Optimal control over a continuous range of homogeneous and heterogeneous innovations with finite life-cycles.

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Heterogeneous and homogeneous innovations under finite life-cycles of production technologies.

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Abstract

Current paper analyses the influence of the length of production technologies life-cycles on the relative intensity of investments of a multi-product monopolist into different types of innovations. This monopolist is developing new versions of the basic product continuously and simultaneously invests into the production technologies of all these new products. In the paper the finite character of these products' life-cycles is assumed. It is demonstrated that under the condition of finite-time life-cycles of new products the heterogeneity of investment characteristics of these new products play a substantial role in the intensity of innovations of both types (variety-enhancing and quality-improving). It is argued, that the heterogeneity of products creates two effects of the length of the life cycle of different directions, while this is not the case for homogeneous products.

Keywords: Heterogeneous Innovations, Economic Dynamics, Product Life-cycle, Distributed Control

JEL codes: C02, L0, O31.

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1 Introduction

The main focus of this paper is on the optimal behaviour of a multi-product monopolist firm in the field of its innovative policy in the presence of time constraints on its activities. Namely, what are the optimal relations between investments into the generation of innovations of different types if such an innovative activity may be carried out only during some limited time.

The multi-product monopolist is modelled as a single planning agent in the industry (market). He/she decides upon investments into the development of new products (variety expansion process in the sense of (?)) and into the improvement of production technology or quality for every of already introduced products (quality improvement in the sense of (?)). The process of variety expansion is described as continuous in time thus yielding infinitely many new potential products. The range of such products is a continuum. The special feature of the framework is that every such a product possesses its own quality characteristic and the quality-improving process is described on two levels: as an aggregate process, which depends on variety expansion (product innovations) and as an ansamble of separate quality processes for each of the products, which does not depend on variety expansion but in turn influences this last. The process of quality-improving innovations in such a framework depends crucially on the investments efficiency of every new product.

In this paper I consider two alternatives. In the first all the products possess the same investments efficiency (homogeneous products). In the second the investments efficiency of the product depends on the position of this product in the products space (heterogeneous products). It turns out that the heterogeneity of investments characteristic creates new effect of the length of the life-cycle which is not observed for homogeneous products, which is referred to as compensation effect. As a result, innovator's incentives are different for these two alternatives and the role of the length of the life-cycle is different. Namely, under homogeneous products assumption variety expansion growth is linear and products range under the control of the innovator is constant. The longer the life-cycle, the wider this range is independently on the stage of the industry's development. At the same time for heterogeneous products this range is varying over time and reduce for mature stages of industry's development, thus creating the negative effect of the length of the life-cycle.

It is thus argued, that if products are characterized by sufficient heterogeneity of their investment characteristics, longer life-cycles (aka patents) may reduce incentives for new product innovations while this is not the case for homogeneous products. The same difference is observed for quality-improving innovations: under homogeneous products assumption the longer life-cycles unambiguously stimulate quality growth for all the products and also the aggregate process innovations stream. At the same time for heterogeneous products the longer life-cycles does not necessarily lead to the increase in the aggregate process innovations although stimulating quality improvements for every separate product.

Description of multi-product innovations follows the ideas of (?), (?), (?) and mainly (?), where such a multi-product monopolist behaviour for limited time-horizon is introduced. In the current paper the infinite planning horizon is adopted instead, as in (?). This allows for modelling long-run behaviour of the agent and for explicit solution for the dynamics of both types of innovations.

Current paper differs from this last one by the following additional assumption. Namely, the improvement of quality for every new product is limited by a finite life-cycle of the product (length of which is identical across products). This finite life-cycle is considered as to describe the effect of patenting: upon the invention of the product, the monopolist is granted the exclusive right to develop and sell this product during some fixed time, τ. After this time passes, the product development becomes common knowledge and no further economic profit may be derived from it.

To stimulate the introduction of new products in heterogeneous products setting the patent's length should be limited by some finite number. Such an argument replicates the argument of Nordhaus in his seminal paper (?) but from completely different grounds: patents should be limited because of the interplay between quality-improving and variety-enhancing innovations and not to stimulate innovations from other agents as in this classic paper.

Main findings of the paper may be summarized as following:

- The intensity of introduction of new products is almost always higher for heterogeneous products setting then for homogeneous ones.
- Qualities of all new products are developed to higher levels under heterogeneous setting also.
- For heterogeneous products two different effects of the length of the life-cycle on the products variety are present, while this is not true for homogeneous products.
- In the absence of competition there is no ground for the limited duration of patents under homogeneous setting while under heterogeneous setting shorter length of the life-cycle creates incentives for further introduction of new products at the mature stage of industry development.

- The interplay between product and process innovations is important only if heterogeneity of investment characteristics of new products is present.

The rest of this paper is organised as following: in the next section a brief review of the state of the arts is made. Then the basic framework of heterogeneous innovations is introduced following the lines of (**?**). After that the model with limited life-cycles is formally described. Then I consider homogeneous and heterogeneous versions of the model and its solution for both cases. The comparison of the influence of the length of the life-cycle on the intensity of product and process innovations in both scenarios is considered in the end of the paper.

2 Finite life-cycles model

2.1 *Objective function and dynamic constraints*

In this section the modification of the basic framework presented above is made that allows for finite life-cycles of all the new products. For this purpose one may treat products' life-cycles as patents: after the expiration of the patent's time the agent cannot use his/her achieved quality level of the given product for profitable activities (e.g. sell this product as a monopolist with monopolistic price). This of course is not true in real economies, but one can imagine the high density of competition on the product market which approaches the perfect one. As soon as the patent expires, all quality development of this product becomes the common knowledge to all the competitors and hence the agent in the model is no longer able to derive non-zero economic profit from it and thus he/she is no longer interested in quality investments in this product. In terms of the model this means that every product from $n(t)$ range has a limited time life-cycle (determined by the patent's length) during which its quality is developed by the agent. After this time development stops. At the same time such a setup would make sense only if at any given time the agent may also invest into the variety expansion thus creating another portion of patented new products. For this the infinite time horizon model with respect to variety expansion process from the previous section is used. Below all these considerations are formulated in a more formal way.
The objective functional of the agent is almost the same as in the basic

framework:

$J^{pat} \stackrel{\text{def}}{=}$

(1)

$$\stackrel{\text{def}}{=} \max_{u(\bullet),g(\bullet)} \int_0^\infty e^{-rt} \left(\alpha u(t) \int_{t_i(0)}^{t_i(0)+\tau} \{q(n,s+t) - \frac{1}{2} g(n,s+t)^2\} ds - \frac{1}{2} u(t)^2 \right) dt$$

Such an objective functional is defined allowing for finite time of development of quality q_i of each new product. To see this, consider the decomposition of the problem into quality growth and variety expansion problems. This is done in the same way, as in (**?**) with necessary modifications described below.

Start with the quality growth part. In the basic model every product is developed in infinite time and hence the value function for the quality growth of each product i is defined from 0 to infinity. Under finite life-cycles conjecture the time of development of every new product i is limited by τ. Hence the value function of quality growth is time-limited and time dependent:

$$V^{pat}(q_i, t) = \max_{g_i(\bullet)} \int_{t_i(0)}^{t_i(0)+\tau} e^{-rs} \{q_i(s) - \frac{1}{2} g_i(s)^2\} ds \tag{2}$$

Note, that this value function depends not only on the number of the product (which implicitly defines dependence on the time of emergence $t_i(0)$ also as the inverse function of i) but also on the length of the life-cycle, τ, which is assumed to be the same for all products.

Second part of the overall value generation consists of the intensity of addition of new products at every time given the expected value of the stream of profit derived from the quality of these newly introduced products. This part may be represented by the integral over all potential stream of quality of the product over its (limited) life-cycle. At the same time the information on the value generated by the development of the quality of the product is already contained in the value function of the quality problem above, so it suffices to integrate over all potential products at the zero time (since every product has zero quality before its emergence and after the expiration of the patent, this is similar to integrating over quality evolution path of every product). Denote by $V^{pat}(0)_{n(t)}$ the value of the quality growth problem above for the boundary product, $i = n(t)$. This value does not depend on the quality level, since this last is zero for the boundary product (just invented one), as it is required by constraints. This value is given by:

$$V^{pat}(0,\tau)_{n(t)} = V^{pat}(q_i, t)|_{i=n(t), q_i=0, t=t_i(0)}. \tag{3}$$

Observe that in this last equation the value is computed at the time of emergence, $t_i(0)$, not at zero time. This is done to simplify the expressions and does not change the value itself, since quality of any product till this time $t_i(0)$ is zero.

Variety expansion generates value through the addition of new products which are then developed through quality growth process. These yield the value function for variety expansion problem in the following form:

$$V^{pat}(n) = \max_{u(\bullet)} \int_0^\infty e^{-rt}\Big(\alpha u(t) \times V^{pat}(0,\tau)_{n(t)} - \frac{1}{2}u(t)^2\Big)dt \tag{4}$$

subject to dynamic and boundary constraints on variety expansion process:

$$\begin{aligned} \dot{n(t)} &= \alpha u(t); \\ n(0) &= n_0. \end{aligned} \tag{5}$$

where the value of $V^{pat}(0,\tau)_{n(t)}$ is given by equation above.

2.2 *General solution for quality growth*

The solution of the problem of quality growth for the finite life-cycles model follow the same steps as for the basic infinite time model. First the Hamilton-Jacobi-Bellman equation for the development of every product i is derived. Assumption of the polynomial form of the associated value function helps to derive optimal investments for every product i. These are then used to solve for the optimal dynamics of quality of product i within the duration of its life-cycle, τ. This last is the function of the efficiency of investments, $\gamma(i)$, and decay rate $\beta(i)$. These two are specified further on, while here the general form of the solution is considered only.

The Hamilton-Jacobi-Bellman equation for the development of every product i depends on t and on the quality level itself. For each i the evolution of the value of quality development is thus defined from the equation:

$$rV^{pat}(q_i,t) + \frac{\partial V^{pat}(q_i,t)}{\partial t} = \max_{g_i(\bullet)}\Big\{q_i - \frac{1}{2}g_i^2 + \frac{\partial V^{pat}(q_i,t)}{\partial q_i} \times (\gamma_i g_i - \beta_i q_i)\Big\},$$

(6)

$$t \in [t_i(0),..,t_i(0)+\tau]$$

One may assume linear form of value function for this problem, but with time-varying coefficients (since time-dependence of the value):

$$V^{ass}(q_i,t) = A_i(t)q_i + B_i(t). \tag{7}$$

Then the first-order condition for every product's quality growth is:

$$-g_i + \frac{\partial V(q_i, t)}{\partial q_i} \times \gamma_i = 0;$$

$$g_i^{pat} = A_i(t) \times \gamma_i. \tag{8}$$

One has a system of 2 differential equations on value function coefficients:

$$\begin{aligned} \dot{A_i}(t) &= (r + \beta_i)A_i(t) - 1; \\ \dot{B_i}(t) &= rB_i(t) - \frac{1}{2}\gamma_i^2 A_i(t)^2 \\ A_i(\tau + t_i(0)) &= 0; \\ B_i(\tau + t_i(0)) &= 0. \end{aligned} \tag{9}$$

Observe that for every product i the value function is different, as coefficients are different due to different boundary conditions and possible dependence of $\beta(\bullet), \gamma(\bullet)$ on the products position, i.

This is a system of first order equations which can be readily solved. First the solution for $A_i(t)$ coefficient as a function of emergence time $t_i(0)$ is obtained:

$$A_i(t) = \frac{1}{(r + \beta_i)}(1 - e^{(r+\beta_i)(t - t_i(0) - \tau)}) \tag{10}$$

Substitution of this into the equation for $B_i(t)$ term yields the second coefficient as function of emergence time also:

$$\dot{B_i}(t) = rB_i(t) - \frac{1}{2}\gamma_i^2 \Big(\frac{1}{(r + \beta_i)}(1 - e^{(r+\beta_i)(t - t_i(0) - \tau)})\Big)^2$$

$$\begin{aligned} B_i(t) &= \frac{\gamma_i^2}{r(r + \beta_i)^2} \times \\ &\times \Big(\frac{r}{\beta_i} e^{(r+\beta_i)(t - (\tau + t_i(0)))} - \frac{1}{2(r + \beta_i)} e^{2(r+\beta_i)(t - (\tau + t_i(0)))} - \frac{(r + \beta_i)^2}{\beta_i(r + 2\beta_i)} e^{r(t - (\tau + t_i(0)))} + \frac{1}{2}\Big). \end{aligned} \tag{11}$$

These calculations provide the form of the value function for quality growth for every product i:

$$V^{pat}(q_i,t) = \frac{1}{(r+\beta_i)}(1-e^{(r+\beta_i)(t-t(0)_i-\tau)}) \times q_i(t)+$$
$$+\frac{\gamma_i^2}{r(r+\beta_i)^2}\times$$
(12)
$$\times\Big(\frac{r}{\beta_i}e^{(r+\beta_i)(t-(\tau+t_i(0)))}-\frac{1}{2(r+\beta_i)}e^{2(r+\beta_i)(t-(\tau+t_i(0)))}-\frac{(r+\beta_i)^2}{\beta(r+2\beta_i)}e^{r(t-(\tau+t_i(0)))}+\frac{1}{2}\Big).$$

The resulting coefficients being inserted into the first order condition (8) yield optimal investments into quality growth:

$$g_i^{pat}(t) = \gamma_i\Big(\frac{1-e^{(r+\beta_i)(t-t_i(0)-\tau)}}{(r+\beta_i)}\Big) \tag{13}$$

Finally one obtains ODE for quality growth:

$$\dot{q_i(t)} = \gamma_i^2\Big(\frac{1-e^{(r+\beta_i)(t-t_i(0)-\tau)}}{(r+\beta_i)}\Big)-\beta_i q_i(t);$$
$$q_i(t_i(0)) = 0. \tag{14}$$

which is the first-order linear ODE with the unique solution:

$$q_i^{pat}(t) =$$
$$=\frac{\gamma_i^2}{(r+\beta_i)(r+2\beta_i)\beta_i}\times$$
$$\times\Big(\beta_i(e^{-(r+\beta_i)(t_i(0)-\tau-t)}-e^{(r+\beta_i)(t-t_i(0)-\tau)})-(r+2\beta_i)(e^{-\beta_i(t-t_i(0))}-1)\Big)$$
(15)

As it can be seen from (15), quality growth for each product depends on time, but only within the boundaries of the patent length and from the patent length itself, and from the time of emergence of the product, $t_i(0)$. It also depends on the investment efficiency and quality decay rates for each individual product, β_i,γ_i. However, to define the location of the evolution path for the certain product in the product's space and in the overall quality improving process, one has to define the time of emergence from the variety expansion part of the problem.

To proceed to the variety expansion part of the model one needs $V^{pat}(0,\tau)_{n(t)}$

defined above. This value depends on the patent's length, investment efficiency and decay rate for the boundary product ($i = n(t)$):

$$V^{pat}(0,\tau)_{n(t)} = \frac{\gamma^2_{n(t)}}{r\beta_{n(t)}(r+2\beta_{n(t)})(r+\beta_{n(t)})^2}\times$$

$$\times\Big(r(r+2\beta_{n(t)})e^{-(r+\beta_{n(t)})\tau} - \frac{1}{2}r\beta_{n(t)}e^{-2(r+\beta_{n(t)})\tau} - (r+\beta_{n(t)})^2e^{-r\tau} + \frac{1}{2}\beta_{n(t)}(r+2\beta_{n(t)})\Big). \tag{16}$$

The general solution for variety expansion cannot be thus obtained, since the essential part of the value generation process of $V^{pat}(n)$ in (4) depends on the $V^{pat}(0,\tau)_{n(t)}$, and this last is defined from the form of the investment efficiency and decay functions as functions of $n(t)$. In what follows two specific scenarios are considered, denoted by the names homogeneous products and heterogeneous products. For this the functions $\beta(i), \gamma(i)$ are specified as follows:

$$\begin{aligned} \beta_{homo}(i) &= \beta = const;\\ \gamma_{homo}(i) &= \gamma = const;\\ \\ \beta_{heter}(i) &= \beta = const;\\ \gamma_{heter}(i) &= \gamma\times\sqrt{N-i}. \end{aligned} \tag{17}$$

The first scenario assumes constant efficiency of investments and quality decay rate for all the products, while in the second scenario some degree of heterogeneity is allowed through the introduction of decreasing investments efficiency across products range. In the next section solutions of the model are described for both scenarios.

3 Solutions for homogeneous and heterogeneous products

3.1 Homogeneous solution

First consider the solution of the model for homogeneous case constant functions in (17)). The particular solution for quality growth and investments into the quality improvements are then similar for all the products i except for the time of the start of these investments.

$$
\begin{aligned}
&q_i^{homo}(t) = \\
&= \frac{\gamma^2}{(r+\beta)(r+2\beta)\beta} \times \\
&\times \Big(\beta(e^{-(r+\beta)(t_i(0)-\tau-t)} - e^{(r+\beta)(t-t_i(0)-\tau)}) - (r+2\beta)(e^{-\beta(t-t_i(0))} - 1) \Big), \\
(18) \quad & t_i(0) + \tau > t \geq t_i(0).
\end{aligned}
$$

This expression is derived from (15) with the constant investment efficiency and quality decay values. To fully define quality evolution paths one has to compute the variety expansion path and time of emergence for every product. This is doe further on. However, consider the shape of dynamics of quality innovations at the Figure 1. Here the function for time of emergence is already inserted intot he expression (18). It can be seen, that the maximal level of quality development is identical across products for homogeneous version of the model. At the same time, for every product i the technology is developed only during the duration of its life-cycle (patent) and is zero afterwards.

For this illustration as well as further on in the paper the following set of numerical values is used (rather arbitrary):

$$
\begin{aligned}
&\alpha = 0.5, \\
&\beta = 0.1, \\
&\gamma = 0.7, \\
&r = 0.05, \\
&N = 100, \\
&n_0 = 0.
\end{aligned}
\tag{19}
$$

For Figure 1 the length of the patent is taken equal to 1.

Since for homogeneous case investment efficiency and quality decay rates are constant, they are also constant for the boundary product $i = n(t)$ and the value $V^{pat}(0,\tau)_{n(t)}$ does not in fact depend on $n(t)$, since $\beta_{n(t)} = \beta, \gamma_{n(t)} = \gamma$ in the expression (16). In fact, this value is constant for any product i and depends only on the length of the patent:

$$
\begin{aligned}
&V^{homo}(0,\tau)_{n(t)} = \frac{\gamma^2}{r\beta(r+2\beta)(r+\beta)^2} \times \\
(20) \quad &\times \Big(r(r+2\beta)e^{-(r+\beta)\tau} - \frac{1}{2} r\beta e^{-2(r+\beta)\tau} - (r+\beta)^2 e^{-r\tau} + \frac{1}{2}\beta(r+2\beta) \Big).
\end{aligned}
$$

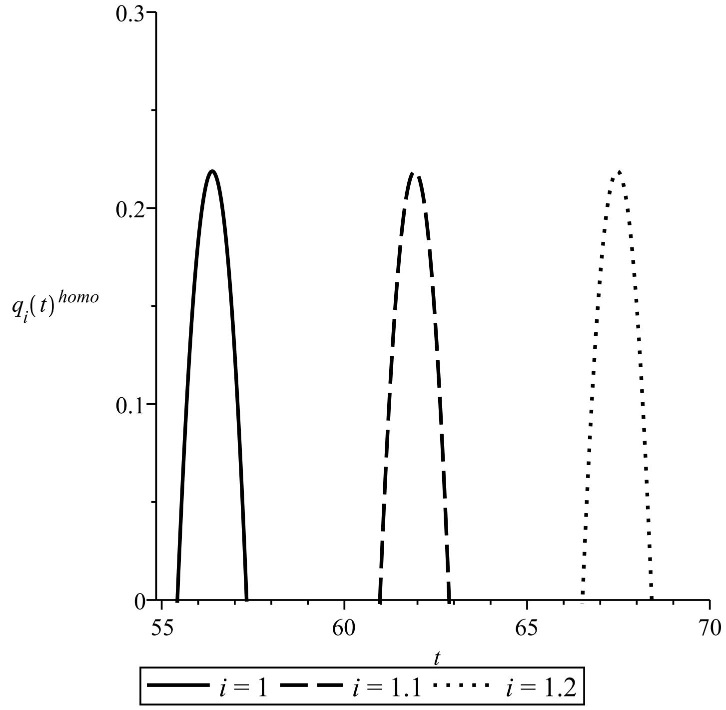

Figure 1: Quality innovations for homogeneous products with limited life cycles

Substitution of this expression into the (4) yield the form of the value function for variety expansion of the homogeneous model. One may then construct the Hamilton-Jacobi-Bellman equation for this value function as follows:

(21)
$$rV^{homo}(n) = \max_{u(\bullet)}\Big\{\alpha u(t) \times V^{homo}(0,\tau)_{n(t)} - \frac{1}{2}u(t)^2 + \alpha u(t) \times \frac{\partial V^{homo}(n)}{\partial n}\Big\}.$$

Assuming quadratic form of the value function for this problem one have first order condition for the optimal control which depends on value function for quality problem:

(22)
$$u^{homo}(t) = \alpha\Big(V^{homo}(0,\tau)_{n(t)} + 2Cn(t) + F\Big)$$

with $V^{ass}(n) = Cn(t)^2 + Fn(t) + E$. Note, that in this part of the problem coefficients are not time-varying, since variety expansion take place in the infinite time.

The system of algebraic equations for coefficients C, F, E is solved in the standard way: insert expression for $u^{homo}(t)$ into the (21) above and regroup coefficients at equal powers of $n(t)$. Hence one arrives to the system of 3 equations with three unknown coefficients, which has a straightforward solution. Substitution for these coefficients into the first order condition (22) yields optimal investments into variety expansion process as a function of $V^{homo}(0, \tau)_{n(t)}$ only, since for this homogeneous case the value function of variety expansion is constant (coefficients C, F are zero) and thus optimal investments into variety expansion are also constant:

$$u^{homo}(t) = \alpha \times V^{homo}(0, \tau)_{n(t)} = const. \tag{23}$$

Then dynamic constraint (5) yields the first-order ODE for $n(t)$:

$$\dot{n(t)} = \alpha^2 \times V^{homo}(0, \tau)_{n(t)} \tag{24}$$

which results in the linear growth of variety:

$$n^{homo}(t) = \alpha^2 \times V^{homo}(0, \tau)_{n(t)} t + n_0. \tag{25}$$

With such law of motion variety expansion paths for different initial ranges n_0 never converge and form different steady states of the whole system. This is illustrated on the Figure 2 (all other parameters held the same as in (19)).

The last point which is necessary to obtain the full characterization of dynamics of the model is the emergence time, $t_i(0)$ for all products $i \in \mathbf{N}$. This time is just an inverse function of variety expansion process, since it is defined from the condition $i = n(t)$ in each case. It is calculated by substitution i for n and $t_i(0)$ for t into the variety expansion and then finding the inverse. Formally:

$$t_i(0) : i \to f(i);$$

$$f(i) = n^{homo}(t)^{-1}|_{n=i};$$

$$t_i^{homo}(0) = \frac{i - n_0}{\alpha^2 V^{homo}(0, \tau)_{n(t)}}. \tag{26}$$

This is just a linear function of the position of the product i relative to the initial range n_0. It does not depend on the maximal available range (industry capacity), N. With this emergence time at hand one may fully define the quality evolution path for each of the products. It thus may be seen, that the process of quality innovations in homogeneous case depends on the position

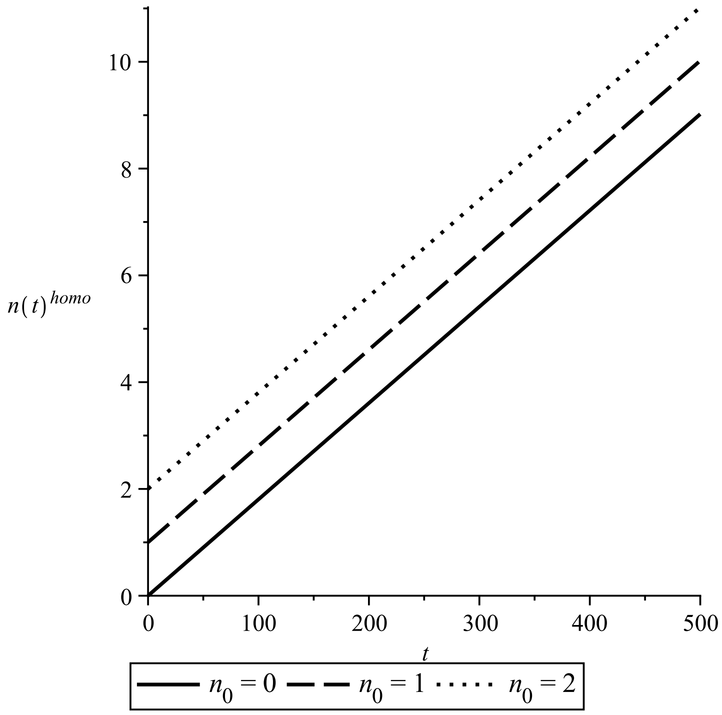

Figure 2: Products variety expansion: homogeneous case

of the product in the available range $n(t)$. However, this dependence affect only the time of the start of investments, but not their intensity. As an effect, all the products are developed up to the same level of quality, as it is demonstrated by Figure 1. Formally one may compute the maximal level of quality by maximizing the function of quality growth w. r. t. to time t and then calculating the associated level of quality. This last turns to be independent on the product's position i:

$$q_i^{hom\bar{o}}(t) = \frac{\gamma}{(r+2\beta)(r+\beta)\beta}\times$$

$$\times\left(\beta + r - \left(\frac{(2\beta+1)e^{(r+\beta)\tau} - \beta}{r+\beta}\right)^{\frac{r+\beta}{r+2\beta}}\beta e^{-(r+\beta)\tau} - \right.$$

$$\left. - \left(\frac{(2\beta+1)e^{(r+\beta)\tau} - \beta}{r+\beta}\right)^{-\frac{\beta}{r+2\beta}}\left(r + 2\beta - \beta e^{-(r+\beta)\tau}\right)\right) \tag{27}$$

Since β, γ are constant and do not depend on the product, the quality development is similar for all the products. One may conclude, that the homogeneous version of the model with limited life-cycles has rather simple structure and does not allow for any non-monotonic effects: range of products in the industry is growing linearly with constant speed, thus giving birth to a constant portion of new quality improving processes at any time t; every product's technology is developed within the life-cycle in the same way as for all other products.

Proposition 1 *With homogeneous products variety expansion process has constant speed, and process innovations are of the same scale and shape for all the products.*

Essentially such a structure may be well described by a two dimensional process, as all the products are the same in their characteristics. However this is not the case for heterogeneous products as it is discussed further on.

3.2 Heterogeneous solution

Now consider the case with decreasing efficiency of investments into quality improving innovations and constant decay rates, denoted $\gamma_{heter}, \beta_{heter}$ in (17). In such a case all the products are different with respect to the process innovations. As a result, maximal quality levels are different and variety expansion process is no longer linear. Effective product's range is non-monotonic and decreases at mature stage of industry development. This case has been considered in more details in the other paper on the subject, (?).

Solution for quality improvements is now looking like:

$$\begin{aligned} & q_i^{heter}(t) = \\ & = \frac{\gamma^2 \times (N-i)}{(r+\beta)(r+2\beta)\beta} \times \\ & \times \left(\beta(e^{-(r+\beta)(t_i(0)-\tau-t)} - e^{(r+\beta)(t-t_i(0)-\tau)}) - (r+2\beta)(e^{-\beta(t-t_i(0))} - 1) \right) \\ (28) \quad & t_i(0)+\tau > t \geq t_i(0). \end{aligned}$$

This differs from the homogeneous solution by the term $(N-i)$ and thus depends not only on the time of emergence of the product i, but also from the position of this product in the products space relative to the maximal industry capacity N. Hence, the higher is the position of the product, the more complicated the product is and the more difficult it is to develop its quality. Thus the maximal achievable quality decreases with the index of the product. This is illustrated by Figure 3 with the same parameter settings as before.

It has to be noted, that every next product has slightly lower maximal

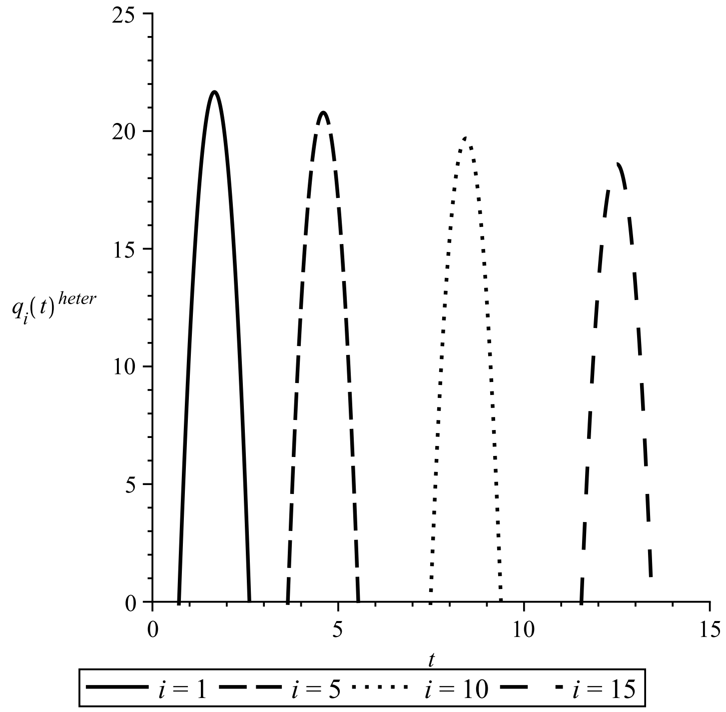

Figure 3: Quality innovations under limited life-cycles for heterogeneous products i

quality, then all the preceding products. This can be seen from the form of the solution (28): the greater is the index of the product i, the lower is its quality:

$$\frac{\partial q^{heter}(i,t)}{\partial i} < 0. \tag{29}$$

For displayed on Figure 3 i values it also can be observed with sufficient changes of i. Note the difference in the density of quality-improving process between homogeneous and heterogeneous versions of the model: with the same parameter settings in heterogeneous case the range of products up to $i = 10$ is developed during $t = 10$, while for homogeneous version the range of $i = 1$ is developed only up to time $t = 60$! The same striking difference is in the maximal attainable levels of quality for every product. In homogeneous version there is no quality decay from product to product, but the maximal level is around 0.2, while for heterogeneous version of this model the maximal attainable level of quality for $i = 1$ is above 20, although declining from product to product.

Proposition 2 *In heterogeneous case process innovations decrease in their maximal level with the increase of the index of the product i, but are (almost) always higher, then for homogeneous case.*

It is the heterogeneity of the space of products which boosts innovative activity of the agent. However I do not claim that *any* form of heterogeneity will boost innovative activity in comparison to the homogeneous case. However the comparison of these two models is sufficient to claim that the form and speed of the dynamics of innovations strongly depends on the level and form of heterogeneity of products characteristics in the industry. This result is in line with (**?**), where the same result is obtained for the model with finite time horizon for both types of innovations.

Proposition 3 *The form and scale of heterogeneity of products characteristics determines the shape and size of quality innovations as an aggregate process.*

The final step of the solution of quality growth problem is the calculation of the value function. For heterogeneous version this value function $V^{pat}(0, \tau)_{n(t)}$ depends on $n(t)$, since the investments efficiency for the boundary product $i = n(t)$ depends on $n(t)$:

$$V^{heter}(0, \tau)_{n(t)} = \frac{\gamma^2 (N - n(t))}{r\beta(r + 2\beta)(r + \beta)^2} \times$$

$$\times \Big(r(r + 2\beta)e^{-(r+\beta)\tau} - \frac{1}{2} r\beta e^{-2(r+\beta)\tau} - (r + \beta)^2 e^{-r\tau} + \frac{1}{2}\beta(r + 2\beta) \Big). \tag{30}$$

Denote

$$V(\tau) = \frac{V^{heter}(0, \tau)_{n(t)}}{(N - n(t))} = \frac{\gamma^2}{r\beta(r + 2\beta)(r + \beta)^2} \times$$

$$\times \Big(r(r + 2\beta)e^{-(r+\beta)\tau} - \frac{1}{2} r\beta e^{-2(r+\beta)\tau} - (r + \beta)^2 e^{-r\tau} + \frac{1}{2}\beta(r + 2\beta) \Big). \tag{31}$$

This does not depend on $n(t)$, but only on the length of the patent, τ as in homogeneous case.

For heterogeneous products the HJB equation for variety expansion problem takes the form:

(32)

$$rV^{heter}(n) = \max_{u(\bullet)}\Big\{\alpha u(t) \times V(\tau) \times (N - n(t)) - \frac{1}{2}u(t)^2 + \alpha u(t) \times \frac{\partial V^{heter}(n)}{\partial n}\Big\}.$$

Assuming the same quadratic form of the value function for this problem as before ($V^{ass}(n) = C^{heter}n(t)^2 + F^{heter}n(t) + E^{heter}$) one has first order condition for the optimal control which depends on value function for quality problem:

$$u^{heter}(t) = \alpha\Big(V(\tau)(N - n(t)) + 2C^{heter}n(t) + F^{heter}\Big) \tag{33}$$

Note, that in this part of the problem coefficients are again not time-varying and hence one may find them from the system of 3 algebraic equations in the same way, as in the homogeneous case above. Substitution for these coefficients into the first order condition (33) yields optimal investments into variety expansion process as a function of $V(\tau), n(t)$:

$$u^{heter}(t) = \frac{2\alpha r(N - n(t))V(\tau)}{r + \sqrt{4\alpha^2 rV(\tau) + r^2}} \tag{34}$$

The associated $n(t)$ dynamics is more complicated then for homogeneous case:

$$\begin{aligned} \dot{n(t)} &= \frac{2\alpha^2 rV(\tau)}{r + \sqrt{4\alpha^2 r \times V(\tau) + r^2}}(N - n(t)) \\ n(0) &= n_0. \end{aligned} \tag{35}$$

This IVP is linear in n and has the closed-form solution:

$$n^{heter}(t) = N + e^{-\frac{2\alpha^2 rtV(\tau)}{r+\sqrt{4\alpha^2 rV(\tau)+r^2}}}(n_0 - N) \tag{36}$$

Observe that instead of linear growth as in homogeneous case, one has exponential growth here, which is (almost) always higher. The shape of dynamics is illustrated for heterogeneous case by Figure 4.

The shape of dynamics of variety expansion is different from homogeneous case and demonstrates convergence of evolution paths with different initial ranges. Compare the intensity of the process with those of the homogeneous model above, at Figure 2. It is important to note, that in heterogeneous case the dynamics is similar in its shape to the infinite-time horizon model of (?), which is the natural limiting case of the finite-time life-cycles model with heterogeneous products considered in this section.

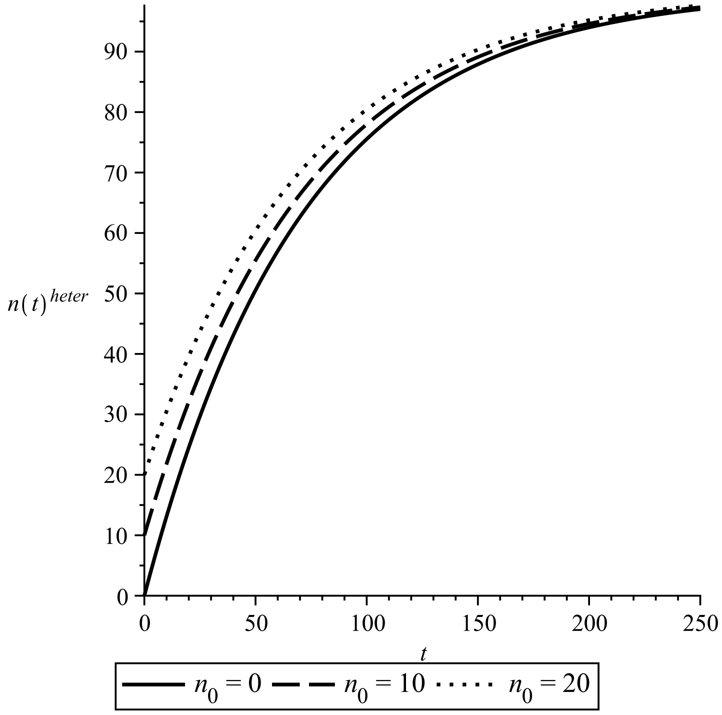

Figure 4: Products variety expansion for heterogeneous products with limited life-cycles

Proposition 4 *For heterogeneous products variety expansion process is converging w. r. t. initially known products range to the same limiting process. The case with $\tau \to \infty$ is a limiting case and coincides with unconstrained monopolist problem.*

In the same way as for homogeneous version of the model one may now define the time of product's emergence as an inverse function of variety expansion:

$$t_i(0) : i \to f(i);$$

$$f(i) = n^{heter}(t)^{-1}|_{n=i};$$

$$t_i^{heter}(0) = -ln\Big(\frac{N-i}{N-n_0}\Big) \times \frac{r + \sqrt{4\alpha^2 rV(\tau) + r^2}}{2\alpha^2 rV(\tau)} \tag{37}$$

This function, demonstrated on Figure 5, shows, that the higher is the index of a product, i, the more time is needed for the introduction of the next product after this one. This is the direct consequence of the slowing rates of variety expansion, as Figure 4 shows. As a result, *density* of quality

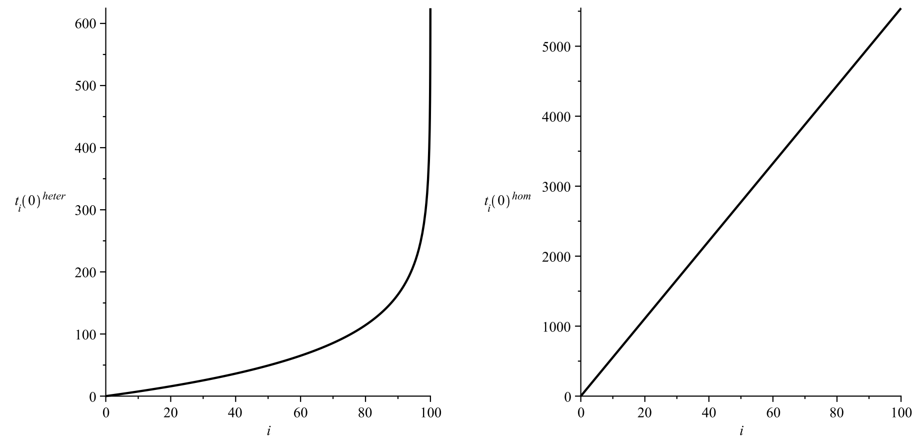

Figure 5: Time of emergence as function of product position in the product's space.

innovations, $q_i(t)$, is decreasing with time, as Figure 3 shows: the distance between evolution paths of technologies for products in the beginning of the products range is shorter, then in the end of it. Note, however, that this is not the case for homogeneous model, where products emergence intensity is constant and does not change over time.

4 Analysis of the model

Now one may analyse the effect of the length of the life cycle on the model for both homogeneous and heterogeneous products. As it will be discussed further on this effects are quiet different.

4.1 Effects of the life-cycle on variety expansion process.

First consider effects of the length of life-cycles on the variety expansion. For this I introduce the notion of effective range of products. This is the range of products which qualities are being developed or supported by investments of the agent at a given moment of time. In the case of unlimited life-cycles this range coincides with the total developed range of products or with products variety in the industry. At the same time, with limited life-cycles at any moment only those products, which have been introduced into the industry not earlier then τ time ago are still under control of the agent. Define:

$$n_\tau = n(t) - n(t-\tau). \tag{38}$$

as this effective range. As it has been said, the case $\tau \to \infty$ coincides with the infinite-time horizon monopolist with unlimited life-cycles. For this case $n_\tau = n(t)$.

A priori one may observe two opposite effects of the life-cycles on the process of variety expansion. The first one should be negative: the shorter is the length of patent (life-cycle), the lesser is the range of products effectively at the agent's disposal at each point in time. Then to maximize the range of products under control at each point in time the agent should invest more in variety expansion with shortening patent length. This effect is denoted as *compensation effect*, because the agent has to compensate the decrease in the effective product's range with additional investments into variety expansion.

At the same time shorter length of the life-cycle limits agent's opportunities to develop products' qualities and thus, decreases incentives to develop new products. This effect is referred to as *potential profit effect*, as it is the changes in potential profit expected from new product, which creates it.

To formally define these two effects, consider first the derivative of the effective product's range with respect to the length of the life-cycle, which includes both effects:

$$\frac{\partial n_\tau}{\partial \tau} = \frac{\partial n^{pat}(t)}{\partial \tau} - \frac{\partial n^{pat}(t-\tau)}{\partial \tau} = PPE + CE. \tag{39}$$

The first effect denotes the expansion of the products range as a response to the increase in patent's length, while the second denotes the increase(decrease) in the effective range n_τ as a result of the changes in the lower bound of this range.

Direct computation of the derivative above for homogeneous and heterogeneous cases reveals, that for homogeneous case the total effect of the patent's length on the effective range is constant:

$$\begin{aligned}
\frac{\partial n_\tau^{homo}}{\partial \tau} &= \alpha^2 \times V^{homo}(0,\tau)_{n(t)} \\
&= \frac{\alpha^2\gamma^2}{r\beta(r+2\beta)(r+\beta)^2} \times \Bigg(\beta r((\beta+r)\tau - \frac{1}{2})e^{-2(r+\beta)\tau} + (r+\beta^2)(r\tau-1)e^{-r\tau} \\
&\quad - r(r+2\beta)((r+\beta)\tau-1)e^{-(r+\beta)\tau} + \frac{1}{2}\beta(r+2\beta)\Bigg).
\end{aligned} \tag{40}$$

From this it can be seen that the response of the effective range of products in homogeneous case is function of parameters and the patent's length itself, but not of time. It can be demonstrated that as a result the effective range positively depends on the length of the patent all the time for homogeneous case. Figure 6 demonstrates the positive dependence of the effective range on patent's length in this case. The same list of parameters as before is used, (19).

This effect is constant in time, since variety expansion for homogeneous

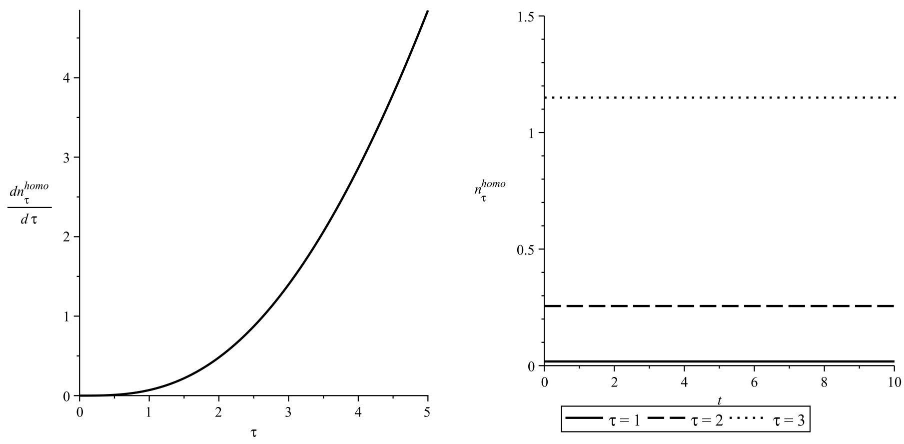

Figure 6: Response of n_τ on patent's length, homogeneous case

products is a linear function of time: both effects, *PPE* and *CE* depend on time by themselves, but the difference between them is always constant. Hence, one may conclude:

Proposition 5 *For homogeneous case effective range is positively affected by the increase in the length of life-cycles and this influence is constant during all stages of the industry development, $PPE - CE = const$*

In heterogeneous case the effect is more complicated and depends on time. The derivative (39) is for this case given by:

$$\frac{\partial n_\tau^{heter}}{\partial \tau} = -\frac{2\alpha r^2(r + \sqrt{4\alpha^2 rV(\tau) + r^2} + 2\alpha^2 V(\tau))}{\sqrt{4\alpha^2 rV(\tau) + r^2} \times (r + \sqrt{4\alpha^2 rV(\tau) + r^2})^2} \times$$

$$(N - n_0)\frac{dV(\tau)}{d\tau} e^{-\frac{2\alpha rV(\tau)}{r+\sqrt{4\alpha^2 rV(\tau)+r^2}}t}((t-1)e^{\frac{2\alpha rV(\tau)\tau}{r+\sqrt{4\alpha^2 rV(\tau)+r^2}}} - 1 - t). \quad (41)$$

The sign of this derivative depends on the sign of expression

$$\mathbf{A} = ((t-1)e^{\frac{2\alpha r V(\tau)\tau}{r+\sqrt{4\alpha^2 r V(\tau)+r^2}}} - 1 - t). \tag{42}$$

This last may be positive or negative depending on relative size of the value function $V(\tau)$. It depends on the length of the life-cycle, τ. For longer life-cycles it is greater then one and the subsequent expression (42) is positive for almost all t's, yielding negative derivative sign, for shorter life-cycles it is negative for most t's, yielding positive derivative sign. Observe also that since this expression depends on time there is always some initial period when it is negative for $t \to 0$ and always positive for $t \to \infty$. In effect this means that changes in patents' length may influence the effective range of products in different directions. This last phenomena is illustrated in Figure 7.

The effective product's range is first increasing with patent's length, but

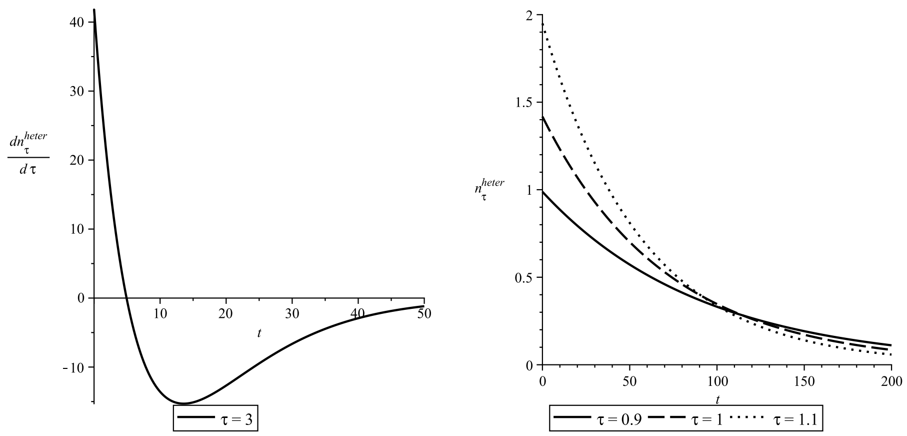

Figure 7: Response of n_τ on patent's length, heterogeneous case

afterwards it decreases. This point to the fact that this effective range is subject to effects of the length of the life-cycle.

It can be seen from the Figure 7, that as length of the life-cycle grows, the effective range is growing at initial stage and decreases afterwards. The time at which the compensation effect outperforms the potential profit effect does not depend on the parameter τ and is defined from the very form of the variety expansion process.

Economic intuition behind this result is clear: at initial stage of development there are a lot of opportunities to develop new versions of the basic product for the monopolist. Hence, increase in the length of the life-cycle gives him/her higher possibilities to develop all these new products and derive profit from them. Thus potential profit effect is high. At the same time the increase in this length affects negatively the rate of investments into variety expansion, but this effect is not significant. As time flows, more products are introduced into the market, but effective range decreases, as the variety expansion process slows down. At this point the effect of expected profit from all of the new products in the additional range from the increase in the length of patent wears down, as there is lesser mass of products in this range. Simultaneously it becomes less important to sustain the given effective range, as it increases from the length of the patent, hence compensation effect is larger. Al this is summarized in the following Proposition:

Proposition 6 *For heterogeneous case effective range of products grows with increase in the length of the patent while industry is in early stage of development, $n(t) << N$, but decreases after the industry enters mature stage of its development, $n(t) \to N$.*

It has to be noted, that this does not mean that the variety expansion process may negatively depend on the length of the life-cycle. The process of variety expansion is always boosted by the increasing length of the life-cycle and infinite-time patent is a limiting case for this dynamics, as it is shown in (**?**). This happens because the effective products range is a characteristic of a speed of variety expansion, not of its overall level. Thus with longer life-cycles the level (stock) of products variety is always increasing, while the rate of their introduction and as a result, the effective patented range, not always increases but only at the initial stage of development while decreasing afterwards.

Compare the dependence on the length of the life-cycle of the overall variety expansion process for homogeneous and heterogeneous cases. The PPE is positive for both cases of the model and depends on time. Thus, variety expansion processes are stimulated by the growth of the length of the patent. However, this influence is different, as Figure 8 illustrates.

In homogeneous case the rise in the length of patent creates diverging paths of variety expansion and the stimulus is much bigger in comparison to the heterogeneous case. In this last variety expansion processes are still converging to the same maximal variety, defined by the industry capacity N. This is not necessary the case for homogeneous process, which, as a linear process, is unbounded from above.

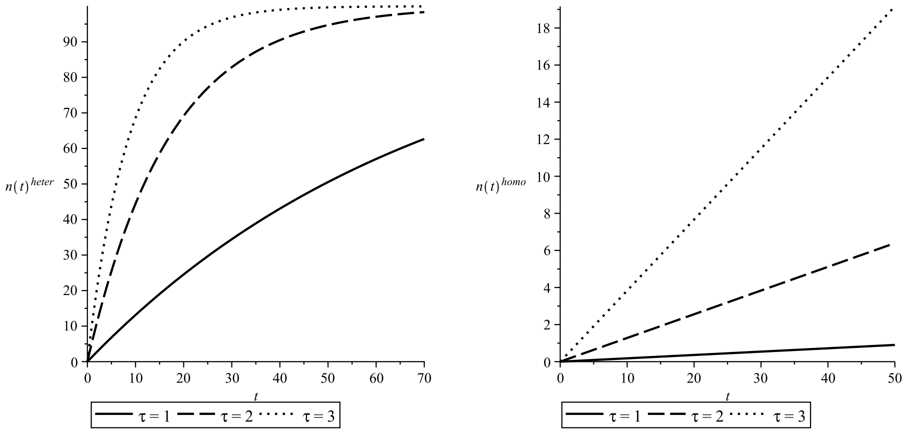

Figure 8: Response of $n(t)$ on patent's length in both cases

4.2 Effects of the life-cycle on the quality innovations

Quality growth essentially depends on the length of the life-cycle of products also. The longer the life-cycle, the closer patent's model quality dynamics is to the infinite-time one. The quality growth displays only the potential profit effect as long as one consider single product quality investments: the longer the life-cycle of the product, the higher is the maximal attainable quality of this product and thus the higher is the expected stream of profits from the development of this product. Hence,

$$\frac{\partial q_i^{pat}(t)}{\partial \tau} > 0. \tag{43}$$

This can be checked by directly computing the derivative of (15) w.r.t. to τ. There is no ambiguity in the effect of the length of the life-cycle onto the development of every separate product i from the effective range of products.

However there are two different effects on the aggregate level of quality development. Observe that at any given time t there is a mass of products under the control of the innovator which qualities might be developed. This mass is given by the effective products range defined above. The level of aggregate (across products) quality development is then given by the quantity, denoted $\mathbf{Q}$:

$$\mathbf{Q} = \int_{n(t-\tau)}^{n(t)} q_i(t)di. \tag{44}$$

I proceed in the same fashion as for the case of variety expansion: calculate the derivative and decompose it. For this usual rules of integration and derivative w.r.t. to the parameter are used. Expressions for this case are very cumbersome and not displayed. The quantity being computed is:

$$\frac{\partial \mathbf{Q}}{\partial \tau} = \int\limits_{n(t-tau)}^{n(t)} \frac{\partial q_i^{pat}(t)}{\partial \tau} di. \tag{45}$$

where I make use of the interchange of the order of differentiation and integration due to Fubini's theorem (and its extensions).

This quantity behaves differently for homogeneous and heterogeneous cases of the model. For homogeneous case the relation is monotonic in its sign, since qualities of all introduced products are being developed till exactly the same level and are essentially the same. This is different in heterogeneous case, where qualities decline along the development of the products range. Hence the same non-monotonic effect as for effective range of products above is observed.

In homogeneous version of the model the aggregate quality innovations do not grow over time for every fixed patent length. This is the direct consequence of similar quality levels for all the products. At the same time, increase in the length of the patent increases this overall quality at each point in time. The responsiveness of quality innovations as an aggregate to patent changes is decreasing over time but always remain positive. Figure 9 illustrates this behaviour of the model.

Observe, that the aggregate quality level as well as responsiveness are

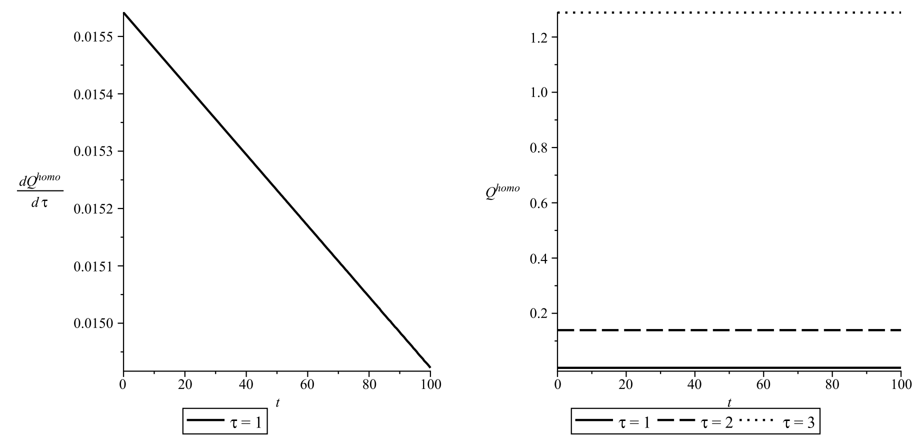

Figure 9: Response of Q on patent's length for homogeneous case

very low. This illustrates an important insight: in homogeneous industry the agent does not have a lot of incentives to increase the quality of a given product, since all the products are similar and he/she may invest in any of them without loss of efficiency and potential profit. At the same time the intensity of new products introduction into the market is also rather low, thus making the aggregate quality innovations process less dense: at any given point in time there are quit a few ongoing quality innovations processes and their intensity is also low. This creates the low level of aggregate innovations.

The situation is different in heterogeneous case. The dynamics of aggregate quality innovations follows the same pattern, as thos for variety expansion (effective range) above: at initial stage of industry development the increase in patent's length boosts quality innovations for all products in existence, but this effect wears down and at mature stage the effect is negative: the longer is the patent, the lower is the aggregate level of quality innovations. This is illustrated by Figure 10.

The derivative sign is initially positive but changes to negative at the

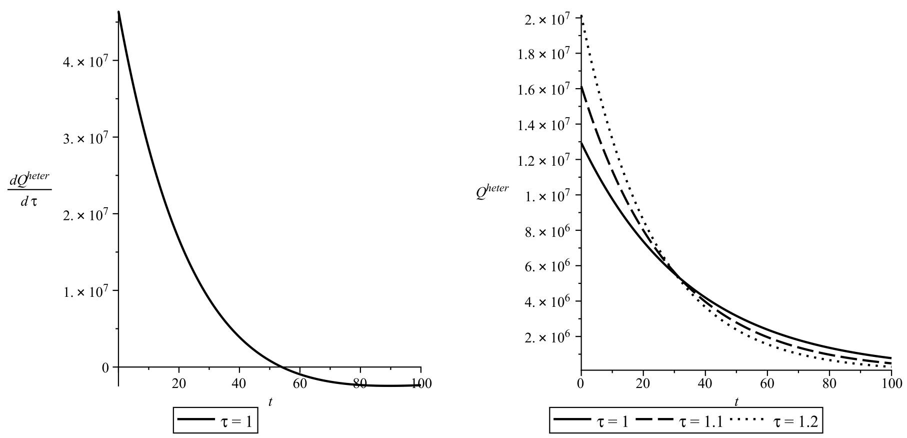

Figure 10: Response of Q on patent's length for heterogeneous case

mature stage of products range development (for higher t). The compensation effect for qualities influences the range of the integral itself, that is, the total quantity of quality improving innovations.

Indeed the total effect of changes in patents length on the overall qualities development is defined from 2 sources: potential profit effect for quality of every single product within the effective range and by the scale of the effective range itself. It is known from above discussion, that this effective range for heterogeneous products tends to shrink along increase of the life-cycle's length for mature stages of development; thus the total range of quality investments shrinks also. The effect of the range's length outweighs the effect of potential profit for every single product and thus the overall behaviour of the $\mathbf{Q}$ is determined by the changes in effective range of products.

In conclusion one may claim, that the influence of the length of the life-cycle on the behaviour of the model differs substantially between homogeneous and heterogeneous cases, being monotonic in the first case and of changing sign in the second. Quality innovations for separate products are unambiguously stimulated by the increase in products life-cycle, but the aggregate quality innovations have the same type of behaviour as variety expansion. The reason for this is that one have numerous quality innovations and each of them has zero mass in the total aggregate, but the effective range under the control of the agent affects all of these innovations as a whole, while not affecting each separate product.

Proposition 7 *The aggregate behaviour of quality innovations w. r. t. changes in patents length is defined by the dynamics of effective range of products, while each separate quality innovation is not affected and is stimulated by the increase in the patent.*

5 Discussion

In this paper I developed a dynamic model of a single agent which is engaged in innovating activity in some industry. The product innovations are governed by th process of products variety expansion and has infinite time horizon. At the same time,every introduced products has a stream of process (or quality) innovations associated with it. These ones are time-limited, as every product has a limited life-cycle within which improvements to its quality may be made. I consider mainly the role of heterogeneity of products characteristics in the effect of the length of this life-cycles on the development of both types of innovations.

To this end homogeneous investment efficiency and heterogeneous one have been considered. It turns out, that under homogeneous investment characteristics of products the influence of the length of the life-cycle on the development of innovative activity is very straightforward: the longer is the life-cycle, the higher is the intensity of variety expansion as well as quality improving innovations for all the products. The same is true for heterogeneous products characteristics (decreasing efficiency of investments across products).

However, the main difference lies in two other introduced concepts: the effective range of products and the aggregate measure of quality innovations across all the existing products. These two reflect the influence of the length of the life-cycle on those products and their development, which are currently in existence for each point in time. It turns out, that for heterogeneous case the dynamics of these two measures is not monotonic: the increase in the length of the life-cycle does not necessary lead to the increase in innovations of both types. This is not true for homogeneous products, where the increase in the length of the life-cycle increase both the effective range and aggregate quality innovations.

It is claimed, that such a difference is a direct consequence of the heterogeneity of products characteristics being introduced into the model. If one would treat the life-cycle of the product as some notion of the patent, then only in heterogeneous case there are grounds totally different from usual competition arguments for the limited time of these patents. Although the stream of innovations is stimulate by longer patents with infinite time length being the limiting case, the range of products actually in existence as well as the overall aggregate quality of these products may be negatively affected by longer patents when the industry is in the mature stage of development (the majority of products are already developed and the products space is dense). In such a case a decrease in patents length would stimulate both the effective range of products and their qualities as a whole although depressing somewhat quality improving processes for every single product. This cannot be claimed for homogeneous products, where the effective range of products remains constant over time and all the products are essentially the same in their qualities. However the heterogeneous case seems to be more realistic, since there are no two completely similar products in any given industry. otherwise they may be treated as a single indistinguishable product.

The main message of the paper is clear: if one would account for heterogeneity of investment characteristics of products and also the dynamic link between aggregate product and quality innovations, one may obtain theoretical ground for the decrease in patenting times without reference to competitive pressure but following purely technological constraints of an innovator himself. At the same time this is not the case for homogeneous industry.