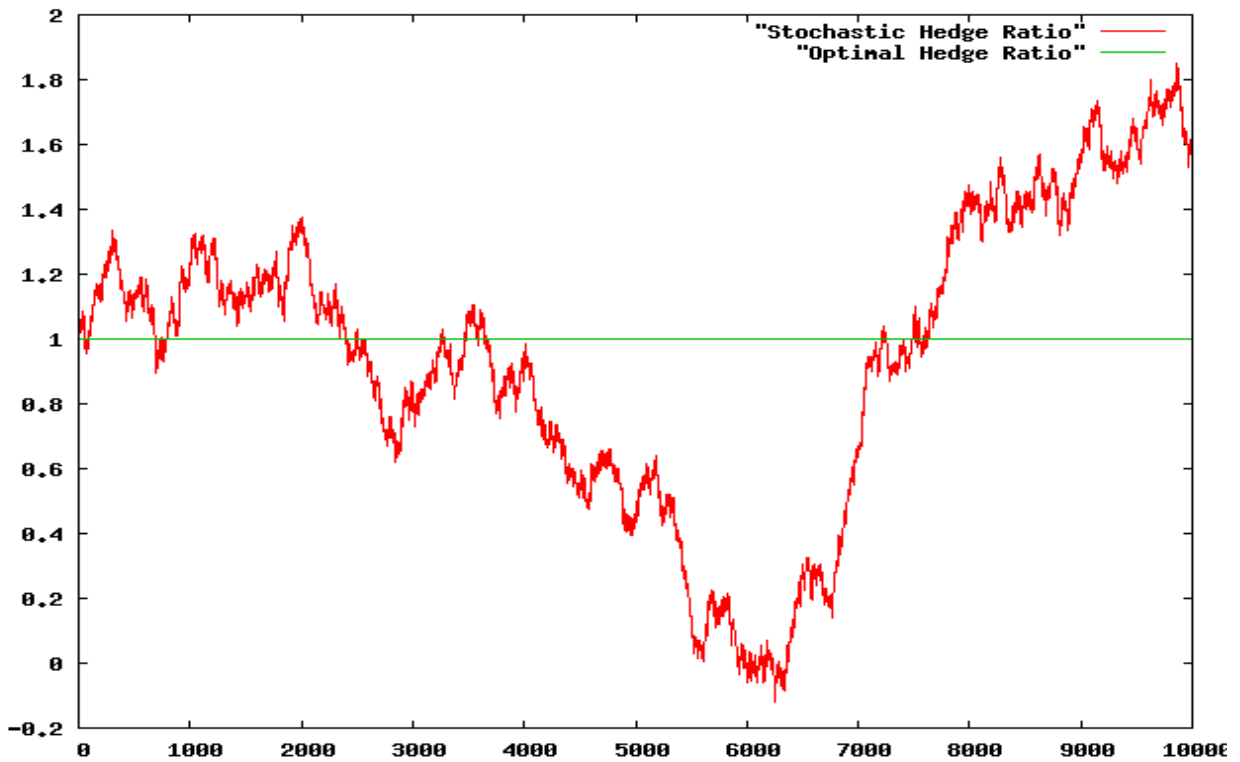
**Proposition 3:** Suppose that the Brownian motion vanishes  $n$  times between  $[0, T]$  at  $t_1, t_2, \dots, t_n$  with  $0 = t_0 \leq t_1 \leq \dots \leq t_n \leq t_{n+1} = T$ . Without loss of generality we assume that  $B_t > 0$  for  $t \in [t_0, t_1]$  and that  $n$  is even, then we have

$$e = \int_0^T |B_t| dt = TB_T + \sum_{k=0}^n (-1)^{k+1} \int_{t_k}^{t_{k+1}} t dB_t \quad (7)$$

For the proof of each proposition see the appendix.

The behavior of the stochastic hedge ratio with the same mean as the constant hedge ratio is simulated and depicted in Figure 1. It is evident from this figure that the stochastic hedge ratio might deviate significantly from its mean value. Thus, investors could improve on their hedging strategies if this deviation is taken into account in the estimation of their optimal hedge ratio.

Figure 1: Simulated trajectory for the stochastic hedge ratio compared to the constant hedge ratio.<sup>3</sup>



It should be mentioned that the simulations are conducted by a program in C++ produced by the authors, which is available on request.

### 3. An Application

In our application, the spot instrument corresponds to the equity market index while the hedging instrument is the futures market index. We use the MSCI price index for the US equity market and the Share Price Index for the US futures market. The dataset consists of weekly observations of spot and future rates of the shares during the period 1999-2009. The source of the data is Reuters.

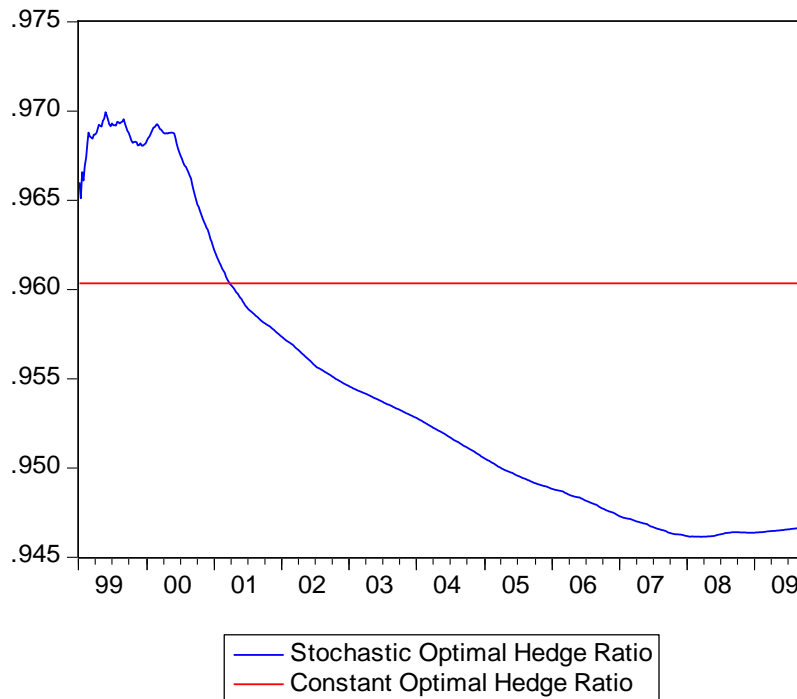
The constant optimal hedge ratio is estimated by the ordinary least squares method. It should be mentioned that we also tested for unit roots and cointegration to avoid the spurious regression

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<sup>3</sup> This simulation is conducted by programming in the C language.

problem. The results, not reported but available on request, revealed that the spot price index as well as the futures index contained one unit root. However, tests for cointegration provided evidence that the linear combination of the two indexes is stationary. Thus, there is a long-run relationship between the two variables. The stochastic hedge ratio is obtained within a state space model which is calculated by the Kalman filter. This state space model is described in the appendix. The estimation results for both hedge ratios, constant as well as stochastic, are depicted in Figure 2. As it is clearly evident from this figure, the optimal hedge ratio is indeed not constant and assuming it to be constant would result in loss of precision in the hedging strategy. In this particular case it also appears that the mean value of the stochastic optimal hedge ratio is not equal to the constant hedge ratio. This in turn implies that allowing for time-varying optimal hedge ratio stochastically is of fundamental importance when the optimal hedge ratio is calculated.

Figure 2: Stochastic and Constant Optimal Hedge Ratios in the US.





#### 4. Conclusions

In this paper we put forward the proposition that an optimal hedge ratio with a stochastic structure will perform differently than an optimal hedge ratio with a constant structure. This is even the case if stochastic optimal hedge ratio has the same mean value as the constant hedge ratio. Some mathematical proofs and simulation results are provided for this proposition. We also apply the stochastic optimal hedge ratio to the US equity market data during 1999-2009 on weekly basis. The stochastic structure of the optimal hedge ratio is deduced via the Kalman filter and it is compared to the constant hedge ratio. The estimation results reveal clearly that the optimal hedge ratio in the US equity market is indeed stochastic and it changes significantly across time. This implies that the best hedging strategy by the investor should not be constant but rather changing across time in the form of rebalancing the underlying portfolio successively.

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## Appendix

### Proof of Proposition 1:

For any time  $t \in [0, T]$  we have  $E[h_t] = [\alpha + B_t] = \alpha + E[B_t] = \alpha$ , since  $\alpha$  is a constant and  $B_t$  follows a centred Gaussian law by definition of the Brownian motion.

### Proof of Proposition 2:

We have  $e := \int_0^T |h_t - \alpha| dt = \int_0^T |\alpha + B_t - \alpha| dt = \int_0^T |B_t| dt > 0$ .

### Proof of Proposition 3:

If we assume that  $B_{t_1} = B_{t_2} = \dots = B_{t_n} = 0$  where  $0 = t_0 \leq t_1 \leq \dots \leq t_n \leq t_{n+1} = T$ , and also that  $B_t > 0$  for  $t \in (0 = t_0, t_1)$  then we have the following:

$B_t < 0$  for  $t \in (t_1, t_2)$ , and

$B_t > 0$  for  $t \in (t_2, t_3)$ ,

$B_t < 0$  for  $t \in (t_3, t_4)$ ,

⋮

$B_t > 0$  for  $t \in (t_{2k}, t_{2k+1})$ ,

⋮

$B_t > 0$  for  $t \in (t_n, t_{n+1} = T)$ , since  $n$  is supposed to be an even integer, and therefore

$$\begin{aligned} e &= \int_0^T |B_t| dt = \int_{t_0=0}^{t_1} |B_t| dt + \int_{t_1}^{t_2} |B_t| dt + \int_{t_2}^{t_3} |B_t| dt \cdots \int_{t_n}^{t_{n+1}=T} |B_t| dt \\ &= \int_0^{t_0} B_t dt - \int_{t_0}^{t_1} B_t dt + \int_{t_1}^{t_2} B_t dt - \int_{t_2}^{t_3} B_t dt + \cdots + \int_{t_n}^T B_t dt = \sum_{k=0}^n (-1)^k \int_{t_k}^{t_{k+1}} B_t dt. \end{aligned}$$

Equation (7) is obtained by applying the *Integration by Parts Formula* for stochastic integration states that for any continuously differentiable function  $f(t)$  we have the following:

$$\int_0^t f(s)dB_s = f(t)B_t - \int_0^t f'(s)B_s ds$$

So, if we take  $f(s) = s$ , we have

$$\int_0^t B_s ds = tB_t - \int_0^t s dB_s \quad \text{and} \quad \int_{t_k}^{t_{k+1}} B_s ds = (t_{k+1}B_{t_{k+1}} - t_k B_{t_k}) - \int_{t_k}^{t_{k+1}} s dB_s$$

and then

$$e = \sum_{k=0}^n (-1)^k \int_{t_k}^{t_{k+1}} B_t dt = \sum_{k=0}^n (-1)^k \left[ (t_{k+1}B_{t_{k+1}} - t_k B_{t_k}) - \int_{t_k}^{t_{k+1}} B_t dt \right] = \left[ TB_T + \sum_{k=0}^n (-1)^{k+1} \int_{t_k}^{t_{k+1}} B_t dt \right].$$

## Kalman Filter

A general formulation of a model within a state space context of the underlying vector  $y$  as a function of the vector  $x$  may be expressed by the following system of equations:

$$y_t = \beta_t' x_t + e_t$$

$$\beta_t = A\beta_{t-1} + v_t$$

Where  $e_t$  and  $v_t$  are zero means error terms that are identically and independently distributed as the following:

$$e_t \sim NID(0, \sigma)$$

$$v_t \sim NID(0, Q)$$

These conditions are required in order to make maxim likelihood inference. In addition, the error terms are assumed to be independent across the equations in the system. That is, we assume that the following condition is fulfilled:

$$E(e_t v_s) = 0, \quad \forall t \text{ and } s.$$

The goal of the state space model is to estimate the underlying parameters  $A$ ,  $P$ , and  $Q$  in order to make inference about the time varying parameter vector, given observation on the data set  $(y_t, x_t)$  at each time period  $t$ . To achieve this goal we need to make use of the Kalman filter. This filter can be described by the following questions:

$$\hat{\beta}_t = A\hat{\beta}_{t-1}$$

$$P_{t|t-1} = AP_{t-1}A' + Q$$

$$\hat{e}_t = (y_t - \beta_t' x_t)$$

$$f_t = x_t' P_{t|t-1} x_t + \sigma$$

$$\hat{\beta}_t = \hat{\beta}_{t-1} + x_t' P_{t|t-1} x_t \left( \frac{\hat{e}_t}{f_t} \right)$$

$$P_t = P_{t|t-1} - P_{t|t-1} x_t x_t' P_{t|t-1} \left( \frac{1}{f_t} \right)$$

Where  $\hat{\beta}_t$  is the optimal estimated value of the time varying parameter vector  $\beta_t$ , which has a variance equal to  $P_t$  at time the given time  $t$ . The one step forward prediction error is signified by  $\hat{e}_t$  and its variance is measured by  $f_t$ . The expression  $t|t-1$  implies that the optimal estimate of the parameter at time  $t$  is conditional on the information that is available at period  $t-1$ . It should be clarified that the optimal prediction of the future observations can be made when the end of the series is reached. Another possibility is also to utilize backward recursion in order to re-estimate the optimal estimator of the time varying parameter vector at all points in time using the full sample as is demonstrated by Harvey (1993).