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Abstract

This paper proposes a reduced form model of dynamic duopoly in the context of heterogeneous innovations framework. Two agents invest into expansion of variety of available products and into the improvement of quality of existing products simultaneously. Every newly introduced product has its own dimension of quality-improving innovations and there is a continuum of possible new products. In the area of quality innovations the costless imitation effect is modelled while in the area of variety expanding innovations agents are cooperating with each other. As a result the specialization of innovative activity is observed. This specialization arises from strategic interactions of agents in both fields of innovative activity and is endogenously defined from the dynamics of the model.

Keywords: innovations, dynamics, multiproduct, technology spillovers, distributed control, differential games

JEL codes: C02, L0, O31.

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1 Introduction

It is widely known, that modern firms have multiple research projects with some of them being more directed onto new products creation while others being directed on the improvement of existing ones. It is also known, that some firms tend to specialize more in products creation while the other in the improvements of existing products. Current paper tries to provide an insight into the mechanism, which defines the direction and degree of specialization of R&D activities of such multiproduct firms.

The main focus of the paper is thus on the modelling of strategic interactions of agents in the field of innovations. To this end the production activities of the agents is not explicitly modelled. Only the investment activities related to the introduction of new products and their further refinement are accounted for. It is assumed, that there exists a continuum of potential products which may be invented (introduced) and each of these products may be further improved through innovations, specific for each such a new product. Creation of new products is related to product innovations and improvement of quality of each new product is related to process innovations in the literature. Both types of innovations require specific types of investments. Such a view on innovations is in line with definition of vertical and horizontal innovations as put in (Rosenkranz 2003). In particular, the notion of variety expanding and quality improving innovations follow the setup of vertical and horizontal innovations of (Peretto and Connolly 2007).

Both agents participate in the joint R&D partnership devoted to the creation of new products and may use freely these newly created products in their further activities. At the same time each new product may be further improved in its characteristics (referred to as "quality" throughout the paper) separately by each of the agents. Such a structure assumes joint lab financing for fundamental research (products creation) while quality improving process is managed privately by each agent. For all the new products this quality characteristic is zero for both agents at the time of product's creation. However, due to possibly different efficiencies of investments into the development of the products qualities, eventually one of the agents may become the leader in this quality development. In this case I allow for undirected technological spillover between agents: at each time when one of the agents has lower quality for a given product, she benefits from partial spillover from the development of this product's quality by the other agent. Thus the model under analysis allows for both symmetric (no leader-follower situation occur) and non-symmetric outcomes with technological spillovers being included as well as for catching-up in quality improvements.

This framework allows to catch several major issues relevant for dynamic interactions in the field of innovations. First, it is shown that the technology spillovers in the form of costless imitation do not lead to the incentive to de-

crease investments into qualities of products for both agents simultaneously (as it should be in the case of only one direction of innovations being considered) if one would account for their cooperation on the more fundamental level of creation of new products. One of the agents, which eventually becomes the leader (due to higher quality investments efficiency), does not benefit from the spillover effect, while the other one positions herself as a follower, reducing her own investments to benefit from this technological spillover created by the activities of the first agent.

Next, in the direction of variety expansion the united efforts of both agents are distributed unevenly. Instead, there is a natural specialization of investment activities of both agents. The agent, who is benefiting from the technology spillovers in process innovations (named "the follower" in the sequel of the paper) puts more investments into the product introduction activities. However, in the simplest case of open-loop strategies the other agent invests non-zero amount into the product creation also. In an effect the agent who is the most efficient in one or another type of innovative activities carries the major burden of investments in this direction while benefiting from the investments of the other agent in the other type of innovations.

The rest of the paper is organized as follows. In the next section a brief summary on relevant studies is presented. Next the formal setup of the model is described. It is solved sequentially through employing the Hamilton-Jacobi-Bellman and Maximum Principle approaches. After obtaining analytical results the dynamics of the optimal investment strategies for both agents are considered and the nature and form of the resulting strategic interactions as well as some practical implications of the findings are discussed.

2 Related Research

One of the first works on the influence of the market structure on the outcome of R&D competition is (Loury 1979). In this pioneering work the discrete single innovation is assumed and n firms compete for being the first to introduce the new product. The first firm which would introduce such a product obtains an exclusive right for its production and hence receives the perpetual stream of profits associated with this product. Attempts to model the strategic interactions of oligopolistic agents via the differential games approach followed in works like (Reinganum 1982). In this paper the author combined static games approach with optimal control one to obtain the dynamic game of R&D competition in a n-firm industry.

There exists broad literature on the effects of imitation on innovation activity. One of the examples is the work of Gallini,(Gallini 1992), but imitation there is

costly and the model is static. Another more recent work on dynamic interactions of R&D firms is that of Judd, (Judd 2003). In this paper the author analyses the multi-stage innovative race between multiple agents with multi-product situation and this is rather close to the suggested approach. He finds out, that there is an ambiguity in the results of a game, namely a given player may increase his expenditure when the other agent is ahead of him, while this is not profitable for him as an imitator.

The recent paper on product and process innovations in differential games framework (Lambertini and Mantovani 2010) assumes fully dynamical model of the duopoly competition of innovating firms. However, this paper does not handle heterogeneity of innovations and hence is reduced to the differential game with two states, while the current paper allows for distributed nature of innovations and all products differ from each other in their investment characteristics. This is more in line with the setup of Lin, (Lin 2004), but with fully dynamic context.

The last feature of the suggested model is the R&D cooperation on the level of products variety expansion (product innovations). It is argued that such a situation is more typical for R&D firms then the full-scale competition on both levels. First such a type of strategic interactions has been considered in (D'Aspremont and Jacquemin 1988), where it is argued that in real economies the majority of R&D activities is performed in the form of joint R&D ventures if one would consider innovations of big enough size.

One may consider the suggested model as an extension and combination of these results from different directions. First, it contributes to the line of literature on strategic interactions between innovating agents in the spirit of (Reinganum 1982), (Judd 2003) and works by Lambertini and co-authors, like (Cellini and Lambertini 2002), (Lambertini and Mantovani 2010). These approaches are extended by considering the distributed parameter model and formulating the fully dynamical differential game with richer strategic sets for both players. The model also assumes explicitly the heterogeneous nature of innovations, that is, all of the new products are different from each other in the degree, to which their quality may be improved and how much efforts this would require.

Next, models of imitation, (Gallini 1992) and R&D joint ventures, (D'Aspremont and Jacquemin 1988), (Rosenkranz 2003) are combined in a single model and it is demonstrated that these two effects are complementary in nature and resulting strategies cannot be optimal in a dynamic context while taking into account only one of them. The paper follows the result of Rosenkranz on efficiency of R&D cartels in terms of innovations output, but extends them by accounting for distributed model with continuum of research projects being pursued simultaneously and allow for technological spillovers also.

The setup with heterogeneous continuous spectrum of innovations have been used in (Belyakov, Tsachev, and Veliov 2011) and (Bondarev 2012). Both papers

do not consider multiple agents and their strategic interactions and are one-agent optimization models. The current paper builds up upon the results of (Bondarev 2012). The same formal framework of innovations is assumed, but the analysis is carried out for two agents instead of one and the focus is on the dynamic interactions between these agents, rather then on the dynamics of innovations for a monopolist. It is demonstrated, that under full cooperation the current model is reduced to the single monopolist's model of this paper. Moreover, if no technological spillover occur (in symmetric case), the model reduces to two independent monopolists problems of the same type. Thus the presented paper is a natural extension of this previous work in the field. The generation of innovations under the assumed partial R&D partnership is higher, then under both full cooperation case and for two independent monopolies.

3 The Model

3.1 Form of strategic interaction between agents

In this paper it is assumed, that agents follow the incomplete cooperation pattern. They cooperate in joint R&D projects for new products creation (thus having the single lab for their projects) while develop these new products separately. At the same time, they may benefit from technological spillovers from each other's activities in quality improvements. This exact situation can rarely be accounted for in real industries. However the main point of the paper is to compare such "what-if" situation with more frequently observed cases of full cooperation in R&D and independent R&D activities.

In the framework of the model full cooperation between agents will lead to the dynamic problem of a single unified agent, as long as market activities is not taken into account. Thus no specialization and technological spillovers may be observed. At the same time, if one would allow for independent products creation activities, no technological spillovers would occur, since at each time both agents would be developing different products and cannot coordinate their imitation activities. The technological spillover effect appears exactly because of joint R&D on the product development level: both agents have similar designs and blueprints at the time of introduction of new product and thus may compare their own quality improving efforts to those of the other agent. Then it would be possible for the technologically backward agent to (almost) costlessly acquire some information on the developments of the more advanced agent.

It is argued, that such a structure of R&D activities would be more beneficial in terms of the total R&D output for both agents, since the specialization of activities occur. This specialization cannot happen both in fully cooperative case and in

the independent research case. This specific form of R&D interactions may be observed in some industries, where initial product development requires a lot of investments, while further development of product lines is not that costly and may be handled by separate individual firms. The most close example would be the pharmaceutical industry, where only few basic active substances exist and are developed by large joint ventures, while every company develops its own design of medicals on the basis of these basic substances.

The suggested model does not include market competition, which, of course, may substantially change the comparative benefits of different forms of R&D co-operation. If one would allow for the agents under consideration to co-exist at the same market, the technological spillover in the quality innovations may be actually more harmful for profits of the technological leader, than under current setup. To this end the agents under consideration are considered to be local regional monopolies which do not compete on the product markets. In the other way around it would be difficult to motivate the joint research lab for products creation. Additional difficulty for consideration of a standard duopoly case comes from the distributed character of R&D activities being considered. It is assume din the model, that there exists a continuum of potential products and all these products are eventually invented and developed. Since all these products are different from each other in their qualities, they would have different market prices and thus to obtain duopoly market equilibrium one should account for the continuum of such markets. Still, the existence of equilibrium for economies with a continuum of goods is the open question. Thus one has to consider either similar prices for all products, thus neglecting the full extent of heterogeneity of products or rather neglect market competition. I use the second alternative to concentrate on the dynamics of innovations and specialization patterns.

3.2 Formal framework

Assume there are two firms in a given industry. The industry is mature and no growth of the demand is expected for existing products variety. There are equilibrium quantities for both firms which depend on the production costs. However, these costs are fixed for mature products and are not subject to change in the result of new process innovations. Both firms act as monopolists in their markets which are separated and cannot be entered by the other firm. In this respect the firms are regional monopolies. This implies that profit of every firm depends only on its own production costs but not of that of the other one. For those products which are already in mature stage, the production, price and profit for each of the firms are constant. Because of this one may abstract from this part of firms' activities in the optimization problem. The objective function of such a reduced optimization

problem for both firms, is then given by:

$$J_j \stackrel{\text{def}}{=}$$

$$\stackrel{\text{def}}{=} \max_{u_j(\bullet), g_j(\bullet)} \int_0^\infty \mathrm{e}^{-rt} \Big[\int_0^{n(t)} \Big(q_j(i,t) - \frac{1}{2} g_j(i,t)^2 \Big) di - \frac{1}{2} u_j(t)^2 \Big] dt. \tag{1}$$

Where:

- j is the label of the firm, $j \in [\{1\}, \{2\}]$;
- $q_j(i,t)$ is the quality of product i at time t for the firm j;
- $g_j(i,t)$ are investments into the development of product i at time t being made by the firm j;
- $n(t)$ is the common for both firms achieved level of the variety of products at time t;
- $u_j(t)$ are investments of the firm j into the expansion of variety of products at time t.

The basic intuition behind (1) is clear: every firm is maximizing the effect from its quality innovations for every of the introduced products at every time t. The range of products, being introduced till time t is given by $n(t)$ and hence the difference in the effect of achieved quality levels and the investments into them is evaluated over this range at any point in time. At the same time the investments into the creation of new products, u_j, negatively influence the total value generated by qualities for the firm while the introduction of new products at some continuous rate enlarges the space of products, for which subsequent quality improvements may be done. Observe that the introduction of the new product per se does not bring the increase in the value for the firm, since it is assumed that such a product has zero level of technology. The objective is to maximize the impact of innovations of both types at the firms value at the infinite time horizon.

This form of a functional assumes that every firm has at each point in time $n(t)+1$ state variables and strategic variables (controls) and this quantity grows in time. However the total potential number of variables is fixed by the maximal potential range of products, N. In terms of value generated by innovations, both firms are completely symmetric and there is no direct influence of qualities of developed products of one form on the value of the other. The interaction between firms is introduced in the dynamics of state variables for both of them.

Assume the process of development of products is continuous in time and yields new products (which are new versions of some basic for the industry product) proportionally to joint investments of both firms with possibly different efficiencies of investments across firms. The range of these new products is limited from above. The variety expansion is thus limited to upgrades of some basic product which defines the industry (e.g. cell phones industry produces different versions of cell phones but not computers) which justifies the bounded range for potential products development.

$$\dot{n(t)} = \alpha_1 u_j(t) + \alpha_{-j} u_{-j}(t), n(t) \in [0,..,N]. \tag{2}$$

Where α_j, α_{-j} are investment efficiencies for variety expansion for both firms respectively and N is the maximal achievable range of products to be invented.

This means both firms have the same underlying variety expansion process while freely choosing the level of efforts they would devote to the development of this variety. Note that this does not exclude the possibility for one or the other firm to have zero investments while benefiting from the investments being made by the other through using achieved variety level.

The introduction of each new product is simultaneous for both firms, since they have joint R&D lab, so they start their investments into these new products qualities simultaneously. It is natural to require that at the time of introduction, denoted by $t_i(0)$ for each product i, the level of quality of this product is zero. At each point in time, each of the firms has to choose optimally the level of investments being made into the development of new products (product innovations) and into the development of quality of already existing products (quality innovations). These investment streams cannot be negative. At the same time the technological spillover effect is possible if one of the firms has lower quality than the other one as it is discussed in Subsec. 3.1. Then for each product i within the range of introduced products the evolution of qualities of both firms is given by:

$$\begin{aligned}
&\dot{q_j(i,t)} = \gamma_j \sqrt{N-i} g_j(i,t) - \beta_j q_j(i,t) + \theta \cdot \max\{0, (q_{-j}(i,t) - q_j(i,t))\};\\
&\dot{q_{-j}(i,t)} = \gamma_{-j} \sqrt{N-i} g_{-j}(i,t) - \beta_{-j} q_{-j}(i,t) + \theta \cdot \max\{0, (q_j(i,t) - q_{-j}(i,t))\};\\
&q_j(i,t) \mid_{i=n(t)} = 0;\\
&q_j(i,0) = 0, \forall i \in [0,..,N];
\end{aligned} \tag{3}$$

Where:

- γ_j are efficiencies of investments into the quality of product i for the firm $\{1,2\}$, constant across products and time;

- β_j are technology decay rates in the absence of investments for both agents;
- θ is the speed of technological spillover, equal for both agents.

The term $\sqrt{N-i}$ in both processes defines the decreasing efficiency of investments into the quality growth for both players across products' range. It is assumed that the higher is the index i of the product, the harder it is to improve its production technology due to the increased complexity of the product. The specific form of this decreasing function is chosen to linearise the resulting dynamical systems. This is done following the same arguments as in (Bondarev 2012).

Two first equations allow for the possibility of technological spillovers in quality development between firms. However, due to the max term in them, only one frim may actually benefit from this spillover and this is the firm which would have lower quality. At the same time, last two equations here are requirements for qualities of any product to be zero at the beginning, at $t = 0$ and at the time of the introduction of every new product, $t_i(0)$, which is defined by the condition $i = n(t)$. With continuity of state variables and the form of dynamics assumed this is sufficient to ensure that any product has zero quality all the time before it is actually introduced (variety expansion process reaches its position in the products space).

Thus for any product i development of quality starts without any spillovers and this spillover may appear only eventually due to different speeds of development of this product. I do not assume any exogenous leader-follower structure here. Rather conditions for one or another firm to be the leader in the development of quality of a given product i are obtained as a part of the solution procedure. The model itself allows for 4 different scenarios for every product development, depending on the configuration of parameters. These are:

- Firm 1 is the leader in quality development of product i;
- Firm 2 is the leader in quality development of product i;
- Both firms have equal levels of quality development and no technological spillover occur;
- Initially one of the firms is the leader in quality development, but after some time the other firm catches up with the leader and becomes the leader itself.

  In these paper I analyse only first 3 scenarios, while the last one is more complicated. The full analysis of this case is the immediate extension of the results which follow.

4 Solution

In this section the solution techniques applied to the problem formulated above are discussed.

4.1 Decomposition of the Problem

Given the basic formulation of dynamical problems of both agents above, it is clear that the optimal solution has to be found in the form of the equilibrium pair of strategies in the differential game framework. From the general point of view the model considered here is the infinite-dimensional one as long as one have the continuum of quality improving innovations associated with each product for every player. This may provide some difficulties in formal construction of the game. However due to the special structure of the dynamic framework being used it is possible to decompose the problem into 'quality' growth problem and variety expansion problem. This can be done due to the fact that quality improving innovations do not depend on the variety expansion except for the time of emergence of new products. Then every such a problem should be the finite-dimensional one and as long as it is of the linear-quadratic form, one may be assured that equilibrium exists for each such a game of quality innovations under the same conditions as in standard linear-quadratic differential game, (Dockner, Jorgensen, Long, and Sorger 2000). Then the results obtained for this game may be used for solution of variety expansion problem which is also the differential game but with only one state, $n(t)$. For this one may rewrite the objective functional of both players (1) in terms of values generated by the quality innovations and by the variety expansion games.

To decompose the value function of the overall model, first we make use of the observation above. Starting from the time of emergence (denoted by $t(0)_i$) value of each product's quality for each firm j is independent of variety expansion process:

$$V_j(q(i)) = \max_{g_{j(i)}} \int_{t(0)_i}^{\infty} \mathrm{e}^{-r(t-t(0)_i)}(q_j(i,t) - \frac{1}{2}g_j(i,t)^2)dt; \quad j \in \{1,2\}, i \in [0,..,N]. \tag{4}$$

where $t(0)_i$ is the time of emergence of the product i, which is defined from the dynamics of the variety expansion process and is similar for both agents due to the form of dynamic constraints (2), (3). With infinite time horizon the problem of quality innovations is the time-autonomous one and hence the time of the products emergence does not influence the value generation process and can be normalized for all products to zero, (Bondarev 2012). For every product i the maximization of

the value of quality innovations may be performed separately and independently of other such problems due to the fact that there is no interaction between quality innovations across products. However, the overall process of quality innovations depends on variety expansion, since this last determines the density of quality innovations. Thus the dependence of these latter from the intensity of introduction of new products takes place only in the aggregate measure.

Proposition 1 *Value functions of quality innovations management game for each separate product i for both firms, $V_j(q(i))$ are invariant to the investments of both firms into new products development, $u_j(t)$. Only the emergence time of this product, $t(0)_i$ depends on variety expansion process.*

Second part of the overall value generation consists of the intensity of introduction of new products at every time given the expected value of the stream of profit derived from the reduced costs of production (which comes from quality innovations) of the newly introduced products. This part may be represented by the integral over all potential stream of quality innovations for each product over it's life-cycle. At the same time this information is already contained in the value function of the 'quality' problem above, so it suffices to integrate over all potential products at initial time. Last observation to be made is that at the moment of the emergence of the new product it's quality is zero, as it is required by (3). These yield the value function for firm j for variety expansion problem in the following form:

$$\begin{aligned} & V_j(n) = \\ (5) \qquad & = \max_{u_j} \int_0^\infty \mathrm{e}^{-rt} \Big((\alpha_j u_j(t) + \alpha_{-j} u_{-j}(t)) \times V_j(q(i)) \mid_{i=n(t)} - \frac{1}{2} u_j(t)^2 \Big) dt, \\ & j \in \{1,2\}, i \in [0,..,N]. \end{aligned}$$

In the last equation value generated by the quality innovations management game for each firm is estimated at the zero technology level for the product next to be invented. Hence one may sequentially solve the quality innovations game for an arbitrary i, then calculate the associated value function at the zero technology level and $i = n(t)$ and use this last as an input for the variety expansion game.

Observe that a decomposition method is valid here since there is no competition on the level of variety expansion. Joint variety expansion process yields the coincidence of emergence times of all new products for both firms. There is no dependence of value creation at the level of each separate quality innovation from the relative speed of variety expansion. Moreover, every firm is able to estimate the potential accumulated value from the quality of each product in the potential products' space, because it may estimate it at zero technology levels not only for

itself but for the other firm also, since the time of emergence is the same. Then the value function for the variety expansion does not depend on quality innovations themselves, but only on the potential value generated by the refinement of the quality for each product as a whole. In the effect this value function although different for both firms (as their quality innovations' value functions are different) is invariant to the future quality innovations associated with every new product i.

Proposition 2 *Value function of the variety expansion game for both players, $V_j(n)$ is invariant to the investments being made into the qualities of all the products except the boundary one, $g_j(i)\mid_{i=n(t)}$. Moreover, it depends only on the total value generated by the quality of this product at the time of its emergence, $V_j(q(i))\mid_{i=n(t)}$.*

In an effect one may observe that there is an influence of quality innovations on the intensity of variety expansion but the inverse effect is almost absent. However, due to cooperation at the level of variety expansion, technological spillovers are possible in quality innovations. One may think of quality and variety expansion innovations as of process and product innovations respectively. Then it is straightforward to relate this form of the interdependence of different types innovations to the empirical literature on the subject, (Faria and Lima 2009) (Kraft 1990). Also it follows the general setup of (Albernathy and Utterback 1978).

4.2 Quality Innovations

Consider first the problem of quality innovations management for each product i for both firms $1,2$. The value function for each such an innovation is given by Eq. (4) for every firm. At the same time the evolution of qualities is given by dynamic constraints(3) for every product i.

Such two problems constitute the differential game with two states, being levels of qualities achieved by both firms for product i, and two controls which are investment strategies of the firms for every i. Note that the formulation of the game is of the same form across all products' qualities and they are independent of each other. Hence solution of this game is valid for any i. The associated pair of HJB equations is dependent on both states for each firm as well as on investments of both firms.

Proposition 3 *For each product $i \in [0,..,N]$ the evolution of qualities of both firms is governed by the 2-state differential game, which has the same form for all products. This game is defined by the pair of objective functions* (4) *and dynamics of qualities* (3) *and includes undirected technological spillover effect. The pair of HJB equations governs all modes of the game. Realization of one or the other mode depends on the direction of technological spillover and its presence.*

The technological spillover effect is modelled through the term $\max\{0,(q_j(i,t)-q_{-j}(i,t))\}$ in the dynamics of both qualities. This is undirected, since any firm may benefit from this spillover, depending on which one will take the lead in development of the product. As it has been discussed above, there are 4 possible situations, with symmetric outcome being the one with no spillover.

I first consider the non-symmetric case with different efficiencies of quality investments. The symmetric case result is obtained along the same lines, but is more difficult to prove, since value functions of both firms are not differentiable in the symmetric situation. Thus I state the result for symmetric case with underlying intuition later on without formal proof. The rigorous treatment of the symmetric case may be found in the PhD thesis of the author.

Constant leadership in qualities

For technological spillover to be present it is necessary for firms to have different investment efficiencies $\gamma_{1,2}$, since they start their investments simultaneously and with zero quality level for any new product. However, if one of the firms has higher efficiency of investments, it will acquire higher quality level almost from the start of the project. It is possible, however, for the other firm to catch up with the leading one later on, if the backward firm has lower decay rates of technology, β_j. I limit myself to the situation with constant leadership, when such a catch up is not possible.

As long as one of the firms has the leadership in the quality innovations, that is, $q_j(i,t) > q_{-j}(i,t)$, its quality dynamics does not depend on the technological spillover, while the other's does. Then subsequent pair of HJB equations may be

written as:

$$rV_j(i) =$$
$$\max_{g_j(\bullet)}\Bigg\{q_j(i,t) - \frac{1}{2}g_j(i,t)^2 + \frac{\partial V_j(i)}{\partial q_j(i,t)}\Bigg(\gamma_j\sqrt{(N-i)}g_j(i,t) - \beta_j q_j(i,t)\Bigg)$$
$$+\frac{\partial V_j(i)}{\partial q_{-j}(i,t)}\Bigg(\gamma_{-j}\sqrt{(N-i)}g_{-j}(i,t) - \beta_{-j}q_{-j}(i,t) - \theta\times\big(q_j(i,t)-q_{-j}(i,t)\big)\Bigg)\Bigg\};$$

$$rV_{-j}(i) =$$
$$\max_{g_{-j}(\bullet)}\Bigg\{q_{-j}(i,t) - \frac{1}{2}g_{-j}(i,t)^2 + \frac{\partial V_{-j}(i)}{\partial q_j(i,t)}\Bigg(\gamma_j\sqrt{(N-i)}g_j(i,t) - \beta_j q_j(i,t)\Bigg)$$
$$+\frac{\partial V_{-j}(i)}{\partial q_{-j}(i,t)}\Bigg(\gamma_{-j}\sqrt{(N-i)}g_{-j}(i,t) - \beta_{-j}q_{-j}(i,t) + \theta\times\big(q_j(i,t)-q_{-j}(i,t)\big)\Bigg)\Bigg\}.$$
(6)

Observe that in this formulation only the firm $-j$, which is called the 'follower' is benefiting from the technological spillover resulting from superior production technology of the other firm j. This can be seen from the form of the dynamic constraint on the dynamics of technologies which is different between firms for constant leadership case and includes the spillover effect only for the follower.

Proposition 4 *The regime with constant technological leadership of one firm is an equilibrium of the quality game if one of the firms has advantage in the efficiency of quality investments and qualities decay at the same rate for both firms:*

$$\gamma_j > \gamma_{-j};$$
$$\beta_j = \beta_{-j} = \beta. \tag{7}$$

Where j could be any of the two firms. I denote the one with the leading efficiency γ_j as the leader, L and the other as the follower, F.

Equilibrium strategies are derived through the HJB equations (6) under the assumption of linear in states value functions of both firms. Formal derivation may be found in the Appendix. Linear value functions yield open-loop equilibrium. The case of closed-loop may also be considered, but the formal treatment is more

complicated. It is sufficient to note here, that for the case of constant leadership open-loop equilibrium coincides with the closed-loop one.

I use the derived (piecewise-constant) optimal strategies to define evolution of qualities for product i for both firms as a pair of differential equations resulting from (3). These equations may be explicitly solved. Formal derivations and explicit evolution of qualities are in the Appendix. The following proposition summarizes main information on the solution of quality game in constant leadership mode.

Proposition 5 (Solution of quality game with constant leadership) *If* (7) *is fulfilled, optimal strategies of the leader and the follower are given by:*

$$g_j^L(i) = \begin{cases} \frac{\gamma_j\sqrt{(N-i)}}{r+\beta}, t : i \geq n(t); \\ 0, t : i < n(t). \end{cases}$$
$$g_{-j}^F(i) = \begin{cases} \frac{\gamma_{-j}\sqrt{(N-i)}}{r+\beta+\theta}, t : i \geq n(t); \\ 0, t : i < n(t). \end{cases}$$

(8)

They are constant in time after the introduction of the product i, but are different across products. For every product i the leader invests more, then the follower, $g_j^L(i) \geq g_{-j}^F(i)$. *For every next product investments are lower for both firms,*

$$g_j^L(i) < g_j^L(i+\varepsilon), g_{-j}^F(i) < g_{-j}^F(i+\varepsilon);$$
$$\forall (N-i) > \varepsilon > 0. \tag{9}$$

Associated quality level of the leader and the follower are defined by the dynamic system after the time of emergence:

$$\begin{cases} \dot{q_j^L}(i,t) = \frac{\gamma_j^2}{\beta+r}(N-i) - \beta q_j(i,t), t : i \geq n(t); \\ \dot{q_{-j}^F}(i,t) = \frac{\gamma_{-j}^2}{\beta+\theta+r}(N-i) + \theta q_j(i,t) - (\beta+\theta) q_{-j}(i,t), t : i \geq n(t). \end{cases}$$

(10)

Quality of the leader is higher then that of the follower after the time of emergence, $q_j^L(i,t) > q_{-j}^F(i,t)$ *and for both firms quality for every next product is lower at any time after the product's emergence:*

$$q_j^L(i,t) < q_j^L(i+\varepsilon,t), q_{-j}^F(i,t) < q_{-j}^F(i+\varepsilon,t);$$
$$\forall (N-i) > \varepsilon > 0, \forall t > t_i(0). \tag{11}$$

One may now compute values, which are generated by the quality development of product i for both firms. The proof, that the leader-follower non-symmetric equilibrium is indeed the stable one amounts to the comparison of the value, which the leader may acquire through imitation activities with the value the leading firm acquires from its equilibrium strategy. This direct comparison shows, that under the strategy of being the leader the technologically superior firm still acquires maximum possible value although not benefiting from the spillover effect. Observe that this result would not necessarily hold under proper duopoly situation, since this will influence prices of products. Thus it is essential to restrict attention to regional monopolies in the sense of market competition. Formal comparison of value functions may be found in the Appendix.

Proposition 6 (Uniqueness and stability of constant leadership mode) *As long as* (7) *holds, there is only one equilibrium in the quality game for each i and it is given by* (8).

In this case the value of quality innovations game is always higher for the firm which leads in investment efficiencies when it invests as a leader in this game, while the opposite holds for the firm which has lower efficiency γ_{-j}:

$$V_j^L(i) > V_j^F(i); \tag{12}$$

$$V_{-j}^L(i) < V_{-j}^F(i). \tag{13}$$

and the equilibrium with constant technological spillover to the benefit of technologically backward firm is stable.

We need values generated by the quality game estimated at the zero quality level and at the initial time. These are estimated future profits from quality development which drive the variety expansion activity. These values are estimated for the boundary product, which will appear next at every moment of time. Denote these values as $V_j^L(0)$ and $V_{-j}^F(0)$ for firm j being the technological leader and firm $-j$, being technologically backwards respectively. Derivation in the Appendix shows that these values are functions of variety expansion process, see Eqs. (B.3), (B.4).

Symmetric case

The symmetric case is not easy to prove, since value functions of both firms are not differentiable along the line of equal qualities for any product, $q_j(i,t) = q_{-j}(i,t)$. Thus standard HJB techniques do not apply for this case. However it may be formally proved, that the symmetric outcome occurs only if efficiencies of investments into qualities for both firms are equal. In this case there are two symmetric

outcomes: with positive imitation speed θ and with zero imitation speed. Here I just state this result without proof. The formal treatment of this symmetric case may be found in the PhD thesis of the author.

Proposition 7 (Solution of quality game in symmetric case) *The symmetric outcome of quality game for each i realises when the following is fulfilled:*

$$\gamma_j = \gamma_{-j} = \gamma, \beta_j = \beta_{-j} = \beta. \tag{14}$$

In this case both firms make similar investments into qualities and there qualities are equal during all the development of any product i. Equilibrium levels of investments are the same as for the technologically backward firm:

$$g_j^{SYM}(i,t) = g_{-j}^{SYM}(i,t) = \frac{\gamma\sqrt{N-i}}{r+\theta+\beta}; \tag{15}$$

With positive speed of potential spillover, $\theta > 0$, qualities of both firms are as low as that of the technologically backward firm in constant technological leadership regime:

$$q_j^{SYM}(i,t) = q_{-j}^{SYM}(i,t) = q^{\theta}(i,t) = q^{F}(i,t),\ \theta > 0; \tag{16}$$

With zero imitation speed both qualities are developed to the level of technologically superior firm in constant leadership setting:

$$q_j^{SYM}(i,t) = q_{-j}^{SYM}(i,t) = q^{0}(i,t) = q^{L}(i,t),\ \theta = 0. \tag{17}$$

Thus the symmetric case solution is described as a particular case of non-symmetric solution and not the other way around. It has to be noted, that under the zero technological spillover speed the solution for each firm is fully the same as that for the single monopoly in (Bondarev 2012).

4.3 Variety Expansion Game

Variety expansion problem is the differential game with one state and two controls. Both firms invest simultaneously in the variety expansion and benefit from the resulting variety on common base thus sharing all the information on this level of innovations. The dynamic problem for both firms is to maximize the potential output of innovations given the costs of investments. Note that the potential profit in this part of the model consists only from the future accumulated profit from development of qualities of newly invented products. Since we limit ourselves to the case of open-loop strategies in the quality innovations game the variety expansion problem is also solved in this class of strategies. The other strategies,

of piecewise-constant and of closed-loop type are considered as an immediate extension.

For the characterization of the open-loop solution for the variety expansion game the maximum principle method is convenient. One may rewrite the problem of variety expansion, using the Eq. (5) for firm j as following:

$$
\begin{aligned}
J_j(n) = &\max_{u_j(\bullet)} \int_0^\infty \mathrm{e}^{-rt}\Big((\alpha_j u_j(t) + \alpha_{-j} u_{-j}(t))V_j(0) - \frac{1}{2}u_j(t)^2\Big)dt; \\
&s.t. \\
&\dot{n(t)} = \alpha_j u_j(t) + \alpha_{-j} u_{-j}(t); \\
&u_j(t), u_{-j}(t) \geq 0, \forall t \geq 0. \qquad (18)
\end{aligned}
$$

And the same type of a problem for firm $-j$ since the variety expansion game is completely symmetric and objectives differ only in the value generated by development of qualities, $V_{j,-j}(0)$, given by Eqs. (B.3), (B.4).

Both value functions from quality innovations game are linear in $n(t)$. Denote the constant part by C_j, C_{-j}:

$$
\begin{aligned}
V_j(0) &= C_j \cdot (N - n(t)); \\
V_{-j}(0) &= C_{-j} \cdot (N - n(t)). \qquad (19)
\end{aligned}
$$

where $C_j, C_{-j} \geq 0$ are given by constant parts of (B.3) and (B.4).

The constant part may vary in size depending on the leadership in the quality innovations game, but the variety expansion is analysed parametrically and then the dynamics corresponding to different regimes of the quality innovations game are compared. This may be done since these constant parts of value functions above do not depend on the state variable and controls nor time. This constitutes the one-state differential game with common state constraint which may be solved using standard techniques. First I construct Hamiltonians of the given problem and derive first-order conditions on controls. Controls appear to be functions of shadow costs of investments. Then the resulting triple of differential equations (for common variety expansion and separate shadow costs for both firms) is solved and this solution is sued to define explicit form of optimal controls of both firms in variety expansion game. Formal details may be found in the Appendix. The resulting optimal investments of both firms are decreasing functions of time, since the exponential term has negative power. Thus, the variety expansion innovations increase the variety of products at a decreasing speed up to the maximal variety N. Investments are completely symmetric except for the terms $\alpha_{j,-j}C_{j,-j}$,

which depend on investment efficiencies and value generated by the quality innovations for the next product to be invented for both firms. Hence relative scale of investments into the variety expansion depends on the outcome of the quality innovations game. The overall evolution of variety expansion is an increasing concave function.

Main features of the resulting solution are summarized in the Proposition below.

Proposition 8 (Solution of variety expansion game) *Investments of both firms into the variety expansion are greater than zero and depend on the value generated by expected quality innovations into the next product. They are decreasing function of time:*

$$u_j(t)^* = \frac{\alpha_j C_j (r+A)}{2\alpha_j^2 C_j + 2\alpha_{-j}^2 C_{-j} + r + A} \cdot (N - n_0) \mathrm{e}^{\frac{1}{2}(r-A)t};$$

$$u_{-j}(t)^* = \frac{\alpha_{-j} C_{-j} (r+A)}{2\alpha_j^2 C_j + 2\alpha_{-j}^2 C_{-j} + r + A} \cdot (N - n_0) \mathrm{e}^{\frac{1}{2}(r-A)t}. \tag{20}$$

With common variety expansion process the available for both firms products variety $n(t)$ is increasing over time with decreasing speed and reaches the maximal available level N in infinite time:

$$n(t)^* = N - (N - n_0) \mathrm{e}^{\frac{1}{2}(r-A)t}. \tag{21}$$

This solution has the same form independent of which regime realises in quality innovations game.

with

$$A = \sqrt{r(r + 4\alpha_j^2 C_j + 4\alpha_{-j}^2 C_{-j})}, \tag{22}$$

being the function of values generated by quality development for both firms.

Observe, that the solution of the variety expansion game has the same form independent of what regime realises in quality game (constant leadership, catch-up or symmetric case), since variety expansion investments depend on constant parts of value functions of both firms in quality game, C_j, C_{-j}. These of course, will be different for different regimes, but they will be still constant and relative intensity of investments of both firm into variety expansion will be affected, but not the analytic form derived here.

It is of interest to compare the dynamics of variety expansion under cooperative investments of both firms and in the case when they invest separately. If

firms would choose not to cooperate in variety expansion, they will have independent ranges of products being defined by their investment efficiencies. These two processes may be obtained by solving the dynamic problem for variety expansion with for every firm with other firm's investments being set to zero. This solution coincides with the one for multiproduct monopoly in (Bondarev 2012). The direct comparison of variety expansion paths shows, that the cooperation in the case of constant leadership is better for both firms.

Proposition 9 (Stability of cooperative equilibrium in variety expansion game) *Denote:*

- $n^{LF}(t)$ *cooperative variety expansion under constant leadership of one of the firms in quality innovations game;*
- $n_j^{MONO}(t), n_{-j}^{MONO}(t)$ *variety expansion being obtained under independent investments;*
- $V_j^{LF}(n), V_j^{MONO}(n)$ *value of variety expansion for firm* j *under cooperative constant leadership mode and independent investments.*

As long as (7) *hold, for any difference in variety expansion investment efficiencies* α_j, α_{-j}*, the cooperative variety expansion rate is higher than the one being obtained under independent investments for both leading and the following firm. Value of variety expansion game under cooperation is also higher than for independent investments for both firms as a result:*

$$\begin{aligned} &\forall 0 \leq t < \infty : \\ &n^{LF}(t) > n_j^{MONO}(t), n^{LF}(t) > n_{-j}^{MONO}(t); \\ &V_j^{LF}(n) > V_j^{MONO}(n), V_{-j}^{LF}(n) > V_{-j}^{MONO}(n). \end{aligned} \tag{23}$$

Thus cooperative variety expansion is a stable equilibrium of variety expansion game with non-zero investments of both firms, defined by open-loop strategies.

Now with both parts of the problem being solved we may analyse the dynamics of the model in more details.

5 Results

5.1 Dynamics of quality innovations

The quality innovations game exhibits a variety of features. First, depending on relations between efficiency and decay parameters of both firms it is possible to

obtain positive imitation (technological spillover) effect and symmetric outcome of the game.

The symmetric outcome appears to have two case, depending on the presence of possibility of technological spillover. It turns out, that if there is a potential for imitation, both firms will try to imitate each other with no success. In this case the presence of imitation possibility has a negative impact on the development of all the products for both firms, since their qualities will be low. At the same time, if there is no possibility for imitation (or firms are unaware of it), the game reduces to two independent optimization problems for both firms and the outcome is the same as if these firms would not interact at all. In this case both firms will have higher qualities than in the presence of imitation effect.

If there are differences between firms in terms of investment efficiencies, but not in decay rates, one of the firms becomes the leader in development of quality of every product i since the introduction of this product (being defined from variety expansion game). In this case the technological leader will have higher quality for every product all the time after appearance of this product, which is the same as if this leading firm would develop the quality without interaction with the other firm. Thus the value of quality innovations for the leading firm is the same as for the monopolist and the symmetric game with no imitation.

At the same time the technologically backward firm will always have lower quality level, but will also invest less than the leader to obtain benefits from the technological spillover. In this way the value, being generated for the follower by quality innovations would be greater, then for the leading firm, since less investments are being made. Direct comparison of resulting value functions demonstrates, that both firms have no incentives to change their positions even if they would have such a possibility. Observe, that positions of the leader and the follower are not predefined as well as optimal investment paths. It is possible for the leader to reduce investments to lower its quality to benefit from technological spillover. However, as the leading firm has higher efficiency of investments the relative gain from imitation and reduction of investments is not enough to compensate the reduction in quality. It is more profitable for the leading firm to remain the leader, although it misses positive potential gain from this spillover. The more is the difference in investment efficiencies γ between firms, the more is the gap between qualities of the leader and the follower and the more benefits the follower obtains. At the same time this means higher efficiency for the leader and thus this firm will still benefit more from direct investments than from potential imitation.

Qualities of both firm for each product i gradually increase (since both invests positive amounts) to the maximum. This maximum is reached at infinite time and is the steady sate level of quality for every firm. Observe, that there are two different possible steady state levels of quality for every firm, one associated with the strategy of the follower and the other with the strategy of the leader. The

stability result means, that the firm which is the leader, has higher steady state value of quality innovations while remaining the leader and the following firm has higher steady state value remaining the follower. This result may change if one allows for different decay rates of qualities. In this case the steady state is not unique and changes in leadership may occur. However, such situation is much more complicated and left over for extension of the current paper. The following proposition summarizes information on the relations of values and quality levels in steady states of the quality game.

Proposition 10 (Steady states of quality game) *There is a unique steady-state level of quality, $\bar{q_j}(i)$, as long as either* (7) *or* (14) *hold. This is the maximal level of quality for each product i. Quality for every firm monotonically grows up to this level being reached in infinite time.*

For each product i this level is higher for every firm in the leadership regime than while being the follower

$$q^{\bar{L}}(i) > q^{\bar{F}}(i), \forall j; \tag{24}$$

However, the firm which is the follower obtains higher value by reaching lower quality level and the opposite for the leader:

$$\begin{aligned} &V^F(q_F^{\bar{F}}(i)) > V^F(q_F^{\bar{L}}(i)); \\ &V^L(q^{\bar{L}}(i)) > V^L(q^{\bar{F}}(i)) \end{aligned} \tag{25}$$

In symmetric case steady state levels of quality and values are similar for both firms,

$$q_j^{SY\bar{M}}(i) = q_{-j}^{SY\bar{M}}(i) = q^{SY\bar{M}}(i), V(q_j^{SY\bar{M}}(i)) = V(q_{-j}^{SY\bar{M}}(i)) = V(q^{SY\bar{M}}(i)). \tag{26}$$

Steady state level of quality are lower with possibility of imitation than in the absence of it, as well as values of the game:

$$q^{\bar{0}}(i) > q^{\bar{\theta}}(i), V(q^{\bar{0}}(i)) > V(q^{\bar{\theta}}(i)). \tag{27}$$

For illustration of the difference in investment policies caused by leader-follower patterns I take the following set of parameters which corresponds to leadership of firm j in quality innovations. Efficiency of investments into variety expansion is assumed to be equal for both firms:

$$SETJL := [\gamma_j = 0.7, \gamma_{-j} = 0.4, \beta = 0.2]; \tag{28}$$

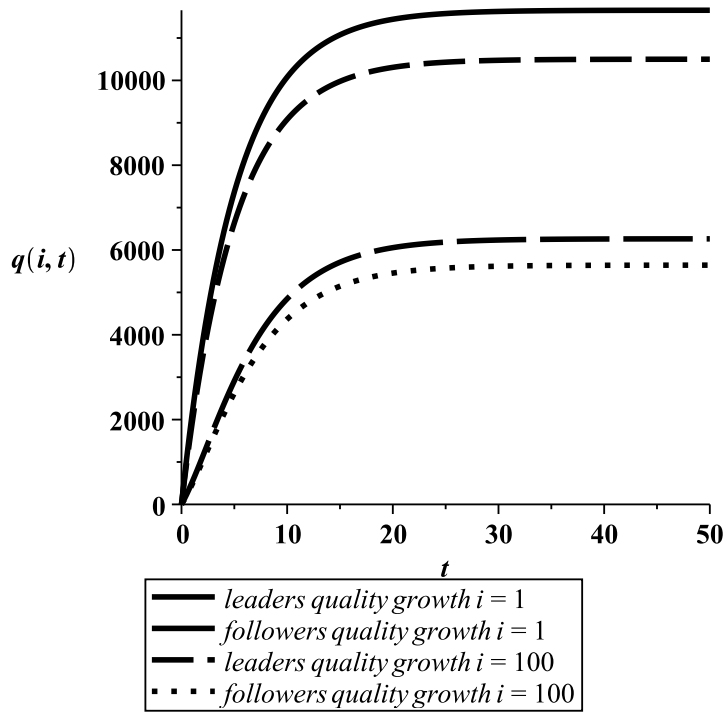

Figure 1: Difference in quality levels for different products and between firms

with

$$[n_0 = 1, \alpha_j = \alpha_{-j} = 0.5, r = 0.01, \theta = 0.15, N = 1000]$$

for both variants.

The evolution of qualities governed by the system (10) is displayed at the Figure 1. It might be seen that leader's 'quality' is always higher then that of the following firm while both firms' technology levels are lower for every next invented product then for the preceding one. This last comes from the assumption of decreasing returns on investments into every next product quality, which is reflected in the $\sqrt{N-i}$ term entering the evolution equations. It can also be seen that quality innovations for each product eventually reach the steady-state level and do not increase further on. This steady state levels are different for the leader and the follower and also differ across products.

5.2 Dynamics of variety expansion

Consider now the shape of the variety expansion. Since it is already demonstrated, that cooperative variety expansion will lead to higher value for both firms under

constant leadership mode, here I compare the variety expansion speed under constant leadership in quality growth and symmetric quality investments. It turns out, that the benefit the follower expects to obtain from technological spillover in the quality development of the next to be introduced product stimulates this firm to invest more into variety expansion. As a result, under positive technological spillover variety expansion speed is higher, than in the absence of such. The next proposition defines the ordering of variety expansion (and associated values of the game) between different regimes.

Proposition 11 (Variety expansion for cooperative firms and monopoly) *Denote:*

- $n^{MONO}(t)$ *variety expansion of a single monopolist with* $\gamma_M = \frac{\gamma_j+\gamma_{-j}}{2}$*;*
- $n_0^{SYM}(t)$ *variety expansion of cooperative firms with symmetric quality investments without possibility of imitation;*
- $n_0^{MONO}(t)$ *variety expansion of both firms with symmetric quality investments without possibility of imitation, when both firms invest independently;*
- $n_\theta^{SYM}(t)$ *variety expansion of cooperative firms with symmetric quality investments with possibility of imitation;*
- $n_\theta^{MONO}(t)$ *variety expansion of both firms with symmetric quality investments with possibility of imitation, when both firms invest independently.*

As long as (7) *holds, the following ordering of variety expansion processes holds for* $t < \infty$*:*

$$n^{LF}(t) > n_0^{SYM}(t) > n_\theta^{SYM}(t) > n^{MONO}(t) = n_0^{MONO}(t) > n_\theta^{MONO}(t). \tag{29}$$

Proof is done by direct verification of each relationship and is straightforward.

At the Figure 2 one may find the variety expansion of the differential game with different modes being considered for both firms and that of the single firm in the market. In this figure the same set $SETJL$ of parameters is adopted for illustration purposes, while the single monopolist's efficiency of quality innovations investments is set in between of the two firms at the level $\gamma_M = \frac{\gamma_j+\gamma_{-j}}{2}$ and all other parameters kept similar.

One may see that under the cooperative investments the variety expansion speed is the highest possible for constant leadership case and all other cases follow the ordering of Proposition 11.

This ordering gives an insight into the interdependence of both part of the model. The variety expansion rate is maximal under positive technological spillover of the quality game, since the cooperative investments are driven by the expected

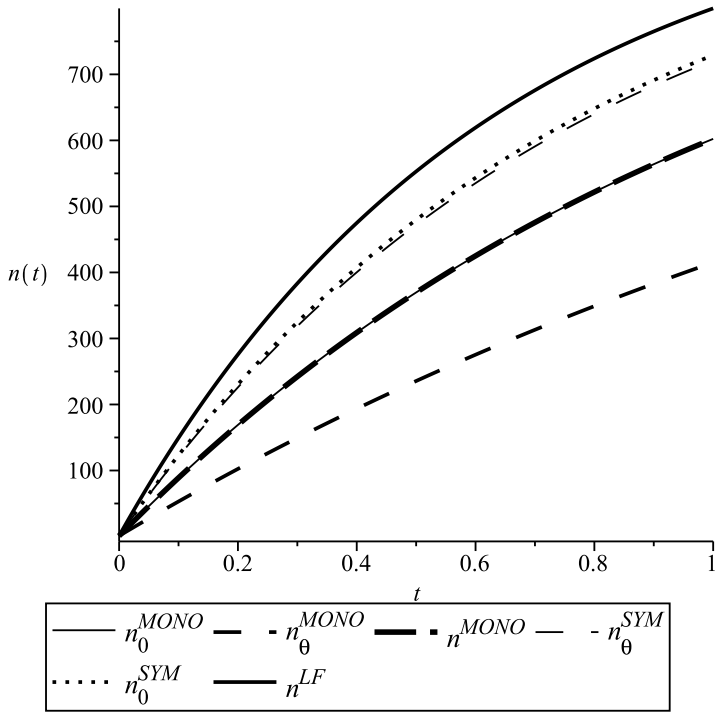

Figure 2: Product innovations for cooperative investments and a single firm

value of the development of the next product for every firm. The sum of these values is maximal if one of the firms use the benefits of technological spillover. In symmetric case cooperative investments are slower, since no one benefits from this spillover effect. Variety expansion is faster if no imitation is possible in symmetric situation, since value of quality game in this case is higher than under symmetric outcome with positive potential spillover. In this last case the possibility of imitation has a negative impact on the overall variety expansion, but this may take place only if no actual imitation occurs.

All cooperative cases yield higher variety expansion rates than monopolistic one, since both firms contribute non-zero amounts to this variety expansion process. The only case, when variety expansion process may be slower, than that of the monopolist, is the case when two conditions are fulfilled: there is possibility for imitation, but firms are symmetric and they choose to invest independently of each other.

Combination of dynamics of both parts of the model, demonstrated at Figures 2 and 1 provides the reconstruction of the full dynamics of the model under constant leadership in qualities. This is displayed at Figure 3.

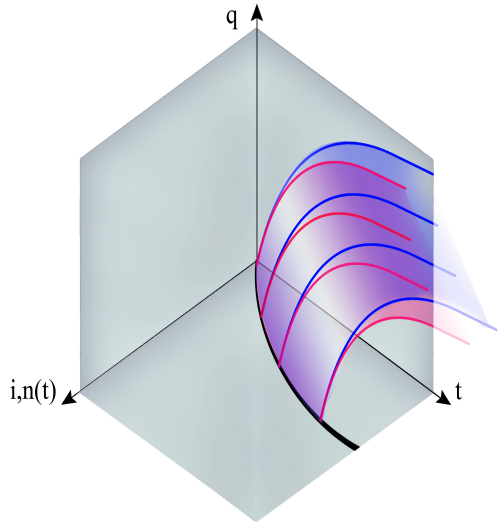

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Figure 3: Variety expansion and quality innovations

From this Figure one may observe the role of variety expansion process. The higher is the speed of introduction of new products, the more dense is the aggregate process of quality innovations. Thus, variety expansion is the generator of quality innovations. One may observe the underlying process of generation of products variety, $n(t)$, together with associated processes of technology improvements for both firms and for different products. The domination in technology levels is preserved along all the range of products to be invented.

All these points to the fact, that positive costless technological spillover or imitation is not harmful for any of the firms. Instead, it allows for specialization of investment activities in the case firms are not symmetric. This specialization yields both higher value of quality improving and variety-expanding innovations. This effect is treated in more details further on.

5.3 Specialization of innovations

Now consider the overall strategic profile and value generated for both firms under the positive technological spillover regime of the model. Here only this regime is considered, since the specialization effect in both directions of quality and varieties may be observed only for this regime. It turns out, that one of the firms invests more into the development of quality innovations while the other one invests more into the creation of new products. Thus the specialization of innovative ac-

tivities is observed. It may be shown, that this effect is robust to parameters value changes as well as to the change in leadership.

To observe this specialization effect, consider first the set of optimal strategies for the quality innovations game for both firms in the case of constant leader, given by Eq. (8). It is straightforward that the leaders' investments are higher than those of the follower for each i, if (7) hold. At the same time leader's value of the quality innovations game is always lower than that of the follower since its investments do not depend on the achieved technology level and are higher than those of the follower. At zero technology level leader's value is lower than that of the follower also. This constitutes the specialization of innovative activities of both firms in the area of quality innovations.

Next consider variety expansion innovations iven by Eq. (20). As it has been noted, the value of the quality innovations game estimated at zero technology level is higher for the follower. Hence,the investments into the variety expansion of the follower are higher than those of the leader, as (C.6) are completely symmetric except for the value functions of the quality innovations game. It follows, that the higher is the difference in values generated for both firms by the quality innovations game, the higher is the difference in variety expansion investments. The firm which is the leader in quality innovations invests less then the follower all the time. The decrease in intensity of product innovations over time is explained through the increasing complexity of the development of process innovations for every next product within the products' range. Since the only source of new value is the value generated by the development of the production technologies for new products through quality innovations, it becomes less attractive to introduce new products in comparison to the development of already existing ones as the process of variety expansion approaches its limit N.

These two effect constitutes the specialization effect, which is summarized in the last proposition of this paper.

Proposition 12 (Specialization of innovations) *As long as* (7) *is fulfilled, and cooperation at variety expansion level takes place, with firm j being the leader and $-j$ being the follower for simplicity, the following specialization of investment activities is observed:*

$$\begin{aligned} &\forall i \in N, \forall t > t_i(0) : g_j^L(i) > g_{-j}^F(i) \geq 0; \\ &\forall t < \infty : u_{-j}^F(t) > u_j^L(t) \geq 0; \end{aligned} \tag{30}$$

so that the firm which benefits from technological spillover invests more into the variety expansion than the other one. This specialization is efficient in the sense, that no other strategic profile may yield higher overall values of innovating activities for both firms and higher rates of introduction of new products and their

improvements. Thus, the positive costless technological spillover is beneficial for technology and value of innovations.

6 Discussion

In this section I discuss the robustness of results being obtained.

First consider the quality innovations part of the model. Every next product has lower maximal quality (production technology) level than preceding ones. This is the direct consequence of the shape of the function $\gamma_j(i) = \gamma_j\sqrt{N-i}$ which defines, how investment efficiency changes from product to product. Since this function is defined to be decreasing, the resulting maximal quality is decreasing also from product to product. However, the fact, that if one of the firms is the leader in quality innovations for one of the products it is also the leader for all other products is robust to the choice of investment efficiency function, as long as this last is a monotonic function of i. If it is not monotonic, for some products one of the firms will be technological leader, and for other products - the other firm. This means that results of Proposition 5 concerning ordering of qualities across products are specific for this specific choice of $\gamma(i)$ function. However, for each product i the same form of investment strategies and evolution of qualities hold, once the leadership of one of the firms is defined. Investments for both firms are constant and higher for the leader than for the follower. For every product i stability result of the solution given by Proposition 6 holds irrespective of the choice of investment efficiency function $\gamma(i)$.

Results of the variety expansion part also depend on the choice of this efficiency function. The decreasing speed of variety expansion is the direct consequence of decreasing efficiency of investments across products. Every next product has lower expected value of innovations and its introduction is thus less profitable. This may change with increasing across products efficiency of investments. The fact, that both firms invest non-zero amounts into variety expansion is not robust. If one would consider closed-loop strategies, the firm, which is the leader in quality innovations will not invest into variety expansion at all, since it will anticipate positive investments of the technological follower. However, in this paper only open loop strategies are analysed. Under open loop both firms invest according to the desired level of variety expansion and they do not take into account the possible reaction of the other firm. Still the ordering of investments in Proposition 12 will continue to hold for closed loop case also with leader's investments being zero.

The suggested model demonstrates, that costless imitation effect may be of benefit not only for the imitator, but for the firm, which is imitated also, if these firms are operating on separated markets (no market competition). Whether or

no this would hold for the proper duopoly with product market competition, is an open question, since innovations would influence price and quantity dynamics. These may influence innovations incentives in turn. Thus the overall dynamics of the model would be more complicated and it is not clear, whether the explicit solution may be achieved with such modification. At the same time treating firms as local monopolies makes the costless imitation effect sustainable, since the leader does not loose any value because of being imitated. This allows for sustainable specialization of investments between firms, described by Proposition 12.

Now one may ask, how robust is this main result of the paper. The specialization effect takes plays as long as there is positive technological spillover to the benefit of one of the firms. This spillover creates differences in the investment patterns of the leader and the follower. For this to happen it is enough for firms to have different efficiencies of investments into quality improvements, $\gamma_j \neq \gamma_{-j}$. Equal decay rates, being assumed by (7) are really not necessary. These identical decay rates are introduced to simplify the notation only. If decay rates would be different, there appears additional regime of the game, namely with changing leadership. It may occur, when one of the firms has higher efficiency of quality investments and also higher decay rates. Then initially this firm will take the leading position in technology development, but later on its growth would slow down due to high decay rates. The other firm would be the follower for some time and then will catch up with the technological leader due to positive technological spillover and lower decay rates. It may be demonstrated, that specialization effect holds in this situation also, with the firm which become the follower in the long run investing more into the variety expansion and less into quality development for all the products.

The case of possible catching up in technologies may also change the relative values of innovations processes for both firms. The broader solution of the model with different decay rates is consider to be an immediate extension of the current work. Another interesting extension would be to consider proper duopoly and check, whether the specialization effect would continue to hold if market competition is allowed to influence innovations.

Appendices

A Derivation of optimal strategies and solutions for the quality game with constant leadership

Using HJB equation from (6) first order conditions for controls yield the following form of them:

$$g_j(i,t)^* = \gamma_j \sqrt{(N-i)} \cdot \frac{\partial V_j(i)}{\partial q_j(i,t)};$$

$$g_{-j}(i,t)^* = \gamma_{-j} \sqrt{(N-i)} \cdot \frac{\partial V_{-j}(i)}{\partial q_{-j}(i,t)}. \tag{A.1}$$

Hence the form of the optimal control is defined by the form of the underlying value function for both firms separately. Assume the following form of value functions for both firms:

$$V_j(i) = A_j q_j(i,t) + B_j q_{-j}(i,t) + C_j. \tag{A.2}$$

Consider first the HJB equation for the leading firm. This firm does not benefit from the imitation effect but its problem is influenced by the imitation effect present for the other firm. The position of the leader is characterized by the condition:

$$\forall i,t : q_j(i,t) > q_{-j}(i,t), \tag{A.3}$$

Of course, as long as one of the firms is the leader in quality innovations, the other is the follower. Inserting (A.2) into the pair of HJB equations (6) and using (7) one obtains the following set of coefficients for the leaders' value function:

$$\begin{cases} A_j = \frac{1}{\beta+r}; \\ B_j = 0; \\ C_j = \frac{1}{2}\frac{\gamma_j^2}{(\beta+r)^2 r}(N-i). \end{cases} \tag{A.4}$$

Hence coefficients for the leader's value function do not depend on the optimal investments of the follower. This set of coefficients corresponds to the linear value function of the leader with the absence of cross-effects and hence the optimal strategy is constant as long as (A.3) holds. Together with first-order conditions on

controls the derived value function of the leader constitutes optimal (piecewise-constant) control for the leader:

$$g_j(i,t) = \begin{cases} \frac{\gamma_j\sqrt{(N-i)}}{r+\beta}, t : i \geq n(t); \\ 0, t : i < n(t). \end{cases}$$

(A.5)

Now consider the problem of the follower. Inserting (A.2) into the second equation in (6) results in a system of equations for coefficients of the followers' value function:

$$\begin{cases} A_{-j} = \frac{1}{\theta+\beta+r}; \\ B_{-j} = \frac{\theta}{(\beta+r)(\beta+\theta+r)}; \\ C_{-j} = \frac{1}{2}\frac{(r^2\gamma_{-j}^2+\beta^2\gamma_{-j}^2+2\gamma_j^2\theta^2+2\theta\beta\gamma_j^2+(2\theta\gamma_j^2+2\beta\gamma_{-j}^2)r)}{r(\beta+r)^2(r+\beta+\theta)^2}(N-i). \end{cases}$$

(A.6)

Here value function of the follower depends on the level of technology achieved by quality innovations of the leader. Since this last is known already, one has the explicit formulation of the value function for the follower. One may derive the optimal investments of the follower according to first order conditions (A.1):

$$g_{-j}(i,t) = \begin{cases} \frac{\gamma_{-j}\sqrt{(N-i)}}{r+\beta+\theta}, t : i \geq n(t); \\ 0, t : i < n(t). \end{cases}$$

(A.7)

Optimal strategies (A.5) and (A.7) yield the ODE system starting from time $t_i(0)$ of the introduction of product i:

$$\begin{cases} \dot{q_j^L}(i,t) = \frac{\gamma_j^2}{\beta+r}(N-i) - \beta q_j(i,t), t : i \geq n(t); \\ \dot{q_{-j}^F}(i,t) = \frac{\gamma_{-j}^2}{\beta+\theta+r}(N-i) + \theta q_j(i,t) - (\beta+\theta)q_{-j}(i,t), t : i \geq n(t). \end{cases}$$

(A.8)

yielding qualities fro both firms as functions of the position of the product, i and time t:

$$q_j^L(i,t) = \frac{\gamma_j^2(N-i)}{\beta(r+\beta)} \cdot (1-\mathrm{e}^{-\beta t});$$

$$q_{-j}^F(i,t) = \left(1 + (N-i)\cdot(\frac{E_1(\mathrm{e}^{(\beta+\theta)t}+1)}{\beta+\theta} - \frac{E_2(\mathrm{e}^{\theta t}+1)}{\theta}\right) \times \mathrm{e}^{-(\beta+\theta)t}. \quad \text{(A.9)}$$

where $E_1, E_2 = f(\gamma_{j,-j}, \beta, \theta)$ are some functions of parameters only. Eqs. (A.5), (A.7), (A.8), (A.9) constitute Proposition 5.

B Value functions for the quality game

Values generated by the quality innovations with constant leadership are computed by inserting (A.4), (A.6) into (A.2) and then using the solution (A.9):

$$V_j^L(i) = \frac{q_j(i,t)}{r+\beta} + \frac{1}{2}\frac{\gamma_j^2}{r(r+\beta)^2}(N-i);$$

$$V_{-j}^F(i) = 2\left(\frac{q_{-j}(i,t)}{\theta+r+\beta} + \frac{\theta}{r+\beta} \times \frac{q_j(i,t)}{\theta+r+\beta}\right) +$$

$$+\left(\frac{\theta\gamma_j^2}{r(r+\beta)(r+\theta+\beta)}\frac{1}{r+\beta} + \frac{1}{2}\frac{\gamma_{-j}^2}{r(r+\theta+\beta)}\right)(N-i). \tag{B.1}$$

From this it might be seen that it is not profitable for the firm which is the leader in process innovations to choose the investments rate lower than optimal. In the constant leadership case it suffices to compare value functions at the start of investments, with zero quality level. The value functions for the firm j at this point are given by:

$$V_j^L(i)|_{q(i)=0} = \frac{1}{2}\frac{\gamma_j^2}{r(r+\beta)^2}(N-i);$$

$$V_j^F(i)|_{q(i)=0} = \left(\frac{\theta\gamma_{-j}^2}{r(r+\beta)(r+\theta+\beta)}\frac{1}{r+\beta} + \frac{1}{2}\frac{\gamma_j^2}{r(r+\theta+\beta)}\right)(N-i). \tag{B.2}$$

for being the leader and being the follower respectively. Direct comparison of these values while (7) holds shows that the first value is always higher than the second one. The opposite holds for the following firm. This proves Proposition 6.

The values being generated for both firms by the boundary product in constant leadership mode, estimated at the initial time are:

$$\text{(B.3)} \quad V_j^L(0)\,|_{(i=n(t))} = \frac{1}{2}\frac{\gamma_j}{(r+\beta)^2 r}(N-n(t));$$

$$\text{(B.4)} \quad V_{-j}^F(0)\,|_{(i=n(t))} = \left(\frac{\theta\gamma_j^2}{r(r+\beta)(r+\theta+\beta)}\frac{1}{r+\beta} + \frac{1}{2}\frac{\gamma_{-j}^2}{r(r+\theta+\beta)}\right)(N-i).$$

These values are used for the solution of the variety expansion problem in the way discussed previously.

C Optimal strategies and solution for the problem of variety expansion

Hamiltonian functions for both firms are obtained using eqs. (18). They are symmetric for both players:

$$\mathscr{H}_j(n,\lambda_j,u_j,u_{-j}) =$$

(C.1)

$$= \left((\alpha_j u_j(t) + \alpha_{-j}u_{-j}(t))C_j(N-n(t)) - \frac{1}{2}u_j(t)^2\right) + \lambda_j(t)(\alpha_j u_j(t) + \alpha_{-j}u_{-j}(t));$$

$$\mathscr{H}_{-j}(n,\lambda_j,u_j,u_{-j}) =$$

(C.2)

$$= \left((\alpha_j u_j(t) + \alpha_{-j}u_{-j}(t))C_{-j}(N-n(t)) - \frac{1}{2}u_{-j}(t)^2\right) + \lambda_{-j}(t)(\alpha_j u_j(t) + \alpha_{-j}u_{-j}(t)).$$

The first-order conditions on the investments into the variety expansion for both firms define investments as functions of co-state variables:

$$\frac{\partial\mathscr{H}_j}{\partial u_j} = 0 : u_j(t)^* = \alpha_j\lambda_j(t) + C_j(N-n(t));$$

$$\text{(C.3)} \qquad \frac{\partial\mathscr{H}_{-j}}{\partial u_{-j}} = 0 : u_{-j}(t)^* = \alpha_{-j}\lambda_{-j}(t) + C_{-j}(N-n(t)).$$

Substituting these into Hamiltonian functions and writing down co-state equations yield the canonical system for the variety expansion game:

$$\dot{\lambda}_j = r\lambda_j - \frac{\partial \mathscr{H}_j}{\partial n(t)} =$$

$$= (r + \alpha_j^2 C_j)\lambda_j(t) + \alpha_{-j}^2 C_j \lambda_{-j}(t) + (\alpha_j^2 (C_j)^2 + \alpha_{-j}^2 C_j C_{-j})(N - n(t));$$

$$\dot{\lambda}_{-j} = r\lambda_{-j} - \frac{\partial \mathscr{H}_{-j}}{\partial n(t)} =$$

$$= (r + \alpha_{-j}^2 C_{-j})\lambda_{-j}(t) + \alpha_j^2 C_{-j} \lambda_j(t) + (\alpha_{-j}^2 (C_{-j})^2 + \alpha_j^2 C_j C_{-j})(N - n(t));$$

$$\dot{n(t)} = \alpha_j^2 \lambda_j(t) + \alpha_{-j}^2 \lambda_{-j}(t) + (\alpha_j^2 (C_j)^2 + \alpha_{-j}^2 (C_{-j})^2)(N - n(t));$$

$$n(0) = n_0;$$

$$\lim_{t\to\infty} e^{-rt}\lambda_j(t) = 0;$$

(C.4)

$$\lim_{t\to\infty} e^{-rt}\lambda_{-j}(t) = 0.$$

Inserting optimal controls into the dynamic constraint for variety expansion together with (C.4) constitutes the system of 3 linear ODEs with one initial condition and two boundary conditions (transversal ones) which is then solved. The

solution is:

$$n(t)^* = N - (N - n_0)\mathrm{e}^{\frac{1}{2}(r - \sqrt{r(r+4\alpha_j^2 C_j + 4\alpha_{-j}^2 C_{-j})})t};$$

$$\lambda_j(t)^* = -\frac{C_j(\alpha_j^2 C_j + \alpha_{-j}^2 C_j)}{2\alpha_j^2 C_j + r + \sqrt{r(r+4\alpha_j^2 C_j + 4\alpha_{-j}^2 C_{-j})}} \cdot$$
$$\cdot 2(N - n_0)\mathrm{e}^{\frac{1}{2}(r - \sqrt{r(r+4\alpha_j^2 C_j + 4\alpha_{-j}^2 C_{-j})})t};$$

$$\lambda_{-j}(t)^* = -\frac{C_{-j}(\alpha_j^2 C_j + \alpha_{-j}^2 C_j)}{2\alpha_j^2 C_j + r + \sqrt{r(r+4\alpha_j^2 C_j + 4\alpha_{-j}^2 C_{-j})}} \cdot$$
$$\cdot 2(N - n_0)\mathrm{e}^{\frac{1}{2}(r - \sqrt{r(r+4\alpha_j^2 C_j + 4\alpha_{-j}^2 C_{-j})})t}. \tag{C.5}$$

Then I use (C.5) to define explicitly investments of both firms as functions of time from (C.3):

$$u_j(t)^* = \frac{\alpha_j C_j(r + \sqrt{r(r+4\alpha_j^2 C_j + 4\alpha_{-j}^2 C_{-j})})}{2\alpha_j^2 C_j + 2\alpha_{-j}^2 C_{-j} + r + \sqrt{r(r+4\alpha_j^2 C_j + 4\alpha_{-j}^2 C_{-j})}} \cdot$$
$$\cdot (N - n_0)\mathrm{e}^{\frac{1}{2}(r - \sqrt{r(r+4\alpha_j^2 C_j + 4\alpha_{-j}^2 C_{-j})})t};$$

$$u_{-j}(t)^* = \frac{\alpha_{-j} C_{-j}(r + \sqrt{r(r+4\alpha_j^2 C_j + 4\alpha_{-j}^2 C_{-j})})}{2\alpha_j^2 C_j + 2\alpha_{-j}^2 C_{-j} + r + \sqrt{r(r+4\alpha_j^2 C_j + 4\alpha_{-j}^2 C_{-j})}} \cdot$$
$$\cdot (N - n_0)\mathrm{e}^{\frac{1}{2}(r - \sqrt{r(r+4\alpha_j^2 C_j + 4\alpha_{-j}^2 C_{-j})})t}. \tag{C.6}$$

Eqs. (C.5) and (C.6) give Proposition 8.

D Value functions of variety expansion game

One also may compute the value function of the variety expansion game as the optimized Hamiltonian function. The value of the variety expansion game for the case of cooperative investments is the respective Hamiltonian function at time

$t = 0$ and optimal co-state and variety values:

$$V_j^{LF}(n) = \frac{1}{r}\mathscr{H}_j(n(0)^*, \lambda_j(0)^*) =$$

$$= C_j \cdot \frac{(N-n_0)^2(2\alpha_{-j}^2 C_{-j} + \alpha_j^2 C_j)}{2\alpha_{-j}^2 C_{-j} + 2\alpha_j^2 C_j + r + \sqrt{r(r + 4\alpha_j^2 C_j + 4\alpha_{-j}^2 C_{-j})}};$$

$$V_{-j}^{LF}(n) = \frac{1}{r}\mathscr{H}_{-j}(n(0)^*, \lambda_{-j}(0)^*) =$$

$$= C_{-j} \cdot \frac{(N-n_0)^2(2\alpha_j^2 C_j + \alpha_{-j}^2 C_{-j})}{2\alpha_{-j}^2 C_{-j} + 2\alpha_j^2 C_{-j} + r + \sqrt{r(r + 4\alpha_j^2 C_j + 4\alpha_{-j}^2 C_{-j})}}. \tag{D.1}$$

In the case of no cooperation the value of the variety expansion does not include the variables of the other firm:

$$V_j^{MONO}(n) = C_j \cdot \frac{(N-n_0)^2 \alpha_j^2 C_j}{2\alpha_j^2 C_j + r + \sqrt{r(r + 4\alpha_j^2 C_j)}};$$

$$V_{-j}^{MONO}(n) = C_{-j} \cdot \frac{(N-n_0)^2 \alpha_{-j}^2 C_{-j}}{2\alpha_{-j}^2 C_{-j} + r + \sqrt{r(r + 4\alpha_{-j}^2 C_{-j})}}. \tag{D.2}$$

Since both C_j and C_{-j} are nonnegative, $V_j^{LF}(n) > V_j^{MONO}(n)$ and $V_{-j}^{LF}(n) > V_{-j}^{MONO}(n)$. The same applies to $n(t)$ comparisons. This proves Proposition 9.

References

Albernathy, W. and J. Utterback (1978). Patterns of industrial innovation. *Technology Review 80*(7), 40–47.

Belyakov, A., T. Tsachev, and V. Veliov (2011). Optimal control of heterogeneous systems with endogenous domain of heterogeneity. *Applied Mathematics and Optimization 64*, 287–311.

Bondarev, A. (2012). The long run dynamics of heterogeneous product and process innovations for a multi product monopolist. *Economics of Innovation and New technology 21*(8), 775–799.

Cellini, R. and L. Lambertini (2002). A differential game approach to investment in product differentiation. *OJournal of Economic Dynamics and Control* (27), 51–62.

D'Aspremont, C. and A. Jacquemin (1988). Cooperative and noncooperative r & d in duopoly with spillovers. *The American Economic Review 78*(5), 1133–1137.

Dockner, E., S. Jorgensen, N. Long, and G. Sorger (2000). *Differential Games in Economics and Management Sciences*. Cambridge University Press,/Cambridge.

Faria, P. and F. Lima (2009). Firm decision on innovation types: Evidence on product, process and organizational innovation.

Gallini, N. (1992). Patent policy and costly imitation. *The RAND Journal of Economics 23*, 52–63.

Judd, K. (2003). Closed-loop equilibrium in a multi-stage innovation race. *Economic Theory N/A*(21), 673–695.

Kraft, K. (1990). Are product and process innovations independent from each other? *Applied Economics 22*(8), 1029–1038.

Lambertini, L. and A. Mantovani (2010). Process and product innovation: A differential game approach to product life cycle. *International Journal of Economic Theory 6*(2), 227–252.

Lin, P. (2004). Process and product r&d by a multiproduct monopolist. *Oxford Economic Papers working paper*(56), 735–743.

Loury, G. (1979). Market structure and innovation. *The Quarterly Journal of Economics 93*, 395–410.

Peretto, P. and M. Connolly (2007). The manhattan metaphor. *Journal of Economic Growth 12*(4), 329–350.

Reinganum, J. (1982). A dynamic game of r and d: Patent protection and competitive behavior. *Econometrica 50*(3), 671–688.

Rosenkranz, S. (2003). Simultaneous choice of process and product innovation when consumers have a preference for product variety. *Journal of Economic Behaviour and Organisation 50*, 183–201.