Imitating the Most Successful Neighbor in Social Networks

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Abstract

We consider a model of observational learning in social networks. At every period, all agents choose from the same set of actions with uncertain payoffs and observe the actions chosen by their neighbors, as well as the payoffs they received. They update their choice myopically, by imitating the choice of their most successful neighbor. We show that in finite networks, regardless of the structure, the population converges to a monomorphic steady state, i.e. one at which every agent chooses the same action. Moreover, in arbitrarily large networks with bounded neighborhoods, an action is diffused to the whole population if it is the only one chosen initially by a non–negligible share of the population. If there exist more than one such actions, we provide an additional sufficient condition in the payoff structure, which ensures convergence to a monomorphic steady state for all networks. Furthermore, we show that without the assumption of bounded neighborhoods, (i) an action can survive even if it is initially chosen by a single agent, and (ii) a network can be in steady state without this being monomorphic.

Keywords: Social Networks, Learning, Diffusion, Imitation.

JEL Classification: D03, D83, D85.

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1. Introduction

1.1. Motivation

A common characteristic of most economic activities is that they are not organized on a centralized and anonymous way (Jackson, 2008). They rather involve bilateral interactions between agents, which also tend to be local in nature. These interactions lead to the formation of social networks, since they affect indirectly not only the direct neighbors of the agents, but also all neighbors of their neighbors, etc. Sometimes, the network may represent how information is transmitted in the population, rather than imposing direct payoff externalities.

Information transmission is a very important aspect, mainly in situations where agents base their decisions on others' experience. Several models analyze the ways in which agents are affected by the information they receive. Recently, there is strong empirical evidence supporting the fact that people tend to imitate successful past behavior (Apesteguia et al., 2007; Conley and Udry, 2010; Bigoni and Fort, 2013). This literature provides evidence that in several dynamic decision problems, the agents seem to behave as if they observe the actions of their neighbors, and they tend to imitate those who did better in the past. There are several reasons that can justify this behavior. On the one hand, agents may not be aware of the mechanisms controlling the outcome of their choices, hence they need to experiment themselves or rely on past experience of those they can observe. On the other hand, in certain environments, Bayesian updating may require calculations which are beyond the computational capabilities of the agents, leading them to adopt simple "rules of thumb" (Ellison and Fudenberg, 1993).

In this paper, we study imitation of successful past behavior in a society with a network structure. In our environment, the network describes the transition of information and not direct interaction, meaning that the decision of one agent does not affect the payoffs of the others. In particular, we study a problem of dynamic decision making under uncertainty, where the agents observe the actions and the realized payoffs of their neighbors. Subsequently, they update their decisions by imitating that neighbor who received the highest payoff in the previous period – which is the so-called "imitate-the-best" updating rule (Alós-Ferrer and Weidenholzer, 2008).

An example that is very relevant to our setting is the parents' decision about which school to send their children to; or their decision about whether to send them to a public school or a private one (also mentioned by Ellison and Fudenberg, 1993). It is apparent that the satisfaction of the parents by such a decision is based mainly on the characteristics of the school itself, rather than on the decisions of other parents. Moreover, the level of satisfaction may be different between parents who chose the same school. It is also commonly observed that the parents make this decision relying mostly on other parents' previous experience. This happens mainly because of the difficulty lying on the identification of the real quality of each school. Furthermore, information received by those who had been extremely satisfied in the past tends to be more influential; an observation that leads directly to our naive updating mechanism. One can also apply this setting to decision problems regarding the adoption of alternative technologies by agents who are not informed about their profitability, or the purchasing of a service, such as the choice of mobile telephone operator.

1.2. Results

Formally, we consider a countable population forming an arbitrary network. In each period, every agent chooses an action from a finite set of alternatives and receives a payoff, which is drawn from a continuous distribution associated with the chosen action. The agents are not aware of the underlying distributions and the payoffs do not depend on the choices of other agents. The draws are independent for every agent, meaning that it is possible that two agents choose the same action and receive different payoffs.[1] After making their own decision, they observe the actions and realized payoffs of all their neighbors. Subsequently, they update their choice, imitating myopically the choice of that neighbor who received the highest payoff in the previous round.

We show that, when the population is finite the network eventually converges with probability one to a monomorphic steady state, meaning that all the agents will eventually choose the same action. However, this action need not be the most efficient. This happens because each action is vulnerable to sequential negative shocks that can lead to its disappearance. Our result is robust to cases where only a fraction of the population revises its action each period, as well as to cases where some agents are experimenting, choosing randomly one of the actions they observed.

The results differ significantly when we let the population become arbitrarily large. First of all, without further restrictions we cannot ensure convergence to a monomorphic steady state. In fact, convergence depends on whether or not the agents have bounded neighborhoods, i.e. on whether there are agents who interact with a non-negligible share of the population.

Assuming bounded neighborhoods, we can ensure the diffusion of a single action, if this is the only one chosen initially by a non-negligible share of the population. If this is not the case, then we provide a counter example, where a network never converges to a steady state. Nevertheless, we

[1] The assumption that the draws are independent between agents facilitates the technical analysis, however looking at the proofs one can see that the results could be extended easily to cases were the draws are partially or completely correlated.

provide a sufficient condition in the payoff structure, which ensures convergence regardless of the network structure. This condition is more demanding than first order stochastic dominance, thus implying that in very large networks the diffusion of a single action is very hard to occur and it demands a very large proportion of initial adopters, or a special network structure, or an action to be much more efficient compared to all others. The fact that our sufficient condition disregards the importance of the network architecture completely, makes it useful mostly for networks with small upper bound in the size of the neighborhoods. The behavior of specific network structures would be a very interesting topic for further research.

Once we drop the assumption of bounded neighborhoods, the properties of the network change significantly. In this case, an action may survive, even if it is chosen initially by only one agent. This happens because this one agent may affect the choice of an important portion of the population; an observation that stresses the role of centrality in social networks. For instance, providing a technology or a product to a massively observed agent, can affect seriously the behavior of the population. Finally, we construct another example where a network is in steady state without this being monomorphic, which contrasts what our result for finite population has established.

Concluding, our extensive study of "imitate-the-best" learning mechanism introduces questions, which intend to act as a trigger for future research. The fact that learning is a natural procedure in societies and that imitation of successful behavior is commonly observable in many aspects of social life, makes the study of the topic important, promising and, at the same time, interesting and fascinating.

1.3. Related Literature

Our work is in line with the literature on observational learning. Banerjee (1992), Ellison and Fudenberg (1995) and Banerjee and Fudenberg (2004) introduce simple decision models where agents tend to rely on behavior observed by others, rather than experimenting themselves. Closer to our work is Ellison and Fudenberg (1993), where agents choose between two technologies, and periodically evaluate their choices. They introduce the concept of an exogenous "window width", which plays a role similar to the network structure in our case. The fact that their updating mechanism involves averaging the performance of several agents makes it significantly different from our "imitate-the-best" rule.

This paper is also related to the literature on learning from neighbors. Large part of this literaure has focused on Bayesian learning (see Gale and Kariv, 2003; Acemoglu et al., 2011) and best-response strategies.[2] On the one hand, Bala and Goyal (1998) study social learning with local interactions, under myopic best-reply. Learning occurs through Bayesian update, using information only about their own neighbors. They provide sufficient conditions for the convergence of beliefs: Neighborhoods need to be bounded in order to ensure convergence to the efficient action, which is an issue that arises also in our problem.[3] Existence of a set of agents which is connected with everybod (often called the "royal family") can be harmful for the society. This happens because, such a set may enforce the diffusion of their action/opinion, even if it not the optimal for the society. On the other hand, Gale and Kariv (2003) study a Bayesian learning procedure, but using a network and payoff structure quite similar to ours. They show that, given the fact that agents information is non-decreasing in time, the equilibrium payoff must be also non-decreasing and since they assume it to be bounded, it must converge. Moreover, in equilibrium identical agents gain the same in expectation. In general, in Bayesian procedures the agents accumulate information through time, which it is not the case in our setting. Hence, it is normal that convergence will be harder to occur, but eventually not impossible.

Learning by imitation has also been studied extensively in the literature. Various models of evolutionary game theory (see Weibull, 1995; Fudenberg and Levine, 1998) are using different types of imitation processes. In particular, Vega-Redondo (1997) provides strong theoretical support to the imitation of successful behavior. He studies the evolution of a Cournot economy, where the agents adjust their behavior by imitating the agent who was the most successful in the previous period. He concludes that, this updating rule leads the firm towards Walrasian equilibrium. Furthermore, Schlag (1998) studies different imitation rules, in an environment with agents with limited memory randomly matched with others from the same population. He concludes that the most efficient mechanism is the one where an agent imitates probabilistically the other's behavior if it yielded higher payoff in the past, with probability being proportional to the difference between the realized outcomes. In Schlag (1999), he extends his framework to infinite populations, where each agent is randomly matched with two agents each time. Even though the settings in those papers are different, they provide additional evidence about the validity of our mechanism.

The literature on imitation in networks focuses mainly on coordination games and prisoner's dilemma games (e.g., Eshel et al., 1998; Alós-Ferrer and Weidenholzer, 2008; Fosco and Mengel, 2010). In these models, uncertainty comes from the lack of information about the choices made by each agent's neighbor. However, for each action profile, the payoffs are deterministic, which is not the case in our environment. Closer to our work is the paper by Alós-Ferrer and Weidenholzer (2008),

[2]Where the agents maximize short-term utility based on observed behavior of their neighbors in the previous period.

[3]By bounded neighborhood we mean that there exists $K > 0$ such that the number of neighbors of every agent i satisfies $k_i \equiv |N_i| \leq K$, $\forall i \in N$.

where the aim is to identify the conditions for contagion of efficient actions in the long-run, problem similar to the one we study.

Our analysis focuses on the diffusion of behavior, as well as the long-run properties of a network. Large part of the literature in diffusion restricts attention to best-response dynamics, and relates the utility of adopting a certain behavior with the number, or proportion, of neighbors who have adopted it already. Morris (2000) shows that maximal contagion occurs for sufficiently uniform local interactions and low neighbor growth. Along the same line, Lopez-Pintado (2008) and Jackson and Yariv (2007) look into the role of connectivity in diffusion, finding that stochastically dominant degree distributions to favor diffusion.

The rest of the paper is organized as follows. In Section 2, we explain the model. In Section 3, we provide the main results for networks with finite population. While, in Section 4 we study networks with arbitrarily large population. Finally, in Section 5 we conclude and provide possible extensions.

2. The Model

2.1. The agents

There is a countable set of agents N, with cardinality n and typical elements i and j, mentioned as *population* of the network.[4] Each agent $i \in N$ takes an action a_i^t from a finite set $A = \{\alpha_1, ..., \alpha_M\}$, at every period $t = 1, 2, ...$Each $a \in A$ yields a random payoff.

Uncertainty is represented by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is a compact metric space, $\mathcal{F}$ is the Borel σ-algebra, and $\mathbb{P}$ is a Borel probability measure. The states of nature are drawn independently in each period and for each agent. Hence, it is possible that two agents choose the same action at the same period and they receive different payoffs.

There is a common stage payoff function $U : A \times \Omega \to \mathbb{R}$, with $U(a, \omega)$ being bounded for every action a and continuous in Ω. We restrict our attention to cases where $\forall a, a' \in A$, $U(a, \Omega)$ is a closed interval in $\mathbb{R}$ and $U(a, \Omega) = U(a', \Omega)$, while at the same time $\mathbb{P}$ has full support.[5]

2.2. The Network

A social network is represented by a family of sets $\mathcal{N} := \{N_i \subseteq N \mid i = 1, \ldots, n\}$, with N_i denoting the set of agents observed by agent i. Throughout the paper N_i is called i's neighborhood, and is

[4]In Section 3 we assume that n is finite, whereas in Section 4 we assume n to be arbitrarily large.

[5]The cases where an action is strictly payoff dominant are uninteresting, because it is trivial that the dominant action will spread to the whole population.

assumed to contain i. The sets $\mathcal{N}$ induce a graph with nodes N, and edges $E = \bigcup_{i=1}^{n} \{(i, j) : j \in N_i\}$. We focus on undirected graphs: as usual, we say that a network is undirected whenever for all $i, j \in N$ it is the case that $j \in N_i$ if and only if $i \in N_j$. In the present setting, the network structure describes the information transmission in the population. Namely, each agent $i \in N$ observes the action and the realized payoff of every $j \in N_i$.

A path in a network between nodes i and j is a sequence $i_1, ..., i_K$ such that $i_1 = i$, $i_K = j$ and $i_{k+1} \in N_{i_k}$ for $k = 1, ..., K-1$. The distance, l_{ij}, between two nodes in the network is the length of the shortest path between them. The diameter of the network, denoted as $d_{\mathcal{N}}$, is the largest distance between any two nodes in the network. We say that two nodes are connected if there is a path between them. The network is connected if every pair of nodes is connected. We focus on connected networks, however the analysis would be identical for connected components of disconnected networks.[6]

2.3. Behavior

In the initial period, $t = 1$, each agent is assigned, exogenously, to choose one of the available actions.[7] After each period, $t = 2, 3, \ldots$ the agents have the opportunity to revise their choices. We suppose that they can observe the choices and the realized payoffs of their neighbors in the previous period. According to these observations, they revise their choice by imitating the action of their neighbor who earned the highest payoff in the previous period. Ties are broken randomly.[8] Formally, for $t > 1$,

$$a_i^{t+1} \in \left\{ \; a \in A \; : \; a = a_k^t \text{ where } k = \text{argmax}_{j \in N_i} U(a_j^t, \omega_j^t) \; \right\} \tag{1}$$

where ω_j^t is the actual state of nature at t, for agent j. Throughout the paper, we refer to this updating rule as "imitate-the-best" (see also Alós-Ferrer and Weidenholzer, 2008).

Notice that, the connections among the agents determine only the information flow and do not impose direct effects in each other's payoffs. An important aspect of this myopic behavior is that the agents discard most of the available information. They ignore whatever happened before the previous round and even from this round they take into account only the piece of information related to the most successful agent. This naive behavior makes the network vulnerable to extreme shocks, which may be very misleading for the society.

[6]A component is a non-empty sub-network N' such that $N' \subset N$, N' is connected and if $i \in N'$ and $(i, j) \in E$ then $j \in N'$ and $(i, j) \in E'$.

[7]We assume that every action in A is chosen initially by some agent .

[8]Ties arise with probability zero, because the agents are countably many, while the states of nature are uncountably many.

2.4. Steady state and efficiency

State of period t is called the vector which contains the action chosen by each agent at this period, i.e. $S^t = (a_i^t, \ldots, a_n^t)$. We denote by $A_t = \{\alpha_k \in A : \exists i \in N \text{ such that } a_i^t = \alpha_k\}$ the subset of the action space, A, which contains those actions that are chosen by at least one agent in period t. Notice, that an action which disappears from the population at a given period, never reappears, hence $A_t \subseteq A_{t-1} \subseteq \cdots \subseteq A_1 \subseteq A$. In a given period t, its state is called *monomorphic* if every agent chooses the same action, i.e. if there exists $k \in \{1, ..., M\}$, such that $a_i^t = \alpha_k$ for all $i \in N$. Also, the population is in *steady state* if no agent changes her action from this period on, i.e. if $S^{t'} = S^t$ for all $t' > t$. Throughout the paper, the idea of convergence refers to *convergence with probability one*. Finally, we call an action *efficient* if it is the one that yields the highest expected payoff. An action is mentioned as *more efficient* than another one if it yields higher expected payoff.

3. Networks with finite population

In this section, we restrict our attention to networks with finite population. We prove that finite networks always converge to a monomorphic steady state, regardless of the initial conditions and the network structure. Moreover, any action can be the one to survive in the long-run.

Before presenting our main results, it is worth mentioning a remark regarding the complete network. In a complete network, each agent is able to observe actions and realized payoffs of every other agent, i.e. $N_i = N$ for all $i \in N$. If the network is complete, then it will converge to a monomorphic steady state from the second period on. Namely, the agent who received the highest payoff in the first period will be imitated by everyone else in the second period, leading to the disappearance of all alternative actions. The probability of two actions giving exactly the same payoff is equal to zero, because of the assumptions regarding the continuity of payoff functions' distributions.

Even when the network is not complete, if there exists a set of agents which is observed by everyone, then the initial action chosen by these agents is likely to affect the behavior of the whole population. For finite population, if one of them receives a payoff close to the upper bound, this will lead to the diffusion of the chosen action from the next round on, irrespectively of this being the most efficient or not (see also "Royal Family" in Bala and Goyal, 1998).

The simple cases mentioned above do not ensure neither the convergence to a steady state, nor that this steady state needs to be monomorphic. The following proposition establishes the fact that if the system reaches a steady state, then it must be the case that all the agents choose identically.

Formally,

Proposition 3.1. *Consider an arbitrary connected network with finite population. If the agents behave under imitate-the-best rule, then all possible steady states are necessarily monomorphic.*

The proof (which can be found in the appendix) is very intuitive. The idea is that when more than one actions are still chosen, then there must exist at least one pair of neighbors choosing different actions. Hence, each one of them faces a strictly positive probability of choosing a different action in the next period. This ensures that at some period in the future, at least one of the two will change her action, meaning that the population is not in steady state. The above proposition is in line with the work of Gale and Kariv (2003), where, although the updating mechanism differs, identical agents end up making identical choices in the long run. Notice that the result is no longer valid if the network becomes arbitrarily large (see Example 4.4 - Two Stars). Nevertheless, we have not ensured yet the convergence of the population to a steady state, but only the fact that if there exists a steady state, then it has to be monomorphic. In principal, all monomorphic states are possible steady states.

In the following theorem we provide the first main result of the paper, which is that any arbitrary network, with finitely many agents, who behave under "imitate-the-best rule", converges to a monomorphic steady state. The intuition behind our result is captured by the following three lemmas, which are also used for the formal proof (all proofs can be found in the appendix).

Lemma 3.1. *Consider an arbitrary connected network with finite population, where more than one action is observed. Each period t, every action $\alpha_k \in A_t$ faces positive probability of disappearing after no more than $d_{\mathcal{N}}$ periods.*

The main idea behind this lemma is that every action is vulnerable to a sequence of negative shocks to the payoffs of its adopters (agents who are currently choosing the action). Regardless of the number and the position of those agents, the probability of disappearance after finitely many periods is strictly positive.

Lemma 3.2. *Suppose that $K-1$ actions have disappeared from the network until period t. Then, there is strictly positive probability that exactly K actions will have disappeared from the network after a finite number of periods τ.*

This result is a direct implication of Lemma 3.1. Its importance becomes apparent if we notice that an action which disappears from the population (not chosen by any agent at a given period) never reappears. τ can take different values depending on the structure of the network and the initial conditions, but it is always bounded above by the diameter of the network $d_{\mathcal{N}}$.

Lemma 3.3. *Suppose that $K-1$ actions have disappeared from the network until period t. Then there is strictly positive probability of convergence to a monomorphic state in the next $T = (M-K+1)d_{\mathcal{N}}$ periods.*

There are many possible histories which lead to convergence to a monomorphic state. Some of them lead to the disappearance of one action every $\tau = d_{\mathcal{N}}$ periods, which is an intersection of events with strictly positive probability of occuring, as shown in Lemma 3.2.

Theorem 3.1. *Consider an arbitrary connected network with finite population. If the agents behave under imitate-the-best rule, then the network will converge with probability one to a monomorphic steady state.*

By Lemma 3.3 we know that with strictly positive probability convergence will occur after a finite number of periods. Analogously, convergence will not occur in the same number of periods with probability bounded below one. Hence, the probability that convergence never occurs is given by the infinite product of probabilities strictly lower than one, which means that convergence to a monomorphic steady state is guaranteed in the long run.

Corollary 3.1. *For a finite network, there is always positive probability of convergence to a sub-optimal action.*

The corollary is apparent from the fact that even the most efficient action faces a positive probability of disappearance, as long as there are more actions chosen in the network. This result points out a weakness of this updating mechanism, which is the inability to ensure efficiency. However, if we the population is arbitrarily large, then we provide sufficient conditions for the diffusion of the most efficient action (see Section 4).

Probabilistic updating

Our results still hold if we relax the assumption that all the agents update their choices in every period and introduce probabilistic updating. Formally, suppose that in every period there is a positive probability, $r > 0$, that an agent will decide to update her choice. The proof is identical to the one of Theorem 3.1, if we multiply the lower bound of Lemma 3.1 by $r^{|I_k^t|}$, where $|I_k^t|$ is the cardinality of the set of agents choosing action α_k in period t. Hence, the network again converges to a monomorphic steady state with probability one, although convergence may occur at a slower rate.

Experimentation

The result holds even under certain forms of experimentation. In particular, we transform the updating rule as following: Every period, each agent imitates her most successful neighbor with probability $(1-\epsilon) \in (0,1)$ and with probability ϵ imitates randomly another of the actions she observes, including her own current choice. Under this updating rule, the proof remains the same, multiplying again the lower bound of Lemma 4.1 by $(1-\epsilon)^{|I_k^t|}$. Notice, that here the experimentation is limited to observed actions, which allows us to ensure that once an action disappears from the network it does not reappear.

4. Networks with arbitrarily large population

In this section, we consider an arbitrarily large population N. Formally, what we mean is a sequence of networks $\{(\{1,\ldots,n\},\mathcal{N}^n)\}_{n=1}^{\infty}$, where every agent i of the n-th network is also an agent of the $(n+1)$-th network, and moreover any pair of agents i,j of the n-th network are connected if and only if they are connected in the $(n+1)$-th network. Roughly speaking, that would mean that the $(n+1)$-th network of the sequence would be generated by adding one additional agent to the n-th network. Notice that, every countably infinite network can be obtained as the limit of such a sequence. Henceforth, with a slight abuse of notation, we write $n \to \infty$.

Before we begin our analysis we need two definitions. First, we say that *an action is used by a non-negligible share of the population* if the ratio between the number of agents using this action and the size of the population does not vanish to zero, as n becomes arbitrarily large. Formally, if $I_k^t = \{i \in N : a_i^t = \alpha_k\}$ is the set of agents who choose α_k at period t, then:

Definition 1. *α_k is used by a non-negligible share of the population at period t, if* $\lim\limits_{n\to\infty} \frac{|I_k^t|}{n} > 0$.

Second, we say that *an agent has unbounded neighborhood*, or equivalently, *an agent is connected to a non-negligible share of the population*, if there exists at least one agent that the ratio between the size of her neighborhood and the size of the population does not vanish to zero as n becomes arbitrarily large. Formally, if $|N_i|$ is called the *degree* of agent i and denotes the number of agents that i is connected with, then:

Definition 2. *Agent i has unbounded neighborhood (is connected to a non-negligible share of the population), if* $\lim\limits_{n\to\infty} \frac{|N_i|}{n} > 0$.

Notice that, this is equivalent saying that for all $K \in \mathbb{N}$ it holds that $|N_i| > K$.

At first glance, one could doubt whether there is a difference between the cases of finite and arbitrarily large networks. Throughout this section we show, why the two cases are indeed different. Intuitively, we expect different behavior, since for arbitrarily large networks there must exist actions chosen by a non-negligible share of the population and for each of these actions, the probability of disappearance in finite time vanishes to zero.

Moreover, the possibility that some agents may be connected with a non-negligible share of the population, makes them really crucial for the long-term behavior of the society. This is a property that changes the results of our analysis, even between different networks with arbitrarily large population. For this reason, we introduce the following assumption. (Keep in mind that the following assumption is used only when it is clearly stated.)

Assumption 1. [Bounded Neighborhood] Exists $K > 0 \in \mathbb{N} : |N_i| \leq K$, for all $i \in N$ ◁

Notice that, Assumption (1) does not hold if there exists an agent who affects a non-negligible share of the population. In the rest of the section, we compare the cases where Assumption 1 holds or does not hold, while stressing the conditions that make the results of Section 3 fail when the population becomes very large.

4.1. Bounded Neighborhoods

In this part we assume that Assumption (1) holds. The main importance of this assumption is that it removes the cases where an agent can affect the behavior of a non-negligible share of the population.

For an arbitrarily large network, an obvious, nevertheless crucial, remark is that there must be at least one action chosen by a non-negligible share of the population. Our analysis will be different when there is exactly one such action and when there are more.

Proposition 4.1. *Consider a network with arbitrarily large population ($n \to \infty$), where there is only one action, α_k, initially chosen by a non-negligible share of the population. If the agents behave under imitate-the-best rule and Assumption (1) holds, then the network will converge with probability one to the monomorphic steady state, where every agent will choose α_k.*

The result of the above proposition is quite intuitive. If an action is chosen by a non-negligible share of the population and also each agent can affect finitely many others, then the probability that this action will disappear in a finite number of periods vanishes to zero. However, this is not the case for the rest of the actions, which face a positive probability of disappearance. Hence, no other action can survive in the long-run. Nevertheless, Proposition 4.1 covers only the case where there is only one such action.

The question that follows naturally is whether the network has the same properties when more than one actions are diffused to a non-negligible share of the population. For a general network and payoff structure, the answer is negative. The negative result is supported by a counter-example. More formally, consider a network with arbitrarily large population behaving under imitate-the-best rule and having bounded neighborhoods. Then, if there are more than one actions chosen by a non-negligible share of the population, we cannot ensure convergence to a monomorphic steady state, without imposing further restrictions to the network or/and payoff structure.

Example 4.1. *Think of the following arbitrarily large network (Figure 1- Linear Network).*

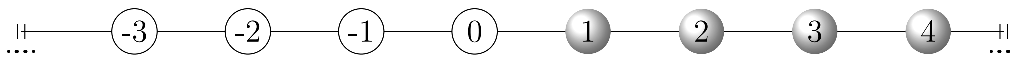
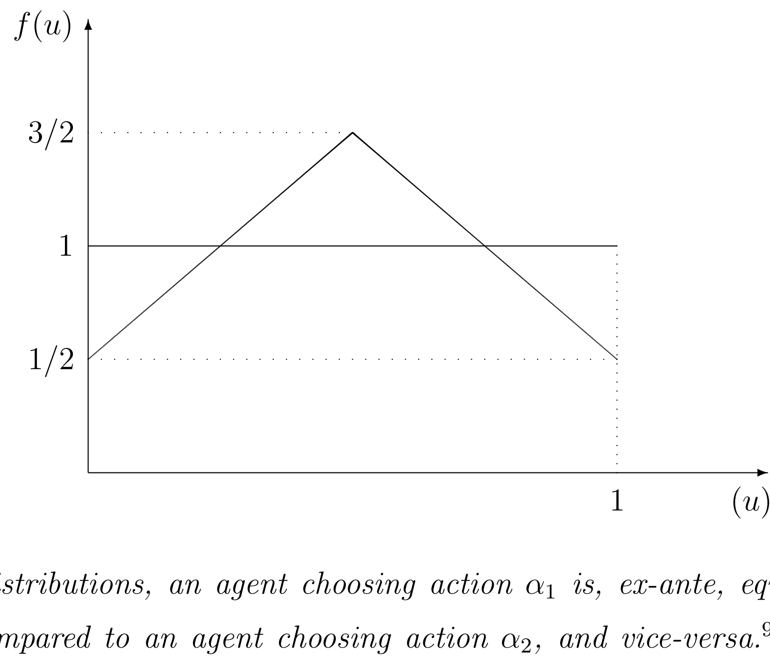


Figure 1: Linear Network

Notice that, all the agents have bounded neighborhoods, since they are connected with exactly two agents and the diameter of the network, $d_{\mathcal{N}} \to \infty$ *as* $n \to \infty$. *At time* t*, there are two actions still present,* α_1 *and* α_2*. Half of the population chooses each action; all the agents located from zero and to the left choose* α_1 *and the rest choose* α_2*. Both actions have the same support of utilities,* $U(\alpha_1, \Omega) = U(\alpha_2, \Omega) \in [0, 1]$*. The distributions of the payoffs are as shown in the following graph.*



For these distributions, an agent choosing action α_1 *is, ex-ante, equally likely to get higher or lower utility compared to an agent choosing action* α_2*, and vice-versa.*[9] *Moreover, notice that the*

[9] $P[X_1 > X_2] = \int_0^1 P[X_1 > x_2] f_2(x_2) dx_2 = \int_0^{1/2} (1 - x_2)(\frac{1}{2} + 2x_2) dx_2 + \int_{1/2}^1 (1 - x_2)(\frac{5}{2} - 2x_2) dx_2 = \frac{1}{2}$

only agents who can change are those in the boundary between the groups using each action. This boundary will be moving randomly in the form of a random walk without drift. With probability $\frac{1}{4}(\frac{1}{4})$ will move one step to the left (right) and with probability $\frac{1}{2}$ will stay at the same position. By standard properties of random walks, this boundary will never diverge to infinity.

The divergence of the boundary is identical to the disappearance of one of the two actions, i.e. convergence to a monomorphic steady state. Since we have shown that this is not possible, it means that this network will never converge to a monomorphic steady state. In fact it will be fluctuating continuously around zero, without ever reaching a steady state.

The negative result of the previous example does not allow us to ensure convergence to a steady state under every structure, when at least two actions are chosen by a non-negligible share of the population. However, there exist sufficient conditions, related to the payoff and network structure, which can ensure it.

In the following proposition, we consider cases where all the agents have bounded neighborhoods and that all the remaining actions are chosen by non-negligible shares of the population. These two facts yield the initial remark that no action will disappear in finite time. Nevertheless, it is shown that it is possible for some action to be diffused to a continuously increasing share of the population, capturing the whole of it in the long-run. Formally, if F_k is the cumulative distribution of payoffs associated with action α_k, then:

Proposition 4.2. *Consider a network with arbitrarily large population, with agents behaving under imitate-the-best rule and having bounded neighborhoods, with upper bound D. Moreover, each of the remaining actions, $\{\alpha_1, ..., \alpha_m\}$ is chosen by a non-negligible share of the population, $\{s_1, ...s_m\}$. If there exists action α_k such that:*
(i) $F_k(u) \leq [F_{k'}(u)]^D$, for all $k' \neq k$, and
(ii) s_k is sufficiently large,
then the network will converge, with probability one, to a monomorphic steady state where every agent chooses action α_k.

Notice that D is an exponent of F.

Our sufficient condition is weaker than first order stochastic dominance, nevertheless we have the advantage of providing a result adequate for every network structure. The important fact in our proof, is that the agents changing to α_k are, in expectation, more than those changing from α_k to some other action. In general, this condition may depend not only on the payoff structure and the initial share, but also on the network structure, which is something we completely disregarded in the present analysis. Hence, if we can construct a condition depending on both network and payoff structure that yields the same result, then we could again ensure convergence. This result can become the benchmark for future research on stronger conditions for specific structures.

Moreover, it is somehow surprising that a condition as strong as first order stochastic dominance may not be sufficient. This happens either because of the complexity of the possible network structures, or because of insufficient initial share of the action of interest. To clarify this argument, we construct the following example.

Example 4.2. *Consider the linear network described in Example 4.1. At time $t = 0$, there are two actions present in the network, α_1 and α_2. Like before, half of the population chooses action α_1 (all the agent located at zero and to the left) and, analogously, the rest choose α_2. Notice that each agent has a neighborhood consisting of two agents apart from herself. Moreover, every period, there are only two agents, one choosing each action, who face positive probability of changing their chosen action.*

The payoffs are such that $F_1 < F_2$ (i.e. action α_1 First Order Stochastically Dominates action α_2), but $F_1 \geq (F_2)^2$.[10] Calling r_t the number of agents who change from action α_2 to action α_1 we get the following:

$$r_t = \begin{cases} 1 & \text{with probability } p_1 \\ 0 & \text{with probability } p_2 \\ -1 & \text{with probability } p_3 \end{cases}$$

FOSD ensures that $p_1 > p_3$. However, the second condition means that $p_1 \leq p_2 + p_3$, or equivalently $p_1 - p_3 < \frac{1}{2}$. The expected value of r_t will be $E[r_t] = (p_1 - p_3)$. So, $\frac{\lim_{\tau\to\infty} \sum_{t=1}^{\tau} r_t + s_1}{\lim_{n\to\infty} n} = p_1 - p_3 + s_1$. In order to get diffusion of α_1 we need $p_1 - p_3 + \frac{1}{2} \geq 1$, which here cannot be the case. Hence, although α_1 is FOSD compared to α_2, α_2 will be chosen by non-negligible shares even in the long run.

On the contrary, notice that if $F_1 \leq (F_2)^2$, then it can be the case (not necessarily) that $p_1 - p_3 \geq \frac{1}{2}$ and action α_1 is diffused to the whole population.

4.2. Unbounded Neighborhoods

In this section we drop the assumption of neighborhoods being bounded. This means that as n grows, there exists at least one agent whose neighborhood grows as well without bounds. If neighborhoods are unbounded, we cannot ensure convergence to a monomorphic steady state, even when the diameter of the network is finite. This happens because a single agent can affect a non-negligible share of the population, meaning that in one period an action can be spread from a finite subset of the population to a non-negligible share of it. This means that we will have more than one actions

[10]For example, let $F_1(u) = u$ and $F_2(u) = u^{\frac{2}{3}}$

chosen by non-negligible shares of the population, which may lead the network not to converge to a monomorphic state. We clarify the above statement with the following example. Moreover, in case of unbounded neighborhoods, it does not hold anymore the result of Proposition 3.1, based on which the steady state has to be necessarily monomorphic. We provide an example where a network is in steady state and there are more than one actions present.

Example 4.3. *Think of the star network (Figure 2 - One Star), that satisfies finite diameter. As the population grows, the neighborhood of agent* 1 *also grows, without bounds.*

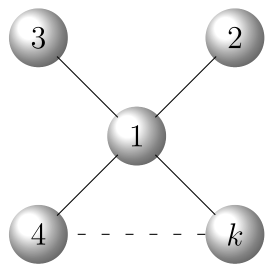


Figure 2: One Star

Suppose there are two actions present in the network, with payoffs same as in the Example 4.1. Initially, all the peripheral agents choose action α_1*, while the central agent chooses* α_2*. It is apparent that, as* $n \to \infty$ *the probability that the central agent will change her action in the second period will go to 1, because for* n *very large, for any realized payoff of the central agent, there will be at least one peripheral agent who received higher payoff than her. However, reversing the argument, as* $n \to \infty$ *the probability that every peripheral agent will receive higher payoff than the central agent goes to 0. Then, for any realized payoff of the central agent, there will be a non-negligible share of peripheral agents who received lower payoff than this. Hence, they will change to* α_2*. This is because, for any realized payoff of* $U(\alpha_2, \cdot) = \hat{u}$ *it holds that* $Pr[U(\alpha_1, \cdot) \leq \hat{u}] = F_1(\hat{u}) > 0$.

A similar argument holds in every period, leading the choice of the central agent to change randomly from the second period on. This leads the network to a continuous fluctuation, where a non-negligible share of the population chooses each action in every period. Obviously, the system will never converge to a steady state.

We have shown that without assuming the neighborhoods to be bounded, we cannot ensure convergence to a steady state. In the following part we provide an example where the network can be in steady state, without this being monomorphic. This is in contrast to Proposition 3.1, where we have shown that for finite population, steady states are necessarily monomorphic. For arbitrarily large population, this may not be the case.

Example 4.4. *Think of the following network (Figure 3 - Two Stars), that satisfies finite diameter and two of the agents have unbounded neighborhoods.*

Figure 3: Two Stars

Suppose there are two actions present in the network, with payoffs same as in Example 4.1. All the agents on the left star of the figure, including the center, choose α_1 *and similarly all the agents on the right star choose* α_2*. As* n *grows, the neighborhoods of the central agents grow undoubdedly. They get connected with more and more agents choosing the same action as them and only one choosing differently. Hence, the probability that they will change action goes to 0, as* n *goes to infinity, hence they will continue acting the same, with probability one.*[11] *The peripheral agents have only one neighbor each, who always chooses the same action as them; so none of them will change her action either. Concluding, this network will be in a steady state where half of the agents choose each action.*

The intuitive conclusion of this observation is that groups of agents who choose the same action and communicate almost exclusively between them can ensure the survival of this action in the long-run, since no agent of this group ever changes her choice.

5. Conclusion

We study a model of observational learning in social networks, where agents imitate myopically the behavior of their most successful neighbor. We focus on identifying if an action can spread in the whole population, as well as the conditions under which this is possible. Our analysis reveals the different properties between finite and arbitrarily large populations.

For networks with finite population, we show that the network necessarily converges to a steady state, and this steady state has to be monomorphic. Moreover, it cannot be ensured that the action diffused is the most efficient. This reveals the fact that small populations are more vulnerable to

[11] As n grows, the extreme case where one of the two centers changes action will occur only if the other center receives the absolute maximum, but by continuity of the distributions of payoffs this will occur with probability 0.

misguidance that can lead to the diffusion of inefficient actions. Furthermore, an action's performance in the initial periods, is crucial for its survival, since a sequence of negative shocks in early periods can lead to early disappearance of the action from the population.

The results differ when the population is arbitrarily large. Under the assumption of bounded neighborhoods, an action is necessarily diffused, regardless of its efficiency, only when it is the only one to be chosen by a non-negligible share of the population. When more such actions exist, we provide a sufficient condition in the payoff structure, which can ensure convergence regardless of the network architecture. The general idea behind this result is that the diffusion of an action in a very large network is quite hard and requires either very special structure, very large proportion of initial adopters, or an action to be much more efficient compared to the rest.

Our sufficient condition is valuable mostly in networks where the largest neighborhood has quite small size. This increases the importance of studying the role of network architecture, rather than the payoff structure, when applying to network where agents have large neighborhoods. The advanced complexity of this problem, makes it hard to deal with its general version. A very interesting and natural extension of the present paper would be to study different payoff structures in specific networks with importance in applied problems.

The role of central agents becomes apparent when we drop the assumption of bounded neighborhoods. Even an action initially chosen by a single agent can survive in the long-run, if this agent is observed by a non–negligible share of the population. This stresses the influential power of massively observed agents in a society. Affecting the decision of a very well–connected agent can become very beneficial for its diffusion.

In the present paper, we have studied some important properties of an "imitate-the-best" mechanism in a social network. However, there are still crucial questions to be answered. Specifically, our analysis refers only to long-run behavior, without mentioning neither the speed of convergence, nor the finite time behavior of the population. Different network structures are expected to have much different characteristics, leaving open space for future research.

6. Appendix

Proof of Proposition 3.1. Suppose that the statement does not hold. This means that at the steady state at least two different actions are chosen by some agent. for any structure, given that the network is connected, there will be at least a pair of neighbors, $i, j \in N$ choosing different actions. Moreover, given that the population is finite, each one of these agents must have a bounded neighborhood. Let $b_{i,1}$ and $b_{i,2}$ be the number of neighbors of i choosing actions α_1 and α_2 respectively.

If we are in a steady state, this means that agent i will never change her choice. However, because of the common support and continuity of payoff functions, as well as boundedness of neighborhoods, it always holds that:[12]

$$\begin{aligned} &Pr[U(j,\omega_j^t) > U(l,\omega_l^t): \text{ for some } j \in b_{i,2} \text{ and for all } l \in b_{i,1}] = \\ &= (1 - \prod_{j \in b_{i,2}} Pr[U(j,\omega_j^t) \leq \hat{u}]) \prod_{l \in b_{i,1}} Pr[U(l,\omega_l^t) \leq \hat{u}] = \\ &= [1 - F_2(\hat{u})^{b_{i,2}}] F_1(\hat{u})^{b_{i,1}} \geq p > 0 \end{aligned} \tag{2}$$

Hence, every period agent i will change her action with strictly positive probability p. The probability that she will never change, given that no other agent in the network does so, is zero, because:

$$\lim_{t\to\infty} \prod_{t=1}^{\infty} (1-p) = \lim_{t\to\infty} (1-p)^t = 0$$

This leads to contradiction of our initial argument. The analysis is identical under the presence of more than two actions. Concluding, we have shown that it is impossible for a connected network with finite population to be in a steady state when more than one actions are still present. □

Proof of Lemma 3.1. Suppose that, at period t there are at least two actions observed in the network. Take all players $i \in N$ such that $a_i^t = \alpha_k$. Let $I_k^t = \{i \in N : a_i^t = \alpha_k\}$ be the set of agents who chose action α_k in period t and $\neg I_k^t = \{i \in N : a_i^t \neq \alpha_k\}$ the set of agents who did not. Also, let $F_k^t = \{i \in I_k^t : N_i \cap \neg I_k^t \neq \emptyset\}$, i.e. the set of agents in I_k^t who have, at least, one neighbor choosing an action different than α_k. Moreover, let the set $NF_k^t = \{i \in I_k^t : N_i \cap F_k^t \neq \emptyset \text{ and } i \notin F_k^t\}$ contain those agents who have neighbors contained inF_k^t, without themselves being contained in it.

Think of the following events: $\hat{b}^t = \{\omega_i^t \in \Omega$, for $i \in N_{i,\alpha_k}^t = \{F_k^t \cup NF_k^t\} : U_i^t(\alpha_k, \omega_i^t) \leq \hat{u}^t\}$, meaning that all agents in N_{i,α_k} get payoff lower than a certain threshold and $\hat{B}^t = \{\omega_j^t \in \Omega$, for $j \in N_i \setminus N_{i,\alpha_k}^t$, where $i \in N_{i,\alpha_k}^t$ and $k' \neq k : U_j^t(\alpha_{k'}, \omega_j^t) > \hat{u}^t\}$, meaning that all the neighbors of the above mentioned agents who choose different action get payoff higher than this threshold.

Given that the states of nature are independent between agents, we can define a lower bound to the probability that all agents who play α_k at time t change their choice in the following period and none of their neighbors changes to α_k. We denote this event as C^t. In fact, $C^t \supset \{\hat{b}^t \cap \hat{B}^t\}$, because the intersection of $\hat{b}^t$ and $\hat{B}^t$ is a specific case included in the set C^t. Hence, $Pr(C^t) \geq Pr(\hat{b}^t \cap \hat{B}^t) = Pr(\hat{b}^t)Pr(\hat{B}^t)$. Notice that we impose independence between $\hat{b}^t$ and $\hat{B}^t$, which holds because the realized payoffs are independent across agents and across periods.

[12] For F_1 and F_2 being the cumulative distribution functions of the payoffs of actions α_1 and α_2 respectively

Remark: $\forall \alpha_k \in A$ and $\omega \in \Omega \Rightarrow Pr[U(\alpha_k, \omega) \leq \hat{u}] \in [b_t, B_t]$ where $b_t, B_t \in (0,1)$. Hence, $Pr[U(\alpha_k, \omega] \leq \hat{u}^t) \geq b_t > 0$ and $Pr[U(\alpha_k, \omega) > \hat{u}^t] \geq 1 - B_t > 0$. This is because of the full support of U and its continuity in Ω.

Using this remark we observe that $Pr(\hat{b}^t) \geq b_t^{|N^t_{i,\alpha_k}|}$ and analogously $Pr(\hat{B}^t) \geq (1 - B_t)^{|N_i \setminus N^t_{i,\alpha_k}|}$[13]. Hence,

$$Pr(C^t) \geq Pr(\hat{b}^t \cap \hat{B}^t) = Pr(\hat{b}^t)Pr(\hat{B}^t) \geq b_t^{|N^t_{i,\alpha_k}|}(1 - B_t)^{|N_i \setminus N^t_{i,\alpha_k}|} > 0$$

Notice that, C^t and C^{t+1} are conditionally independent, because the realization or not of C^t does not give any extra information about the realization of C^{t+1}.

Let $D^t_{\alpha_k}$ denote the event that, starting from period t, action α_k will disappear in the next $d_{\mathcal{N}}$ periods. One of the possible histories that leads to disappearance of action α_k is the consecutive realization of the events $C^{t+\tau}$ for $\tau \geq 0$, until all the agents who were using α_k at t have changed their choice. The number of periods needed depends on the structure of the network. More specifically it is at most equal to the diameter of the network, which cannot be greater than $n - 1$. Hence, we can construct a strictly positive lower bound for the probability of occurrence of the event $D^t_{\alpha_k}$:

$$\begin{aligned} Pr(D^t_{\alpha_k}) \geq Pr(\bigcap_{\tau=0}^{d_{\mathcal{N}}-1} C^{t+\tau}) = [\prod_{\tau=1}^{d_{\mathcal{N}}-1} Pr(C^{t+\tau} \mid C^{t+\tau-1})]Pr(C^t) \geq \\ \geq \prod_{\tau=0}^{d_{\mathcal{N}}-1} b_{t+\tau}^{|N^{t+\tau}_{i,\alpha_k}|}(1 - B_{t+\tau})^{|N_i \setminus N^{t+\tau}_{i,\alpha_k}|} = \dot{B} > 0. \end{aligned}$$

□

Proof of Lemma 3.2. Denote as K_t the following set of histories:

$$K_t = \{\text{at time } t \text{, exactly } K \text{ actions have disappeared from the network}\}$$

It is enough to show that $Pr(K_{t+\tau} \mid (K-1)_t) > 0$ where $\tau < \infty$ (in fact, at most $\tau = d_{\mathcal{N}}$). Using the definition of $D^t_{\alpha_k}$ from Lemma 3.1 we get:

$$Pr[K_{t+\tau} \mid (K-1)_t] = \sum_{\alpha_k \in A_t} \{Pr[D^{t+d_{\mathcal{N}}-1}_{\alpha_k} \mid (K-1)_t]\} - Pr[\bigcup_{m=1}^{M-K-1} (K+m)_{t+\tau} \mid (K-1)_t] \qquad (3)$$

where M is the total number of possible actions. Namely, the above expression tells that the probability of exactly one more action disappearing in the next τ periods equals the sum of probabilities of disappearance of each action, subtracting the probability that more than one them will

[13] Recall that $\mid N^t_{i,\alpha_k} \mid$ denotes the amount of agents belonging to this set

disappear in the given time period. Analogously:

$$Pr[(K+1)_{t+\tau} \mid (K-1)_t] = \sum_{\alpha_k,\alpha_{k'} \in A_t}^{k \neq k'} \{P[D_{\alpha_k}^{t+d_{\mathcal{N}}-1} \bigcap D_{\alpha_{k'}}^{t+d_{\mathcal{N}}-1} \mid (K-1)_t]\} - Pr[\bigcup_{m=2}^{M-K-1} (K+m)_{t+\tau} \mid (K-1)_t]$$

$$Pr[(K+1)_{t+\tau} \mid (K-1)_t] + Pr[\bigcup_{m=2}^{M-K-1} (K+m)_{t+\tau} \mid (K-1)_t] = \sum_{\alpha_k,\alpha_{k'} \in A_t}^{k \neq k'} Pr[D_{\alpha_k}^{t+d_{\mathcal{N}}-1} \bigcap D_{\alpha_{k'}}^{t+d_{\mathcal{N}}-1} \mid (K-1)_t]$$

But:

$$Pr[(K+1)_{t+\tau} \mid (K-1)_t] + Pr[\bigcup_{m=2}^{M-K-1} (K+m)_{t+\tau} \mid (K-1)_t] = Pr[\bigcup_{m=2}^{M-K-1} (K+m)_{t+\tau} \mid (K-1)_t]\}$$

$$\Rightarrow Pr[\bigcup_{m=2}^{M-K-1} (K+m)_{t+\tau} \mid (K-1)_t]\} = \sum_{\alpha_k,\alpha_{k'} \in A_t}^{k \neq k'} Pr[D_{\alpha_k}^{t+d_{\mathcal{N}}-1} \bigcap D_{\alpha_{k'}}^{t+d_{\mathcal{N}}-1} \mid (K-1)_t]$$

Hence, by equation (2):

$$Pr[K_{t+\tau} \mid (K-1)_t] = \sum_{\alpha_k \in A_t} \{Pr[D_{\alpha_k}^{t+d_{\mathcal{N}}-1} \mid (K-1)_t]\} - \sum_{\alpha_k,\alpha_{k'} \in A_t}^{k \neq k'} \{Pr[D_{\alpha_k}^{t+d_{\mathcal{N}}-1} \bigcap D_{\alpha_{k'}}^{t+d_{\mathcal{N}}-1} \mid (K-1)_t]\}$$

By Lemma 3.1, we have shown that the first summation is strictly larger than zero, and we just need to show that it is also strictly larger than the second one. Notice that the first summation is weakly larger trivially, because for any events A, B, C it holds that $Pr[A \mid C] \geq Pr[A \cap B \mid C]$. If the equality was possible, this would mean that the disappearance of any action would lead necessarily to the disappearance of another action. However, this cannot be the case because of the independence of payoff realizations across agents. Hence we have shown that: $Pr[K_{t+\tau} \mid (K-1)_t] \geq \ddot{B} > 0$, which completes our argument. □

Proof of Lemma 3.3. The event of "convergence to a monomorphic state" is the same as telling that $M-1$ actions will have disappeared after T periods. One possible way of this to happen is if one action disappears every $\tau = d_{\mathcal{N}}$ periods. Namely:

$$\{(M-1)_{t+T}\} \supset \{\{(K)_{t+\tau} \mid (K-1)_t\} \cap \{(K+1)_{t+2\tau} \mid (K)_{t+\tau}\} \cap \cdots \cap \{(M-1)_{t+T} \mid (M-2)_{t+T-\tau}\}\}$$

However, notice that the event on the right hand side of consists of other independent events, because the states of nature are independent across periods. Hence, recalling the result of Lemma 3.3, we construct the following relation.

$$\begin{aligned} P_t[(M-1)_{t+T}] &\geq P_t[\{(K)_{t+\tau} \mid (K-1)_t\} \cap \{(K+1)_{t+2\tau} \mid (K)_{t+\tau}\} \cap \cdots \cap \{(M-1)_{t+T} \mid (M-2)_{t+T-\tau}\}] \\ &= P_t[(K)_{t+\tau} \mid (K-1)_t] P_{t+\tau}[(K+1)_{t+2\tau} \mid (K)_{t+\tau}] \ldots P_{t+T-\tau}[(M-1)_{t+T} \mid (M-2)_{t+T-\tau}] \\ &\geq \ddot{B}^{(M-K-1)} = \underline{C} > 0 \end{aligned}$$

This means that the probability of convergence in finite time is strictly positive. Moreover, notice that, for all $K < M-1 \Rightarrow \underline{C} < 1$. This remark is trivial because if $K = M-1$, the system has already converged, nevertheless we will use this to prove the following theorem. □

Proof of Theorem 3.1. Lemma 3.3 shows us that the probability that the network will NOT converge to a monomorphic steady state in the next T periods, given that it has not converged until the current period, is bounded below 1. Formally:

$$P_t\{[(M-1)_{t+T}]^c \mid [(M-1)_t]^c\} \leq 1 - \underline{C} < 1$$

The event that the network will never converge is just the intersection of all the events where the network does not converge after $t+T$ periods, given that it has not converged until period t.

$$\{\text{The Network never Converges}\} = \{NC\} = \bigcap_{i=0}^{\infty}\{[(M-1)_{t+(i+1)T}]^c \mid [(M-1)_{t+iT}]^c\}$$

However, again these events are independent. Namely, the event of no convergence until $t+2T$ given no convergence until $t+T$ is independent of the event of no convergence at $t+T$ given no convergence until t: $\{[(M-1)_{t+2T}]^c \mid [(M-1)_{t+T}]^c\} \perp \{[(M-1)_{t+T}]^c \mid [(M-1)_t]^c\}$.

Hence, we can transform the above expression in terms of probabilities:

$$\begin{aligned} P[\{NC\}] &= \lim_{s\to\infty}\prod_{i=0}^{s} P\{[(M-1)_{t+(i+1)T}]^c \mid [(M-1)_{t+iT}]^c\} \\ &\leq \lim_{s\to\infty}(1-\underline{C})^s = 0 \end{aligned}$$

So, the network will converge with probability one to a monomorphic steady state. □

Proof of Proposition 4.1. Let $I_k^t = \{i \in N : a_i^t = \alpha_k\}$ be the set of agents who choose α_k at time t and analogously $\neg I_k^t = \{i \in N : a_i^t \neq \alpha_k\}$ the set of agents who do not. By construction of the problem, it holds that $\lim_{n\to\infty}\frac{|I_k^t|}{n} > 0$, hence $|I_k^t| \to \infty$ as $n \to \infty$. On the other hand, $\lim_{n\to\infty}\frac{\neg|I_k^t|}{n} = 0$, hence only finitely many agents do not choose α_k. For the rest of the notation recall Lemma 3.1.

Notice that, every action $k' \neq k$ is chosen by a finite number of agents, so the longest distance, $L_{k'}$, between an agent choosing k' and the closest agent choosing $k'' \neq k'$ must have finite length, $l \leq L_{k'}$. Hence, for all $k' \neq k$, the result of Lemma 3.1 still holds, if we substitute the diameter $d_{\mathcal{N}}$ by the maximum of all these distances, say L.

$$Pr(D_{\alpha_{k'}}^t) \geq Pr(\bigcap_{\tau=0}^{L} C^{t+\tau}) = \prod_{\tau=0}^{L} Pr(C^{t+\tau} \mid C^{t+\tau-1}) \geq \prod_{\tau=0}^{L} b_{t+\tau}^{|N_{i,\alpha_{k'}}^{t+\tau}|}(1-B_{t+\tau})^{|N_i \setminus N_{i,\alpha_{k'}}^{t+\tau}|} = \dot{B} > 0$$

Notice, as well, the importance of the assumption for bounded neighborhoods. If this did not hold, then we could not ensure that the above product would be strictly positive. Moreover, the expression does not hold for the action α_k. The bounded neighborhoods' assumption can hold only if the diameter of the network grows without bounds as n grows. Given that α_k is the only action chosen by non-negligible share of the population, L_k must also grow without bounds. Hence, for any finite number of periods τ there exists n large enough, such that action α_k faces a zero probability of disappearance before period $t+\tau$. More intuitively, this happens because each one of the agents choosing a different action can affect the choice only of a finite number of agents, each period. Hence, it is not possible that a non-negligible share of the population will stop choosing α_k in a finite time period.

Subsequently, the result of Lemma 3.2 still holds, with some appropriate modification. Namely $\tau = L$ and $Pr[K_{t+\tau} \mid (K-1)_t, I_k^{t+\tau} \neq \emptyset] \geq \ddot{B} > 0$, meaning that action α_k cannot disappear from the network.

With similar reasoning, we get the modification of Lemma 3.3, which tells that there is positive probability of convergence to a monomorphic steady state in finite time, given that action α_k will not disappear. But, this is equivalent to the case where every agent chooses action α_k, denoted as $\{CA_k\}$. Formally, $P_t[\{CA_k\}] \equiv P_t[(M-1)_{t+T} \mid I_k^{t+T} \neq \emptyset] \geq \ddot{B}^{(M-K-2)} = \underline{C} > 0$.

Finally, as in Theorem 3.1, we get a similar expression, showing that the probability that agents' behavior will not converge to the same action, given also that action α_k is still present, it is bounded below 1. However, this is equivalent to the event of "Not converging to action α_k". This is because action α_k cannot disappear, hence if the system converges to a single action, this has to be α_k. Namely:

$$
\begin{aligned}
P[\{NCA_k\}] &\equiv P_t\{[(M-1)_{t+T}]^c \mid [(M-1)_t]^c, I_k^{t+T} \neq \emptyset\} \leq 1 - \underline{C} < 1 \\
P[\{NCA_k\}] &= \lim_{s\to\infty} \prod_{i=0}^{s} P\{[(M-1)_{t+(i+1)T}]^c \mid [(M-1)_{t+iT}]^c, I_k^{t+T} \neq \emptyset\} \\
&\leq \lim_{s\to\infty} (1-\underline{C})^s = 0
\end{aligned}
$$

Which means that the network will converge with probability one, to a monomorphic steady state, where every agent chooses the action α_k. □

Proof of Proposition 4.2. Notice that the following equivalence holds: $F_k(u) \leq [F_{k'}(u)]^D \Leftrightarrow Pr(U(\alpha_k, \Omega) \geq \hat{u}) \geq 1 - Pr(U(\alpha_{k'}, \Omega) \leq \hat{u})^D$, for all $k' \neq k$ and $\hat{u} \in U$. Before convergence occurs, there is at least one pair of agents such that $a_i = \alpha_k$ and $a_j = \alpha_{k'}$, with $k' \neq k$. Condition (i) ensures that for all the agents who observe at least one neighbor (including themselves) choosing α_k at period t (i.e. $\forall i \in N$ such that, $\exists j \in N_i : a_j^t = \alpha_k$), it holds that $P(a_i^{t+1} = \alpha_k) = p_1 > \frac{1}{2}$. Since this holds for each agent we can construct the following variable, which we assume, in the first part of the proof, to be a finite number. Let:

$$r_{t+1} = \{\#i \in N : a_i^{t+1} = \alpha_k, a_i^t = \alpha_{k'}, k' \neq k\} - \{\#j \in N : a_j^{t+1} = \alpha_{k'}, a_j^t = \alpha_k, k' \neq k\}$$

In words, this represents the difference between the number of agents changing from any action $\alpha_{k'}$ to α_k, minus those that change from α_k to any $\alpha_{k'}$. The expected value of r_t at each round will be, at every time period t:

$$E_n[r_t] = \sum_{i:\exists j \in N_i}^{a_j^t = \alpha_k} [p_1 - (1 - p_1)] = \sum_{i:\exists j \in N_i}^{a_j^t = \alpha_k} [2p_1 - 1] \geq r > 0, \text{ for all } t$$

The last inequality is direct implication of condition (i) and the assumption about finiteness of r_t. Since for every agent it is more probable to change to α_k, we expect that more agents will change from other actions towards α_k than the opposite.

In order to prove convergence, we need to show that in the long-run and as the population grows it holds that:

$$\sum_t r_t + s_k n \to n \Leftrightarrow \lim_{n\to\infty} \frac{\lim_{t\to\infty} \sum_t r_t}{n} + s_k \geq 1 \tag{4}$$

By the weak law of large numbers, applied for the population, we get with probability one:

$$\lim_{t\to\infty} \sum_{\tau=1}^{t} r_\tau = \lim_{t\to\infty} \sum_{\tau=1}^{t} E_n[r_t] \geq \lim_{t\to\infty} rt$$

Which makes equation (6) equal to:

$$\lim_{n\to\infty} \frac{\lim_{t\to\infty} \sum_t r_t}{n} + s_k \geq \frac{\lim_{t\to\infty} rt}{\lim_{n\to\infty} n} + s_k = r + s_k \geq 1$$

The equality holds because $\lim_{t\to\infty} t \equiv \lim_{n\to\infty} n$. Finally, the last inequality comes from condition (ii) that requires s_k to be sufficiently large. Sufficiency depends on r, which itself depends on the distributions of the payoffs. Obviously, the more efficient is action α_k, the smaller initial population share is needed in order to ensure convergence.

Recall that we assumed r to be a finite number. if r is a share of the population, it is apparent that the result still holds. Even more, the result holds without any restriction on s_k, because the limit we calculated diverges to infinity. □

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